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AND THE

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L. E. J. BROUWER / E. W. BETH / A. HEYTING

EDITORS

Intuitionism

An Introduction

SECOND REVISED EDITION

A. HEYTING

RTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM

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AND

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INTUITIONISM

AN INTRODUCTION

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University of Amsterdam

1966

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PREFACE

In order to prevent the reader from wasting his time in useless attempts to solve supposed riddles, I warn him that the persons of the dialogue are not caricatures of living or deceased persons, much less their doubles. They are pegs to hang ideas on, and nothing else. To a certain extent this is even true for Int, who represents the position of intuitionism. For the sake of clearness I made him speak sometimes in a somewhat more absolute way than I should have done if I had freely expressed my own opinions. The discussion is strictly limited to intuitionism; other conceptions of mathematics are touched on only in so far as they lead to objections against intuitionism. I reject any reproach for incomplete exposition of other points of view.

It was necessary to give proofs in great detail, even where they differ only by small additions from the well-known classical ones. There was no other way to indicate in which places these additions had to be made. In the course of the book, as the reader is supposed to develop a feeling for the specifically intuitionistic difficulties, I have gradually adopted a more condensed style.

I thank all those have contributed to improve the book, among them Dr. Paul Gilmore, Prof. Leon Henkin and Mr. William Tait, who read parts of the manuscript and suggested many linguistic improvements, Mr. J. J. de Iongh and Mr. F. van de Oudeweetering, who accurately revised the manuscript. Mr. de Iongh indicated many corrections and clarifications in the text.

In many places of the book the reader will find old-fashioned reasonings which lack generality and which are more clumsy than the modern methods. This has different reasons. In the first place, the powerful methods often make an excessive use of indirect proof, so that it is almost impossible to introduce them in intuitionistic mathematics. In the second place, the very general modern theories proceed by the axiomatic method. Now this method can only work well, if some concrete theories exist, from which the axiomatic theory can be constructed by generalization.

For instance, general topology could only be developed after the topology of euclidean spaces was known in some detail. As a matter of fact, almost no part of intuitionistic mathematics has been investigated deeply enough to admit the construction of a general axiomatic theory. Thus in this book I had to confine myself to the most elementary case of integration; when this will be better known than it is at present, it will become possible to construct an axiomatic theory on the subject. Even in the case of algebra, where axiomatization is possible at this moment, it seemed better to treat the concrete example of the real number field, in view of the fact that the book is meant as an introduction.

Probably in some cases I used antiquated methods because I did not know the modern ones. One of the aims of the book is, to enable working mathematicians to decide, which of their results can be proved intuitionistically. Intuitionism can only flourish, if mathematicians, working in different fields, become actively interested in it and make contributions to it. In order to build up a definite branch of intuitionistic mathematics, it is necessary in the first place to have a thorough knowledge of the corresponding branch of classical mathematics, and in the second place to know by experience where the intuitionistic pitfalls lie. I try in this book to teach the latter; I hope that some of my readers will give a more satisfactory treatment of details than I could, or that they will treat other theories intuitionistically. The "reading suggestions" are intended to help them; they indicate the most important intuitionistic work on some special subjects.

References have been made in the following way.
"Th. 2" refers to theorem 2 of the same section.
"Def. 2" refers to definition 2 of the same section.
"6.2.1, Th. 2" refers to theorem 2 of section 6.2.1.

A. HEYTING

Amsterdam, August 1955.

PREFACE TO THE SECOND EDITION

In this reprint I have made only minor changes and additions. Only section 6.5 has been partly rewritten. It would have been tempting to add chapters on the relations of intuitionism to the theory of recursive functions and to the recent Russian investigations on constructive analysis; however, it seemed better to confine the content to one subject.

Amsterdam, March 1965. A. HEYTING

Persons of the dialogue: Class, Form, Int, Letter, Prag, Sign.

I

DISPUTATION

CLASS. How do you do, Mr. Int? Did you not flee the town on this fine summer day?

INT. I had some ideas and worked them out at the library.

CLASS. Industrious bee! How are you getting along?

INT. Quite well. Shall we have a drink?

CLASS. Thank you. I bet you worked on that hobby of yours, rejection of the excluded middle, and the rest. I never understood why logic should be reliable everywhere else, but not in mathematics.

