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Springer-Verlag, 175 Fifth Avenue, New York, NY 10010/USA

ISBN 3-540-**08443**-6
ISBN 0-387-**08443**-6

Lecture Notes in Economics and Mathematical Systems
Managing Editors: M. Beckmann and H. P. Künzi

Operations Research

151

Christoph A. Schneeweiß

Inventory-Production Theory

Springer-Verlag
Berlin Heidelberg New York

and Mathematical Systems

:h and Mathematical Economics, Vol. 16–59: Lecture
ʒystems)

der Informationstheorie. VI, 116 Seiten. 1969.

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Lecture Notes in Economics and Mathematical Systems
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Operations Research

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Inventory-Production Theory

A Linear Policy Approach

Springer-Verlag
Berlin Heidelberg New York 1977

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Library of Congress Cataloging in Publication Data

Schneeweiss, Ch
Inventory-production theory.

(Lecture notes in economics and mathematical systems ; 151 : Operations research)
Bibliography: p.
Includes index.
1. Production planning--Mathematical models.
2. Inventory control--Mathematical models. 3. Linear programming. I. Title. II. Series: Lecturenotes in economics and mathematical systems ; 151.
TS176.S36 658.7'87 77-13254

AMS Subject Classifications (1970): 90-02, 90B30, 49C99

ISBN 3-540-08443-6 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08443-6 Springer-Verlag New York Heidelberg Berlin

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2142/3140-543210

P R E F A C E

This treatise tries to give a coherent account of part of the work that has been done during recent years at the "Fachrichtung Operations Research" of the Free University Berlin. It is dealing with what shall be called the "linear policy approach" to stochastic inventory-production problems: The treatise investigates the efficiency of certain types of inventory-production models in which the optimal non-linear ordering or production policy is replaced by a linear one.

A linear policy not only reduces the computational burden considerably, but also allows to a certain extent for the existence of dynamic certainty equivalents. I.e., roughly speaking, the optimal production policy remains optimal if one replaces the stochastic sequence of demand by its forecasts. Thus, if for a stochastic dynamic optimization problem only forecasts are available the question arises whether a linear policy can be shown to be more adaquate than a non-linear policy.

Part of this account has been written during the author's stay at the University of Cambridge. He is particularly grateful for the hospitality of the Control Engineering Group of the University Engineering Department and to Clare Hall, the college to which the author was an Associated Member during winter 1974/75.

Furthermore the author wants to express his gratitude to the members of the Fachrichtung Operations Research of the Free University of Berlin for numerous stimulating discussions. Special acknowledgements are due to Dr. Karl Inderfurth for many valuable comments and to Mrs. E. Weber who did the irksome task of typing the manuscript.

Berlin, März 1977

Ch. Schneeweiß

INVENTORY-PRODUCTION THEORY

A linear Policy Approach

INTRODUCTION

The term inventory-production theory is not well defined. It comprises e.g. such models like cash balance models, production smoothing models and pure inventory models. We shall here mainly be concerned with stochastic dynamic problems and shall give exact definitions in the next section. Most of our work will concentrate on cash balance models. However, production smoothing situations and pure inventory problems will also be investigated.
Since we are faced in principle with dynamic stochastic situations a dynamic programming approach would be appropriate. This approach, however, due to computational restraints, is limited to only but the simplest models. Therefore, in practice, one ruduces stochastics just in taking forecasts of demand and then treating the problem as a deterministic optimization problem. In addition one often introduces certain safety stocks to safeguard the system from possible forecasting errors. In general, this procedure is suboptimal. However, there exists one particular situation when a separation in a forecasting procedure and a subsequent optimization of the remaining deterministic model is not suboptimal. This is known as the linear-quadratic model, i.e. a model having linear system equations and a quadratic cost criterion. For this type of model H. A. Simon [23] and later H. Theil [25] have shown that the above separation property holds. In fact, Simon's and Theil's results are nothing else but what has later and more generally become known to control engineers as Kalman's famous separation principle. The above forecasts, being minimum mean square, Theil called dynamic certainty equivalents.

Thus we shall first be concerned in Chap.2 with linear-quadratic models and shall show the certainty equivalence property in applying dynamic programming. The linear-quadratic approach has been applied by Holt et al. [7] to problems in production and workforce smoothing. However, in most real situations cost dependences will not be quadratic. Fitting quadratic functions to nonquadratic costs could therefore lead to seriously suboptimal results.

Consequently we shall develop in Chap.3 a theory of linear-non-quadratic models. For these models it will be shown that in the main the separation theorem still holds. However, the system has still to be linear which particularly has as a consequence that we can only deal with linear production policies. Thus this approach will in general only be an approximation to the optimal case for which, of course, also non-linear decision rules have to be taken into account.

The effect of the restriction to linear decision rules will be studied in detail in Chap.4. Uncorrelated and exponentially correlated sequences of demand will be investigated and we shall particularly consider the important case of cost dependences having production set-up costs.

As has already been mentioned above the linear-nonquadratic approach allows to a certain respect for the existence of dynamic certainty equivalents. Also we know that in general this approach is suboptimal. On the other hand one could replace the stochastic sequence of demand by its forecasts from the outset and then apply a deterministic dynamic programming approach. In fact, it turns out that (for the cases we studied [see Chap. 5]) this approach which is usually to be met in practice is inferior to the linear-nonquadratic approach*). This again indicates that in view of the stochastics involved linear decision rules should seriously be taken into consideration.

Chap.6 finally deals with the pure inventory case. I.e. in this case production is only allowed to take on positive values. The main economic idea is to regard an inventory problem as a particular production smoothing model and to optimize this model using the linear-nonquadratic approach. Our results show that at least in case of no-set up costs the LNQ - approximation is nearly optimal. Thus also in this case taking forecasts and applying a linear decision rule turns out to be a reasonable procedure. Enlarging the inspection period these results hold at least approximately also for set-up costs.

*) Henceforth also called LNQ - approach

Summarizing the results leads us to the following conclusion. The linear-nonquadratic approach allows to a certain respect for the existence of dynamic certainty equivalents thus reducing the information needed mainly to a sequence of demand forecasts (see p. 41). For situations incurring no set-up costs this approach is not only nearly optimal for the cash balance but also for the pure inventory case. The deterministic approach usually applied in practical problems can at least in cases having no set-up costs be considerably more suboptimal than the linear-nonquadratic approach. For models incurring set-up costs non-linearities have to be taken into account more seriously. Depending on the relative size of the cost parameters involved and the variance of the stochastic sequence of demand the LNQ - approach can in many cases still be applied successfully.

Chapter 1

THE GENERAL MODEL

We shall be concerned with the following stochastic dynamic inventory-production model

(1) x_k: stock on hand at the beginning of period k, (k=1,2,...,N+1)

$x_k \in \mathbb{R} \; \forall k$, x_1 : initial stock; N: planning horizon

(2) u_k: production decision at the beginning of period k which results in a shipment in this period

$u_k \in U_k$ (k=1,2,...,N), where (1) $U_k := \mathbb{R}$ (cash balance case)

(2) $U_k := \mathbb{R}_+$ (inventory case)

(3) r_k: stochastic demand in period k. All conditional expectations subsequently used are assumed to exist. $\{r_k\}$ is not to be influenced by the initial value x_1.

Inventory balance equation

(4) $$x_{k+1} = x_k + u_k - r_k \tag{1.1}$$

(5) Cost criterion

(5) $$C = E\left\{ \frac{1}{N} \sum_{k=1}^{N} \{I(x_{k+1}) + P(u_k)\} \middle| x_1 \right\} \Rightarrow \min. \tag{1.2}$$

$I(\cdot)$ and $P(\cdot)$ denoting inventory and production costs respectively.

Remarks

Ad (1) The general linear-quadratic and linear-nonquadratic approach is not restricted to the one dimensional case. However, in the numerical calculations to be reported in subsequent chapters we are confined to a one-comodity-one-inventory situation. Thus $\mathbb{R}$ will generally be $\mathbb{R}_1$.

<u>Ad (2)</u> In most of the treatise we will be concerned with the case in which u_k is not restricted, i.e. u_k is allowed to take any real value. Only in our discussion of a pure inventory problem we will have to restrict u_k to be non-negative.

<u>Ad (3)</u> We shall mainly concentrate on two stochastic demand sequences

(1) White Gaussian noise with mean $E\{r_k\} = 0$ and variance $E\{r_k^2\} = \sigma^2$

(2) Gauss-Markov process with mean $E\{r_k\} = 0$ and covariance $E\{r_k \cdot r_{k+i}\} = \sigma_{rr}^2 \, a^{|i|}, \; i = 0, \pm 1, \ldots, 0 < a < 1.$

<u>Ad (2) and (3)</u>

In most cases discussed below u_k and r_k are considered to be deviations from certain given mean levels.

<u>Ad (4)</u> The balance Equ.(1.1) describes the back-logging case. In one subsection leadtimes will also be taken into account.

<u>Ad (5)</u> In the next chapter Equ.(1.2) will be taken to be quadratic, i.e. $I(\cdot)$ and $P(\cdot)$ will be assumed to be quadratic functions. However, in the chapters to follow costs will be assumed to be piecewise linear and particularly of the form

$$I(x) = \begin{cases} h\,x & \text{for } x \geq 0 \\ -v\,x & \text{for } x < 0 \end{cases} \quad \text{and} \quad P(u) = \begin{cases} P + pu & \text{for } u > 0 \\ 0 & \text{for } u = 0 \\ Q - qu & \text{for } u < 0 \end{cases} \quad (1.3)$$

h, v, p and q being positive parameters describing proportional inventory carrying (h), stock out (v), production (p) and production deminishing (q) costs respectively; moreover P and Q (both positive) denote set-up costs. (In some sections we shall also consider costs not having exactly the structure of (1.2)).

In most of what follows we will be concerned with the asymptotic situation $N \to \infty$. Only the next chapter will consider the finite horizon case.

The situation we basicly have in mind may be described as follows.

There is a production facility producing one good (or an aggregate of several similar goods) which is stored and sold. According to reasons outside the scope of the production and inventory manager a mean production is fixed to meet (long term) mean demand. The task of the production-inventory manager now is to cope with short term sales fluctuations. I.e. the production level has to be chosen higher or lower than the long term "normal" level such that the total costs of the system be minimum. The costs given in (1.3) thus describe a penalty for not being in a long term equilibrium situation.

A cash balance problem would be an adaquate illustration of our main model. In this case one has to pay interests for holding (too much) cash (hx) and penalty interest costs (-vx) for being out of cash. Moreover P(u) would represent linear transfer costs for liquidation ($u > 0$) and investment ($u < 0$) of cash.*) Accordingly $\{r_k\}$ would describe a stochastic sequence of net expenses (of cash).

Not all of the models discussed below will be of this basic structure. In Sec.3.3.3 we shall consider a model taking into account costs for <u>changing</u> a production level: $p|u_k - u_{k-1}|$, whereas costs (1.3) prescribe costs only for being out of an equilibrium situation. Models taking into account the above production changing costs are usually denoted as "pure" production smoothing models. Besides cash-balance and production smoothing models we also shall investigate (pure) inventory problems, where u_k can only take positive values and the costs attached are to be interpretated as ordering costs.

*) In this case the above mentioned mean production would be identically zero.

Chapter 2

THE LINEAR-QUADRATIC MODEL

The main object of this treatise is to investigate the capability of linear decision rules for models defined in the last chapter. As will be shown later (Chap.3) this ultimately amounts to a reduction of the non-quadratic cost structure (1.3) to a quadratic one. Thus it seems to be reasonable to study first the quadratic case in some detail.
This will be done by first investigating a finite horizon model (Sec.2.1). In optimizing this model by dynamic programming we can easily introduce the notion of dynamic certainty equivalence leading in the next section to a discussion of the forecasting problem. Sec.2.3 will then study an example being of some relevance for our later investigations. Finally, Sec.2.4 will discuss the steady state case $N \to \infty$ which is of particular importance for the whole of our approach. This is because the non-quadratic model to be investigated in the next chapter will be shown to be reducible to a steady state quadratic model.

2.1 Finite Horizon Case

Given the general model of chapter 1 with x_k, $u_k \in \mathbb{R}_1$,
N finite and

$$C = E\left\{\frac{1}{N} \sum_{k=1}^{N} (x_{k+1}^2 + \rho^2 u_k^2) \,|\, x_1\right\} \Rightarrow \min, \qquad (2.1)$$

we shall minimize (2.1) using a dynamic programming approach (see e.g. [21]).
Defining a value function $f_k(x^k, r^{k-1})$, where e.g. x^k denotes as usual $(x_k, x_{k-1}, \ldots, x_1)$, Bellman's functional equation is readily given by

$$f_k(x^k, r^{k-1}) = \min_{u_k \in \mathbb{R}_1} E \frac{1}{N+1-k}\left\{x_{k+1}^2 + \rho^2 u_k^2 + (N-k) f_{k+1}(x^{k+1}, r^k) \,|\, r^{k-1}, x_1\right\} \qquad (k=1,\ldots,N) \qquad (2.2)$$

$$f_{N+1}(x^{N+1}, r^N) \equiv 0$$

Solving this equation we start at k = N working back to k = 0.

Step 1 (k = N)

$$f_N(x^N,\ r^{N-1}) = \min_{u_N} E\left\{ (x_N+u_N-r_N)^2 + \rho^2 u_N^2 \ |r^{N-1},\ x_1\right\}$$

$$= \min_{u_N} E\left\{ x_N^2+2x_Nu_N+(1+\rho^2)u_N^2-2(x_N+u_N)\hat{r}_{N-1}^{(1)}+n_N^2\right\} \qquad (2.3)$$

where $n_N^2 \ : = E\left\{ r_N^2 | r^{N-1}\right\}$ (2.4)

and $\hat{r}(1)_{N-1} \ : = E\left\{ r_N | r^{N-1}\right\}$ (2.5)

(Note that according to our above assumption there is no dependence on the initial value x_1)

Determining the optimal decision u_N^* we immediately obtain from differentiating (2.3) with respect to u_N

$$u_N^* = u_N^*(x_N, r^{N-1}) = -\frac{1}{1+\rho^2}(x_N-\hat{r}_N)$$

or (2.6)

$$u_N^* = S_N(x_N+Q_N)$$

where for later reference we defined

$$S_N \ : = -\frac{1}{1+\rho^2} \qquad (2.7)$$

and

$$Q_N \ : = -\hat{r}_{N-1}(1) \qquad (2.8)$$

Determination of $f_N(x^N,\ r^{N-1})$

Substituting (2.6) in (2.3) yields

$$f_N(x_N,\ r^{N-1}) = -S_N\ \rho^2\ x_N^2 + T_N\ x_N + R_N \qquad (2.9)$$

with

$$T_N \ : \ = 2\rho^2\ S_N\ Q_N \qquad (2.10)$$

and

$$R_N \ : \ = S_N\ Q_N^2 + n_N^2 \qquad (2.11)$$

Step 2 (k = N-1)

Bellman's functional equations (2.2) yield

$$f_{N-1}(x^{N-1}, r^{N-2}) = \min_{u_{N-1}} E\,\frac{1}{2}\left\{ (x_N^2+\rho^2 u_{N-1}^2+f_N(x^N, r^{N-1})\,|\,r^{N-2}, x_1\right\} \qquad (2.12)$$

Substituting (2.9) and the inventory balance equation (1.1) into (2.12) and introducing the abriviation

$$W_N \ : 1-\rho^2 S_N \qquad (2.13)$$

one obtains

$$2f_{N-1}(x^{N-1},r^{N-2}) = \min_{u_{N-1}} E\Big\{(x_{N-1}+u_{N-1}-r_{N-1})^2 W_N+\rho^2 u_{N-1}^2$$
$$+ (x_{N-1}+u_{N-1}-r_{N-1})\hat{T}_N + \hat{R}_N | r^{N-2},x_1\Big\}$$

$$= \min_{u_{N-1}} \Big\{W_N(x_{N-1}^2+2x_{N-1}+u_{N-1}^2)$$
$$- 2(x_{N-1}+u_{N-1})W_N \hat{r}_{N-2}^{(1)}+\eta_{N-1}^2 W_N$$
$$+ (x_{N-1}+u_{N-1})\hat{T}_{N-1}+\hat{R}_{N-1}-D_{N-1}+\rho^2 u^2_{N-1}\Big\} \qquad (2.14)$$

where η_{N-1}^2 and $\hat{r}_{N-2}(1)$ are defined in analogy to (2.4) and (2.5) and

$$\hat{T}_{N-1} := E\{T_N | r^{N-2}\}, \hat{R}_{N-1} := E\{R_N | r^{N-2}\}, D_{N-1} := E\{T_N r_{N-1} | r^{N-2}\} \qquad (2.15)$$

For the optimal decision one obtains

$$u_{N-1}^* = S_{N-1}(x_{N-1}+Q_{N-1}) \qquad (2.16)$$

where

$$S_{N-1} := -\frac{W_N}{W_N+\rho^2} \qquad (2.17)$$

and

$$Q_{N-1} := \hat{r}_{N-2}^{(1)} + \frac{1}{2W_N}\hat{T}_{N-1}. \qquad (2.18)$$

<u>Determination of $f_{N-1}(x^{N-1},\gamma^{N-2})$</u>

Analog to step 1 one has from (2.14)

$$f_{N-1}(x^{N-1},r^{N-2}) = -S_{N-1}\rho^2 x_{N-1}^2 + T_{N-1}x_{N-1} + R_{N-1} \qquad (2.19)$$

where we introduced the definitions

$$T_{N-1} := -2\rho^2 S_{N-1} Q_{N-1} \qquad (2.20)$$

$$R_{N-1} := W_N S_{N-1} Q_{N-1}^2 - 2 S_{N-1} Q_{N-1} W_N \hat{r}_{N-2}^{(1)} + \eta_{N-1}^2 W_N$$
$$+ S_{N-1} Q_{N-1} \hat{T}_{N-1} - D_{N-1} + \hat{R}_{N-1} \qquad (2.21)$$

In general we obtain the following algorithm for the <u>optimal policy</u>

$$u_k^* = S_k(x_k + Q_k) \qquad (k = 1,\dots,N) \tag{2.22}$$

$$S_k := -\frac{W_{k+1}}{\rho^2 + W_{k+1}}$$

$$W_k := 1 - \rho^2 S_k$$

$$Q_k := \hat{r}_{k-1}(1) + \frac{1}{2W_{k+1}} \hat{T}_{k+1}$$

$$\hat{T}_k := E\left\{T_{k+1} \mid r^{k-1}\right\}$$

$$T_k := -2\rho^2 S_k Q_k$$

The starting values being (2.7) and (2.8):

$$S_N := -\frac{1}{1+\rho 2}$$

$$Q_N := -\hat{r}_{N-1}(1)$$

Resubstituting some of the above definitions which have only been introduced to simplify the derivation, algorithm (2.22) can be written more compactly

$$u_k = S_k(x_k + Q_k) \qquad (k = 1,2,\dots,N) \tag{2.22}$$

$$S_k = \frac{\rho^2 S_{k+1} - 1}{1+\rho^2(1-S_{k+1})} \tag{2.23}$$

$$Q_k = -\hat{r}_{k-1}(1) + \gamma_{k+1} E\left\{Q_{k+1} \mid r^{k-1}\right\} \quad (k = 2,3,\dots,N) \tag{2.24}$$

where γ_k is defined by

$$\gamma_k := \frac{\rho^2 S_k}{1-\rho^2 S_k} \tag{2.25}$$

(Note that it is reasonable to define $\hat{r}_0(1) := E\{r_1\}$ and $Q_{N+1} \equiv 0$). The above algorithm constitutes an interesting and important result. It can be seen that the optimal decision only depends on conditional means

$$\hat{r}_k(i) := E\left\{r_{k+i} \mid r^k\right\} \tag{2.26}$$

of the stochastic demand sequence. I.e. the probability distributions of the random variables are reduced to their first conditional moments. Otherwise stated: having no more information about the sequence of demand than just the above mean values we would be able to derive the same results as if we knew the total probability structure. This important result can be made even more explicit in solving (2.24) and substituting in (2.22).
Let us start with

$$Q_{N-2} = -\hat{r}_{N-3}(1) + \gamma_{N-1}\, E\left\{Q_{N-1} \mid r^{N-3}\right\} \tag{2.27}$$

Similarly, for N-3 one obtains

$$\begin{aligned} Q_{N-3} &= -\hat{r}_{N-4}(1) + \gamma_{N-2}\, E\left\{Q_{N-2} \mid r^{N-4}\right\} \\ &= -\hat{r}_{N-4}(1) + \gamma_{N-2}\, E\left\{\hat{r}_{N-3}(1) + \gamma_{N-1}\, E\left\{Q_{N-1} \mid r^{N-3}\right\} \mid r^{N-4}\right\} \\ &= -\hat{r}_{N-4}(1) - \gamma_{N-2}\,\hat{r}_{N-4}(2) + \gamma_{N-2}\,\gamma_{N-1}\, E\left\{Q_{N-1} \mid r^{N-4}\right\} \end{aligned} \tag{2.28}$$

where the relation

$$E\left\{\hat{r}_{N-3}(1) \mid r^{N-4}\right\} = E\left\{E\left\{r_{N-2} \mid r^{N-3}\right\} \mid r^{N-4}\right\} = E\left\{r_{N-2} \mid r^{N-4}\right\} =: \hat{r}_{N-4}(2)$$

has been used. (2.29)

Substituting the initial condition (2.8) (or $Q_{N+1} \equiv 0$), (2.28) becomes

$$\begin{aligned} Q_{N-3} &= -\hat{r}_{N-4}(1) - \gamma_{N-2}\,\hat{r}_{N-4}(2) + \gamma_{N-2}\,\gamma_{N-1}\, E\left\{-\hat{r}_{N-2}(1) \right. \\ &\quad \left. + \gamma_N\, E\left\{-\hat{r}_{N-1}(1) \mid r^{N-2}\right\} \mid r^{N-4}\right\} \\ &= -\hat{r}_{N-4}(1) - \gamma_{N-2}\,\hat{r}_{N-4}(2) - \gamma_{N-2}\,\gamma_{N-1}\,\hat{r}_{N-4}(3) \\ &\quad - \gamma_{N-2}\,\gamma_{N-1}\,\gamma_N\,\hat{r}_{N-4}(4). \end{aligned}$$

Finally, for period k one obtains

$$\begin{aligned} Q_k &= -r_{k-1}(1) - \gamma_{k+1}\,\hat{r}_{k-1}(2) - \gamma_{k+1}\,\gamma_{k+2}\,\hat{r}_{k-1}(3) - \dots \\ &\quad - \gamma_{k+1}\cdots\gamma_{N-1}\, r_{k-1}^{(N-k)} - \gamma_{k+1}\cdots\gamma_{N-1}\,\gamma_N\,\hat{r}_{k-1}^{(N-(k-1))} \end{aligned} \tag{2.30}$$

This can be written more compactly in the form

$$Q_k = - \sum_{i=1}^{N-(k-1)} w_i \hat{r}_{k-1}(i) \tag{2.31}$$

where the weighting factors w_i are defined by

$$w_i := \prod_{j=1}^{i-1} \gamma_{k+j} \qquad i = 2,3,\ldots,N-(k-1) \tag{2.32}$$

and

$$w_1 = 1$$

As we shall see below, $1 \geq w_i \geq w_{i+1} \geq 0 \forall i$ (2.33)

Resubstituting (3.31) into (3.22) finally yields

$$u_k^* = s_k \left(x_k - \sum_{i=1}^{N-(k-1)} w_i \hat{r}_{k-1}(i) \right), \quad (k = 1,\ldots,N). \tag{2.34}$$

Equ.(2.34) constitutes a remarkable result. The conditional mean values $\hat{r}_{k-1}(i) := E\{r_{k-1+i} | r^{k-1}\}$ can be interpreted as forecasts i periods ahead based on the information gained up to period k-1. Thus the optimal production decision not only depends on the present amount of stock on hand but also on a decreasingly weighted sequence of demand forecasts up to planning horizon N. These forecasts Theil called <u>dynamic certainty equivalents.</u> The certainty equivalence property says that one would have obtained the same optimal policy result (2.34) if one had replaced from the outset the stochastic sequence of demand by a sequence of conditional means. (The reader should himself convince of this property in looking again at the derivation of the above algorithm (see also [21])). I.e. for linear-quadratic models the procedures of taking forecasts and optimizing the system can be separated without loss of optimality.

As we could see, this extraordinary property generally holds for linear-quadratic models having <u>additive</u> (non stationary) stochastic disturbances. Therefore, whenever possible one would try to cast a real inventory-production problem into a linear-quadratic model. This was done by Holt et.al for a problem in production planning and workforce smoothing [7]. In many situations, however, it turns out not to be reasonable to approximate costs by quadratic functions. We therefore shall develop in the next chapter a theory which still allows to a certain respect for the existence of dynamic certainty equivalents for which, however, costs have no longer to be quadratic.

It should be noticed that the existence of dynamic certainty equivalents not only diminishes the computational load considerably but that also it reduces to a high extend the amount of information needed. Thus, in practice, a stochastic dynamic optimization problem is always treated as a deterministic problem with the stochastic demand sequence being replaced by its forecasts. As we know, this procedure usually is suboptimal depending on how well the separation property holds. Starting from a situation for which dynamic certainty equivalents exist all subsequent chapters may be regarded as a discussion of the separation property of various important inventory-production models.

Concluding, let us look at the forecasts occuring in (2.34) in some detail. Let us consider two examples which will be of some importance in subsequent sections.

Example 1

Let $\{r_k\}$ be a sequence of independent random variables with mean $E\{r_k\} = 0 \;\forall k$. This implies $\hat{r}_{k-1}(i) := E\left\{r_{k-1+i} \,|\, r^{k-1}\right\} = E\left\{r_{k-1+i}\right\} = 0$ so that the optimal policy reduces to

$$u_k^* = S_k \; x_k \qquad (k = 1,\ldots,N) \tag{2.35}$$

This result was to be expected since in case of a white noise disturbance no forecasts are possible.

Example 2

Let $\{r_k\}$ be a Gauss-Markov process generated by the autoregressive scheme $r_{k+1} = a\, r_k + \varepsilon_k$, where ε_k is white Gaussian noise and $0 < a < 1$. In this case

$$\hat{r}_{k-1}(i) : = E\left\{r_{k-1+i} \,|\, r^{k-1}\right\} = E\left\{a^i \; r_{k-1} + \sum_{j=1}^{i} a^{j-1} \varepsilon_{k-1+i-j} \,|\, r_{k-1}\right\} =$$

$$= a^i \; r_{k-1} \tag{2.36}$$

Hence (2.34) reduces to

$$u_k^* = S_k\,(x_k - r_{k-1} \sum_{i=1}^{N-(k-1)} W_i \; a^i) = S_k\,(x_k - a_k \; r_{k-1}) \tag{2.37}$$

where

$$a_k := \sum_{i=1}^{N-(k-1)} W_i \; a^i$$

<u>Remark:</u> Introducing a state vector by (x_k, r_{k-1}) this result was to be expected since in this case (2.37) may be written

$$\begin{pmatrix} u_k \\ 0 \end{pmatrix}^{*} = \begin{pmatrix} S_k & -a_k \; S_k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ r_{k-1} \end{pmatrix} \tag{2.38}$$

which shows the same structure as (2.35) for the simple case of a Markov sequence of 0-th order. (See also appendix to this chapter)

2.2 Least Square Forecasts

Before persuing the implication of the optimal policy (2.34) further, let us investigate some properties of the forecasts $\hat{r}_k(i)$. First, we show $\hat{r}_k(i)$ to be a minimum mean-square forecast. Secondly, in Sec.2.2.2, we derive a recursive procedure to calculate $\hat{r}_k(i)$.

2.2.1 Least Square Property of $\hat{r}_k(i)$

We defined $\hat{r}_k(i)$ by the conditional mean

$$r_k(i) = E\left\{r_{k+i} \mid r^k\right\} \tag{2.39}$$

We now show that $\hat{r}_k(i)$ has the property of minimizing the mean square prediction error ([24] p.164).

Let $\hat{x}_k(i)$ be a forecast of r_{k+i}, then the prediction error is defined by $r_{k+i} - \hat{x}_k(i)$, and the mean quadratic error is given by

$$E\left\{\left(r_{k+i} - \hat{x}_k(i)\right)^2 \mid r^k\right\} \tag{2.40}$$

Let us now determine $x_k(i)$ such that (2.40) be minimized. Introducing a density function*) $f(r_{k+i} \mid r^k)$ equ.(2.40) may be written

$$E\left\{\left(r_{k+i}-\hat{x}_k(i)\right)^2 \mid r^k\right\} = \int_{-\infty}^{\infty} \left(r_{k+i}-\hat{x}_k(i)\right)^2 \; f(r_{k+i} \mid r^k) dr_{k+i}$$

*) Note that for convenience our notation does not distinguish between random variable and adjoined realization.

Differentiating with respect to $\hat{x}_k(i)$ and setting equal to zero we obtain

$$- 2 \int_{-\infty}^{\infty} \left(r_{k+i}-\hat{x}_k(i)\right) \ f \ (r_{k+i}|r^k) \ dr_{k+i} = 0$$

or

$$\hat{x}_k(i) = \int_{-\infty}^{\infty} r_{r+1} \ f(r_{k+i}|r^k) \ dr_{k+i} = E\left\{r_{k+i}|r^k\right\} \qquad (2.41)$$

which clearly shows $\underline{\hat{x}_k(i) = \hat{r}_k(i)}$

i.e. mean conditional forecasts have the property of being mean square optimal.

2.2.2 Recursive Calculation of $\hat{r}_k(i)$

The calculation of forecasts may be performed recursively; i.e. knowing $\hat{r}_k(1)$ we would like to calculate $\hat{r}_{k+1}(1)$ using the old forecast $\hat{r}_k(1)$. A typical recursive formula of this kind is the well known exponential smoothing recursive equation

$$\hat{r}_{k+1}(1) = \alpha \ r_{k+1} + (1-\alpha) \ \hat{r}_k(1) \qquad (2.42)$$

(We shall soon return to this equation in a somewhat more general context).

It is an essential of our general model of chapter 1 that the stochastic structure of the demand sequence has already been determined from past data. As we shall show later many time series $\{r_k\}$ occurring in inventory problems may be represented by the following set of equations*)

---

*) Note that setting $n_{k+1}\equiv 0$ and $A_k=A$, $C_k=C$, $B_{k+1}=B \ \forall k$ (see p. 16) one obtains the usual ARMA (autoregressive-mooving average) input-output representation of a stochastic process:

Let

$$\underline{\xi}_{k+1} = A \ \underline{\xi}_k + C \ \underline{\varepsilon}_k$$

$$\underline{r}_{k+1} = B \ \underline{\xi}_{k+1}$$

Using the time shifting operator z^{-1} : $z^{-1}y_k =: y_{k-1}$ one obtains from the first equation

$$\underline{\xi}_{k+1} = \left[I-z^{-1}A\right]^{-1} C \ \underline{\varepsilon}_k$$

from which the ARMA-representation follows immediately

$$\left[zI-A\right] B^{-1} \ \underline{r}_k = C \ \underline{\varepsilon}_k$$

(See also appendix to this chapter)

$$\underline{\xi}_{k+1} = A_k \, \underline{\xi}_k + C_k \, \underline{\varepsilon}_k \qquad \text{("state" equation)} \qquad (2.43)$$

$$\underline{r}_{k+1} = B_{k+1} \, \underline{\xi}_{k+1} + \underline{n}_{k+1} \qquad \text{("output" equation)} \quad (k = 0,1,2,\ldots)$$

where the bar denotes column vectors

$\underline{\xi}_k$: n - vector	A_k : nxn-matrix
$\underline{r}_k$: r - vector	B_k : rxn-matrix
$\underline{\varepsilon}_k$: p - vector	C_k : nxp-matrix
$\underline{n}_k$: p - vector	

$\{\underline{\varepsilon}_k\}$ and $\{\underline{n}_k\}$ representing Gauss - sequences

with
$$E\{\underline{\varepsilon}_k\} = E\{\underline{n}_k\} = \underline{0} \ \forall \ k$$
$$E\{\underline{\xi}_k \, \underline{\xi}_j'\} = Q_k \, \delta_{kj}$$
$$E\{\underline{n}_k \cdot \underline{n}_j'\} = R_k \, \delta_{kj}$$
$$E\{\underline{n}_{kx} \cdot \xi_j'\} = \underline{\underline{0}} \ \forall \ k \text{ a and } j$$

$\underline{0}$ represents a p-nullvector, Q_k and R_k are known positive definite matrices and $\underline{\underline{0}}$ denotes a pxp-null-matrix. Moreover

$$E\{\underline{\varepsilon}_k \, \underline{\xi}_0'\} = E\{\underline{n}_k, \underline{\xi}_0'\} = \underline{0} \ \forall \ k$$

In fact, the general scheme (2.43) describes a large number of models which have been proposed and estimated in inventory theory [15, 26]. One of the most prominent models is that of Theil and Wage [26] which is given by

$$\left.\begin{aligned} \xi_{k+1} &= \xi_k + \Psi_k \\ \Psi_{k+1} &= \Psi_k + \varepsilon_k \end{aligned}\right\} \quad \text{("process")} \qquad (2.44)$$

$$r_k = \xi_k + \zeta_k \qquad \text{(observation)}$$

(all variables being scalars). Clearly, (2.44) is a special case of (2.43) with

$$\underline{\xi}_k : \begin{pmatrix} \xi_k \\ \Psi_k \end{pmatrix}, \ \underline{r}_k := r_k, \ A_k := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ C_k := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ B_k := (1,0)$$

The model (2.44) has a simple intuitive meaning. ξ_k may be interpreted as a trend variable and Ψ_k as a change in trend. $\{\varepsilon_k\}$ then

describes pernament disturbances (being processed to the next period k+1), whereas η_k describes transient disturbances.
An even more simple model is that of a (non-stationary) random walk

$$\xi_{k+1} = \xi_k + \varepsilon_k$$
$$r_k = \xi_k + \eta_k$$

for which we shall show exponential smoothing will be a least square forecasting procedure.
Let us now develop forecasting formulae for a time series represented by (2.43). This can easily be done by using Kalman's recursive estimation scheme which is also known as Kalman filter [24].
The Kalman filter gives a recursive formula for a predictor $\hat{\underline{\xi}}_{k+1} := E\{\underline{\xi}_{k+1} | \underline{r}^k\}$ which minimizes the quadratic expression $E\left\{(\hat{\underline{\xi}}_{k+1}-\underline{\xi}_{k+1})'(\hat{\underline{\xi}}_{k+1}-\underline{\xi}_{k+1})\right\}$, where $\hat{\underline{\xi}}_{k+1}-\underline{\xi}_{k+1}$ represents the estimation error.
Actually, we are not interested in the best predictor, $\hat{\underline{\xi}}_{k+1}$, but in the best forecast $\hat{\underline{r}}_{k+1}$ of the sequence. However, $\hat{\underline{r}}_{k+1}$ can easily be obtained from $\hat{\underline{\xi}}_{k+1}$. Knowing from (2.41) that a least square estimator can be represented by

$\hat{\underline{\xi}}_{k+1} = E\{\underline{\xi}_{k+1} | \underline{r}^k\}$ and $\hat{\underline{r}}_{k+1} = E\{\underline{r}_{k+1} | \underline{r}^k\}$*), we obtain from the observation equation of (2.43)

$$\hat{\underline{r}}_{k+1} = B_{k+1}\, E\{\underline{\xi}_{k+1} | \underline{r}^k\} + E\{\underline{\eta}_{k+1} | \underline{r}^k\} = B_{k+1}\, \hat{\underline{\xi}}_{k+1} \tag{2.45}$$

i.e. the forecast $\hat{\underline{r}}_{k+1} := \hat{\underline{r}}_k(1)$ can be expressed by the predictor $\hat{\underline{\xi}}_{k+1}$ which in turn will be derived from Kalman's algorithm below.
From (2.43) then follow all further predictions $\hat{\underline{r}}_k(i)$, $(i > 1)$.

This can easily be seen ([24] p.171) by taking conditional expectations for $k := k+i-1$

$$E\{\underline{\xi}_{k+i} | \underline{r}^k\} = A_{k+i-1}\, E\{\underline{\xi}_{k+i-1} | \underline{r}^k\}$$

*) Note that $\hat{\underline{r}}_{k+1}$ is an abriviation for our more precise notation $\hat{\underline{r}}_k(1)$.

or equivalently

$$\hat{\underline{\xi}}_k(i) = A_{k+i-1}\,\hat{\underline{\xi}}_k(i-1) \qquad (i = 2,3,\ldots)$$

Solving for $\hat{\underline{\xi}}_k(i)$ we obtain

$$\begin{aligned}\hat{\underline{\xi}}_k(i) &= A_{k+i-1}\,\hat{\underline{\xi}}_k(i-1)\\ &= A_{k+i-1}\,A_{k+i-2}\,\hat{\underline{\xi}}_k(i-2)\\ &= A_{k+i-1}\,A_{k+i-2}\cdots A_{k+1}\,\hat{\underline{\xi}}_k(1)\end{aligned}$$

Premultiplying by B_{k+i} finally yields

$$\hat{r}_k(i) = B_{k+1}\,\hat{\underline{\xi}}_k(i) = B_{k+i}\,A_{k+i-1}\,A_{k+i-2}\cdots A_{k+1}\,\hat{\underline{\xi}}_k(1)$$

In the simple case where all matrices are time invariant (stationarity!) and B is the unit matrix this reduces to (see also Equ.(2.36))

$$\hat{\underline{r}}_k(i) = A^{i-1}\,\hat{\underline{r}}_k(1) \tag{2.46}$$

The Kalman filter which in principle can be derived by dynamic programming minimizing a quadratic functional (representing the prediction error) [13] is given by the following set of equations

$\hat{\underline{\xi}}_k := E\{\underline{\xi}_k | \underline{r}^{k-1}\}$ (predictor)

$\underline{\xi}_k^* := E\{\underline{\xi}_k | \underline{r}^k\}$ (estimator)

$P_k^* := \mathrm{cov}\{\underline{\xi}_k | \underline{r}^k\}$ (conditional covariance-matrix of $\underline{\xi}_k$ reprenting the quadratic estimation error)

$P_k := \mathrm{cov}\{\underline{\xi}_{k+1} | \underline{r}^k\}$ (quadratic (mean conditional) forecasting error)

$$\underline{\xi}_{k+1}^* = \hat{\underline{\xi}}_{k+1} + K_{k+1}\left(\underline{r}_{k+1} - B_{k+1}\,\hat{\underline{\xi}}_{k+1}\right) \qquad (k = 0,1,\ldots,N) \tag{2.47}$$

$$\hat{\underline{\xi}}_{k+1} = A_k\,\underline{\xi}_k^* \tag{2.48}$$

$$K_{k+1} = P_{k+1}^*\,B_{k+1}'\,R_{k+1}^{-1} \tag{2.49}$$

$$P^{*-1}_{k+1} = P^{-1}_{k+1} + B'_{k+1} R^{-1}_{k+1} B_{k+1} \tag{2.50}$$

$$P_{k+1} = A_k P^*_k A'_k + C_k Q_k C'_k \tag{2.51}$$

starting with ξ^*_o and P^*_o.

The algorithm essentially consists of two parts. The first part (Equs.(2.47) and (2.48)) generates a recursive improvement of the estimator $\underline{\xi}^*_{k+1}$ relying on the estimator $\underline{\xi}^*_k$ one period before.

This improvement is due to new measurement data (of the time series) at k+1 : $\underline{r}_{k+1}$. Equ.(2.47) then compares $\underline{r}_{k+1}$ with its forecasts $\hat{\underline{r}}_{k+1} = B_{k+1} \hat{\underline{\xi}}_{k+1}$ forming the difference $\underline{r}_{k+1} - H_{k+1}\underline{\xi}_{k+1}$ which is multiplied by the (so-called) gain matrix K_{k+1}. This recursive scheme, obtained by a quadratic optimization procedure is very similar to the exponential smoothing formula (2.42). In deed, in Sec. 2.3, we shall retrieve (2.42) from a Kalman filtering procedure applied to a particular time series.

The second part of the algorithm then determines the gain matrix K_{k+1} recursively.

To illustrate the above algorithm let us again consider the simple random walk

$$\begin{aligned} \xi_{k+1} &= \xi_k + \varepsilon_k \\ r_{k+1} &= \xi_{k+1} + \eta_{k+1} \end{aligned} \tag{2.52}$$

For this example we readily identify

$A_k = C_k = B_k = 1$ and $Q_k = \sigma_\varepsilon^2$, $R_k = \sigma_\eta^2$, where σ_ε^2 and σ^2_η are the variances of $\{\varepsilon_k\}$ and $\{\eta_k\}$ respectively. Hence, the Kalman filter reduces to

$$\xi^*_{k+1} = \hat{\xi}_{k+1} + K_{k+1}\left(r_{k+1} - \hat{\xi}_{k+1}\right) \ \left(k=0,1,2,\ldots,N\right) \tag{2.53}$$

$$\hat{\xi}_{k+1} = \xi^*_k \tag{2.54}$$

$$K_{k+1} = P^*_{k+1} \frac{1}{\sigma_\eta^2} \tag{2.55}$$

$$P^{*-1}_{k+1} = P^{-1}_{k+1} + \frac{1}{\sigma_\eta^{\ 2}} \qquad (2.56)$$

$$P_{k+1} = P^{*}_{k} + \sigma_\varepsilon^{\ 2} \qquad (2.57)$$

In view of (2.52) Equ.(2.53) can also be written

$$\hat{r}_{k+2} = (1 - K_{k+1})\ \hat{r}_{k+1} + K_{k+1}\ r_{k+1} \qquad (2.58)$$

or, more precisely

$$\hat{r}_{k+1}(1) = \hat{r}_k(1) + K_{k+1}\ (\hat{r}_{k+1} - r_k(1)) \qquad (2.59)$$

showing a close analogy to the exponential smoothing formula (2.42). Thus the Kalman filter equations lead to a recursive updating of the forecasts of a demand sequence (being modelled by (2.43).) Hence, in principle, the aim of this section is reached. However, let us in addition look at the dependence on the starting values $\underline{\xi}^*_o$ and $\underline{P}^*_o$. It seems to be plausible to set $\underline{\xi}^*_o := E\{\underline{\xi}_o\}$ and $\underline{P}^*_o := \{\underline{\xi}_o\ \underline{\xi}_o'\}$. This choice implies the Kalman estimates to be unbiased [24]. The initial values represent the a-priori information one has with respect to the initial predictor $\underline{\xi}^*_o$ and the mean quadratic forecasting error $\underline{P}^*_o$. It can be shown that this information is "dieing out" if $k \to \infty$ and the adjoined (deterministic) system

$$\underline{\xi}_{k+1} = A_k\ \underline{\xi}_k$$

$$\underline{r}_{k+1} = B_{k+1}\ \underline{\xi}_{k+1}$$

is "observable" (i.e., roughly speaking, if

$$\text{rank}\ [H_1,\ H_2\Phi_{o,o}, \ldots, H_N\Phi_{o,N-2}, \ldots] \neq 0$$

where

$$\Phi_{o,k} := A_o \cdot A_1 \cdots A_k) \text{ (see e.g. [12]).}$$

Let us again consider the simple random walk example (2.52). From (2.56) and (2.57) we obtain

$$P^{*-1}_{k+1} = \left(P^*_k + \sigma^2_\varepsilon\right)^{-1} + \frac{1}{\sigma_\eta^{\ 2}}$$

or

$$P^{*}_{k+1} = \frac{\sigma_\eta^{\,2} \, (P^{*}_{k} + \sigma_\varepsilon^{\,2})}{\sigma_\eta^{\,2} + \sigma_\varepsilon^{\,2} + P^{*}_{k}}$$

Letting $k \to \infty$ the asymptotic error covariance matrix $P_{asymp} =: P^{*}$ clearly is

$$P^{*} = (1-\alpha)\, \sigma_\eta^{\,2} \tag{2.60}$$

where

$$\alpha := 1 + \frac{1}{2} \frac{\sigma_\varepsilon^{\,2}}{\sigma_\eta^{\,2}} \left(1 - \sqrt{1 + 4\,\frac{\sigma_\eta^{\,2}}{\sigma_\varepsilon^{\,2}}}\right)$$

Hence, in view of (2.55) the "asymptotic gain factor" K is given by

$$K = 1-\alpha$$

Finally, (re)substituting $K = 1-\alpha$ into (2.59) yields the exponential smoothing equation

$$\hat{r}_{k+1}(1) = \alpha\, r_{k+1} + (1-\alpha)\, \hat{r}_k(1)$$

or, in closed form

$$\hat{r}_{k+1}(1) = (1-\alpha) \sum_{j=o}^{\infty} \alpha^{j}\, r_{k-j}$$

where the linear operator $(1-\alpha) \sum_{j=o}^{\infty} \alpha^{j} \ldots$ could be denoted as the (input-output form of the) asymptotic Kalman forecasting filter for a random walk sequence one step ahead.
Note that, in view of (2.46), $\hat{r}_{k+1}(i) = \hat{r}_{k+1}(1)$.

(In applying this theory to real data one would first identify the structure of the system (2.43). However, instead of estimating the system parameters one would try to optimize right ahead the parameters in the forecasting formula.)

2.3 An ideal Situation

Let us again return to our main problem in Sec. 2.1. As we already mentioned above inventory-production problems are often solved in practice by first taking forecasts of demand and then optimizing the remaining deterministic problem. This optimization is repeated whenever new information gives rise to new forecasts. Moreover, in the absence of seasonal fluctuations often a simple exponential smoothing formula is used in avaluating demand forecasts.

As we have seen in preceeding sections such a procedure is indeed possible. However, we are restricted to the following situation:

1. The cost criterion has to be quadratic.
2. There are no capacity constraints to be taken into account.
3. The demand sequence has to be a random walk (with known variances σ_ε^2 and σ_η^2).

In all what follows we shall try to reduce more realistic models to this simple situation.

We shall particularly be concerned with (1). The existence of constraints will not be considered until Chap.6. Since other forecasting methods than exponential smoothing are known and, in fact, amply used we will not be confined to a random walk process. However, since forecasts have to be certainty equivalents they should always be conditional means.

2.4 Infinite Horizon Case

Our main concern in later chapters will be the infinite horizon case. Therefore we shall now consider the quadratic model of Sec.2.1 for $N \to \infty$. More precisely, we shall consider the general model of Chap.1 with x_k, $u_k \in \mathbb{R}_1$, $\{r_k\}$ being a stationary stochastic sequence with known conditional means and

$$C := \lim_{N\to\infty} \frac{1}{N} \; E\left\{ \sum_{k=1}^{N} \left(x_{k+1}^2 + \rho^2 u_k^2 \right) \mid x_1 \right\} \to \min.$$

Since we have already solved the finite horizon case we have solely to study the limiting behavior of the finite horizon policy (2.34). Starting with γ_k, defined by (2.25), we shall now write more carefully

$$\gamma_k^N := \frac{\rho^2 S_k^N}{1-\rho^2 S_k^N}$$

denoting by N the finite (N) horizon case. Similarly, we have instead of (2.23)

$$S_k^N = \frac{\rho^2 S_{k+1}^N - 1}{1 + \rho^2 (1-S_{k+1}^N)}$$

Now,

defining $S := \lim_{N \to \infty} S_k^N$ we observe

$$S = \frac{\rho^2 S - 1}{1 + \rho^2 (1-S)} ,$$

or

$$S = \frac{1}{2\rho^2} \left(1 - \sqrt{1+4\rho^2}\right)$$

and

$$\gamma := \frac{\rho^2 S}{1 - \rho^2 S} = \frac{1}{2\rho^2} \left(1 + 2\rho^2 - \sqrt{1 + 4\rho^2}\right) = S + 1 < 1 \tag{2.61}$$

with γ being the limiting value of γ_k^N. Hence, (2.34) becomes

$$u_k^*(x_k) = (\gamma-1) \left(x_k - \sum_{i=o}^{\infty} \gamma^i \hat{r}_{k-1}(i+1)\right) , \quad (N \to \infty) \tag{2.62}$$

Note that in deriving (2.62) we have to assume $\left\{ \gamma^i \hat{r}_{k-1}(i+1) \right\}$ to decrease sufficiently fast, (guaranteeing the convergence of the above infinite sum). This is usually the case for stationary sequences met in practice.

Next, we shall derive expressions for total asymptotic costs. This could be done by simply letting $N \to \infty$ in the dynamic programming algorithm for the value function $f(x^k, r^{k-1})$. However, it seems to be analytically easier to observe that C is nothing else but a variance criterion

$$C = \sigma_x^{2} + \rho^2 \sigma_u^2 \tag{2.63}$$

Thus we have only to determine the variances of the stationary processes $\{x_k\}$ and $\{u_k\}$.*)

We shall consider two cases being of importance for later reference

(1) White noise **)

As we already know from (2.35) and (2.61) $u_k^*(x_k)$ reduces to

$$u_k^* = (\gamma-1)\; x_k . \tag{2.64}$$

Hence, $\sigma_u^{2} = (\gamma-1)^2\sigma_x^{2}$ (2.65)

Further, from (2.64) and the balance equation it follows

$$x_{k+1} = \gamma x_k - r_k$$

or

$$x_k = -\sum_{j=1}^{\infty} \gamma^j\, r_{k-j}$$

implying

$$\sigma_x^{2} = \frac{1}{1-\gamma^2}\,\sigma^2 \tag{2.66}$$

Finally, from (2.65)

$$\sigma_u^{2} = \frac{1-\gamma}{1+\gamma}\,\sigma^2 \tag{2.67}$$

These results could have also been obtained by a Wiener-filtering approach [17] minimizing the variance criterion (2.63) (see also [18]).

Hence the minimal costs are

$$C^* = \left[\frac{1}{1-\gamma^2} + \rho^2\,\frac{1-\gamma}{1+\gamma}\right]\sigma^2 = \frac{1}{1-\gamma}\,\sigma^2 \tag{2.68}$$

---

*) This idea will frequently be used for more complicated situations in later chapters

**) See Chapter 1

(2) Gauss - Markov sequence *)

Since $\hat{r}_{k-1}(i+1) := E\left\{ r_{k+i} | r^{k-1} \right\} = a^{i+1}\ r_{k-1}$ Equ.(2.62) reduces to

$$u_k^* = (\gamma-1)\ \left(x_k - \frac{a}{1-\gamma a}\ r_{k-1}\right). \tag{2.69}$$

Let us derive $\sigma_x{}^2$ and $\sigma_u{}^2$ using z-transforms. With $\Phi_{xx}(z)$ and $\Phi_{uu}(z)$ being the spectral density functions of $\{x_k\}$ and $\{u_k\}$ respectively one has (see e.g. [10])

$$\sigma_x{}^2 = \frac{1}{zuj}\ \oint \Phi_{xx}\ (z)\ z^{-1}\ d_z \tag{2.70}$$

$$\sigma_u{}^2 = \frac{1}{zuj}\ \oint \Phi_{uu}\ (z)\ z^{-1}\ d_z \tag{2.71}$$

where $j := \sqrt{-1}$ and $\oint$ denotes the integration on the unit circle. Let us first express $\Phi_{xx}(z)$ and $\Phi_{uu}(z)$ by $\Phi_{rr}(z)$, which for a Gauss-Markov sequence is given by**)

$$\Phi_{rr}(z) = \frac{(1-a^2)\ \sigma_{rr}^2}{(z-a)\ (z^{-1}-a)} \tag{2.72}$$

From the balance equation and (2.69) one obtains using z-transforms

$$x(z) = -\ \frac{\eta+z}{(z-\gamma)}\ \frac{1}{2}\ r(z) \tag{2.73}$$

with

$$\eta := \frac{a(\gamma-1)}{1-a\gamma}\ .$$

Similarly,

$$u(z) = -\ \frac{(\gamma-1)}{S-\gamma}\ \left[(1+\frac{a}{1-a\gamma}\ )\ z - \frac{a}{1-a\gamma}\right] \frac{1}{z}\ r(z)\ . \tag{2.74}$$

---

*) See Chap.1

**) Note that a z-transform of a sequence $\{y_k\}$ is defined by

$Z\ \{y_k\} := \sum_{k=0}^{\infty} y_k\ z^{-k} =: y(z)$. Obviously, $Z\{y_{k-1}\}=z^{-1}y(z)$.