INT. We have spoken about that subject before. The idea that for the description of some kinds of objects another logic may be more adequate than the customary one has sometimes been discussed. But it was Brouwer who first discovered an object which actually requires a different form of logic, namely the mental mathematical construction [L. E. J. Brouwer 1908]. The reason is that in mathematics from the very beginning we deal with the infinite, whereas ordinary logic is made for reasoning about finite collections.

CLASS. I know, but in my eyes logic is universal and applies to the infinite as well as to the finite.

INT. You ought to consider what Brouwer's program was [L. E. J. Brouwer 1907]. It consisted in the investigation of mental mathematical construction as such, without reference to questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them. That this point of view leads immediately to the rejection of the principle of excluded middle, I can best demonstrate by an example.

Let us compare two definitions of natural numbers, say k and l.

I. k is the greatest prime such that $k-1$ is also a prime, or $k=1$ if such a number does not exist.

II. l is the greatest prime such that $l-2$ is also a prime, or $l=1$ if such a number does not exist.

Classical mathematics neglects altogether the obvious difference in character between these definitions. k can actually be calculated ($k=3$), whereas we possess no method for calculating l, as it is not known whether the sequence of pairs of twin primes p, $p+2$ is finite or not. Therefore intuitionists reject II as a definition of an integer; they consider an integer to be well defined only if a method for calculating it is given. Now this line of thought leads to the rejection of the principle of excluded middle, for if the sequence of twin primes were either finite or not finite, II would define an integer.

CLASS. One may object that the extent of our knowledge about the existence or non-existence of a last pair of twin primes is purely contingent and entirely irrelevant in questions of mathematical truth. Either an infinity of such pairs exist, in which case $l=1$; or their number is finite, in which case l equals the greatest prime such that $l-2$ is also a prime. In every conceivable case l is defined; what does it matter whether or not we can actually calculate the number?

INT. Your argument is metaphysical in nature. If "to exist" does not mean "to be constructed", it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. We have no objection against a mathematician privately admitting any metaphysical theory he likes, but Brouwer's program entails that we study mathematics as something simpler, more immediate than metaphysics. In the study of mental mathematical constructions "to exist" must be synonymous with "to be constructed".

CLASS. That is to say, as long as we do not know if there exists a last pair of twin primes, II is not a definition of an integer, but as soon as this problem is solved, it suddenly becomes such a definition. Suppose on January 1, 1970 it is proved that an infinity of twin primes exists; from that moment $l=1$. Was $l=1$ before that date or not? [Menger 1930].

INT. A mathematical assertion affirms the fact that a certain mathematical construction has been effected. It is clear that before the construction was made, it had not been made. Applying this remark to your example, we see that before Jan. 1, 1970 it had not been proved that $l=1$. But this is not what you mean. It seems to me that in order to clarify the sense of your question you must again refer to metaphysical concepts: to some world of mathematical things existing independently of our knowledge, where "$l=1$" is true in some absolute sense. But I repeat that mathematics ought not to depend upon such notions as these. In fact all mathematicians and even intuitionists are convinced that in some sense mathematics bear upon eternal truths, but when trying to define precisely this sense, one gets entangled in a maze of metaphysical difficulties. The only way to avoid them is to banish them from mathematics. This is what I meant by saying that we study mathematical constructions as such and that for this study classical logic is inadequate.

CLASS. Here come our friends Form and Letter. Boys, we are having a most interesting discussion on intuitionism.

LETTER. Could you speak about anything else with good old Int? He is completely submerged in it.

INT. Once you have been struck with the beauty of a subject, devote your life to it!

FORM. Quite so! Only I wonder how there can be beauty in so indefinite a thing as intuitionism. None of your terms are well-defined, nor do you give exact rules of derivation. Thus one for ever remains in doubt as to which reasonings are correct and which are not [R. Carnap 1934, p. 41; 1937, p. 46] [W. Dubislav 1932, p. 57, 75]. In daily speech no word has a perfectly fixed meaning; there is always some amount of free play, the greater, the more abstract the notion is. This makes people miss each other's point, also in non-formalized mathematical reasonings. The only way to achieve absolute rigour is to abstract all meaning from the mathematical statements and to consider them for their own sake, as sequences of signs, neglecting the sense they may convey. Then it is possible to formulate definite rules for deducing new statements from those already known and to avoid the uncertainty resulting from the ambiguity of language.