Hence, from (2.70), (2.71) and (2.72)

$$\sigma_x^2 = \frac{(1-a^2)\ \sigma_r^2}{2\pi j} \oint z^{-1} \frac{\eta+z}{(z-\gamma)(z-a)} \frac{\eta+z^{-1}}{(z^{-1}-\gamma)(z^{-1}-a)} dz \tag{2.75}$$

$$\sigma_u^2 = \frac{(1-a^2)\ \sigma_r^2\ (\gamma-1)^2}{2\pi j} \oint z^{-1} \frac{\left[(1+\frac{a}{1-a\gamma})\ z-\frac{a}{1-a\gamma}\right]}{(z-\gamma)(z-a)} \frac{\left[(1+\frac{a}{1-a\gamma})^{-1}-\frac{a}{1-a\gamma}\right]}{(z^{-1}-\gamma)(z^{-1}-a)} dz \tag{2.76}$$

Using, e.g., the table in [10] one finds the results

$$\sigma_x^2 = \frac{\sigma_r^2}{(1-\gamma^2)(1-a\gamma)^3}\left\{(1-a^2)+\gamma(-3a+3a)+\gamma^2(2a+2a^2-4a^3)+\gamma^3(-2a^2+2a^3)\right\} \tag{2.77}$$

$$\sigma_u^2 = \frac{\sigma_r^2\ (\gamma-1)^2}{(1-\gamma^2)(1-a\gamma)^3}\left\{(1+2a-2a^3)+\gamma(-3a-2a^2+4a^3)+\gamma^2(a^2-2a^3)+\gamma^3a^3\right\} \tag{2.78}$$

Hence the optimal costs are found to be

$$Q = \frac{\sigma_r^2}{(1-a\gamma)^3(1-\gamma^2)}\left\{(1-a^2)+\gamma(1-a+a^3)-a\gamma^2-a^2\gamma^3+a^3\gamma^4\right\} \tag{2.79}$$

2.5 Appendix to Chapter 2

2.5.1 State Space Representation and Separation Theorem

Up to now we have not made use of a state space representation. For our special linear model this implies that the system (plant) equation is represented by a first order stochastic difference equation having white noise disturbances ε_k.

$$\underline{x}_{k+1} = A_k \underline{x}_k + B_k \underline{u}_k - C_k \underline{\varepsilon}_k \tag{2.80}$$

A state space representation has the advantage that the optimal policy is immediately given by

$$\underline{u}_k = \phi_k (\underline{x}_k) \tag{2.81}$$

i.e. the optimal decision does only depend on the "last" state. This is intuitively obvious, since the state of a system in time k contains all information about the system up to k. For a linear-quadratic model (2.81) specializes to the linear relationship (with matrix L_k)

$$\underline{u}_k = L_k \underline{x}_k \tag{2.82}$$

Thus, knowing the state space representation of a system an optimization is a comparatively simple task.
We shall here discuss two problems. First we shall derive a state space representation for an ARMA-process and secondly the relationship between the property of certainty equivalence and Kalman's separation principle will be investigated.

(a) Deriving a State Space Representation (2.80)

In Chapter 1 we had the plant equation

$$x_{k+1} = x_k + u_k - r_k \tag{2.83}$$

with $\{r_k\}$ being a general Auto-Regressive Moving Average (ARMA)-process. In deriving a state space representation for this situa-

tion let us first consider $\{r_k\}$ separately.
Let $\{r_k\}$ be represented by

$$r_k+a_1r_{k-1}+\cdots+a_\nu r_{k-\nu} = c_o\varepsilon_k+c_1\varepsilon_{k-1}+\cdots+c_{\nu-1}\varepsilon_{k-\nu+1} \tag{2.84}$$

where at least $a_\nu \neq o$ and $c_o \neq 0$. (I.e. in deriving a state space representation we assume the AR-polynomial being of higher degree than the MA-polynomial. This will be sufficient for our purposses). Introducing a "lag-operator" L by $Ly_k := y_{k-1}$, Equ.(2.84) may also be written

$$r_k = \left(\frac{c_o + c_1L + \cdots + c_{\nu-1}L^{\nu-1}}{1 + a_1 L + \cdots + a_\nu L^\nu}\right)\varepsilon_k \tag{2.85}$$

Defining

$$W_k := \left(\frac{1}{1+a_1 L + \cdots + a_\nu L^\nu}\right) \varepsilon_k \tag{2.86}$$

i.e.

$$W_k + a_1 W_{k-1} + \cdots + a_\nu W_{k-\nu} = \varepsilon_k, \tag{2.87}$$

(2.85) can be written

$$r_k = c_o W_k + c_1 W_{k-1} + \cdots + c_{\nu-1} W_{k-\nu+1}$$
$$= \left[c_o, c_1, \cdots, c_{\nu-1}\right] \begin{pmatrix} W_k \\ W_{k-1} \\ \vdots \\ W_{k-\nu+1} \end{pmatrix} \tag{2.88}$$

Defining

$$\underline{r}_k := \begin{pmatrix} W_k \\ W_{k-1} \\ \vdots \\ W_{k-\nu+1} \end{pmatrix} \tag{2.89}$$

and

$$H := (c_o, c_1, ---, c_{\nu-1})$$

Equ.(2.88) can be written

$$r_k = H \underline{r}_k \tag{2.90}$$

Particularly, in case of a pure AR-process, i.e. $C_o = 1$ and $C_j = 0, \forall j \neq 0$:

$$r_k = W_k \text{ and } r_k + a_1 r_{k-1} + --- + a_\nu r_{k-\nu} = \varepsilon_k,$$

$\underline{r}_k$ reduces to

$$\underline{r}_k = \begin{pmatrix} r_k \\ r_{k-1} \\ \cdot \\ \cdot \\ \cdot \\ r_{k-\nu+1} \end{pmatrix} \tag{2.91}$$

Instead of (2.87) one could also write

$$W_k = - a_k W_{k-1} ----- a_\nu W_{k-\nu} + \varepsilon_k$$

or,

$$\begin{pmatrix} W_k \\ W_{k-1} \\ \cdot \\ \cdot \\ W_{k-\nu+1} \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 & ----- & -a_\nu \\ 1 & 0 & -- & 0 \\ & 1 & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{k-1} \\ W_{k-2} \\ \cdot \\ \cdot \\ \cdot \\ W_{k-\nu} \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \varepsilon_k \tag{2.92}$$

Using $\underline{r}_k$ as difined in (2.89) and abriviations Φ and Γ for the above matrices, (2.92) can be written more compactly

$$\underline{r}_k = \Phi \, \underline{r}_{k-1} + \Gamma \, \varepsilon_k, \tag{2.93}$$

(which together with (2.90) : $r_k = H\,\underline{r}_k$)

gives a state space representation of the ARMA-process (2.84). Note that (2.90) is only <u>one</u> possible representation of the above ARMA-process. Other representations may be derived [28]. Comparing (2.90) with (2.43) one (again) realizes that the ARMA process (2.84) is a special case of the process defined by (2.43).

We are now in a position to derive a state space representation for our original problem (2.83).
Defining a state vector

$$\underline{x}_{k+1} := \begin{pmatrix} x_k \\ \underline{r}_k \end{pmatrix} \qquad (2.94)$$

(2.83) together with (2.93) may be written

$$\begin{pmatrix} x_k \\ \underline{r}_k \end{pmatrix} = \begin{pmatrix} 1 & -H \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} x_{k-1} \\ \underline{r}_{k-1} \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} \varepsilon_k \qquad (2.95)$$

or

$$\underline{x}_{k+1} = A\,\underline{x}_k + B\,u_k + C\,\varepsilon_k \qquad (2.96)$$

with

$$A := \begin{pmatrix} 1 & -H \\ 0 & \Phi \end{pmatrix}; \quad B := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } C := \begin{pmatrix} 0 \\ \Gamma \end{pmatrix}.$$

Equ.(2.96) is the state space representation of (2.83) we were looking for. It is an equation of the type given in (2.80).

(b) Separation Theorem

Let us next show that for our situation of complete measurability dynamic certainty equivalence and the separation theorem amount to the same.

(1) Input - Output representation

Consider the plant equation

$$x_{k+1} = x_k + u_k - r_k$$

with $\{r_k\}$ being an ARMA process. Roughly speaking dynamic certainty equivalence then says: being in time k it is optimal to replace all variables by their means conditional on the information up to k:

$$\hat{x}_{k+1}(i) = \hat{x}_k(i) + u_{k+i} - \hat{r}_{k-1}(i) \quad (i=1,2,\ldots,N-k) \tag{2.97}$$

and to optimize with respect to the deterministic criterion

$$Q = \sum_{i=1}^{N-k-1} \left\{ \hat{x}^2_{k+1}(i) + \rho^2 u^2_{k+i} \right\} \Rightarrow \min. \tag{2.98}$$

(2) State Space representation

Instead of (2.83) we now consider (2.93).
The separation property says: replace the state vector by its conditional mean and optimize with respect to (2.98).
Hence, form (2.95) we have

$$\begin{pmatrix} \hat{x}_k(i) \\ \hat{\underline{r}}_k(i) \end{pmatrix} = \begin{pmatrix} 1 & -H \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} \hat{x}_{k-1}(i) \\ \hat{\underline{r}}_{k-1}(i) \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \tag{2.99}$$

or

$$\hat{x}_k(i) = \hat{x}_{k-1}(i) - H\,\hat{\underline{r}}_{k-1}(i) + u_k \tag{2.100}$$

$$\hat{\underline{r}}_k(i) = \Phi\,\hat{\underline{r}}_{k-1}(i)$$

the second equation being of no relevance for our present investigation.
Since, because of (2.90), $H\,\hat{\underline{r}}_{k-1}(i) = \hat{r}_{k-1}(i)$, Equ.(2.100) reduces to (2.97) which was to be shown.

2.5.2 Optimal Policies for ARMA-Processes

In Sec. 2.4 we derived optimal results only for the white noise and the Gauss-Markov case. These results were obtained by value iteration of Dynamic Programming. Equivalently one can start with the variance criterion (2.63) and derive necessary and sufficient optimality conditions known as Wiener-Hopf equations [17]. These equations have to be solved yielding the optimal production policy. In (discrete) Wiener-Theory this solution is performed in the z-transform domain [17], (as defined in Sec. 2.4). Obtaining analytic results for more complicated stationary processes than the two above examples (white noise and Gauss-Markov Process) which will be used throughout our later investigations is extremely difficult. These difficulties can, however, be overcome by solving the Wiener-Hopf equation not in the frequency but in the time domain [3]. This involves the solution of an infinite system of linear equations. I.e. the inversion of an infinite dimensional matrix has to be found. It can in fact be shown that for arbitrary ARMA-processes analytic results can be obtained in a straight foreward way.
Let us express the linear decision rule by

$$u_k = \sum_{j=1}^{\infty} G_j \, r_{k-j} \tag{2.101}$$

with weighting factors G_j which have to be optimized.
Similarly, since the system is linear, one has

$$x_k = \sum_{j=1}^{\infty} H_j \, r_{k-j} \tag{2.102}$$

with $\sum_{j=1}^{\infty} |H_j| < \infty$ from stability requirements.
One can easily show [3] (by substituting (2.102) in the balance equation) that

$$u_k = \sum_{j=1}^{\infty} (H_{j+1} - H_j) \, r_{k-j} \tag{2.103}$$

with $H_1 = -1$. The problem now is to determine $H_j \; \forall \; j$.
Substituting (2.102) and (2.103) in the variance criterion (2.63): $C = \sigma_x^2 + \rho^2 \sigma_u^2$ one obtains C as a function of H_j (j=1,2,...).

Differentiating with respect to $H_j (j=1,2,...)$ yields the time domain Wiener-Hopf equation

$$\sum_{j=1}^{\infty} H_j R_{i-j} + \rho^2 \sum_{j=1}^{\infty} (H_{j+1} - H_j)(R_{i-j-1} - R_{i-j}) = 0 \qquad (2.104)$$

with R_{k-j} being the covariance

$$R_{k-j} := \text{cov}(r_k, r_j)$$

of the stationary sequence $\{r_k\}$.
This infinite set of linear equations represents the necessary and sufficient conditions for $H_j \; \forall \; j$ to be optimal. In solving (2.104) for H_j, (2.104) may be rewritten as an infinite dimensional matrix equation

$$\begin{pmatrix} R_1 & R_0 & R_{-1} & --- \\ R_2 & R_1 & R_0 & --- \\ R_3 & R_2 & R_1 & --- \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} (1-\rho^2) & -\rho^2 & 0 & --- \\ -\rho^2 & (1+2\rho^2) & -\rho^2 & \\ & -\rho^2 & (1+2\rho^2) & -\rho^2 \\ 0 & & & \\ \vdots & & & \\ \vdots & & & \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \qquad (2.105)$$

The rows of (2.105) can easily be identified as the Wiener-Hopf equations (2.104).
In solving (2.105) for $(H_1, H_2, ---)'$ one has to find the inverse matrices of the "ρ^2-matrix" and the infinite covariance-matrix

$$R := \begin{pmatrix} R_0 & R_1 & --- \\ R_{-1} & R_0 & --- \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{pmatrix}$$

If R^{-1} exists $(H_1, H_2, ---)'$ can be isolated and one finds [3]

$$\begin{pmatrix} H_2 \\ H_3 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \frac{1}{\sqrt{1+4\rho^2}} \begin{pmatrix} \Theta^1 & \Theta^o & \Theta^1 & --- \\ \Theta^2 & \Theta^1 & \Theta^o & --- \\ \Theta^3 & \Theta^2 & \Theta^1 & --- \\ \cdot & \cdot & \cdot & \diagdown \\ \cdot & \cdot & \cdot & \diagdown \\ \cdot & \cdot & \cdot & \diagdown \end{pmatrix} \left(\alpha \cdot \begin{pmatrix} \bar{\rho}_1 \\ \bar{\rho}_2 \\ \bar{\rho}_3 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} + \begin{pmatrix} 1+\rho^2 \\ -\rho^2 \\ 0 \\ 0 \\ \cdot \\ \cdot \end{pmatrix} \right) \tag{2.106}$$

with $\Theta := \dfrac{\sqrt{1+4\rho^2} - 1}{\sqrt{1+4\rho^2} + 1}$, $\alpha := \dfrac{\Theta \cdot \rho^2 - \rho^2 - 1}{\sum_{i=1}^{\infty} \Theta^{i-1} \cdot \bar{\rho}_i}$

and $\begin{pmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \end{pmatrix}$ being the first row of R^{-1}. ($\bar{z}$ denotes the conjugate complex of z). Equ. (2.106) is the first and most important step in deriving $(H_1, H_2, \ldots)'$. It gives H_j as a function of the inverse R^{-1} of the covariance matrix of the $\{r_k\}$ sequence. The derivation of R^{-1} is not easy but nevertheless it can be found by a straight foreward procedure. The inverse of the "ρ^2-matrix" which defines the economic structure of the problem, however, can only be found by "trial and error". It should be mentioned that for finding the inverse of the ρ^2-matrix its special structure (and consequently the special structure of the inventory-production problem) was of great help.

The second step now consists in the calculation of the first row of R^{-1}. If $\{r_k\}$ is an ARMA-process defined by

$$a_o\, r_i + \ldots + a_k\, r_{i-k} = c_o\, \varepsilon_i + \ldots + c_j\, \varepsilon_{i-j}$$

and if the polynomial $c_o\, x^j + \ldots + c_j\, x^o$ has j different zeros $w_1, \ldots, w_j$, we find

$$
\begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \vdots \end{pmatrix} = \frac{a_o}{c_o} \begin{pmatrix} \bar{a}_o & & 0 \\ \vdots & \bar{a}_o & \\ \vdots & \vdots & \bar{a}_o \\ \bar{a}_k & \vdots & \vdots \\ & \bar{a}_k & \vdots \\ 0 & & \bar{a}_k \end{pmatrix} \cdot \begin{pmatrix} \bar{w}_1^{\,o} & \cdots\cdots & \bar{w}_j^{\,o} \\ \bar{w}_1^{\,1} & \cdots\cdots & \bar{w}_j^{\,1} \\ \vdots & & \vdots \\ \vdots & & \vdots \end{pmatrix}
$$

$$
\cdot \begin{pmatrix} \bar{w}_1^{\,o} & \cdots\cdots & \bar{w}_j^{\,o} \\ \vdots & & \\ \vdots & & \\ \bar{w}_1^{\,j-1} & \cdots & \bar{w}_j^{\,j-1} \end{pmatrix}^{-1} \begin{pmatrix} \bar{c}_o & & 0 \\ \vdots & \ddots & \\ \vdots & & \ddots \\ \bar{c}_{j-1} & \cdots\cdots & \bar{c}_j \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \qquad (2.107)
$$

Equs. (2.106) and (2.107) show that H_j $(j=1,2,\ldots)$ can be calculated analytically for arbitrary ARMA-processes. The only problem remaining is the inversion of two finite dimensional matrices. Since the MA-part is for practical problems of low dimension (at most $j=5$) the optimal production policy can be derived for all one item inventory production problems. For further results and examples see [3].

Chapter 3

THE LINEAR NON-QUADRATIC MODEL

With the preliminary studies in the last chapter we are now in a position to deal with the main object of this treatise. Let us return to the general model defined in Chap.1. For this model we now give optimal solutions within the class of linear production policies. As will be shown in the sequel, restricting admissible production policies to be linear has as an important advantage that many results of the quadratic theory still hold ; in particular its property of allowing for the existence of dynamic certainty equivalents.*) The crucial linearity assumption will be studied in detail in later chapters.
Let us proceed as follows. First we define the model we shall be concerned with and subsequently a general derivation of the linear non-quadratic approach will be presented.**) Then we investigate some special types of important cost functions and finally two different stochastic demand sequences will be studied in some detail.

3.1 The general Linear Non-Quadratic Model

Let us specialize the general model of Chap.1 in the following way. Constituents (1), (2) and (3) remain almost the same. We require $\{r_k\}$ to be a stationary Gaussian random sequence and $u_k \in \mathbb{R}$. Besides the balance equation

$$x_{k+1} = x_k + u_k - r_k \tag{3.1}$$

we now assume the policy to be linear, i.e.

$$u_k = - \sum_{k'=0}^{\infty} G_{k'} \cdot (x_{k-k'} - \mu_x) \tag{3.2}$$

where μ_x is an unknown parameter and $\{G_k\}$ denotes a sequence of

---

*) But see remark on p. 41

**) Hitherto referred to as "LNQ"-approach.

likewise unknown weighting factors. It will be shown that $\mu_x = E\{x_k\}\ \forall k$. Thus $(x_{k-k'}-\mu_x)$ in (3.2) has the intuitive meaning of a deviation of inventory from its mean value.
Assuming (3.2), the optimization problem thus reduces to the problem of determining $\{G_k\}$ and μ_x. Equs.(3.1) and (3.2) form a linear system which is shown as a block diagram in Fig. 3.1

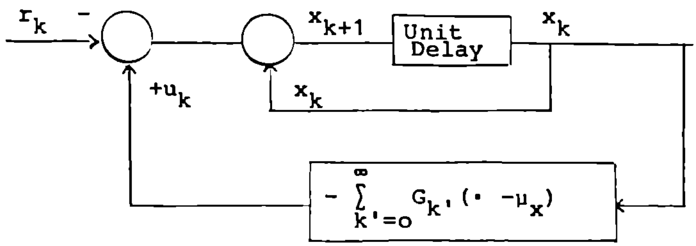


Fig. 3.1 The linear system (3.1) and (3.2)

The cost criterion is given by

$$C = E\{I(x) + P(u)\} \tag{3.3}$$

with I(x) and P(u) representing inventory and production costs respectively.
The optimization problem now reads as follows. Find $\{G_k\}$ and μ_x such that the expected costs C be minimum.

Remarks

Since the above assumptions are essential for the whole approach we should already now give some comments particularly on the economic relevance of the model.

(a) The most serious assumption of the model is its restriction to linear policies (Equ. (3.2)). As will be seen in later chapters the validity of this assumption largely depends on the special cost structure of the model. For costs having relatively small (fixed production) set up costs, assumption (3.2) will turn out not to be too restrictive.

Furthermore, system (3.1), (3.2) takes into account an infinite series of past values of inventory; i.e. together with the stationarity assumption of $\{r_k\}$ we are restricting our investigations to a steady state situation. This assumption is not unusual in theory. Markovian decision processes, e.g., are also considering only the steady state situation. In fact, it turns out that in general the steady state is reached only after a few periods. (See, e.g., footnote p. 65)

(b) Assuming stationarity of the series of demand seems in general not to be restrictive for a short term inventory-production model. If, however, important deterministic components have to be taken into account, the above model has to be modified. The second assumption, i.e. the assumption that $\{r_k\}$ be a Gaussian random sequence seems also not to be too restrictive. First, many demand sequences encountered in practice are, indeed, found to be Gaussian or at least nearly Gaussian. Secondly, as will be shown later, numerical results are fairly insensitive with respect to the normality postulate. (See Sec. 3.6 and Chap. 6)

(c) The cost criterion describes expected costs per period. Instead of (3.3) one can write

$$C = \lim_{N \to \infty} E\left\{ \frac{1}{N} \sum_{k=1}^{N} (I(x_{k+1}) + P(u_k)) \middle| x_1 \right\} \tag{3.3a}$$

The existence of the above limit is guaranteed by the stationarity of $\{u_k\}$ and $\{x_k\}$. This, in turn, is warranted by at least one sequence of weighting factors $\{G_k\}$.
Not all functions $I(\cdot)$ and $P(\cdot)$ will be admissible. Certain economically reasonable assumptions will be required. (See the derivation of Equs. (3.8) and (3.9) below.)

3.2 The general Solution

Having stated and discussed the various assumptions of the general model we will be concerned with, let us now develop a general solution procedure.

Consider Equ.(3.3). Because of the linearity of the system equations (3.1) and (3.2) it follows from the assumption of $\{r_k\}$ being stationary and Gaussian that $\{u_k\}$ and $\{x_k\}$ are also stationary Gaussian processes. This implies that C is solely a function of the variances, covariances, and mean values of $\{u_k\}$ and $\{x_k\}$, i.e.

$$C = F\ (\sigma_x^{\ 2},\ \sigma_u^{\ 2},\ \sigma_{ux},\ \mu_x,\ \mu_u) \tag{3.4}$$

However, in view of the special (additive) structure of the costs, F does not depend on the covariance σ_{ux}. Moreover, from (3.1) it follows $\mu_u\ :=E\ \{u_k\}\ = 0$*). Hence, (3.4) reduces to

$$C = F\ (\sigma_x^{\ 2},\ \sigma_u^{\ 2},\ \mu_x). \tag{3.5}$$

C is a function of μ_x and via $\sigma_x^{\ 2}$ and $\sigma_u^{\ 2}$ of the weighting sequence $\{G_k\}$. Hence, a necessary condition for C to be minimum is given by

$$\frac{\partial C}{\partial \mu_x} = 0 \tag{3.6}$$

and

$$\delta_G\{C\} = \frac{\partial C}{\partial \sigma_x^{\ 2}}\ \delta_G\ \{\sigma_x^{\ 2}\} + \frac{\partial C}{\partial \sigma_u^{\ 2}}\ \delta_G\ \{\sigma_u^{\ 2}\} = 0 \tag{3.7}$$

where $\delta_G\{\cdot\}$ denotes the first variation with respect to $\{G_k\}$. $\delta_G\{\cdot\}$ may also be replaced by (infinitely many) differentiations with respect to G_k. (See also Sec. 2.5.2)

Let us now in additon assume that $\frac{\partial C}{\partial \sigma_x^{\ 2}}$ and $\frac{\partial C}{\partial \sigma_u^{\ 2}}$ be positive (an assumption, which in many cases seems to be economically most reasonable). Hence, (3.7) can be written

$$\delta_G\ \{\sigma_x^{\ 2}\} + \Theta^2\ \delta_G\{\sigma_u^{\ 2}\} = 0 \tag{3.8}$$

*) This also implies together with (3.2) $\mu_x=E\{x_k\}$, so that using the symbol μ_x for the parameter in (3.2) from the outset is justified.

where

$$\Theta^2 := \frac{\dfrac{\partial C}{\partial \sigma_u^2}}{\dfrac{\partial C}{\partial \sigma_x^2}}\Bigg|_{G^*} \tag{3.9}$$

and $\Theta^2 > 0$. The index G^* at (3.9) says that Θ^2 is defined for a weighting sequence $\{G_k^*\}$ for which (3.8) is fulfilled.

Equ.(3.8) represents an important result. Treating for the time being Θ^2 as a fixed parameter, the left hand side of Equ.(3.8) represents the first variation of the quadratic criterion $\sigma_x^2 + \Theta^2 \sigma_u^2$. Hence, the originally non-quadratic criterion has been reduced to a quadratic one.
In solving the above optimization problem we may now proceed as follows. First one solves for fixed Θ^2 the (adjoined) quadratic optimization problem. This can be done by solving a Wiener-Hopf equ. [21] (see also Sec.2.5.2) by value iteration of dynamic programming as shown in Chap.2 (c.o.Equ.(2.62)). One obtains (mean square) optimal variances σ_u^2 and σ_x^2 which of course depend on Θ. In a second step one then substitutes $\sigma_u^2(\Theta)$ and $\sigma_x^2(\Theta)$ into (3.9) which together with (3.6) allows for determining the optimal values of μ_x and Θ^2. In case that <u>several</u> values of Θ^2 occur one has to take that Θ^2 which results in the lowest value of C.

Let us summarize the results thus far obtained stating the following <u>proposition</u>.

> Subject to the above assumptions a dynamic non-quadratic optimization problem can be separated into two parts. The first part involves a <u>quadratic</u> dynamic optimization procedure (taking into account the dynamics of the model). The remaining part then accounts for the special non-quadratic features of the model's criterion requiring only a solution of Equs. (3.9) and (3.6).

Since, as we know from Chap.2, for linear quadratic models dynamic certainty equivalents do exist, the above makes sure the existence of certainty equivalents also in the non-quadratic case. However, with respect to the above derivation this statement should be interpreted carefully. Dynamic certainty equivalents exist only in the sense that one can find an adjoined quadratic problem. In deriving this adjoined quadratic problem the knowledge of variance and covariance of the disturbance sequence was necessary. It is therefore not correct to say that also in the LNQ-Problem dynamic certainty equivalents exist in the sense that only the sequence of forecasts has to be known. However, in the stationary case, we consider, variances and covariances have only to be fixed once and determine together with the cost parameters certain constants in the optimal policy. Hence the only information which is really used is the sequence of forecasts which more or less may be regarded as dynamic certainty equivalents of the LNQ-problem. These arguments will become more clear by the examples given in the next section.

Remarks

(1) The above procedure may be regarded as a rational procedure of fitting quadratic functions to the non-quadratic cost functions I(x) and P(u). In fact, we are not fitting quadratic functions, e.g., by a mean square type of procedure (as was done by [7]), but with respect to the cost criterion of the model.

(2) The procedure we used in deriving an optimal policy may be considered as a special example of a far more general problem which is known as the reduction of a composed functional in a dynamic optimization problem ([1], see also [17]).

(3) For spècial cost functions and demand sequences it will not be necessary to use the above "two-step" procedure. Knowing the structure of the optimal policy one can immediately derive an explicit expression for F $(\sigma_x^2, \sigma_u^2, \mu_x)$ and can optimize F with respect to the unknown parameters in the production policy. (See e.g. [14]) This procedure will be illustrated in later sections.

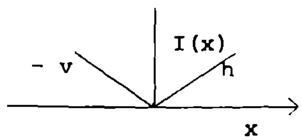
3.3 Special Cost Functions

According to the proposition stated in the last section the effect particular cost functions exert on the optimization can be investigated independently from assumptions on the demand sequence. Thus we shall first study special types of important cost functions and then, in Sec. 3.4, different kinds of demand sequences will be investigated.
The special feature of different cost functions only affects Equs. (3.6) and (3.9). Hence, let us determine these equations for particular costs.

3.3.1 Piecewise Linear Costs

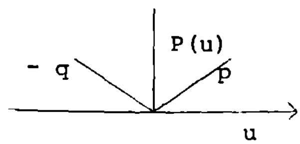
Consider the following piecewise linear cost functions specializing the general criterion (3.3)

$$I(x) := \begin{cases} h\,x & \text{for } x \geq 0 \\ -\,v\,x & \text{for } x < 0 \end{cases} \qquad h > 0,\ v > 0 \tag{3.10}$$



and

$$P(u) := \begin{cases} p\,u & \text{for } u \geq 0 \\ -\,q\,u & \text{for } u < 0 \end{cases} \qquad p \geq 0,\ q \geq 0 \tag{3.11}$$



The cost parameters u, v, p and q may be interpreted as "costs per item and period". h x represents proportional stock holding costs, whereas - v x gives " penalty costs " if one is out of stock. (Note that "negative stock" means back logged orders.) p u and - q u represent costs for producing more and less than at a given mean production level.

Deriving specific formulae for (3.6) and (3.9) we first have to express the expected costs C by (3.10) and (3.11). From (3.5) one immediately has

$$C = \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} I(x) \exp\left\{ - \frac{1}{2\sigma_x^2} (x-\mu_x)^2 \right\} dx$$

$$+ \frac{1}{\sqrt{2\pi}\,\sigma_u} \int_{-\infty}^{\infty} P(u) \exp\left\{ \frac{1}{2\sigma_u^2} u^2 \right\} du$$

$$= - v\mu_x + (h+v) \left\{ \mu_x \,\phi(y) + \sigma_x \,\phi'(y) \right\} + \frac{p+q}{\sqrt{2\pi}} \sigma_u \qquad (3.12)$$

where

$$y := \frac{\mu_x}{\sigma_x} \quad \text{and} \quad \phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left\{ - \frac{1}{2} y'^2 \right\} dy' \qquad (3.13)$$

(For a derivation of (3.12) see also [17, p. 180 f.f.])

From (3.12) we obtain the following partial derivatives

$$\frac{\partial C}{\partial \mu_x} = - v + (h+v)\, \phi(y) \qquad (3.14)$$

$$\frac{\partial C}{\partial \sigma_x^2} = \frac{h+v}{2} \frac{1}{\sigma_x} \phi'(y) \qquad (3.15)$$

$$\frac{\partial C}{\partial \sigma_u^2} = \frac{p+q}{2\sqrt{2\pi}} \frac{1}{\sigma_u} \qquad (3.16)$$

Substituting these expressions in (3.6) and (3.9) yields

$$- v + (h+v)\, \phi(y) = 0 \qquad (3.6a)$$

$$\sqrt{2\pi}\, (h+v)\, \phi'(y)\, \sigma_u\, \Theta^2 - (p+q)\, \sigma_x = 0 \qquad (3.9a)$$

or, indicating the dependance of the variances on Θ

$$\mu_x = \sigma_x(\Theta)\, \phi^{-1}\left(\frac{v}{v+h}\right) \qquad (3.17)$$

$$\Theta^2 = \frac{\beta}{\alpha} \frac{\sigma_x(\Theta)}{\sigma_u(\Theta)} \qquad (3.18)$$

where

$$\alpha := \sqrt{2\pi}\, \phi'(y) = e^{-\frac{1}{2} y^2}, \quad y := \phi^{-1}\left(\frac{v}{h+v}\right) \quad \text{and} \quad \beta := \frac{p+q}{h+v} \qquad (3.19)$$

Equs. (3.17) and (3.18) are expressions (3.6) and (3.9) for the above special cost structure. (It now remains to calculate σ_x (Θ) and σ_u (Θ) for special demand sequences : see next section). Another important cost function, closely related to (3.10) and (3.11), is given by (3.10) and instead of (3.11) one has

$$P(u) = \begin{cases} P + pu & \text{for } u > 0 \\ 0 & \text{for } u = 0 \\ Q - qu & \text{for } u < 0 \end{cases} \tag{3.20}$$

P and Q represent (fixed) production set-up costs. As one can easily see in this case one has to add to the costs (3.12) the (constant) term $\frac{P+Q}{2}$. This expression, of course, will not affect our basic equations (3.17) and (3.18). Hence one obtains the same linear policy irrespective of the presence of set-up costs. This already indicates that for problems having set-up costs the linear policy will in general be a poor approximation (but see Sec. 4.2).

3.3.2 Probability Constraints

Often it will be extraordinary difficult to specify the negative branch of the cost function (3.10). What seems to be easier, however, is to specify a certain service level. There are many definitions of different kinds of service levels used in literature [11] and industry. We shall here concentrate on a service level defined by the probability of being out of stock; i.e. Prob $\{x \leq o\} = l_\alpha$, where l_α is called "α-service level". Instead of (3.10) and (3.11) we now have

$$I(x) = hx \quad \text{for } x \geq 0 \tag{3.21}$$

$$\text{Prob } \{x \leq 0\} = l_\alpha \tag{3.22}$$

$$P(u) = \begin{cases} pu & \text{for } u \geq 0 \\ -qu & \text{for } u < 0 \end{cases} \tag{3.23}$$

Since stock on hand is a Gaussian random variable, (3.22) can also be written

$$1 - \phi(y) = l_\alpha \tag{3.22a}$$

Hence, in view of (3.12) mean costs are given by

$$C = h \left\{ \mu_x \, \phi(y) + \sigma_x \, \phi'(y) \right\} + \frac{p+q}{\sqrt{2\pi}} \sigma_u + \lambda \, (1-\phi(y) - 1_\alpha) \qquad (3.24)$$

where λ is a Lagrange parameter.
Differentiating with respect to λ, μ_x, σ_x^2 and σ_u^2 now leads to

$$\frac{\partial C}{\partial \lambda} = 1 - \phi(y) - \alpha \qquad (3.25)$$

$$\frac{\partial C}{\partial \mu_x} = h \, \phi(y) - \lambda \, \phi'(y) \, \frac{1}{\sigma_x} \qquad (3.26)$$

$$\frac{\partial C}{\partial \sigma_x^2} = \frac{1}{2} h \, \phi'(y) \, \frac{1}{\sigma_x} + \lambda \, \phi'(y) \, \frac{\mu_x}{2\sigma_x^3}$$

$$\frac{\partial C}{\partial \sigma_u^2} = \frac{p+q}{2\sqrt{2\pi}} \, \frac{1}{\sigma_u}$$

Hence, necessary conditions imply

$$y = \phi^{-1}(1-\alpha) \qquad (3.27)$$

$$\lambda = h \, \sigma_x \, \frac{\phi(y)}{\phi'(y)} \qquad (3.28)$$

$$\Theta^2 = \frac{\dfrac{\partial C}{\partial \sigma_u^2}}{\dfrac{\partial C}{\partial \sigma_x^2}} = \frac{p+q}{\sqrt{2\pi}\, h} \, \frac{1}{\phi'(y) + y\phi(y)} \, \frac{\sigma_x}{\sigma_u}$$

$$= \frac{p+q}{\sqrt{2\bar{u}}\, h} \, \frac{1}{\phi'(\phi^{-1}(1-\alpha)) + (1-\alpha)\phi^{-1}(1-\alpha)} \, \frac{\sigma_x}{\sigma_u}$$

or

$$\Theta^2 = \frac{\beta}{\delta} \, \frac{\sigma_x(\Theta)}{\sigma_u(\Theta)} \qquad (3.29)$$

with

$$\delta := \phi'(\phi^{-1}(1-\alpha)) + (1-\alpha) \, \phi^{-1}(1-\alpha) \qquad (3.30)$$

and

$$\beta := \frac{p+q}{\sqrt{2\bar{u}}\, h}$$

Like (3.17) and (3.18) Equs. (3.28) and (3.29) are specifications of the general expressions (3.6) and (3.9). The similarity of the necessary conditions (3.17), (3.18) and (3.28), (3.29) is obvious. A direct analogy, however, could be constructed if we replaced the proportional penalty costs by constant costs, i.e. if we had instead of (3.21) and (3.22)

$$I(x) = \begin{cases} h\ x & x \geq 0 \\ p_f & x < 0 \ (p_f\text{: fixed penalty costs}) \end{cases} \quad (4.32)$$

In this case the expected costs are given by

$$C = p_f(1-\phi(y)) + h\ (\mu_x\ \phi(y) + \sigma_x\ \phi'(y)) + \frac{p+q}{\sqrt{2\pi}}\ \sigma_u^{\ 2} \quad (3.22)$$

with the necessary condition

$$\frac{\partial C}{\partial \mu_x} = h\ \phi(y) - p_f\,\phi'(y) \cdot \frac{1}{\sigma_x} = 0 \quad (3.33)$$

leading to

$$p_f = h\ \sigma_x\ \frac{\phi(y)}{\phi'(y)} \quad (3.34)$$

Hence, comparing (3.34) with (3.28), λ turns out to be identical with , i.e. λ represents constant inventory penalty costs.

3.3.3 A Production Smoothing Problem

A cost function which does no longer fit completely the general pattern defined in Chap. 1 is given by ([14])

$$I(x_k) = \begin{cases} h\ x_k & \text{for } x_k \geq 0 \\ -v\ x_k & \text{for } x_k < 0 \end{cases} \quad (3.35)$$

$$P(u_k, u_{k-1}) = p\,|u_k - u_{k-1}| \tag{3.36}$$

In contrary to our former production costs Equ.(3.36) describes costs to be dependent on the last production level, which for a genuine production smoothing problem seems to be a reasonable assumption.
Defining

$$D_k := u_{k+1} - u_k \tag{3.37}$$

expected costs are given by (see (3.12))

$$\begin{aligned} C &= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} I(x) \exp\left\{-\frac{1}{2\sigma_x^2}(x-\mu_x)^2\right\} dx \\ &+ \frac{1}{\sqrt{2\pi}\,\sigma_D} \int_{-\infty}^{\infty} p\,|D| \exp\left\{\frac{1}{2\sigma_D^2} D^2\right\} dD \\ &= -v\mu_x + (h+v)\left\{\mu_x\,\phi(y) + \sigma_x\,\phi'(y)\right\} + \frac{2p}{\sqrt{2\pi}}\sigma_D \end{aligned} \tag{3.38}$$

Following exactly the same arguments as in Sec. 3.3.1 one obtains instead of (3.17) and (3.18)

$$\mu_x = \sigma_x\;(\Theta)\;\phi^{-1}\left(\frac{v}{v+h}\right) \tag{3.39}$$

$$\Theta^2 = \frac{\beta}{\alpha}\;\frac{\sigma_x\;(\Theta)}{\sigma_D\;(\Theta)} \tag{3.40}$$

where β is now given by $\beta := \frac{2p}{h+v}$

3.4 Special Stochastic Demand Sequences

As we know from our proposition stated in Sec. 3.2 the specification of Equs. (3.6) and (3.9) is only one step towards the solution of the total problem. Two important steps still remain. First, we have to solve the quadratic problem giving us σ_x and σ_u as functions of Θ. Secondly, Equ.(3.9) has to be solved for Θ^2.

We shall concentrate on two sequences of demand. One will be an uncorrelated sequence of random variables (white noise) and the other an exponentially correlated sequence of demand (Gauss - Markov sequence). Both sequences have already been introduced in Chap.2.

Also we shall concentrate on the case of piecewise linear costs dealt with in Sec. 3.3.1. Hence, Sec. 3.4 will derive solutions of (3.17) and (3.18) in the cases of white noise (Sec.3.4.1) and of a Gauss-Markov sequence (Sec. 3.4.2).

3.4.1 Non - Correlated Demand

Solving the adjoined quadratic variation problem (3.8) for the white noise case we arrive at formulas already given in Sec.2.4. These formulas may again be summarized for better reference. The (mean square) optimal policy as a function of Θ is given by

$$u_k(\Theta) = (\gamma(\Theta) - 1)\, x_k \tag{3.41}$$

and the variances are known to be (see (2.66) and (2.67))

$$\sigma_x^2(\Theta) = \frac{1}{1-\gamma^2(\Theta)}\,\sigma^2 \tag{3.42}$$

$$\sigma_u^2(\Theta) = \frac{1-\gamma(\Theta)}{1+\gamma(\Theta)}\,\sigma^2 \tag{3.43}$$

where

$$\gamma(\Theta) = \frac{1}{2\Theta^2}\left(1+2\Theta^2 - \sqrt{1+4\Theta^2}\right). \tag{3.44}$$

Substituting (3.42) and (3.43) into (3.18) one obtains

$$\Theta^{*2} = \frac{\beta}{\alpha}\;\frac{1}{1-\gamma(\Theta^*)}$$

or, in view of (3.44)

$$\Theta^{*2} = \frac{\beta\,(\beta+\alpha)}{\alpha^2} \tag{3.45}$$

Hence,

$$\gamma^* := \gamma(\Theta^*) = \frac{\beta}{\alpha+\beta} \tag{3.46}$$

which implies

$$\sigma_x^{*2} := \sigma_x^2 \; (\Theta^*) = \frac{(\alpha+\beta)^2}{\alpha(\alpha+2\beta)} \; \sigma^2 \tag{3.47}$$

$$\sigma_u^{*2} := \sigma_u^2 \; (\Theta^*) = \frac{\alpha}{\alpha+2\beta} \sigma^2 \tag{3.48}$$

and from (3.17)

$$\mu_x^* := \mu_x \; (\Theta^*) = \sigma_x^* \; \phi^{-1} \left(\frac{v}{h+v}\right) \tag{3.49}$$

Finally, in view of (3.41) and (3.2), the optimal ("non-quadratic") policy is given by

$$u_k^* := u_k(\Theta^*) = \left(\gamma(\Theta^*)-1\right) \left(x_k-\mu_x^*\right) = -\frac{\alpha}{\alpha+\beta} (x_k-\mu_x^*) \tag{3.50}$$

The optimal costs can now be calculated from (3.12). Rewriting (3.12) one obtains for the optimal costs C^*

$$C^* = \mu_x \left[-v + (h+v) \; \phi(y^*)\right] + (h+v) \; \phi'(y^*) \; \sigma_x^* + \frac{p+q}{\sqrt{2\pi}} \sigma_u^*$$

which, taking advantage of the necessary condition (3.6a), reduces to

$$C^* = (h+v) \; \phi'(y^*) \; \sigma_x^* + \frac{p+q}{\sqrt{2\pi}} \; \sigma_u^*$$

or, in view of definitions (3.19)

$$C^* = \frac{h+v}{\sqrt{2\pi}} \left[\alpha \; \sigma_x^* + \beta \; \sigma_u^*\right] \tag{3.51}$$

Substituting (3.47) and (3.48) one finally has

$$C^* = \frac{h+v}{\sqrt{2\pi}} \; \sqrt{\alpha(\alpha+2\beta)} \; \sigma \tag{3.52}$$

Policy (3.50) may be illustrated as follows (Fig.3.2)

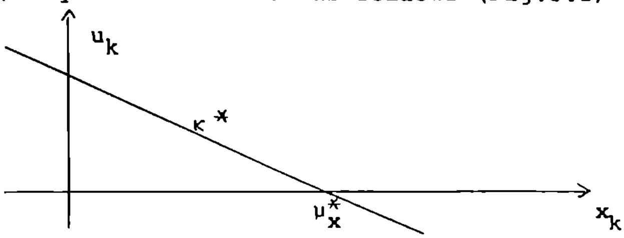


Fig. 3.2 Linear Policy (3.50)

The parameter μ_x^* may be interpreted as an optimal "dynamic safety stock" [19]. The slope κ^* has been used as an abriviation, since from (3.50)

$$\kappa^* = - \frac{\alpha}{\alpha+\beta} \tag{3.53}$$

Note that $-1 \leq \kappa^* \leq 0$

In discussing the above results somewhat further let us first consider some special cases of cost parameters.

(1) $h = v$ implies $\mu_x^* = 0$, i.e. the optimal safety inventory should be zero; a result which was reasonably to be expected.

(2) $h = v = p = q$ implies $u_k^* = -\frac{1}{2} x_k$ which reflects the complete summetry of the model.

(3) $p = q = 0$ implies slope $\kappa^* = -1$

(4) $p \to \infty$ or $q \to \infty$ implies $u_k^* \to 0$

Next, let us give some results of a sensitivity analysis of some of the above quantities with respect to certain cost parameters.

(1) For increasing p (and the same holds for q) it can be shown [8]

$$\frac{\partial \kappa^*}{\partial p} > 0 \text{ and } \frac{\partial \mu^*}{\partial p} \begin{cases} > 0 \text{ for } v > h \\ = 0 \text{ for } v = h \\ < 0 \text{ for } v < h \end{cases}$$

(2) For increasing h one obtains

$$\frac{\partial \kappa^*}{\partial h} < 0 \text{ and } \frac{\partial \mu^*}{\partial h} < 0$$

(3) For increasing v one finds

$$\frac{\partial \kappa^*}{\partial v} < 0 \text{ and } \frac{\partial \mu^*}{\partial v} > 0$$

3.4.2 Exponentially Correlated Demand

As in the non-correlated case the results for the adjoined quadratic problem have already been given in Sec. 2.4. Again (2.78) and (2.79) have (as functions of Θ) to be substituted into (3.17) and (3.18) from which Θ^* and μ_x^* and all other optimal quantities of interest follow. The calculation of Θ^* can, of course, no longer be performed analytically. However, numerically it is an easy and straight forward task to determine zeros of the polynomical in Θ or in λ (Θ) given in (3.18)

Let us now summarize the analytic results of Sec. 2.4.
In view of (3.2) and (2.62) the ("non-quadratic") optimal policy is given by

$$u_k^* = \left(\gamma(\Theta^*)-1\right) \left\{ x_k-\mu_x^* - \sum_{i=0}^{\infty} \gamma^i(\Theta^*)\ \hat{r}_{k-1}\ (i+1)\right\} \qquad (3.54)$$

Note that the white noise case and the Gauss-Markov case essentially differ in the last term in (3.54) which gives an exponentially weighted sequence of forecasts of future demand.*) As we already know $\hat{r}_{k-1}(i+1)$ are called <u>dynamic certainty equivalents</u> (of the adjoined quadratic problem). Hence, also in the non-quadratic case dynamic certainty equivalents exist in the sense discussed at the end of Sec. 3.2.
The optimal costs are again given by (3.51).

$$C^* = \frac{h+v}{\sqrt{2\pi}} \left[\alpha\ \sigma_x\ (\Theta^*) + \beta\ \sigma_u\ (\Theta^*)\right] \qquad (3.55)$$

with the optimal variances (see (2.78) and (2.79))

$$\sigma_x^2(\Theta^*) = \frac{\sigma_r^2}{\left(1-\gamma^2(\Theta^*)\right)\left(1-a\gamma(\Theta^*)\right)^3} \Big\{(1-a^2) + \gamma(\Theta^*)(-3a+3a^2) + \gamma^2(\Theta^*)(2a+2a^2-4a^3) + \gamma^3(\Theta^*)(-2a^2+2a^3)\Big\} \qquad (3.56)$$

---

*) Note that μ_x^* and $\gamma(\Theta^*)$ are of course not the same as in (3.50)

$$\sigma_u^{\,2}(\Theta^*) = \frac{\sigma_r^{\,2}\ (\gamma(\Theta^*)-1)^2}{(1-\gamma^2(\Theta^*))\ (1-a\gamma(\Theta^*))} \Big\{(1+2a-2a^3)+\gamma(\Theta^*)\ (-3a-2a^2+4a^3) + \gamma^2(\Theta^*)\ (a^2-2a^3) + \gamma^3(\Theta^*)\,a^3\Big\} \qquad (3.57)$$

Numerical results are given in the following table [20]

Table 3.1

$\sigma^2 = (1-a^2)\ \sigma_r^{\,2} = 1,\ v = 2h,\ p = q$

	a = 0.1			a = 0.5			a = 0.9		
p = q	μ_x^*	$1-\gamma^*$	c^*	μ_x^*	$1-\gamma^*$	c^*	μ_x^*	$1-\gamma^*$	c^*
0,0 h	0,43	1.00	1.091	0,43	1.00	1.091	0,43	1.00	1.091
0,1 h	0,43	0.94	1.175	0,43	0.95	1.210	0,43	0.97	1.302
1,0 h	0,48	0.60	1.787	0,47	0.69	2.170	0,44	0.81	3.174
10,0 h	0,97	0.12	4.730	1,34	0.14	7.769	0,80	0.34	20.756

Remark

In a recent paper Gaalman [6] studied the multi-variate case by modern control-analytic methods. Since one of our main objects is the investigation of the validity of the linear decision rule approximation we shall concentrate on scalar models with white noise and Gauss-Markov processes as demand sequences. This limitation is necessary since DP solutions which will be presented in the next chapter usually cannot be derived for higher dimensional state spaces and more complicated disturbance processes.

3.5 A direct Approach solving a LNQ-Problem

A slightly more direct approach to the LNQ-problem can be applied by exploiting from the outset the knowledge of the structure of the optimal linear policy. In fact, knowing the type of the stochastic demand sequence the optimal policy is given immediately. For an uncorrelated sequence, e.g., one has

$$u_k = \kappa \ (x_k - \mu) \tag{3.58}$$

with parameters κ and μ which have to be determined by optimizing the cost criterion of the problem.
For an autoregressive process of order ν, e.g., one has

$$u_k = \alpha + \alpha_o \ x_k + \alpha_1 \ r_{k-1} + \text{---} + \alpha_\nu \ r_{k-\nu} \tag{3.59}$$

(See Appendix to Chap.2)

The general procedure of the "direct approach" now is as follows. First, calculate variances σ_x^2 and σ_u^2 as functions of the unknown policy parameters, say α, α_o,---, α_ν. This could in principle be done by solving a stationary Matrix Riccati Equation. Secondly, substitute these variances into the cost criterion C $(\sigma_x^2, \sigma_u^2, \mu_x)$. Finally, optimize C with respect to α, α_o,---, α_ν.
The "direct approach" appears to be somewhat less sophisticated than the theory developed in Sec.3.2. It should, however, be clear that Sec. 3.2 provides a deeper insight into the general nature of the problem.
We shall illustrate the procedure for an uncorrelated sequence of demand leading to the general linear policy (3.58). Two models will be considered. First, in Sec. 3.5.1, the model described in Sec. 3.3.1 having piecewise linear costs will again be investigated and secondly, in Sec. 3.5.2, the production smoothing problem of Sec. 3.3.3 will be studied.

3.5.1 Piecewise linear Costs

Following the 3 steps indicated in the last section we first calculate σ_x^2 and σ_u^2 as functions of κ. From Sec. 2.4 we readily have (setting $\gamma-1 = \kappa$)

$$\sigma_x^2(\kappa) = -\frac{1}{\kappa(\kappa+2)}\sigma^2 \tag{3.60}$$

and

$$\sigma_u^2(\kappa) = -\frac{\kappa}{\kappa+2}\sigma^2 \tag{3.61}$$

Substituting, in a second step, these variances into the cost criterion (3.12) gives

$$C = -v\mu + (h+v)\left\{\mu\ \Phi(y) + \sigma_x(\kappa)\ \Phi'(y)\right\} + \frac{p+q}{\sqrt{2\pi}}\sigma_u(\kappa) \tag{3.62}$$

where $y := \frac{\sigma_x(\kappa)}{\mu}$ and, of course, $\mu = E(x_k) =: \mu_x$

Finally, differentiating (3.62) with respect to the two policy parameters μ and κ and setting equal to zero, one obtains (see derivation of (3.17) and (3.18))

$$\mu = \sigma_x(\kappa)\ \Phi^{-1}\left(\frac{v}{h+v}\right) \tag{3.63}$$

and

$$\frac{dC}{d\kappa} = \frac{\partial C}{\partial\sigma_x^2}\frac{d\sigma_x^2}{d\kappa} + \frac{\partial C}{\partial\sigma_u^2}\frac{d\sigma_u^2}{d\kappa}$$

$$= -(h+v)\ \Phi'(y)\cdot\sigma_x^2(\kappa)\ \frac{(\kappa+1)}{\kappa(\kappa+2)} + \frac{p+q}{\sqrt{2\pi}}\sigma_u(\kappa)\frac{1}{\kappa(\kappa+2)} = 0$$

or

$$\kappa^* = -\frac{\alpha}{\alpha+\beta} \tag{3.64}$$

with α and β defined by (3.19). Substituting κ^* in (3.63) one obtains the optimal μ

$$\mu^* = \frac{(\alpha+\beta)\ \sigma}{\sqrt{\alpha(\alpha+2\beta)}}\ \Phi^{-1}\left(\frac{v}{v+h}\right) \tag{3.65}$$

Hence, the optimal policy is given by

$$u_k = \kappa^*(x_k-\mu^*) \tag{3.66}$$

with κ^* and μ^* determined by (3.64) and (3.65) from which optimal costs follow immediately (see also (3.53))

3.5.2 Production Smoothing Problem

The dynamics of the production smoothing model as defined in Sec. 3.3.3 is given by the two equations

$$\begin{aligned} x_{k+1} &= x_k + u_k - r_k \\ u_{k+1} &= u_k + D_k \end{aligned} \tag{3.67}$$

or, in matrix notation

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} D_k - \begin{bmatrix} 1 \\ 0 \end{bmatrix} r_k \tag{3.68}$$

Defining $\begin{pmatrix} x_k \\ u_k \end{pmatrix}$ as an (extended) state vector, the optimal policy is immediately given by (see Appendix to Chap.2)

$$D_k = \kappa_1 u_k + \kappa_2 (x_k-\mu) \tag{3.69}$$

where μ turns again out to be mean inventory. Policy (3.69) says that a charge-over in production depends on stock on hand and on the last production level.