INT. I see the difference between formalists and intuitionists mainly as one of taste. You also use meaningful reasoning in what Hilbert called metamathematics, but your purpose is to separate these reasonings from purely formal mathematics and to confine yourself to the most simple reasonings possible. We, on the contrary, are interested not in the formal side of mathematics, but exactly in that type of reasoning which appears in metamathematics; we try to develop it to its farthest consequences. This preference arises from the conviction that we find here one of the most fundamental faculties of the human mind.

FORM. If you will not quarrel with formalism, neither will I with intuitionism. Formalists are among the most pacific of mankind. Any theory may be formalized and then becomes subject to our methods. Also intuitionistic mathematics may and will be thus treated [R. Carnap 1934, p. 44; 1937, p. 51].

CLASS. That is to say, intuitionistic mathematics ought to be studied as a part of mathematics. In mathematics we investigate the consequences of given assumptions; the intuitionistic assumptions may be interesting, but they have no right to a monopoly.

INT. Nor do we claim that; we are content if you admit the good right of our conception. But I must protest against the assertion that intuitionism starts from definite, more or less arbitrary assumptions. Its subject, constructive mathematical thought, determines uniquely its premises and places it beside, not interior to classical mathematics, which studies another subject, whatever subject that may be. For this reason an agreement between formalism and intuitionism by means of the formalization of intuitionistic mathematics is also impossible. It is true that even in intuitionistic mathematics the finished part of a theory may be formalized. It will be useful to reflect for a moment upon the meaning of such a formalization. We may consider the formal system as the linguistic expression, in a particularly suitable language, of mathematical thought.

If we adopt this point of view, we clash against the obstacle of the fundamental ambiguousness of language. As the meaning of a word can never be fixed precisely enough to exclude every possibility of misunderstanding, we can never be mathematically sure that the formal system expresses correctly our mathematical thoughts.

However, let us take another point of view. We may consider the formal system itself as an extremely simple mathematical structure; its entities (the signs of the system) are associated with other, often very complicated, mathematical structures. In this way formalizations may be carried out inside mathematics, and it becomes a powerful mathematical tool. Of course, one is never sure that the formal system represents fully any domain of mathematical thought; at any moment the discovering of new methods of reasoning may force us to extend the formal system.

FORM. For several years we have been familiar with this situation. Gödel's incompleteness theorem showed us that any consistent formal system of number-theory may be extended consistently in different ways.

INT. The difference is that intuitionism proceeds independently of the formalization, which can but follow after the mathematical construction.

CLASS. What puzzles me most is that you both seem to start from nothing at all. You seem to be building castles in the air. How can you know if your reasoning is sound if you do not have at your disposal the infallible criterion given by logic? Yesterday I talked with Sign, who is still more of a relativist than either of you. He is so slippery that no argument gets hold of him, and he never comes to any somewhat solid conclusion. I fear this fate for anybody who discards the support of logic, that is, of common sense.

SIGN. Speak of the devil and his imp appears. Were you speaking ill of me?

CLASS. I alluded to yesterday's discussion. To-day I am attacking these other two damned relativists.

SIGN. I should like to join you in that job, but first let us hear the reply of your opponents. Please meet my friend Prag; he will be interested in the discussion.

FORM. How do you do? Are you also a philosopher of science?

PRAG. I hate metaphysics.

INT. Welcome, brother!

FORM. Why, I would rather not defend my own position at the moment, as our discussion has dealt mainly with intuitionism and we might easily confuse it. But I fear that you are wrong as to

intuitionistic logic. It has indeed been formalized and valuable work in this field has been done by a score of authors. This seems to prove that intuitionists esteem logic more highly than you think, though it is another logic than you are accustomed to. INT. I regret to disappoint you. Logic is not the ground upon which I stand. How could it be? It would in turn need a foundation, which would involve principles much more intricate and less direct than those of mathematics itself. A mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever. One may very well know whether a reasoning is sound without using any logic; a clear scientific conscience suffices. Yet it is true that intuitionistic logic has been developed. To indicate what its significance is, let me give you an illustration. Let A designate the property of an integer of being divisible by 8, B the same by 4, C the same by 2. For $8a$ we may write $4 \times 2a$; by this mathematical construction P we see that the property A entails B $(A \to B)$. A similar construction Q shows $B \to C$. By effecting first P, then Q (juxtaposition of P and Q) we obtain $8a = 2 \times (2 \times 2a)$ showing $A \to C$. This process remains valid if for A, B, C we substitute arbitrary properties: If the construction P shows that $A \to B$ and Q shows that $B \to C$, then the juxtaposition of P and Q shows that $A \to C$. We have obtained a logical theorem. The process by which it is deduced shows us that it does not differ essentially from mathematical theorems; it is only more general, e.g. in the same sense that "addition of integers is commutative" is a more general statement than "$2+3=3+2$". This is the case for every logical theorem: it is but a mathematical theorem of extreme generality; that is to say, logic is a part of mathematics, and can by no means serve as a foundation for it. At least, this is the conception of logic to which I am naturally led; it may be possible and desirable to develop other forms of logic for other purposes.