The variances of x_k and D_k which have to be substituted into the optimal costs (3.38) may be calculated as follows:

Defining $\bar{x}_k = x_k-\mu$ Equ. (3.67) may be written

$$\bar{x}_{k+1} = \bar{x}_k + u_k - r_k$$

$$u_{k+1} = u_k + \kappa_1 u_k + \kappa_2 \bar{x}_k$$

Using z-transforms, as defined in Sec. 2.4 (see footnote), one has

$$\begin{aligned} z\,x(z) &= x(z) + u(z) - r(z) \\ z\,u(z) &= (1+\kappa_1)\,u(z) + \kappa_2\,x(z) \end{aligned} \tag{3.70}$$

or,

$$x(z) = \frac{(1+\kappa_1) - z}{z^2 - (2+\kappa_1)z + (1+\kappa_1-\kappa_2)}\, r(z) \tag{3.71}$$

and, in view of (3.37)

$$D(z) = \frac{(1-z)\ \kappa_2}{z^2 - (2+\kappa_1)z + (1+\kappa_1-\kappa_2)}\ r(z) \tag{3.72}$$

As in Sec. 2.4 (see (2.75) and (2.76)) the variances are now given by

$$\sigma_x^{\ 2} = \sigma^2 \frac{1}{2\pi j} \oint z^{-1} \frac{(1+\kappa_1)-z}{z^2-(2+\kappa_1)z+(1+\kappa_1-\kappa_2)} \frac{(1+\kappa_1)-z^{-1}}{z^2-(2+\kappa_1)z^{-1}+(1+\kappa_1+\kappa_2)}\, dz \tag{3.73}$$

$$\sigma_D^{\ 2} = \sigma^2 \frac{\kappa_2^{\ 2}}{2\pi j} \oint z^{-1} \frac{1-z}{z^2-(2+\kappa_1)z+(1+\kappa_1-\kappa_2)} \frac{1-z^{-1}}{z^2-(2+\kappa_1)z^{-1}+(1+\kappa_1-\kappa_2)}\, dz \tag{3.74}$$

Using a table, as given in [10], one readily obtains

$$\sigma_x^{\ 2} = \frac{(2+2\kappa_1+\kappa_1^{\ 2})(2+\kappa_1-\kappa_2)-2\ (1+\kappa_1)(2+\kappa_1)}{(\kappa_2-\kappa_1)\left[(2+\kappa_1-\kappa_2)^2 - (2+\kappa_1)^2\right]}\ \sigma^2 \tag{3.75}$$

and

$$\sigma_D^{\ 2} = \frac{-\ 2\kappa_2^{\ 2}}{(\kappa_2-\kappa_1)\left[(2+\kappa_1-\kappa_2)^2 - (2+\kappa_1)^2\right]}\ \sigma^2 \tag{3.76}$$

These expressions have to be substituted into the expected costs (3.38) which then are to be optimized with respect to μ, κ_1 and κ_2. This may again be done analytically as in the case of Sec.3.5.1. Or, alternatively, one can take necessary conditions $\frac{\partial C}{\partial \kappa_1} = 0$ and $\frac{\partial C}{\partial \kappa_2} = 0$ and has to solve these equations numerically. A third procedure could be applied by optimizing $C = C(\mu, \kappa_1, \kappa_2)$ numerically without relying on the necessary conditions.

3.6 Appendix to Chapter 3

The Normality Condition

In deriving the best linear policy in Sec. 3.2 we assumed the demand sequence to be Gaussian. We shall now show that in cases when this assumption does not hold the effect on our results may in general be disregarded.

Let us proceed as follows. First we derive an integral equation for the stationary probability distribution F(x) of inventory X assuming a general (non-Gaussian) demand sequence. This integral equation can in general not be solved analytically. Consequently, we represent F(x) by a Gram Charlier expansion. It turns out that generally only the first terms involving mean and variance are of importance. Screwness and curtosis usually can be disregarded so that only the "normal part" of a demand probability distribution is of importance.

The derivation can only be sketched here. For a more comprehensive presentation see [4].

Let us again consider the general model stated in Sec. 3.1 with costs (3.20) and an independent non-Gaussian stationary sequence of demand with mean $E\{r_k\} = \mu$. The following discussion will be restricted to the case P = Q = 0. For more general results see [4]. We know that a linear policy has the general structure

$$u(x) = \kappa(x - \bar{\mu}) \tag{3.77}$$

Hence, the balance equation may be written

$$x_{k+1} = \kappa_1 x_k - \kappa\bar{\mu} - r_k \tag{3.78}$$

where $\kappa_1 = 1 + \kappa$

For the mean values of x_k, μ_k, one has

$$\mu_{k+1} = \kappa_1 \mu_k - \kappa\bar{\mu} - \mu \tag{3.79}$$

or

$$\mu_{k+1} = - \frac{1-\kappa_1^{\;k}}{1-\kappa_1} \quad (\kappa\bar{\mu} + \mu) + \kappa_1^{\;k} \mu_1 \tag{3.80}$$

Assuming $-1 < \kappa_1 < 1$, limiting mean stock is given by

$$\mu_\infty := \lim_{k \to \infty} \mu_k = \bar{\mu} + \frac{\mu}{\kappa} \tag{3.81}$$

(Note that for $\mu = 0$ one again finds the result of Sec. 3.2 : $\mu_\infty = \bar{\mu} = \mu_x$).
Since we are only interested in a stationary analysis it is convenient to introduce reduced variables

$$x_k^{\;o} := x_k - \mu_k \text{ and } r_k^{\;o} := r_k - \mu \tag{3.82}$$

resulting in

$$x_{k+1}^{o} = \kappa_1 \; x_k^{\;o} - r_k^{\;o} \tag{3.81}$$

Since x_k and r_k are stochastically independent (3.81) gives us immediately a relation between the probability distribution functions $F_{k+1}^{o}(x)$ and $F^{o}{}_{k}(x)$ of $x^{o}{}_{k+1}$ and x_k respectively,

$$F_{k+1}^{o}(x) = \int_{-\infty}^{\infty} F_k^{o} \quad \frac{x+z}{\kappa_1} \quad \varphi^{o}(z)dz \tag{3.82}$$

In the homogenous convolution integral equation (3.82) $\varphi^{o}(z)$ is the probability density of $r^{o}{}_{k}$.
Letting $k \to \infty$ Equ. (3.82) becomes [4]

$$F^{o}(x) = \int_{-\infty}^{\infty} F^{o} \quad \frac{x+z}{\kappa_1} \quad \varphi^{o}(z) \; dz \tag{3.83}$$

where $F^{o}(x)$ is the limiting distribution function of x. Generally, integral equation (3.83) cannot be solved analytically. However, if $\varphi^{o}(z)$ is a Gaussian density, $F^{o}(x)$ is also Gaussian and the

results of earlier sections can be confirmed.
One may try to solve (3.83) approximately. This can be achieved by a Gram Charlier expansion. Let $F^s(x) = F^o(x \cdot \sigma_x)$ be the standardized distribution function of x. Hence the Gram Charlier expansion is defined by

$$F^s(x) = \sum_{i=o}^{\infty} \frac{C_i(F^s)}{i!} \phi_N^{(i)}(x) \tag{3.84}$$

where $\phi_N^{(i)}$ is the i-th derivation of the standardized normal distribution function. For the first 4 coefficients one has, e.g.,

$$\begin{aligned} &C_o = 1, \; C_1 = C_2 = 0 \\ &C_3(F^s) = -\gamma_1^{F^s} \qquad \text{(screwness of } F^s(x)) \\ &C_4(F^s) = \gamma_2^{F^s} \qquad \text{(curtosis of } F^s(x)) \end{aligned} \tag{3.85}$$

A suitable approximation might already be

$$F^s(x) \approx \phi_N(x) + \frac{C_3(F^s)}{3!} \phi_N^{(3)}(x) + \frac{C_4(F^s)}{4!} \phi_N^{(4)}(x) \tag{3.86}$$

Up to now screwness and curtosis of $F^s(x)$ are still unknown. They will be determined by expanding both sides of the integral equation (3.83) in a Gram Charlier series. It can be shown

$$C_3(F^s) = -\frac{(1-\kappa_1^2)^{\frac{3}{2}}}{1-K_1^3} \; C_3(\phi^s) \tag{3.87}$$

$$C_4(F^s) = \frac{1-K^2}{1+K_1^2} \; C_4(\phi^s) \tag{3.88}$$

where $C_3(\phi^s)$ and $C_4(\phi^s)$ are (negative) screwness and curtosis of the standardized distribution function $\phi^s(r)$ of demand. (For coefficients C_i, $i > 4$, similar results can be obtained [4]). Equs. (3.87) and (3.88) show an interesting and important result. First one has

$$|C_i(F^s)| \leq |C_i(\phi^s)| \qquad (i = 3, 4) \tag{3.89}$$

i.e. if for a demand distribution screwness and curtosis are of no great importance, the same holds a fortiori for the stationary distribution function of x. Hence $F^s(x) \approx \phi_N(x)$ would be a good approximation and, as our numerical results will show, the same reasonable approximation holds for the costs.

Secondly, as one observes from (3.87) and (3.88) for $K_1 \approx 1$ $F^s(x)$, and hence $F^o(x)$, is nearly Gaussian irrespective of the form of the distribution function of demand. $K_1 \to 1$ implies $K \to 0$, i.e. for flat linear decision rules (see e.g. Fig. 3.2) $\phi_N(x)$ represents a good approximation; or, stated in terms of cost parameters, p and q should be larger than h and v.

Average costs and optimality conditions

Let us now study the effect of the second and third term in (3.86) on the optimal costs. Mean average costs, as a function of μ and κ are given by

$$C(\mu, \kappa) = E\{L(x) + P(u)\}$$

Substituting for the costs Equ. (3.20) (with P = Q = O) and for the distribution function, $F^o(x)$ from (3.86), one finds

$$\begin{aligned}\frac{1}{\sigma x} C(\bar{\mu},\kappa) \approx\ & (p+q)(-\kappa)\ \varphi_N(0)\{1 - \frac{1}{24} C_4(F^s)\} \\ & + (h+v)\{\bar{\mu}_\infty \phi_N(\bar{\mu}_x) + \varphi_N(\bar{\mu}_\infty)\ [1+ \frac{1}{6}\ \bar{\mu}_\infty\ C_3(F^s) \\ & - \frac{1}{24} C_4(F^o)\ (1-\bar{\mu}^2_\infty)]\} \\ & - v\ \bar{\mu}_\infty,\end{aligned} \tag{3.90}$$

with $\varphi_N(x)$ being the Gaussian density function and $\bar{\mu}_\infty := \frac{\mu_\infty}{\sigma_\infty}$ (σ_∞ : standard deviation of $F^O(x)$).
Differentiating $C(\bar{\mu},K)$ with respect to $\bar{\mu}$ and K, or, equivalently, with respect to $\bar{\mu}_\infty$ and K_1, finally yields the approximative necessary conditions

$$\frac{v}{v+h} \approx \Phi_N(\bar{\mu}_\infty) + \varphi_N(\bar{\mu}_\infty)\{ \frac{1}{6} C_3(F^S)(1-\bar{\mu}_\infty^2) + \frac{1}{24} C_4(F^S)\bar{\mu}_\infty(3-\bar{\mu}_\infty^3)\} \tag{3.91}$$

and

$$\frac{p+q}{v+h} \varphi_N(0)(1-K_1)\left\{ 1 - \frac{1}{24} C_4(F^S) \frac{K_1{}^2+4K_1+1}{1+K_1{}^2} \right\}$$
$$\approx K_1 \varphi_N(\bar{\mu}_\infty) \left\{ 1 + \frac{1}{6} C_3(F^S) \bar{\mu}_\infty\left[\bar{\mu}_\infty^2 - \frac{3}{1+K_1+K_1{}^2}\right] + \right. \tag{3.92}$$
$$\left. + \frac{1}{24} C_4(F^S)\left[\bar{\mu}_\infty{}^4 - 2\bar{\mu}_\infty^2 - 1 + \frac{4(1-\bar{\mu}_\infty^2)}{1+K_1{}^2} \right]\right\}$$

(As one easily realizes, for normally distributed demand, i.e. $\Phi(v) = \Phi_N(v)$, these conditions reduce to (3.63) and (3.64) respectively)
Conditions (3.91) and (3.92) have to be solved numerically giving approximately optimal values for the policy parameters K and μ. Optimal costs are not determined by (3.90) but by a value iteration procedure. Hence costs are exact with respect to the approximative policy $u = K(x-\mu)$.
Table 3.2 gives the values of K and μ for a forth order Gram Charlier expansion (i.e., taking into account screwness and curtosis). The demand distribution was taken to be a standardized Beta-distribution with parameters a = 10 and b = 2, having the finite support [-8.062; 1.612], mean 0.833, standard deviation 0.103, screwness -0.921 and curtosis 0.789. K and μ are given as a function of cost parameters $\frac{v}{v+h}$ and $\frac{p+q}{v+h}$. In Table 3.2 we chose $\frac{v}{v+h} = 0,333$.

Table 3.2 not only shows the forth order values μ and κ but also μ^o and κ^o being calculated from a second order Gram Charlier expansion which corresponds to a Gaussian approximation

$\frac{p+q}{h+v}$	μ	κ	μ^o	κ^o	D
0.25	- 0.274	- 0.799	- 0.442	- 0.785	0.6
0.50	- 0.301	- 0.664	- 0.461	- 0.646	0.4
0.75	- 0.330	- 0.567	- 0.483	- 0.549	0.3
1.00	- 0.359	- 0.493	- 0.506	- 0.477	0.3
2.00	- 0.462	- 0.322	- 0.593	- 0.313	0.2
3.00	- 0.550	- 0.238	- 0.672	- 0.233	0.1

Table 3.2 Comparison with Gaussian approximation

of the demand distribution. The last column in Table 3.2 shows the relative deviations in costs: $D := \frac{C(\mu^o,\kappa^o) - C(\mu,\kappa)}{C(\mu,\kappa)} \, 100 \, [\%]$.

Even for demand distributions of considerable screwness it is encouraging to notice that a Gaussian approximation does extremely well. This result is also affirmed by the investigations for the pure inventory case in Chap. 6. Hence our assumption of demand to be a Gaussian random series is by no means restrictive.
More detailed and comprehensive results for other values of cost parameters and further Beta-Distributions are given in [4].
The values for D are always of the same order (which shows the strong smoothing property of the linear policy).
In addition, [4] also investigates the suboptimality of the two approximations with respect to the optimal (Dynamic Programming) solution (see also Chap. 4). It can be seen that the Gram Charlier approximation gives satisfactory results. Generally, deviations are not larger than 5 or at most 10 %.

Chapter 4

COMPARISON WITH OPTIMAL DYNAMIC PROGRAMMING SOLUTIONS

In Chap. 3 we established the so called LNQ-approach to inventory production problems having no quadratic performance criterion. This approach, as we know, is suboptimal because of the linearity assumption (3.2). It will now be our main concern to study the effects of this crucial assumption in some detail. I.e. we shall compare the linear approach with the exact solution which will be derived by a dynamic programming procedure. We refer to the models stated in Chap. 1 and Sec. 3.1. For easier reference let us state completely the model we will be concerned with in this Chapter.

(1) $x_k \in \mathbb{R}_1$: stock on hand, $k = 1,2,...$

(2) $u_k \in \mathbb{R}_1$: production (deviation)

(3) $\{r_k\}$: demand sequence (white noise or Gauss-Markov sequence)

(4) $x_{k+1} = x_k + u_k - r_k$

(5) $C = \lim_{N \to \infty} E\left\{\frac{1}{N} \sum_{k=1}^{N} \{I(x_{k+1}) + P(u_k)\} | x_1\right\} \Rightarrow \min.$

where

$$I(x) = \begin{cases} h\,x & \text{for } x \geq 0 \\ -v\,x & \text{for } x < 0 \end{cases} \qquad (4.1)$$

$$P(u) = \begin{cases} P + pu & \text{for } u > 0 \\ 0 & \text{for } u = 0 \\ Q - q\,u & \text{for } u < 0 \end{cases} \qquad (4.2)$$

First, in Sec. 4.1, we shall concentrate on the white noise case having no production set-up costs; i.e. P=Q=O. Then, in Sec. 4.2, P and Q will be taken to be non-zero. Finally, Sec. 4.3 is devoted to the Gauss-Markov case.

4.1 Piecewise Linear Costs (no set up costs: P = Q = O)

Let us first derive a dynamic programming (DP) solution for the above model and let us then, in Sec.4.1.2, compare the LNQ- and the DP-approach numerically. (See also [8])

4.1.1 Dynamic Programming Solution

The DP-solution can only be sketched here (For a more comprehensive representation see [8]). However, the main ideas of deriving the optimal policy and optimal costs will be given.

Defining

$$y_k = x_k + u_k$$

Bellman's functional equation for a finite horizon situation may readily be stated to be

$$f_k(x) = \min_y E\left\{\frac{1}{k}\left[P(y-x) + I\ (y-r) + (k-1)\ f_{k-1}(y-r)\right]\right\} \qquad (4.3)$$

$$f_O(x) \equiv O$$

Here $f_k(x)$ denotes the value function and k the number of steps up to horizon N.

The optimal policy (and optimal costs) for $N \to \infty$ can now be found by applying a value iteration procedure.

Let us first look for the optimizing decision at step k.

Applying the expectation operator E, Equ.(4.3) can also be written

$$f_k(x) = \min_y \frac{1}{k}\left[P(y-x) + H_k(y)\right] \qquad (4.3)$$

where

$$H_k(y) := E\ \{I\ (y-r) + (k-1)\ f_{k-1}\ (y-r)\} \qquad (4.4)$$

In view of (4.2) (P = Q = O), (4.3) may further be reformulated yielding

$$f_k(x) = \min \begin{Bmatrix} \frac{1}{k} \min\limits_{y \geq x} \left\{p(y-x) + H_k(y)\right\} \\ \\ \frac{1}{k} \min\limits_{y<x} \left\{- q(y-x) + H_k(y)\right\} \end{Bmatrix} \qquad (4.5)$$

Now, for the above costs (4.1) and (4.2) it can be shown that expressions $p(y-x)+H_k(y)$ and $-q(y-x) + H_k(y)$ are convex. Hence, there exist optimizing parameters s_k and s_k' defined by

$$\left.\frac{\partial H_k}{\partial y}\right|_{y=s_k} = -p \quad \text{and} \quad \left.\frac{\partial H_k}{\partial y}\right|_{y=s_k'} = q \tag{4.6}$$

The optimal decision is therefore given by

$$y_k(x) = \begin{cases} s_k & \text{for } x < s_k \\ x & \text{for } s_k \le x \le s_k' \\ s_k' & \text{for } x > s_k' \end{cases} \tag{4.7}$$

and the optimal costs are found to be

$$f_k(x) = \begin{cases} \frac{1}{k}\left[p(s_k-x) + H_k(s_k)\right] & \text{for } x < s_k \\ \frac{1}{k} H_k(x) & \text{for } s_k \le x \le s_k' \\ \frac{1}{k}\left[-q(s_k'-x) + H_k(s_k')\right] & \text{for } x > s_k' \end{cases} \tag{4.8}$$

Hence, letting $N\to\infty$, the <u>optimal policy</u>, written again in terms of u_k, turns out to be *)

$$u_k(x) = \begin{cases} s-x_k & \text{for } x < s \\ 0 & \text{for } s \le x \le s' \\ s'-x_k & \text{for } x > s' \end{cases} \tag{4.9}$$

---

*) The convergence of the value iteration to the optimal policy depends, of course, on the values of cost parameters h, v, p and q. However, it was found that in most cases convergence took less then 20 periods. This again shows that an asymptotic ($N\to\infty$) model can in many real situations be regarded as a reasonable approximation to the finite case.

Equ.(4.9) is illustrated by Fig.4.1 below

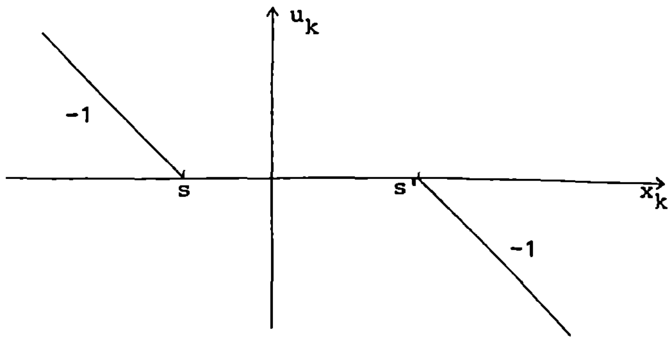


Fig. 4.1 Optimal cash balance policy

The optimal costs could in principle be calculated in forming the limit $N \to \infty$ of value function (4.8). Convergence, however, turns out to be rather poor. Thus, knowing the optimal policy, optimal costs were calculated via the limiting distribution of x. This turned out to be more practicable.

Let c(x) be costs per period and be F(x) the probability distribution of (the asymptotic) x, then expected optimal costs are given by

$$C = \int_{-\infty}^{\infty} c(x) \quad dF(x) \tag{4.10}$$

Let us now determine c(x) and F(x).

Because of (4.9) the balance equation $x_{k+1} = x_k + u_k - r_k$ may be written

$$x_{k+1} = \begin{cases} x_k + (s-x_k) - r = s-r & \text{for } x_k < s \\ x_k - r & \text{for } s \le x_k \le s' \\ x_k + (s'-x_k) - r = s'-r & \text{for } x_k > s \end{cases} \tag{4.11}$$

Hence, expected*) one-period costs are given by

$$c(x_k) = \left\{ \begin{array}{ll} p(s-x_k) & \text{for } x_k < s \\ 0 & \text{for } s \le x_k \le s' \\ -q(s'-x_k) & \text{for } x_k > s' \end{array} \right\} + E\{I(x_{k+1})\} \qquad (4.12)$$

Defining $L(y) := E\{I(y-r)\}$ and dropping the index k (4.12) may finally be written

$$c(x) = \left\{ \begin{array}{ll} p(s-x) + L(s) & \text{for } x < s \\ L(x) & \text{for } s \le x \le s' \\ -q(s'-x) + L(s') & \text{for } x > s' \end{array} \right. \qquad (4.13)$$

The calculation of the asymptotic probability distribution F(x) is more complicated and, for Gaussian demand sequences, can only be accomplished numerically.
Let us first discretize x and r

$$x \in \left\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right\} \text{ where } x^{(i)} = x_{min} + (i-1)\,dx, \quad (i = 1,\ldots,n)$$

$$r \in \left\{r^{(-q)}, r^{(-q+1)}, \ldots, r^{(q)}\right\} \text{ where } r^{i} = r_{min} + (i-1)\,dr, \quad (i = 1,\ldots,q)$$

with dx and dr being appropriate step intervals.
Hence, in view of (4.11), transition probabilities may be defined by

$$P_{ij} := \text{Prob}\left\{x_{k+1} = x_{k+1}^{(j)} \,|\, x_k = x_k^{(i)}\right\} = \left\{ \begin{array}{ll} \varphi_d(s_d - x^{(j)}) & \text{for } x^{(i)} < s_d \\ \varphi_d(x^{(i)} - x^{(j)}) & \text{for } s_d \le x^{(i)} \le s_d' \\ \varphi_d(s_d' - x^{(j)}) & \text{for } x^{(i)} > s_d' \end{array} \right.$$

with φ_d being a discretized Gaussian density function and $s_d \approx s$.

---

*) The expected value has to be taken into account, since inventory costs are attached to the <u>end</u> of the period.

Obviously, $\left\{x_k^{(i)}\right\}$ defines an (ergodic) Markov-chaim. Hence stationary probabilities are given by the well-known formulae

$$\pi_j = \sum_{i=1}^{n} p_{ij}\ \pi_i,\ (j=2,---,n),\ \sum_{j=1}^{n} \pi_j = 1 \qquad (4.15)$$

Together with appropriately discretized costs $c\ (x^{(i)})$, (4.10) can finally be written giving the optimal total costs

$$C = \sum_{j=1}^{n} c(x^{(j)})\ \pi_j \qquad (4.16)$$

4.1.2 Numerical Results

We are now in a position to compare both, the DP and the linear approach. Let us first consider the optimal policies. Fig.4.2 shows both policies in their relation to one another.

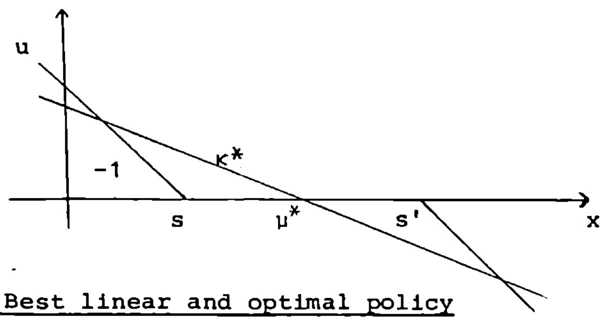


Fig. 4.2 Best linear and optimal policy

The important economic difference between both policies results from the fact that for the linear policy there is no region in which no production action has to be taken. However, if variance of x is large enough (depending in turn on σ and the cost parameters) i.e. at least $\sigma_x \approx s'-s$, then the difference between both policies should not be too important. In fact, numerical results for the optimal costs support this suggestion.

To be more specific, let us denote optimal costs incurred by the linear approach (Equ.(3.52)) by C_L^*. Hence, a relative cost deviation D may be defined by

$$D := \frac{C_L^* - C^*}{C_L^*} \, 100 \, [\%] \qquad (4.17)$$

Table 4.1 below shows D for several values of cost parameters.*)

P=Q \ V	0.5 h	h	1.5 h	2 h	3 h
0.1 h	0.5	0.2	0.2	0.2	0.2
0.5 h	3.7	2.4	2.0	1.7	1.4
1.0 h	6.5	4.8	4.1	3.7	3.1
2.0 h	9.3	7.6	6.9	6.5	6.0
10.0 h	12.9	11.5	11.2	11.8	12.6
20.0 h	13.2	12.0	12.3	12.9	14.2

Table 4.1 D as defined in (4.17)

(Note that equating p and q has only been done for convenience of presentation. It is of no major relevance for the results.)
Table 4.1 clearly shows that for the linear approach deviations from the (overall) optimal policy are considerably small. Even for large production costs compared to inventory costs deviations are not much larger than 10 %. Also, the increase of the deviation slows down.
Considering the fact that for the linear approach the important certainty equivalence property holds (c.o.Chap.3), and that the computational load is considerably smaller than for the (overall) optimal policy one might be well advised to use an LNQ-approach.