It is the mathematical logic which I just described that has been formalized. The resulting formal system proves to have peculiar properties, very interesting when compared to those of other systems of formal logic. This fact has led to the investigations to which Mr. Form alluded, but, however interesting, they are tied but very loosely to intuitionistic mathematics.

LETTER. In my opinion all these difficulties are imaginary or artificial. Mathematics is quite a simple thing. I define some signs and I give some rules for combining them; that is all.

FORM. You want some modes of reasoning to prove the consistency of your formal system.

LETTER. Why should I want to prove it? You must not forget that our formal systems are constructed with the aim towards applications and that in general they prove useful; this fact would be difficult to explain if every formula were deducible in them. Thereby we get a practical conviction of consistency which suffices for our work. What I contest in intuitionism is the opinion that mathematics has anything to do with the infinite. I can write down a sign, say α, and call it the cardinal number of the integers. After that I can fix rules for its manipulation in agreement with those which Mr. Class uses for this notion; but in doing this I operate entirely in the finite. As soon as the notion of infinity plays a part, obscurity and confusion penetrate into the reasoning. Thus all the intuitionistic assertions about the infinite seem to me highly ambiguous, and it is even questionable whether such a sign as $10^{10^{10}}$ has any other meaning than as a figure on paper with which we operate according to certain rules [J. Dieudonné 1949].

INT. Of course your extreme finitism grants the maximum of security against misunderstanding, but in our eyes it implies a denial of understanding which it is difficult to accept. Children in the elementary school understand what the natural numbers are and they accept the fact that the sequence of natural numbers can be indefinitely continued.

LETTER. It is suggested to them that they understand.

INT. That is no objection, for every communication by means of language may be interpreted as suggestion. Also Euclid in the 20th proposition of Book IX, where he proved that the set of prime numbers is infinite, knew what he spoke about. This elementary notion of natural numbers, familiar to every thinking creature, is fundamental in intuitionistic mathematics. We do not claim for it any form of certainty or definiteness in an absolute sense, which would be unrealizable, but we contend that it is sufficiently clear to build mathematics upon.

LETTER. My objection is that you do not suppose too little, as

Mr. Class thinks, but far too much. You start from certain principles which you take as intuitively clear without any explanation and you reject other modes of reasoning without giving any grounds for that discrimination. For instance, to most people the principle of the excluded middle seems at least as evident as that of complete induction. Why do you reject the former and accept the latter? Such an unmotivated choice of first principles gives to your system a strongly dogmatic character.

INT. Indeed intuitionistic assertions must seem dogmatic to those who read them as assertions about facts, but they are not meant in this sense. Intuitionistic mathematics consists, as I have explained already to Mr. Class, in mental constructions; a mathematical theorem expresses a purely empirical fact, namely the success of a certain construction. "$2+2=3+1$" must be read as an abbreviation for the statement: "I have effected the mental constructions indicated by "$2+2$" and by "$3+1$" and I have found that they lead to the same result." Now tell me where the dogmatic element can come in; not in the mental construction itself, as is clear by its very nature as an activity, but no more in the statements made about the constructions, for they express purely empirical results.

LETTER. Yet you contend that these mental constructions lead to some sort of truth; they are not a game of solitaire, but in some sense must be of value for mankind, or you would be wrong in annoying others with them. It is in this pretence that I see the dogmatic element. The mathematical intuition inspires you with objective and eternal truths; in this sense your point of view is not only dogmatic, but even theological [H. B. Curry 1951, p. 6].