4.2 Piecewise Linear Costs (including set-up costs: P and/or Q $\neq$ 0)

We shall now consider the case when P and Q are not equal to zero. Again the optimization problem stated above will be solved without restricting the class of admissible policies to be linear. In Sec. 4.2.2 we shall than compare the (overall) optimal and the LNQ-costs.

*) Note that D does not depend on σ and the dependence on costs is only through cost ratios [8].

4.2.1 Optimal Solution

Again we refer for more detailed results to [8]. As to the author's knowledge in case of $P \neq 0$, $Q \neq 0$ and a Gaussian white noise demand sequence no analytical results as to the structure of the optimal production policy could be obtained. However, it seems to be most likely that as in the pure inventory case (for Gaussian white noise) both set-up costs induce an (s,S)-structure illustrated in Fig. 4.3.

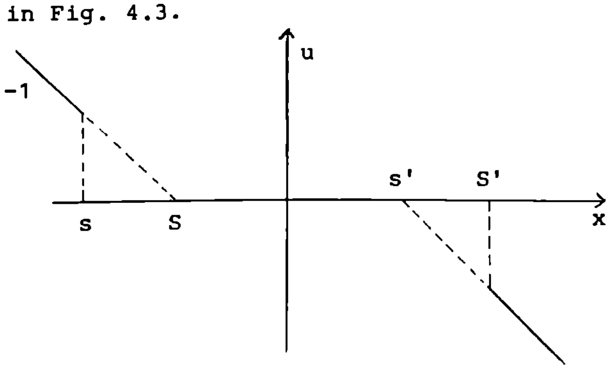


Fig. 4.3 Optimal DP-policy for $P \neq Q$ and $Q \neq 0$

Numerical results which we will present subsequently can be shown to fit Fig. 4.2 fairly well.

Since we are mainly interested in a comparison of optimal costs we shall follow the ideas already used in calculating optimal costs for P=Q=0. The main difference now is that the appropriate analogue to (4.10) has to be optimized.
Following [8] let us again discretize x_k, y_k and r_k

$$x \; \varepsilon \left\{ x^{(1)}, x^{(2)}, ---, x^{(n)} \right\}, \text{ where } x^{(j)} = x_{min} + (j-1)\, dx, \quad (j=1,---,n)$$

$$y \; \varepsilon \left\{ y^{(1)}, y^{(2)}, ---, y^{(n)} \right\}, \text{ where } y^{(l)} = y_{min} + (l-1)\, dy, \quad (l=1,---,n)$$

$$r \; \varepsilon \left\{ r^{(1)}, r^{(2)}, ---, r^{(n)} \right\}, \text{ where } r^{(s)} = r_{min} + (s-1)\, dr, \quad (s = 0, 1, ---, n)$$

with dx, dy and dr being appropriate step intervals. Hence, since

$$x_{k+1}^{(j)} = y_k^{(l)} - r_k^{(s)}$$

the transition probabilities for decision l are given by

$$p_{ij}^{l} = \begin{cases} \text{Prob}\left\{ r^{(s)} \geq y^{(l)} - x^{(j)} \right\} & \text{for } j=1 \\ \text{Prob}\left\{ r^{(s)} = y^{(l)} - x^{(j)} \right\} & \text{for } j=2,3,---,n-1 \\ \text{Prob}\left\{ r^{(s)} \leq y^{(l)} - x^{(j)} \right\} & \text{for } j=n \end{cases} \tag{4.18}$$

Similary to (4.12) the expected discretized one-period costs are given by

$$c_{il} = \begin{cases} -px^{(i)} + P + px^{(l)} + L(x^{(l)}) & \text{for } x^{(i)} < x^{(l)} \\ L(x^{(l)}) & \text{for } x^{(i)} = x^{(l)} \\ q\, x^{(i)} + Q - q\, x^{(l)} + L(x^{(l)}) & \text{for } x^{(i)} > x^{(l)} \end{cases} \tag{4.19}$$

Now, as is well known for the white noise case (Markov-Theorem)

$$y_k = d_k\ (x_k)\ ,$$

or, discretized, for the steady state case

$$y^{(l)} = d\ (x^{(i)}) =: d\ (i)\ , \tag{4.20}$$

where d(.) is called a decision function. Hence, as in Sec.4.1.1, steady state probabilities and optimal costs are given by

$$\pi_j = \sum_{i=1}^{n} \pi_i\ p_{ij}^{d(i)},\ (j=2,---,n),\ \sum_{j=1}^{n} \pi_j = 1 \tag{4.21}$$

and

$$C = \sum_{j=1}^{n} c_{jd(j)}\ \pi_j \tag{4.22}$$

Hence, the optimization problem now consists in finding an optimal decision function (4.20). According to [5] this may be performed by the Simplex-algorithm of Linear Programming.

Setting

$$\pi_j = \sum_{l=1}^{n} x_{jl}$$

the problem (4.21), (4.22) may readily be seen to be a Linear Programming problem of the following structure.

Objective function:

$$C = \sum_{i=1}^{n} \sum_{l=1}^{n} C_{il} \, x_{il} \Rightarrow \min.$$

Side conditions:

$$\sum_{i=1}^{n} \sum_{l=1}^{n} x_{il} = 1$$

$$\sum_{l=1}^{n} x_{jl} - \sum_{i=1}^{n} \sum_{l=1}^{n} x_{il} \, p_{ij}^{l} = 0 \; ; \; j=2,3,---,n$$

Non-negativity constraints

$$x_{jl} \geq 0 \qquad (j,l=1,---,n)$$

Solving this problem the computational load may be reduced considerably by exploiting the special structure of the problem. The results are given in the next section.

4.2.2 Numerical Results

Since, as we know from Sec.3.3.1, the optimal linear policy is not affected by the presence of production set-up costs, a comparison can readily be performed taking into account (see (3.52))

$$C_L^* = \frac{h+v}{\sqrt{2\pi}} \quad \sqrt{\alpha(\alpha+2\beta)} \; \sigma + \frac{P+Q}{2} \; . \qquad (4.23)$$

Again (over-all) optimal costs can be shown (by a simple dimensional analysis) only to depend on cost ratios $\frac{p}{h}$, $\frac{q}{h}$, $\frac{v}{h}$.

Similarly, (for $E\{r_k\} = 0$) only the ratios $\frac{P}{\sigma}$ and $\frac{Q}{\sigma}$ affect C. Hence, choosing the special case v = 3h, the following cost deviations, as defined in (4.17) for $P = Q \neq 0$, can be calculated (see Table 4.2)

P=Q \ p=q	0	h	5 h	10 h
0.0	0	3	10	13
0.1 h σ	1	6	12	14
0.5 h σ	13	18	22	22
1.0 h σ	31	34	34	32
10.0 h σ	286	274	232	200
100.0 h σ	1637	1608	1499	1387

Table 4.2 D as defined by (4.17) for $P = Q \neq 0$

Note that, again for convenience of presentation, we have chosen p=q and Q=P.
As can readily be seen the results are no longer as favorable as in the non-set-up cost case. This is because set-up costs enlarge considerably the region in which no action is taken (see Fig. 4.2)
If, however, σ is sufficiently large, again the LNQ-approach turns out to be a fairly good approximation. This was to be expected, since in the presence of a considerably fluctuating demand sequence production actions will have to be taken in almost each period.
The bad performance of the linear approach, however, can be improved considerably if we enlarge the length of the (inspection) period. This allows for a certain region about μ_x in which generally no actions will be taken. Or, otherwise stated, allowing for a larger period, e.g. aggregating l original periods, amounts to a larger σ ; in fact $\sigma_l^2 := l\sigma_x^2$.
Consequently, if no restraints with respect to the inspection period have to be regarded one might introduce the length of the period as an additional optimization parameter [8].

4.3 Piecewise Linear Costs - Gauss-Markov Case

Up to now we only considered models with non-correlated demand. Optimal DP results were comparatively easy to obtain. Now we shall investigate the Gauss-Markov case. Here optimal DP results have not been known until recently [8a]. This is due to the fact that in the Gauss-Markov case the state space has to be enlarged by one further demension which, as is well known, generally results in considerable computational difficulties. Only the no-set-up cost case shall be discussed here.
Again we give first in Sec. 4.3.1 the DP-result and in Sec. 4.3.2 linear and optimal DP-policy will be compared.

4.3.1 Dynamic Programming Solution

The optimal policy will be obtained in two steps. First the general structure of the policy will be derived with still unknown parameters. In a second step these parameters will be determined.
The optimal policy will for the Gauss-Markov case generally be given by

$$u_k = u_k\ (x_k,\ r_{k-1}) \tag{4.24}$$

For notational simplicity let us drop all indices and define

$$r_{k-1} =: d \tag{4.25}$$

so that

$$u = u\ (x,\ d) \tag{4.26}$$

Similarly to (4.3) Bellman's functional equation can be written

$$f_k(x,d) = \frac{1}{k} \min_{y} \{P(y-x)+L(y,d)+(k-1) \int_{-\infty}^{\infty} f_{k-1}(y-r,r)f(r|d)dr$$

$$f_o(x,d) \equiv 0 \qquad\qquad k=1,2,... \tag{4.27}$$

with conditional expected inventory costs

$$L(y,d) = E\{I(y-r)|d\} = h\int_{-\infty}^{y}(y-r)f(r,d)dr - v\int_{y}^{\infty}(y-r)f(r|d)dr \tag{4.28}$$

and the conditional probability density $f(r|d)$. Since a Gauss-Markov sequence is given by the autoregressive scheme

$$r_k = a\, r_{k-1} + \varepsilon_k \quad (0 \leq a < 1) \tag{4.29}$$

with $\{\varepsilon_k\}$ being white Gaussian noise with probability density $\varphi(\varepsilon)$, $f(r|d)$ can be expressed by

$$f(r|d) = \varphi(r-a\,d) \tag{4.30}$$

As in Sec. 4.1.1 (4.4) let us define

$$H_k(y,d) = L(y,d) + (k-1)\int_{-\infty}^{\infty} f_{k-1}(y-r,r)\ f(r|d)\ dr \tag{4.31}$$

which results in (c. O. (4.5))

$$f_k(x,d) = \frac{1}{k}\ \min\begin{Bmatrix} \min\limits_{y\geq x}\{P(y-x) + H_k\ (y,d)\} \\ \\ \min\limits_{y<x}\{-\ q(y-x) + H_k\ (y,d)\} \end{Bmatrix} \tag{4.32}$$

Again one can show [8a] the convexity of the above expressions from which one concludes that the optimal policy has an (s,s')-structure, i.e. for $k \to \infty$ one has

$$u^*(x,d) = \begin{cases} s(d) - x & \text{for} \quad x < s(d) \\ 0 & \text{for} \quad s(d) \leq x \leq s'(d) \\ s'(d) - x & \text{for} \quad x > s'(d) \end{cases} \tag{4.33}$$

and

$$s(d) \leq s'(d)$$

In the above mentioned second step it now remains to determine s(d) and s'(d). The general structure of the optimal policy (for all cost parameters) is illustrated in Fig. 4.4

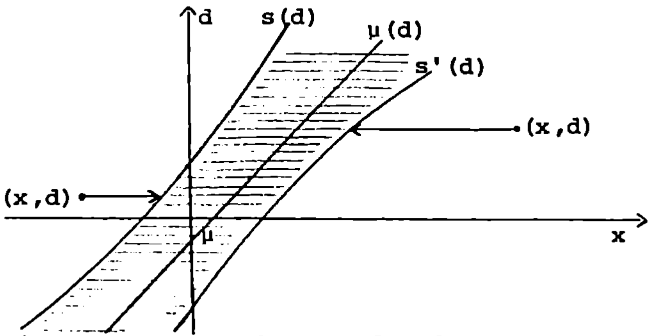


Fig.4.4 Structure of optimal policy

Note that s(d) and s'(d) are monotone in d; similarly the distance s'(d) - s(d) is monotone in |d|.
In the hatched area no action has to be taken. If, however, a point (x,d) is on the left (or right) of the s(d) (s'(d)) line, an amount given by the distance from the s(d) (s'(d)) line has to be produced as shown in Fig. 4.4 (The straight line in Fig. 4.4 will be explained in the next section)

4.3.2 Numerical Results

According to (3.54) and (2.69) the optimal linear policy may be written

$$u_L^* = (\gamma^*-1) \ (x - (\mu^* + \frac{a}{1-\gamma^* a} d) \tag{4.34}$$

or, using a slightly simpler notation

$$u_L^* = \kappa^* \ (x - \mu(d)) \tag{4.35}$$

with $\kappa^* := \gamma^*-1$ and $\mu(d) = \mu^* + \frac{a}{1-\gamma^* a} d$

(Note that for the uncorrelated case (a=0) we find again $\mu(d)=\mu^*$) $\mu(d)$ is illustrated in Fig. 4.4. As in the white noise case (c.o. Fig. 4.2) $\mu(d)$ is situated between s(d) and s'(d).

The relative cost deviation between the optimal DP- and the LNQ-approach may again be measured by $D := \frac{C_L - C}{C_L} 100$ [%]. Table 4.3 gives D for some typical cost parameter constellations.

	v = h				v = 2 h			
p=q	a=0	a=0.1	a=0.5	a=0.9	a=0	a=0.1	a=0.5	a=0.9
0.1 h	0,2	0,2	0,2	0,1	0,2	0,1	0,1	0,1
1.0 h	5	5	5	3	4	4	4	3
10.0 h	12	12	12	12	12	12	13	11

Table 4.3 Cost deviations D

Table 4.3 shows that the influence of the strength of correlation, represented by the value of a, is insignificant. It exhibits that the linear decision rule works in the Gauss-Markov case just as well as in the uncorrelated case. This result may be extended to higher order autoregressive processes. It shows that the deviation of the two policies does not depend primarily on the special type of demand process but on the special cost structure of the problem

Chapter 5

COMPARISON WITH DETERMINISTIC APPROXIMATIONS

In Chapter 3 we derived best linear decision rules in the presence of non-quadratic criteria. These decision rules have been shown to posses the "certainty equivalence property", i.e. in case of linear decision rules it was shown to be not suboptimal to reduce the stochastic sequence of demand to a sequence of (optimal) demand forecasts. (However, to be precise, recall remark on p. 41).
Restricting admissible policies to be linear one has to put up with a loss of optimality. This loss was studied in the last chapter for non-correlated and exponentially correlated demand sequences for which (over all) optimal results were presented.
In this chapter we now study another suboptimal procedure. This procedure is usually met in practice. Instead of restricting the class of admissible policies to be linear and hence allowing for the separation property, one replaces from the outset the sequence of demand by its forecasts and then uses a (non policy restricted) rolling horizon optimization procedure [8b].
Our main object is to compare both suboptimal policies in case of piecewise linear costs. It will be shown that even if we introduce optimal safety stocks the deterministic approach is for all parameter constellations less favorable than the LNQ-approach. This clearly indicates that the deterministic approach usually applied in practice should not be used without taking into account a possible application of a linear decision rule approach. I.e. in spite of obvious shortcomings of a linear decision rule there is its important advantage of allowing for the existence of the separation property.
Let us proceed as follows. First we consider the white noise case, i.e. we assume the sequence of demand $\{r_k\}$ to be a sequence of independent random variables. Then, in Sec.5.2, we will study the exponentiallly correlated case for which a derivation of the "deterministic policy" is given in the appendix.

5.1 White Noise Case

The main idea of the deterministic approximation to a stochastic dynamic optimization problem can be summarized as follows: First one reduces the given stochastic sequence of demand to a sequence of demand forecasts. Then one calculates an optimal production policy with respect to an adjoined "deterministic" criterion. This optimization procedure has to be repeated after each period using updated forecasts. Thus in each period only the first decision will be taken into account. Finally, this sequence of first decisions will be used as an (approximately optimal) policy for the original stochastic case.

Let us now describe the procedure in detail. Suppose for the time being we are at time k = 0. Taking forecasts implies that we have to replace $\{r_k\}$ by a sequence of conditional means $\{r_k\} \to \{\hat{r}_o(k)\}$, where, for the white noise case,

$\hat{r}_o(k) := E\,\{r_k | r_o\} = E\,\{r_k\} = 0.$

Now, since the initial values are assumed to be finite and furthermore the total amount of forecasted demand, $\sum_{k=1}^{\infty} \hat{r}_o(k)$, is zero, projected costs can always be determined to be finite. Therefore, in view of (3.3a), a reasonable adjoined "deterministic" criterion would be

$$C_D = \sum_{k=1}^{\infty} \left\{ P(\bar{u}_k) + I\ (\bar{x}_k + \bar{u}_k) \right\} \Rightarrow \min. \qquad (5.1)$$

where a bar has been introduced in order to distinguish these variables from the stochastic situation.

The first decision of the above deterministic dynamic optimization problem is readily given by

$$U = U_D\ (x_o) = -\ x_o$$

or, for arbitrary starting time k

$$u_k = U_D\ (x_k) = -\ x_k \qquad (5.2)$$

This simple decision function is depicted in Fig. 5.1

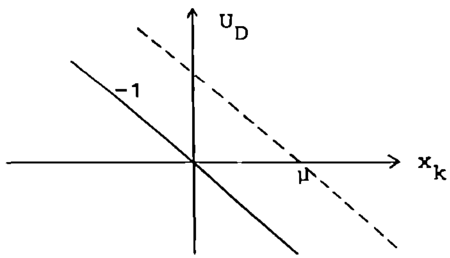


Fig. 5.1 Decision rule (5.2)

Equ. (5.2) is evident. It simply is a consequence of the fact that future demand is 0 and production costs are linear. Not compensating initial stock completely at the first possible occasion would incur additional inventory (shortage or carrying) costs.

Decision (5.2) has now to be substituted into the "expectation criterion" (3.3) with P(u) and I (x) representing piecewise linear costs. In comparing the $U_D(x_k)$ and the NLQ-policy we have to consider the costs incurred in both cases. However, it is obvious that the LNQ-approximation is always better than the deterministic approach. This is already due to the fact that $U_D(x_k)$ is a linear policy which, of course, cannot lead to better results than the best linear policy. Also the introduction of an optimal safety stock to be discussed subsequently cannot change this general qualitative result. This is because a safety stock(μ) does only shift the line in Fig. 5.1 to the left or the right (depending on the values of the cost parameters).

5.1.1 Numerical Results

To be specific let us define C_D^* to be the value of the cost criterion (3.12) if the "optimal deterministic" production policy (5.2) is applied. Comparing (5.2) with (3.50) one finds $\mu_x^* = 0$ and $\gamma(\Theta^*) = 0$ from which in view of (3.42) and (3.43) it follows $\sigma_u = \sigma$ and $\sigma_x = \sigma$. Hence expression (3.12) reduces to

$$C_D^* = (h+v+p+q) \frac{\sigma}{\sqrt{2\pi}} \tag{5.3}$$

These costs may now be compared with the costs C incurred by the LNQ-approximation. Some characteristic results are shown in Table 5.1 where D_C measures the deviation in percentages defined by

$$D_C = \frac{C_D^* - C_L^*}{C_L^*} \cdot 100 \; [\%] \qquad (5.4)$$

v / p=q	h: C_D^*	h: C_L^*	h: D_C	2•h: C_D^*	2•h: C_L^*	2•h: D_C	$D_C^{\mu *}$
0•h	0,798	0,798	0	1,197	1,090	9,8	0
0,1•h	0,878	0,874	0,4	1,277	1,168	9,3	0,3
0,5•h	1,197	1,128	6,1	1,596	1,435	11,2	3,8
h	1,596	1,382	15,5	1,995	1,712	16,5	10,3
2 •h	2,394	1,784	34,2	2,793	2,161	29,2	24,3
10 •h	8,777	3,656	140,0	9,176	4,312	112,8	110,3
20 •h	16,756	5,109	228,0	17,154	6,000	185,9	184,1

Table 5.1 Costs and cost deviations

If, e.g. p=q=v=h, the LNQ-approach leads to costs being D_C= 15,5 % smaller than the costs due to the deterministic approach. For p=q=10h and v=2h this deviation already is 112,8 %. Note that giving only values for p=q is entirely for convenience of presentation. Essentially the same results can be obtained for p≠q.
In practice, however, the deterministic approach would usually not be applied without taking advantage of certain safety stocks. These safety stocks are generally taken to be proportional to the variance of the demand sequence. Mathematically, as already mentioned, safety stocks merely move the straight line in Fig. 5.1 to the right (or the left) whereas the slope remains -1.
Let μ be a safety stock. Hence Equ.(5.2) reads

$$u_k = - (x_k - \mu) \qquad (5.5)$$

Substituting into (3.12) yields

$$C_D^\mu = - v\mu + (h+v)\left\{\mu \; \phi\left(\frac{\mu}{\sigma_x}\right) + \sigma_{xx} \; \phi'\left(\frac{\mu}{\sigma_x}\right)\right\} + \frac{p+q}{\sqrt{2\pi}} \sigma_u \qquad (5.6)$$

Moreover, substituting (5.5) into the balance equation $x_{k+1} = x_k + u_k - r_k$ one obtains

$$x_{k+1} = \mu - r_k$$

Hence, $\mu = E\{x_k\}$ and $\sigma_x = \sigma_u = \sigma$

Thus an optimal safety stock is given by the necessary condition

$$\frac{\partial\ C_D^{\mu}}{\partial\ \mu} = -\ v + (h+v)\ \phi\ (\frac{\mu}{\sigma}) = 0$$

or

$$\mu^* = \sigma\ \phi^{-1}\left(\frac{v}{h+v}\right) \tag{5.7}$$

Resubstituting (5.7) into (5.6) we obtain the optimal costs

$$C_D^{\mu *} = \sigma\left\{(h+v)\ \phi'(m) + \frac{p+q}{\sqrt{2\pi}}\right\}$$

$$= \frac{\sigma}{\sqrt{2\pi}}\left\{(h+v)\ e^{-\frac{1}{2}m^2} + p+q\right\} \tag{5.8}$$

where

$$m := \frac{\mu^*}{\sigma} = \phi^{-1}\ (\frac{v}{h+v}) \tag{5.9}$$

Comparing these costs with costs (4.11) incurred without taking into account a safety stock one obtains

$$C_D^* - C_D^{\mu *} = \frac{\sigma}{\sqrt{2\pi}}\ (h+v)\ (1 - e^{\frac{1}{2}m^2}) \geq 0 \tag{5.10}$$

Hence, the right hand side of the equality in (5.10) gives the cost reduction due to the optimal safety stock. Note that this reduction does not depend on production costs p and q.

The last column of Table 5.1 gives the relative cost deviation $D_C^{\mu *}$ of the deterministic approximation and the LNQ-approach in case of an optimal safety stock. (Note that for h=v : μ=0).

As can clearly be seen, for small production costs (p and q) compared to inventory costs (h and v) a considerable improvement is achieved. However, for p and q being large the results are almost identical with the former case not accounting for a safety stock. As production becomes more and more expensive the presence of a safety stock is of decreasing significance on costs; they are essentially production costs.

5.2 Gauss - Markov Case

As in the white noise case of the last section we have to determine the optimal first decision of a deterministic optimization problem. Since we now have

$$r_o\ (k) = E\ \{r_k | r_o\} = a^k\ r_o$$

total demand $\sum_{k=1}^{\infty} r_o\ (k)$ is again finite and it is reasonable to take the adjoined deterministic criterion

$$C_D: = \sum_{k=1}^{\infty} \left\{ P(\bar{u}_k) + I\ (\bar{x}_k + \bar{u}_k - a^k r_o) \right\} \Rightarrow \min.$$

Solving the deterministic dynamic optimization problem one can take advantage of the fact that demand has a very regular structure. We have therefore been able to derive analytic results for the first decision. The results we obtained (see Appendix to this chapter) can be summarized as follows:

$$U_D(x_k,\ r_{k-1}) = \begin{cases} s(r_{k-1}) - x_k & \text{for } x_k < s(r_{k-1}) \\ 0 & \text{for } s(r_{k-1}) \leq x_k \leq s'(r_{k-1}) \\ s'(r_{k-1}) - x_k & \text{for } x_k > s'(r_{k-1}) \end{cases} \tag{5.11}$$

where

$$s(r_{k-1}) := \begin{cases} \sum_{i=1}^{n''} a^i\ r_{k-1} & \text{for } r_{k-1} < 0 \\ a\ r_{k-1} & \text{for } r_{k-1} \geq 0 \end{cases} \tag{5.12}$$

$$s'(r_{k-1}) := \begin{cases} a\ r_{k-1} & \text{for } r_{k-1} < 0 \\ \sum_{i=1}^{n'} a^i\ r_{k-1} & \text{for } r_{k-1} \geq 0 \end{cases} \tag{5.13}$$

$$n': = \left[\frac{p+q}{h}\right]_+ ; \quad n'': = \left[\frac{p+q}{v}\right]_+ , \tag{5.14}$$

$[z]_+$ denoting the "smallest integer $\geq z$"

Fig. 5.2 illustrates U_D for $r \geq 0$

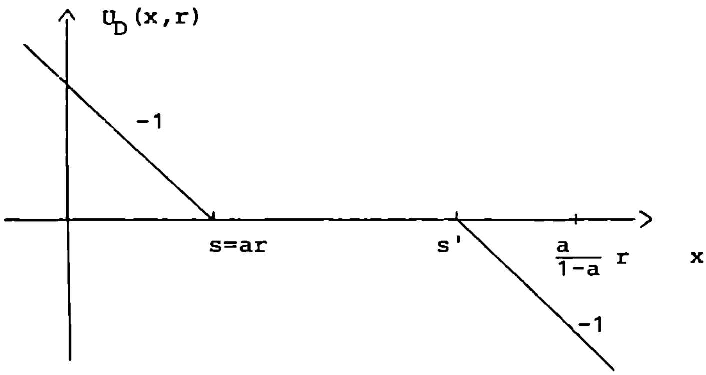


Fig. 5.2 Policy $U_D(x,r)$ defined by (5.11)

Note that the simple policy (5.2) can also be derived from (5.11) setting a=0. For the special case p+q≤min[h;v] it follows from (5.14) that the piecewise linear policy (5.11) degenerates to a linear policy given by

$$U_D(x_k, r_{k-1}) = -(x_k - a\, r_{k-1}) \tag{5.15}$$

This result is entirely reasonable since, when production costs are small compared to inventory carrying and stock out costs there will be no region in which stock will not be adjusted.

As in the white noise case we now have to substitute U_D into the "expectation criterion" (3.12). Because of the structure of the policy we have to distinguish two cases

(a) Linear policy (5.15) ($p+q \leq \min\,[h,v]$)

Comparing (5.15) with (3.54) yields $\mu^*=0$ and $\gamma(\Theta^*)=0$, from which, in view of (3.56) and (3.57), it follows $\sigma_x{}^2 = \sigma_r{}^2\,(1-a^2) = \sigma^2$ and

$$\sigma_u{}^2 = (1+2a-2a^3)\,\sigma_r{}^2 = \frac{1+2a-2a^3}{1-a^2}\,\sigma^2$$

Hence Equ.(3.12) reduces to

$$C_D^* = \frac{\sigma}{\sqrt{2\pi}}\left[(h+v) + (p+q)\sqrt{\frac{1+2a-2a^3}{1-a^2}}\right] \tag{5.16}$$

(Note that setting a=0 one again obtains (5.3)).

(b) Piecewise linear policy (5.11) ($p+\sigma > \min [h,v]$)

In this case analytical results can no longer be obtained. Instead we have to calculate optimal costs numerically. This can in principle be done by first discretizing x and r and then (taking into account (5.9)) calculating stationary probabilities for x and u. These probabilities will then be used to evaluate the adjoined expected costs. This procedure will now be described in more detail. Let us first discretize x and r.

$x \in \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ where $x^i = x_{min} + (i-1)\, d_x, (i=1,\ldots,n)$

$r \in \{r^{(1)}, r^{(2)}, \ldots, r^{(q)}\}$ where $r^i = r_{min} + (i-1)\, d_r, (i=1,\ldots,q)$

with d_x and d_r being appropriate step intervals. Defining state variables $z_k := \{x_k, r_{k-1}\}$, (k=1,2,...), the sequence $\{z_k\}$ constitutes a Markov-process having transition probabilities given by

$$P_{(i,l)(j,m)} := \text{Prob}\left\{x_{k+1} = x^{(m)},\ r_k = r^{(j)} \mid x_k = x^{(l)},\ r_{k-1}=r^{(i)}\right\}$$

$$= \text{Prob}\left\{x_{k+1}=x^{(m)} \mid x_k=x^{(l)}, r_k=r^{(j)}, r_{k-1}=r^{(i)}\right\} \cdot$$

$$\cdot \text{Prob}\left\{r_k= r^{(j)} \mid x_k=x^{(l)}, r_{k-1}=r^{(i)}\right\}$$

Since

$$\text{Prob}\left\{x_{k+1}=x^{(m)} \mid x_k=x^{(l)}, r_k=r^{(j)}, r_{k-1}=r^{(i)}\right\} = \begin{cases} 1 \text{ if } x^{(m)}=x^{(l)}+U_D(x^{(l)},r^{(i)}) - r^{(j)} \\ 0 \text{ if } x^{(m)} \neq x^{(l)}+U_D(x^{(l)},r^{(i)}) - r^{(j)} \end{cases}$$

and r_k is independent of x_k, $P_{(i,l)(j,m)}$ reduces to

$$P_{(i,l)(j,m)} = \begin{cases} \text{Prob}\left\{r_k=r^{(j)} \mid r_{k-1}=r^{(i)}\right\} \text{ for } x^{(m)}=x^{(l)}+U_D(x^{(l)},r^{(i)}) - r^{(j)} \\ 0 \qquad \text{elsewhere} \end{cases} \tag{5.17}$$

The transition probabilities Prob $\{r_k=r^{(j)} \mid r_{k-1}=r^{(i)}\}$ are given by the demand structure and can easily be calculated by discretizing the conditional normal distribution having mean value $a\ r^{(i)}$ and variance $\sigma_r^2(1-a^2) = \sigma^2$.

Since the above defined Markov-chain is ergodic the stationary probabilities $\Pi_{j,m}$ are given by the well-known formula

$$\Pi_{j,m} = \sum_{i=1}^{q} \sum_{l=1}^{n} \Pi_{i,l}\ P_{(i,\ )(j,m)} \quad \text{for } (j=1,---,q; \text{ and } m=1,\ldots,n) \tag{5.18}$$

and

$$\sum_{j=1}^{q} \sum_{m=1}^{n} \Pi_{j,m} = 1$$

Hence, the expected (optimal deterministic policy) costs can be written

$$C_D^* = \sum_{j=1}^{q} \sum_{m=1}^{n} C_{j,m}\ \Pi_{j,m} \tag{5.19}$$

where

$$C_{j,m} = P_{j,m} + I_{j,m} \tag{5.20}$$

More explicitely, production costs $P_{j,m}$ are given by

$$P_{j,m} = \begin{cases} p\left[s(r^{(j)}) - x^{(m)}\right] & \text{for } x^{(m)} < s(r^{(j)}) \\ 0 & \text{for } s(r^{(j)}) \leq x^{(m)} \leq s'(r^{(j)}) \\ -q\left[s'(r^{(j)}) - x^{(m)}\right] & \text{for } x^{(m)} > s'(r^{(j)}) \end{cases} \tag{5.21}$$

and the expected*) inventory carrying and stock out costs conditional on x (stock on hand after replenishment) and r (last period's demand) are known to be

$$L(r,x) = h \int_{-\infty}^{x} (x-\rho)\ d\ \phi_r(\rho) - v \int_{x}^{\infty} (x-\rho)\ d\ \phi_r(\rho)$$

$$= (h+v)\ \left[(x-\mu_r)\ \phi_r(x) + \sigma^2\ \phi_r'(x)\right] - v(x-\mu_r)$$

where $\mu_r = a\ r$

and

$$\phi_r(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2}\left(\frac{\rho-\mu_r}{\sigma}\right)^2\right\} d\rho$$

---

*) The expected value with respect to $d\ \phi(r)$ has to be taken into account since inventory costs are attached to the <u>end</u> of a period.

Hence, inventory costs for policy (5.11) are given by

$$I_{j,m} = \begin{cases} L\ (r^{(j)},\ s\ (r^{(j)})) & \text{for} \quad x^{(m)} < s\ (r^{(j)}) \\ L\ (r^{(j)},\ x^{(m)}) & \text{for} \quad s(r^{(j)}) \le x^{(m)} \le s'(r^{(j)}) \\ L\ (r^{(j)},\ s'(r^{(j)})) & \text{for} \quad x^{(m)} > s'(r^{(j)}) \end{cases} \tag{5.22}$$

5.2.1 Numerical Results

Because of the two cases (a) and (b) the numerical comparisons will be performed separately for each case

(a) Linear policy (5.15) $(p + 1 \le \min |h;v|)$.

In this case again the LNQ-approximation must lead to better results. For a = 0.1, a = 0.5 and a = 0.9 the deviations D_C (see Equ. (5.4)) are given in Table 5.2 up to p = q = 0.5 h

	a = 0.1			a = 0.5			a = 0.9		
v	h	2h		h	2h		h	2h	
p=q	D_C	D_C	$D_C^{\mu*}$	D_C	D_C	$D_C^{\mu*}$	D_C	D_C	$D_C^{\mu*}$
0 h	0,0	9,8	0,0	0,0	9,8	0,0	0,0	9,8	0,0
0,1 h	0,4	9,3	0,3	0,3	9,0	0,2	0,1	8,2	0,1
0,5 h	5,9	10,9	3,8	3,0	8,8	2,4	0,8	5,4	0,5

Table 5.2 Cost deviations D_C and $D_C^{\mu*}$

Introducing optimal safety stocks as we have done in the white noise case readily leads fo formulae similar to (5.7), (5.8) and (5.10). In fact, (5.7) remains the same and the optimal costs are given by

$$C_D^{\mu*} = \frac{\sigma}{\sqrt{2\pi}} \left\{ (h+v)\ e^{-\frac{1}{2} m^2} + (p+q) \sqrt{\frac{1+2a - 2a^3}{1 - a^2}} \right\}$$

which in view of (5.16) leads to the same cost difference as in (5.10). This, (in addition to the white noise case) shows that the improvement in costs obtained by introducing a safety stock does not depend on the value of the correlation parameter a. The corresponding (relative) deviations from the LNQ-approximation $D_C^{\mu *}$ can be found in the last columns of Table 5.2

(b) Piecewise linear policy (5.11) (p + q > min [h; v]).

Also in this case the numerical results show (see Table 5.3) that the LNQ-approach leads to better results than the deterministic approximation. For comparatively low correlations (a ≤ 0.5) the cost deviations are considerable and almost of the same magnitude as in the uncorrelated case. For high correlations (a = 0.9) differences are substantially smaller.

	a = 0.1			a = 0.5			a = 0.9		
v	h	2h		h	2h		h	2h	
p=q	D_C	D_C	$D_C^{\mu *}$	D_C	D_C	$D_C^{\mu *}$	D_C	D_C	$D_C^{\mu *}$
h	15	16	10	9	11	6	1	4	1
2h	33	28	24	20	17	14	2	4	2
10h	138	111	109	90	71	69	8	7	7
20h	225	183	182	154	123	122	17	12	12

Table 5.3 Cost deviations D_C and $D_C^{\mu *}$

As in the former cases an optimal safety stock has been introduced. This can no longer be done analytically. Instead we introduced a safety stock μ into the optimal policy (5.11) writing

$$U_D(x_k, r_{k-1}, \mu) = \begin{cases} s + \mu - x_k & \text{for } x_k < s + \mu \\ 0 & \text{for } s + \mu \le x_k \le s' + \mu \\ s' + \mu - x_k & \text{for } x_k > s' + \mu \end{cases}$$

Optimizing numerically with respect to the parameter μ leads to the results shown in the last columns of Table 5.3. As was to be expected (see also Table 5.1 for the white noise case) the introduction of an optimal safety stock does not improve substantially the deterministic policy for (production) cost values equal or larger then h.

5.3 Appendix to Chapter 5 :

Derivation of the "deterministic" policy (5.11)

In deriving (5.11) we shall not use any of the known algorithms. Because of the special structure of the demand sequence we shall use a direct approach.

Let us define $x := x_1$ initial stock on hand

$r := r_o$ initial demand

$\hat{r}(j) := \hat{r}_o(j) = E\{r_j | r_o\} = a^j r$

First we observe that the sum of initial stock and total production must be equal to total demand

$$x + \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \hat{r}_k = \frac{a}{1-a} r \tag{5.23}$$

otherwise inventory costs will amount to infinity in the course of time.

In what follows, we shall first consider the case $r \geq 0$. Furthermore, we have to distinguish several cases depending on the particular value the initial stock takes on.

(1) $x \leq ar = \hat{r}(1)$

Since $ar \leq \frac{a}{1-a} \cdot r$ it follows from (5.23) : $\sum_{k=1}^{\infty} u_k = \frac{a}{1-a} r - x > 0$; i.e. for each strategy we get production costs not smaller than $p\,(\frac{a}{1-a} r - x)$. Clearly, the strategy minimizing inventory costs is given by a sequence of production decisions which exactly meet demand: $u_k = \hat{r}(k) - x_k \;\forall k$. For this strategy, which shall be called S_1, no inventory costs occur. Hence, total costs for S_1 are given by $K_1 = p(\frac{a}{1-a} r - x)$.

More generally : If $x_k \leq \hat{r}(k)$, then the optimal (remaining) policy is given by

$$\{u_k = \hat{r}(k) - x_k,\; u_{k+1} = \hat{r}(k+1),\; \ldots\}.$$

(2) $ar < x \le \frac{a}{1-a} \cdot r = \sum_{j=1}^{\infty} \hat{r}(j)$

Subcase 2.1 : $ar < x < ar + a^2 r$

In this case two strategies S_1 and S_o have to be taken into account. Furthermore we investigate a mixed strategy S_α :

Strategy	Period 1	Period 2	Period 3	...
S_1	$u_1 = ar - x$	$u_2 = a^2 r$	$u_3 = a^3 r$	...
	$x_2 = 0$	$x_3 = 0$	$x_4 = 0$	...
S_o	$u_1 = 0$	$u_2 = -(x-ar)+a^2 r$	$u_3 = a^3 r$	...
	$x_2 = x - ar$	$x_3 = 0$	$x_4 = 0$	...
S_α	$u_1 = \alpha(ar-x)$	$u_2 = -(1-\alpha)(x-ar) + a^2 r$	$u_3 = a^3 r$	...
	$x_2 = (1-\alpha)(x-ar)$	$x_3 = 0$	$x_4 = 0$	...

The associated costs are given by

$$K_1 = q(x-ar) + pa^2 r + C_3, \text{ where } C_3 := p\frac{a^3}{1-a}\, r$$

$$K_o = h(x-ar) + p(ar + a^2 r - x) + C_3$$

$$K_\alpha = \alpha q(x-ar) + (1-\alpha)\, h(x-ar) + pa^2 r - (1-\alpha)\, p(x-ar) + C_3$$

It can easily be shown that the mixed strategy S_α is suboptimal: $K_\alpha = \alpha K_1 + (1-\alpha)K_o$; hence $K_\alpha \ge \min\, [K_1;K_o]$. Similarly, all other possible strategies can be shown to be suboptimal.
There remain S_1 and S_o for which one obtains
$K_o = K_1 + [h-(p+q)]\ (x-ar)$, i.e. the choice between the two strategies depends on the values of the cost parameters. Since $x-ar > 0$ one obtains S_1 for $p+q \le h$ and S_o for $p+q > h$.

<u>Subcase 2.2 : $ar + a^2r < x \leq ar + a^2r + a^3r$</u>

In this case 3 strategies have to be investigated:

Strategy	Period 1	Period 2	Period 3	Period 4	...
S_1	$u_1 = ar-x$	$u_2 = a^2r$	$u_3 = a^3r$	$u_4 = a^4r$	...
	$x_2 = 0$	$x_3 = 0$	$x_4 = 0$	$x_5 = 0$	...
S_2	$u_1 = ar+a^2r-x$	$u_2 = 0$	$u_3 = a^3r$	$u_4 = a^4r$	...
	$x_2 = a^2r$	$x_3 = 0$	$x_4 = 0$	$x_5 = 0$	...
S_o	$u_1 = 0$	$u_2 = 0$	$u_3 = ar+a^2r+a^3r-x$	$u_4 = a^4r$	...
	$x_2 = x-ar$	$x_3 = x-ar-a^2r$	$x_4 = 0$	$x_5 = 0$	...

Also in this case one can show that mixed strategies are suboptimal. The associated costs are given by

$$K_1 = q(x-ar) + pa^2r + pa^3r + C_4, \quad \text{where } C_4 := p\,\frac{a^4}{1-a}\,r$$

$$K_2 = q(x-ar-a^2r) + pa^3r + h\,a^2r + C_4$$

$$K_o = p(ar+a^2r+a^3r-x) + h(x-ar) + h(x-ar-a^2r) + C_4$$

Hence

$$K_2 = K_1 + \left[h-(p+q)\right]\, a^2r$$

$$K_o = K_2 + \left[2h-(p+q)\right]\,(x-ar-a^2r)$$

Since $x-ar-a^2x > 0$ we obtain for the optimal policy

Cost region	Optimal Strategy
$p+q < h$	$S_1 : u_1 = ar-x$
$h < p+q \leq 2h$	$S_2 : u_1 = ar+a^2r-x$
$p+q > 2h$	$S_o : u_1 = 0$

Subcase 2.n : $ar+a^2r + \dots + a^nr < x \le ar + \dots + a^{n+1}r$

Analogue to the preceeding cases one obtains

Cost region	Optimal Strategy
$p+q \le h$	$S_1 : u_1 = ar-x$
$h < p+q \le 2h$	$S_2 : u_1 = ar+a^2r-x$
$2h < p+q \le 3h$	$S_3 : u_1 = ar+a^2r+a^3r-x$
|	|
|	|
|	|
$(n-1)h < p+q \le nh$	$S_n : u_1 = ar+a^2r+--+a^nr-x$
$p+q > nh$	$S_o : u_1 = 0$

(3) $x = ar+a^2r+.. \quad = \frac{a}{1-a}\,r$

In this case the results of subcase 2.n also apply. The last line for p+q > nh has of course to be dropped.

(4) $x > \frac{a}{1-a}\,r = \sum_{j=1}^{\infty} \hat{r}(j)$

One has almost the same situation as in case (3). There occur only additional costs for reducing the initial stock x to the amount $\frac{a}{1-a}\,r$.

Let us summarize our results thus far obtained in cases (1) to (4). For $r \ge 0$ one obtains the following "deterministic" optimal policy

Stock on hand	Cost region	$U_D(r,x) := u_1$
$x \le ar$	(no constraints)	$ar-x$
$\sum_{j=1}^{n} a^j r < x \le \sum_{j=1}^{n+1} a^j r$	$p+q \le h$	$ar-x$ $(k=n=1)$
	$(k-1)h < p+q \le k \cdot h$	$\sum_{j=1}^{k} a^j r$; $(k=2,3,..n)$
	$p+q > nh$	0
$x > \frac{a}{1-a} r = \sum_{j=1}^{\infty} a^j r$	$p+q \le h$	$ar-x$
	$(k-1)h < p+q \le k \cdot h$	$\sum_{j=1}^{k} a^j r - x$; $(k=2,3,..)$

This policy may be rewritten in the following way

$$U_D(r,x) = \begin{cases} ar-x & \text{for } x \le ar \\ \left.\begin{array}{ll} \sum_{j=1}^{n'} a^j r - x & \text{for } x > \sum_{j=1}^{n'} a^j r \\ 0 & \text{otherwise} \end{array}\right\} & \text{for } ar < x \le \frac{a}{1-a} r \\ \sum_{j=1}^{n'} a^j r - x & \text{for } x > \frac{a}{1-a} r \end{cases} \tag{5.24}$$

where $n' = \left[\frac{p+q}{h}\right]_+$, with $[z]_+$ denoting the "smallest integer > z"

A similar result holds for $r < 0$ where essentially h has to be replaced by v. Hence, one obtains the optimal (s,s')-policy (5.11).

Chapter 6

COMPARISON WITH AHM-INVENTORY MODELS

Up to now we considered cash balance and production smoothing models. In these models the decision variable u_k could take on any real value. We now investigate models with decision variables u_k being restricted to positive values : $u_k \geq 0$, i.e. we will study pure inventory models for which orders can, of course, only be positive.

Inventory models not only differ in the range of the decision variable but also in the probability distribution of demand r_k. Since demand is a positive random variable the probability distribution has to be restricted on $r \geq 0$. This implies that the mean value μ_r of demand has always to be positive. Although it sometimes appears to be necessary to study the case $\mu_r \neq 0$ also in cash holding problems it is only for pure inventory models that we investigate probability distributions having $\mu_r \geq 0$. Furthermore the restriction on $r \geq 0$ also implies that a Gaussian probability distribution will no longer be useful to describe (positive) demand. Instead we will have to choose e.g. a beta-distribution being defined on a finite (positive) interval. These two properties of a pure inventory problem, i.e. $u_k \geq 0$ and $r_k \geq 0$, seem to prevent a description by our LNQ-Theory developed in Chap.3. However, the situation is not as bad as it seems. Consider an item having, e.g., a demand sequence 778, 890, 1100, 1120, 850, ... This case, which is quite common, can be considered as a smoothing problem. Let μ_r be the mean value of the above demand. To describe the inventory problem by a smoothing model we have just to introduce new demand values as deviations from mean demand $r'_k := r_k - \mu_r$; and the order quantity will be interpreted as a deviation from a mean production μ_u (which meets μ_r) $u'_k := u_k - \mu_u$. Thus a pure inventory problem might be described by a smoothing model for which, as we know, an LNQ-Approach provides at least in the case of absent set-up costs a fully sufficient approximation.

We consider two cases. First, in Sec. 6.1, we investigate the comparatively simple case where set-up costs have not to be taken into account. Next, in Sec. 6.2, problems with set-up costs will be discussed. For easier reference let us define the AHM-Inventory model which will be approximated by a LNQ-Model. Specializing the model we introduced in Chap.1 we have

(1) x_k : stock on hand at the beginning of period k, (k=1,2,...)

$x_k \in \mathbb{R}\ \forall k$, x_1: initial stock

(2) u_k : reordering decision at the beginning of period k which results in a shipment in this period

$u_k \in \mathbb{R}_+$

(3) r_k : stochastic demand in period k; identically and independently distributed with $r_k \geq 0$

<u>Inventory balance equation</u>

(4) $x_{k+1} = x_k + u_k - r_k$

<u>Cost criterion</u>

(5) $$C = \lim_{N\to\infty} \frac{1}{N} \; E\left\{ \sum_{k=1}^{N} \{I(x_{k+1}) + P(u_k)\} \,|\, x_1 \right\}$$

with

$$I(x) = \begin{cases} h\ x & \text{for } x \geq 0 \\ -\ v\ x & \text{for } x < 0 \end{cases}$$

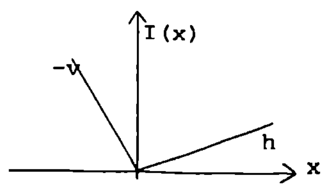


Fig.6.1

and

$$P(u) = \begin{cases} P + p\ u & \text{for } u > 0 \\ 0 & \text{for } u = 0 \end{cases}$$

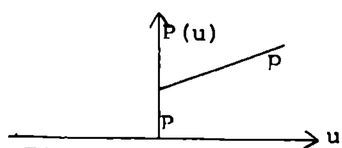


Fig.6.2

h, v, p and P being positive

6.1 No-set-up cost case (P = 0)

This case is well-known in literature. Let the probability distribution function of r_k be F(r). Then the (stationary) ordering policy is given by an (S,S)-policy

$$u_k = \begin{cases} S - x_k & \text{for } x_k < S \\ 0 & \text{for } x_k \geq S \end{cases} \tag{6.1}$$

with

$$S = F^{-1} \left(\frac{v}{h+v}\right) \tag{6.2}$$

For a better understanding of the arguments to follow u_k is depicted in Fig. 6.3

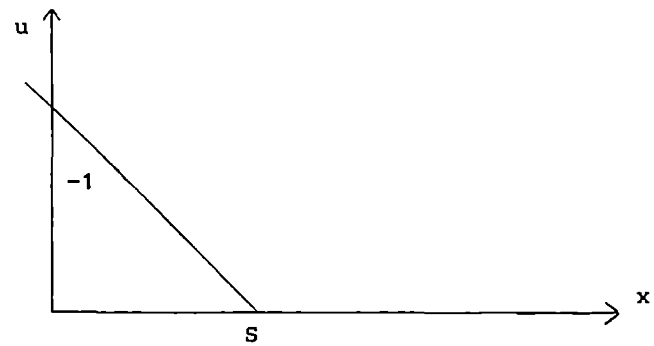


Fig. 6.3 (S,S)-policy

Fig. 6.1 shows a piecewise linear policy. However, once inventory x is smaller than S it cannot become again greater than S. Thus, possibly except after a few initial periods x is restricted to the range $x \leq S$; i.e. (6.1) describes in fact a linear policy.

6.1.1 The LNQ-Approach

Let us now derive policy (6.1) using the LNQ-Approach. Consider again Fig. 6.2 with P = 0 and define a new variable $u' := u - \mu_u$; where, from the balance equation; $\mu_u = \mu$.

Fig. 6.4 shows the ordering costs in the u'-coordinate system.

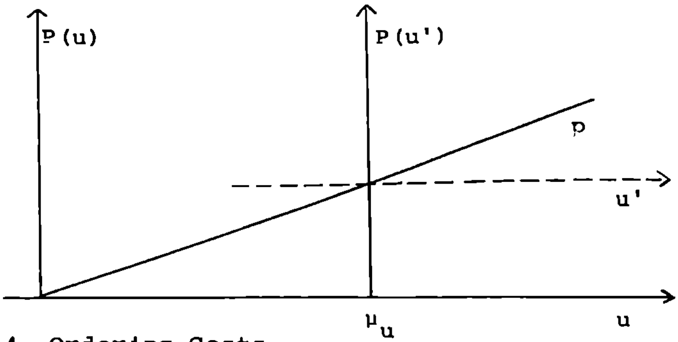


Fig. 6.4 Ordering Costs

Refering to (3.11) this clearly indicates that the inventory problem may be approximated by a (linear) smoothing problem with $-q = p$.

As we know from Chap.3 the LNQ-Approach relies on the assumption of normally distributed demand. Hence, let us assume r_k having the probability distribution function $\phi_{\mu,\sigma}(\rho)$, with

$$\phi_{\mu,\sigma}(\rho) := \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\rho} \exp\left\{ -\frac{1}{2}\frac{(r-\mu)^2}{\sigma^2} \right\} dr \tag{6.3}$$

or, $r'_k \sim \phi_{o,\sigma}(\rho')$, with standardized $r' := r-\mu$, $\rho' = \rho-\mu$

$$\phi_{o,\sigma}(\rho') := \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\rho'} \exp\left\{ -\frac{1}{2}\frac{r'^2}{\sigma^2} \right\} dr' \tag{6.3a}$$

(It will be this normality requirement which will be studied numerically towards the end of this section).

Let us now derive results similar to Sec.3.3.1 for the case $p = -q$ (Note that in Chap.3 we restricted all cost parameters to be positive).

From (3.17) we have

$$\mu_x^* = \sigma_x^* \, \phi^{-1}_{o,1} \left(\frac{v}{v+h}\right) \tag{6.4}$$

Further, one finds, according to (3.19), $\beta = 0$, which, from (3.46), implies $\gamma^{*} = 0$. Hence, because of (3.42) and (3.43), $\sigma_x^{*} = \sigma_u^{*} = \sigma$. Substituting in (6.4) one has

$$\mu_x^{*} = \sigma \, \phi_{0,1}^{-1} \left(\frac{v}{v+h}\right) \tag{6.5}$$

and the optimal policy is given by (c.o.(3.50))

$$u'_k = -(x_k - \mu_x^{*}) \tag{6.6}$$

Instead of (6.5) one can also write

$$\phi_{0,1}\left(\frac{\mu_x^{*}}{\sigma}\right) = \frac{v}{v+h} \tag{6.7}$$

or

$$\phi_{0,\sigma}(\mu_x^{*}) = \frac{v}{v+h} \tag{6.8}$$

or

$$\mu_x^{*} = \phi_{0,\sigma}^{-1}\left(\frac{v}{v+h}\right) \tag{6.9}$$

Resubstituting the original variable $r_k = r_k' + \mu$, Equ.(6.8) becomes

$$\phi_{\mu,\sigma}(\mu_x^{*}+\mu) = \frac{v}{v+h} \tag{6.10}$$

which finally yields

$$\mu_x^{*} + \mu = \phi_{\mu,\sigma}^{-1}\left(\frac{v}{v+h}\right). \tag{6.11}$$

Similarly, resubstituting $u_k = u_k' + \mu_u = u_k' + \mu$, Equ.(6.6) becomes

$$u_k = -(x_k - (\mu_x^{*}+\mu)). \tag{6.12}$$

Equs.(6.11) and (6.12) constitute the result we were looking for. Defining

$$S_N := \mu_x^{*} + \mu,$$

Equs.(6.11) and (6.12) are formally identical to (6.2) and (6.1) respectively in case of a <u>normally distributed demand</u>.

Let us illustrate the optimal linear policy (6.12)

$$u_k = -(x_k - S_N) \tag{6.12a}$$

in Fig. 6.5

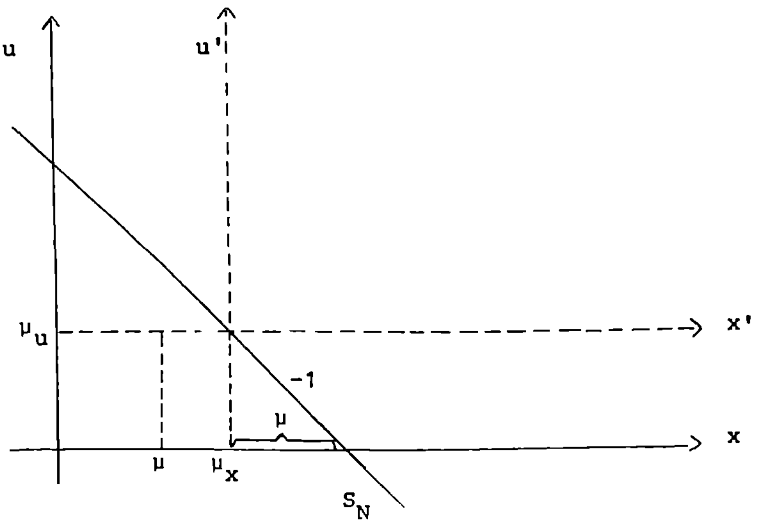


Fig. 6.5 Best linear policy (6.12a)

One will readily discover the smoothing policy of Fig. 3.2 (with slope - 1) in the u'-coordinate system. The variables x and u fluctuate about the point (μ_x, μ_u). However, compared to Fig. 6.3 it is not possible to prevent u_k from becoming negative. This is due to the fact that demand, being Gaussian, can in principle take on any real value, i.e. demand can also be negative. In Fig. 6.3 and in policy (6.1) this could not happen since we worked with a correct probability distribution, i.e. with a distribution being non-zero only for $r \geq 0$.

However, working with Gaussian distributions has some advantages. First it gives a theoretical insight into the relationship between (policy-linear) smoothing models and pure inventory models. Hence the results on certainty equivalence also hold at least approximately for the inventory case (having no set-up costs). The second advantage, closely related to the former, is a computational one. Often one is not able to estimate the probability distribution function F (r) nor its $(\frac{v}{h+v})$ - quantile which is necessary to determine S. It often seems much simpler to estimate mean and variance and to assume demand to be Gaussian. In that case we have the situation depicted in Fig. 6.5.

If, however, actual demand is not Gaussian, the question arrises how well the actual probability distribution is approximated by a Gaussian having the same mean and variance. This problem simply boils down to a determination of $(\frac{v}{h+v})$-quantiles and their impact on optimal costs.

6.1.2 Comparison of Demand Distributions

Assume demand to be beta-distributed with a probability density function

$$\varphi_B(r) = \begin{cases} \frac{1}{\text{Beta }(a,b)\ (b_u-b_l)} \left(\frac{r-b_l}{b_u-b_l}\right)^{a-1} \left(\frac{b_u-r}{b_u-b_l}\right)^{b-1} & \text{for } r\varepsilon(b_l,b_u) \\ 0 & \text{elsewhere} \end{cases} \tag{6.14}$$

where a and b are positive parameters and

$$\text{Beta }(a,b) := \int_0^1 t^{a-1}\ (1-t)^{b-1}\ dt\ ;$$

b_l and b_u are lower and upper bounds respectively. As an example $\varphi_B(r)$ is illustrated in Fig. 6.6 for parameter values a = 2 and b = 10

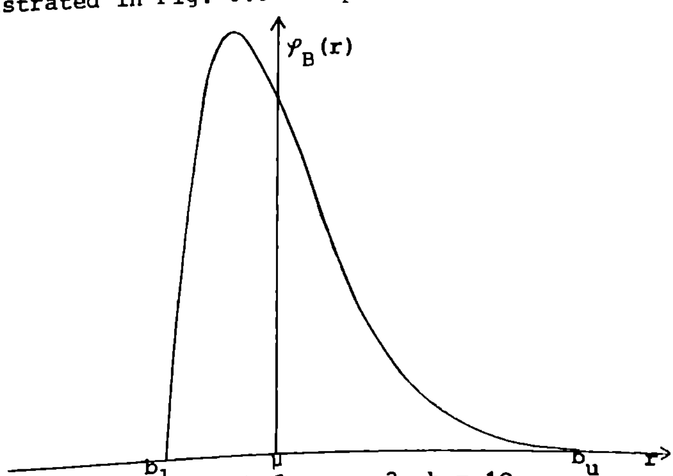


Fig. 6.6 Beta-density (6.14) for a = 2, b = 10

The shape of $\varphi_B(r)$ changes as a and b vary. Hence it is obvious that almost all demand distributions occurring in practice can be represented by a suitable beta-distribution.

Our aim now is to compare optimal results obtained by a beta-distribution of demand with the results we get from an approximating normal-distribution $\varphi_N(r)$ having the same mean and standard deviation. An example is given below (Fig. 6.7) for a = 10 and b = 2

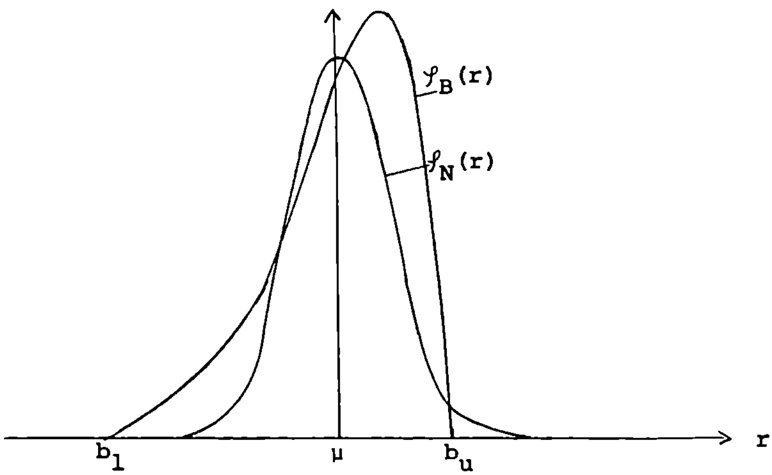


Fig. 6.7 Beta and approximating normal density

In studying the degree of suboptimality of the Gaussian approximation we have to compare optimal costs C^* using the beta-distribution and optimal costs C_N from the Gaussian approximation. The relative deviation will be measured (again) by

$$D_C = \frac{C_N - C^*}{C^*} \tag{6.15}$$

It is well known that mean costs (per period) are given by

$$C = \int_{-\infty}^{S} C(x) \varphi_B (S-x) \, dx \tag{6.16}$$

where C(x) defines total costs per period and x denotes, as usual, inventory at the beginning of the period. C(x) is the sum of ordering costs p(S-x) and inventory costs L(S)

$$C(x) = p(S'-x) + L(S) \tag{6.17}$$

$$L(s) := h \int_{b_l}^{S} (S-r) \varphi_B(r) \, dr + v \int_{S}^{b_u} (r-S) \varphi_B(r) \, dr$$

$$= h \left[S \, \Phi_B(S) - \Psi_B(S)\right] + v \left[\mu - \Psi_B(S) - S(1-\Phi_B(S))\right]$$

$$= (h+v) \left[S \, \Phi_B(S) - \Psi_B(S)\right] + v \, (\mu-S) \tag{6.18}$$

with

$$\Phi_B(S) := \int_{b_l}^{S} \varphi_B(r) \, dr \tag{6.19}$$

and

$$\Psi_B(S) := \int_{b_l}^{S} r \, \varphi_B(r) \, dr \tag{6.20}$$

Because of the finite support of the beta-distribution, (6.16) may be written

$$C = \int_{b_l}^{b_u} C(S-r) \varphi_B(r) \, dr \tag{6.21}$$

Substituting (6.18) yields

$$C = \int_{b_l}^{b_u} \{p \, r + (h+v) \left[S \, \Phi_B(S) - \Psi_B(S)\right] + v(\mu-S)\} \cdot \varphi_B(r) \, dr$$

$$= (p+v) \, \mu + (h+v) \left[S \, \Phi_B(S) - \Psi_B(S)\right] - v \, S \tag{6.22}$$

Optimal costs can now be found by differentiating C with respect to S

$$\frac{\partial C}{\partial S} = (h+v)\left[\Phi_B(S) + S\,\varphi_B(S) - S\,\varphi_B(S)\right] - v \qquad (6.23)$$

Setting equal to zero we find the well known formula for the optimal S (c.o. (6.2)) which will be denoted by S^*:

$$\Phi_B(S^*) = \frac{v}{h+v} \qquad (6.24)$$

and for the optimal costs $C = C^*$ we finally find

$$C^* = (p+v)\mu - (h+v)\,\Psi_B(S^*)\,. \qquad (6.25)$$

Note that for the normal-distribution we found by a different argument (c.o. (6.11) and (6.13))

$$\phi_{\mu,\sigma}(S_N) = \frac{v}{v+h} \qquad (6.26)$$

The costs we now have to compare are optimal costs C^* and the costs one incures if S^* is replaced by the suboptimal parameter S_N, i.e. (c.o. (6.22))

$$C_N := (p+v)\mu + (h+v)\left[S_N\,\phi_B(S_N) - \Psi_B(S_N)\right] - v\,S_N \qquad (6.27)$$

Substituting (6.25) and (6.27) into (6.15) finally yields

$$D_C = \frac{C_N - C^*}{C^*} = \frac{(h+v)\left[S_N \cdot \Phi_B(S_N) - \Psi_B(S_N) + \Psi_B(S^*)\right] - v\,S_N}{(p+v)\mu - (h+v)\,\Psi_B(S^*)} \qquad (6.28)$$

<u>Numerical Results</u>

Numerical results for D_C were obtained for different cost parameters (p, h and v) and different values of a and b. It appears [9] that in most cases $D_C < 5\,\%$. Only in cases of a rather oblique beta-distribution the deviation becomes larger.*)

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*) This is in accordance with the results we derived in Sec. 3.6 for the "cash-balance" situation.

6.2 Set-up Cost Case ($P \neq 0$)

The case with fixed ordering costs $P \neq 0$ being present is far more complicated. This is due to the fact that optimal results cannot easily be obtained, i.e. it is for general demand distributions extremely intricate to calculate values s and S for the optimal (s,S)-policy. However, in recent years, remarkable progress has been made in developing faster algorithms for Markovian Decision Processes (see e.g. [2] and the references given in [2]) which can be used to determine the above policy parameters s and S.
Still the numerical burden in determining optimal DP-policies is considerable. Therefore we are again looking for an approximating linear policy which, as we know from the last section, reduces to the problem of finding an approximating (S,S)-policy.
Obviously, for large P (and consequently for small variance of demand) a linear policy will be a poor approximation. However, as already mentioned at the end of Sec. 4.2.2, fairly satisfying results can be obtained if one enlarges the inspection period by aggregating, say T periods to one large period. The resulting policy will be called S(T)-policy.
Thus, the next subsection will give a short derivation of an optimal S(T)-policy and subsequently Sec. 6.2.2 will present numerical results comparing the (s,S)-policy with its approximating S(T)-policy. (For further results see [22])

6.2.1 Derivation of an optimal S(T)-policy

As mentioned earlier, one of the main differences between an (s,S)- and an (S,S)-policy is given by the fact that an (S,S)-policy places an order each period whereas an (s,S)-policy has a region in which no orders are placed. I.e. there exists for an (s,S)-policy a mean order cycle being larger than the inspection period (i.e. the order cycle of the adjoined (S,S)-policy). Thus it seems to be quite natural (and in many real world situations possible and desirable) to enlarge the inspection period for the (S,S)-policy and to use as an approximating policy an S(T)-policy as defined above.
To derive such a policy let us introduce an index n defining the number of the aggregated inspection period, henceforth called ordering cycle. Starting point is the model defined at the be-

ginning of this chapter.
Aggregated demand in cycle n now is

$$\bar{r}_n = \sum_{i=1}^{T} r_{(n-1)T+i} \tag{6.29}$$

and the balance equation is given by

$$\bar{x}_{n+1} = \bar{x}_n + \bar{u}_n - \bar{r}_n \tag{6.30}$$

where the bar denotes "cycle variables".
The S(T)-policy now reads

$$\bar{u}_n = \begin{cases} S - \bar{x}_n & \text{for } \bar{x}_n \leq S \\ 0 & \text{for } \bar{x}_n > S \end{cases} \tag{6.31}$$

The ordering costs for cycle n are

$$P(\bar{u}_n) = \begin{cases} P + p\,\bar{u}_n & \text{for } \bar{u}_n > 0 \\ 0 & \text{for } \bar{u}_n = 0 \end{cases} \tag{6.32}$$

and the expected holding and shortage costs are given by (c.o.(6.16))

$$L_T(S) = \sum_{i=1}^{T} \{h \int_0^S (S-t)\,dF_i + v \int_S^\infty (t-S)\,dF_i\} \tag{6.33}$$

where F_i is the probability distribution function of

$$\bar{r}(i) := \sum_{j=1}^{i} r_j$$

(If F' is the probability density function of r_k, than the probability density function F_i' of $\bar{r}(i)$ is given by the i-fold convolution of F' : $F_i' = (F')^{*i}$)
Equ. (6.33) says that holding and shortage costs are still assumed to be charged at the end of each (elementary) inspection period. Hence the expected total costs of one <u>cycle</u> are now given by

$$\bar{C}(S,T) = L_T(S) + T\,p\,\mu + P \tag{6.34}$$

and for one period one has

$$C(S,T) := \frac{1}{T}\,\overline{C}(S,T) = \frac{1}{T}\,(L_T(S)+P) + p\,\mu \qquad (6.35)$$

This expression has to be minimized with respect to S and T. Since the last term in (6.35) $p\,\mu$ does not depend on the optimization parameters, we drop $p\,\mu$ and define

$$C'(S,T) := C(S,T) - p\,\mu = \frac{1}{T}\,(L_T(S)+p) \qquad (6.36)$$

As in (6.18), let us reformulate $L_T(S)$ and write

$$L_T(S) = (h+v) \sum_{i=1}^{T} \{S\,F_i(S) - \int_o^S t\,dF_i(t)\} + T\,v\,(\frac{T+1}{2}\,\mu - S) \quad (6.37)$$

Minimizing C(S,T) with respect to S and T we first look (by partial differentiation) for a stationary minimum with respect to S (which as we know from convexity properties exists). As in (6.23) we have

$$\frac{\partial C'(S,T)}{\partial S} = \frac{1}{T}\,(h+v) \sum_{i=1}^{T} F_i(S) - v \qquad (6.38)$$

and for the minimizing $S = S^*(T)$ we find

$$\sum_{i=1}^{T} F_i(S^*(T)) = \frac{v}{h+v}\,T \qquad (6.39)$$

Finally it remains an optimization with respect to T. Since T should not be too large (at most T=10) $C'(S^*(T),T)$ is minimized by simply letting T = 1,2,... and taking that $T = T^*$ which leads to the lowest costs $C'(S^*(T^*),T^*)$.

6.2.2 Numerical Results

In deriving numerical results one has to determine $F_i(S)$ which is a complicated convolution integral of elementary distribution functions. However, as we know from our investigations in previous sections, esp. from Sec. 6.1.2, demand distributions may readily be approximated by normal probability distributions without much loss of optimality. For these distributions $F_i(S)$ can easily be obtained.

Of course the Gaussian distribution is not the only one for which $F_i(S)$ may be calculated without difficulties. Another distribution is the Poisson-distribution which shall now be used to compare the S(T)-policy with the optimal (s,S)-policy.

Poisson				optimal policy		(S,S)-policy		S(T) - policy		
K	v	h	μ	s	S	S	D%	T	S	Δ%
8	9	1	4	3	11	7	31	2	10	12.0
8	9	1	36	37	44	44	0	1	44	0.0
8	99	1	4	6	14	9	20	2	14	14.2
8	99	1	36	45	51	51	0	1	51	0.0
32	9	1	4	2	18	7	123	4	18	13.2
32	9	1	36	32	44	44	0	1	44	0.0
32	99	1	4	6	21	9	99	3	19	20.5
32	99	1	36	42	51	51	0	1	51	0.0
64	4	1	9	1	35	11	122	4	33	4.4
64	4	1	16	5	48	19	71	3	46	2.3
64	4	1	25	12	53	29	40	2	52	0.9
64	4	1	36	23	74	41	25	2	74	0.1
64	4	1	49	34	100	55	12	2	100	0.0
64	4	1	64	45	131	71	1	2	131	0.0
64	9	1	4	1	24	7	206	6	25	11.7
64	9	1	9	5	37	13	109	4	38	6.8
64	9	1	16	11	52	21	62	3	52	3.2
64	9	1	25	19	56	32	35	2	56	0.6
64	9	1	36	30	79	44	21	2	79	0.3
64	9	1	49	42	106	58	9	2	106	0.3
64	9	1	64	55	74	74	0	1	74	0.0
64	99	1	1	1	13	4	423	9	14	35.9
64	99	1	2	3	19	6	279	7	20	27.7
64	99	1	4	5	27	9	176	5	28	20.3
64	99	1	9	11	42	17	92	3	37	12.5
64	99	1	16	19	52	26	51	3	61	7.0
64	99	1	25	29	64	37	28	2	65	1.7
64	99	1	36	41	88	51	16	2	90	1.7
64	99	1	49	55	116	66	5	2	119	1.8
64	99	1	64	71	82	83	0	1	83	0.0

Table 6.1 Comparison of policies

Table 6.1 gives for different values of cost parameters P, v, h and for different values of mean demand μ the optimal parameters of the compared policies and the adjoined cost deviations. These deviations are again (as in previous sections) measured by

$$D := \frac{C^*_{S(T)} - C^*}{C^*} \, 100 \; [\%] \qquad (6.40)$$

where $C^*_{S(T)} = C'(S^*(T^*),T^*) + p\,\mu$ (c.o.(6.36)) are the optimal costs of the S(T)-policy.
Additionally in Table 6.1, a simple (S,S)-policy has been investigated. This policy can be considered as a naive approximation of the (s,S)-policy without enlarging the inspection period. Results have simply been obtained by setting P = O.
Accordingly

$$\Delta := \frac{C_S^* - C^*}{C^*} \, 100 \; [\%] \qquad (6.41)$$

measures deviations from the optimal costs.

Table 6.1 shows that a simple (S,S)-policy may be a good approximation for a (s,S)-policy if $\mu \cdot h >> P$. This result has also been found for Beta-distributions by [9]. In these cases the expected order cycle using an optimal (s,S)-policy is nearly 1. The optimal T of an S(T)-policy is 1. If on the other hand $\mu \cdot h < P$ an (S,S)-policy turns out to be a poor approximation.
In all cases being computed, an S(T)-policy considerably improved an ordinary (S,S)-policy. If $\mu \cdot h \approx P$ and $\frac{\sigma}{\mu} \cdot 100 \leq 30$ an S(T)-policy is a good approximation for an (s,S)-policy. (For more detailed results see [22]).
Summarizing, our investigations show that also in the pure inventory case with $P \neq 0$ a linear decision rule turns out to be a fairly good approximation.

SUMMARY AND CONCLUDING REMARKS

Starting point of our investigations of a linear policy approach to inventory-production models was the linear-quadratic theory presented in Chapter 2. These models are characterized by the linearity of their plant and feedback equations and by the assumption of a quadratic performance criterion. One of the outstanding properties of linear-quadratic models is the small amount of information being required. Due to the validity of the principle of dynamic certainty equivalence total stochastics of the additive inviroment (sequence of demand) can be reduced to conditional mean forecasts without any loss of optimality. This property has been exploited by Holt et. al. in their work on production smoothing and work-force planning.
Generally, however, linear-quadratic models will not occur in real inventory-production systems. For a back-log situation (to which we confined our investigations) the plant equation (balance equation) of the model per definition is linear, however, there will be no quadratic cost dependances and furthermore capacity constraints will generally be present. However, for a smoothing situation it seems to be appropriate to approximate non-quadratic costs by quadratic ones and, in addition, since one has a short term feedback control, fluctuations would not be too large so that capacity constraints need not be taken into account explicitely. Thus Holt et. al. approximated costs by quadratic ones and found an optimal linear policy. This approximation was performed by a least square regression procedure. Obviously such an approximation is not unique. This is due to the fact that one does not know in which region costs should be approximated. If, e.g., inventory fluctuations are very large, a large region should be used to fit a quadratic function. Obviously, the approximation region depends on the cost functions to be approximated. The "optimal" linear policy found with respect to the approximated costs then depends on the goodness of fit. In order to study the influence of the quadratic approximation on optimal costs we had to construct a definite situation. Starting point

were well-defined piece-wise linear cost criteria which had to be fitted by quadratic functions uniquely. This was done by the so called LNQ-approach. The main point in this approach is the assumption of a linear feedback equation. With this assumption the model could be reduced to a linear-quadratic one (see Sec.3.2). However, the non-quadratic costs were approximated in an optimal way such that a best linear policy was derived with respect to the non-quadratic cost criterion. (It should be mentioned that the LNQ-approach and the procedure of Holt et. al. can differ considerably. (See [8])).

Of course, to derive best linear policies, further assumptions are necessary. First we had to assume policies to be stationary. This assumption turned out not to be too restrictive for smoothing situations. Assuming demand to follow a stationary stochastic process it could be shown that the asymptotic stage ($N\to\infty$) was reached only after a few periods. I.e. because of the feedback structure of the policy initial conditions were found to die out rapidly. The second assumption, i.e., the normality assumption of demand, could be shown not to be too restrictive.
This was shown for the cash balance case by a Gram Charlier development in Sec. 3.6 aud for the pure inventory case in Sec. 6.1.

As could be shown in Sec. 3.2 the LNQ-approach still essentially maintains the principle of certainty equivalence. Although a knowledge of the variance of the demand sequence is necessary to calculate the optimal policy it does not, as discussed in Sec.3.2, play an important role. Summarizing, applying the LNQ-approach one actually needs conditional mean forecasts of demand processes being fairly stationary (but not necessarily Gaussian).

Forecasts used in practice, however, will generally not be conditional means. They are usually exponential smoothing forecasts which only in special situations, as discussed in Sec. 2.3, coincide with conditional mean forecasts. The effect of these non-optimal forecasts on the optimal performance is being studied.

Since one of our main objects was the investigation of the validity of the linearity assumption we concentrated on scalar models with white noise and Gauss-Markov processes as demand sequences. This limitation was necessary since Dynamic Programming solutions usually cannot be derived for higher dimensional state spaces and more complicated disturbance processes. However, regarding the LNQ-approach, general ARMA- processes were taken into account in Sec. 2.5 and the multivariate case is treated in a recent paper by Gaalman [6].

Studying the validity of the linear policy assumption it could be shown that in case of no set-up costs (Sec. 4.1) the LNQ-approach is not inferior than at most 10 %. For set-up costs being present results are less favorable depending on the relation of demand variance and cost parameters. However, enlarging the inspection period, results can be improved considerably [8].
These investigations were performed for the case of uncorrelated demand. An extension to the exponentially correlated case (and zero set-up costs) is presented in Sec. 4.3. It turned out that the suboptimality of the LNQ-approach does not depend on the degree of correlation and, more generally, on the type of autoregressive process being involved.
LNQ-policies are suboptimal but well adapted to stochastic situations. Thus is appeared to be most interesting to compare this suboptimal policy with other suboptimal approaches. An often used suboptimal policy is the so called DDO-approximation discussed in Chap. 5. This approximation postulates the existence of the principle of certainty equivalence reducing the stochastic optimization problem to a deterministic one in which by a rolling horizon procedure forecasts are updated after each period. It could be shown that the suboptimal LNQ-policy is for all cost and demand parameters better than the DDO-policy. Even if we introduce an optimal safety stock the LNQ-procedure is still superior. However, for comparatively high inventory costs a safety stock leads to a considerable improvement of the deterministic approach. On the other hand, for low inventory costs also a safety stock cannot improve the considerable insufficiency of the DDO-approximation.

The important result of Chap. 5 sheds some light on the following general problem. In dynamic production theory two main approaches exist. One is a linear (stochastic) control theoretic approach and the other a deterministic mathematical programming (usually Linear Programming) approach. The control theoretic procedure is better adapted to stochastic situations whereas by the deterministic approach constraints can be incorporated more directly. The findings of Chap. 5 together with the results of Chap. 6 are more in favor of the linear control theoretic approach. They show that the effect of stochastics should not be neglected.

Finally, in Chap. 6, we studied the pure inventory case. With no set-up costs being present the inventory problem turns out to be the limiting case of a production smoothing problem. The result of the LNQ-theory coincides essentially with the well-known S-policy result of the AHM-theory. This shows the applicability of the LNQ-approach also in case of constraints on the decision variable. If fixed ordering costs are present the linear approximation works no longer as good as in the non set-up cost case. However, as shown in Sec. 6.2, enlarging the inspection period results can be improved considerably.

The investigation of suboptimal policies for dynamic stochastic inventory-production models is a broad and fascinating field of great practical importance. Studying linear (suboptimal) policies is just one, but an essential facet of the whole problem.

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