INT. In the first instance, my mathematical thoughts belong to my individual intellectual life and are confined to my personal mind, as is the case for other thoughts as well. We are generally convinced that other people have thoughts analogous to our own and that they can understand us when we express our thoughts in words, but we also know that we are never quite sure of being faultlessly understood. In this respect, mathematics does not essentially differ from other subjects; if for this reason you consider mathematics to be dogmatic, you ought to call any human reasoning dogmatic. The characteristic of mathematical thought

is, that it does not convey truth about the external world, but is only concerned with mental constructions. Now we must distinguish between the simple practice of mathematics and its valuation. In order to construct mathematical theories no philosophical preliminaries are needed, but the value we attribute to this activity will depend upon our philosophical ideas.

SIGN. In the way you treat language you put the clock back. Primitive language has this floating, unsteady character you describe, and the language of daily life is still in the main of the same sort, but as soon as scientific thought begins, the formalization of language sets in. In the last decades significists have studied this process. It has not yet come to an end, for more strictly formalized languages are still being formed.

INT. If really the formalization of language is the trend of science, then intuitionistic mathematics does not belong to science in this sense of the word. It is rather a phenomenon of life, a natural activity of man, which itself is open to study by scientific methods; it has actually been studied by such methods, namely that of formalizing intuitionistic reasoning and the signific method, but it is obvious that this study does not belong to intuitionistic mathematics, nor do its results. That such a scientific examination of intuitionistic mathematics will never produce a complete and definite description of it, no more than a complete theory of other phenomena is attainable, is clearly to be seen. Helpful and interesting as these meta-intuitionistic considerations may be, they cannot be incorporated into intuitionistic mathematics itself. Of course, these remarks do not apply to formalization inside mathematics, as I described it a few moments ago.

PRAG. Allow me to underline what Mr. Sign said just now. Science proceeds by formalization of language; it uses this method because it is efficient. In particular the modern completely formalized languages have appeared to be most useful. The ideal of the modern scientist is to prepare an arsenal of formal systems ready for use from which he can choose, for any theory, that system which correctly represents the experimental results. Formal systems ought to be judged by this criterion of usefulness and not by a vague and arbitrary interpretation, which is preferred for dogmatic or metaphysical reasons.

INT. It seems quite reasonable to judge a mathematical system by its usefulness. I admit that from this point of view intuitionism has as yet little chance of being accepted, for it would be premature to stress the few weak indications that it might be of some use in physics [J. L. Destouches 1951]; in my eyes its chances of being useful for philosophy, history and the social sciences are better. In fact, mathematics, from the intuitionistic point of view, is a study of certain functions of the human mind, and as such it is akin to these sciences. But is usefulness really the only measure of value? It is easy to mention a score of valuable activities which in no way support science, such as the arts, sports, and light entertainment. We claim for intuitionism a value of this sort, which it is difficult to define beforehand, but which is clearly felt in dealing with the matter. You know how philosophers struggle with the problem of defining the concept of value in art; yet every educated person feels this value. The case is analogous for the value of intuitionistic mathematics.

FORM. For most mathematicians this value is affected fatally by the fact that you destroy the most precious mathematical results; a valuable method for the foundation of mathematics ought to save as much as possible of its results [D. Hilbert 1922]. This might even succeed by constructive methods; for definitions of constructiveness other than that advocated by the intuitionists are conceivable. For that matter, even the small number of actual intuitionists do not completely agree about the delimination of the constructive. The most striking example is the rejection by Griss of the notion of negation, which other intuitionists accept as perfectly clear [H. Freudenthal 1936A] [G. F. C. Griss 1946, p. 24; 1946A]. It seems probable, on the other hand, that a somewhat more liberal conception of the constructive might lead to the saving of the vital parts of classical mathematics.

INT. As intuitionists speak a non-formalized language, slight divergences of opinion between them can be expected. Though they have arisen sooner and in more acute forms than we could foresee, they are in no way alarming, for they all concern minor points and do not affect the fundamental ideas, about which there is complete agreement. Thus it is most unlikely that a wider conception of constructiveness could obtain the support of in-

tuitionists. As to the mutilation of mathematics of which you accuse me, it must be taken as an inevitable consequence of our standpoint. It can also be seen as the excision of noxious ornaments, beautiful in form, but hollow in substance, and it is at least partly compensated for by the charm of subtle distinctions and witty methods by which intuitionists have enriched mathematical thought.

FORM. Our discussion has assumed the form of a discussion of values. I gather from your words that you are ready to acknowledge the value of other conceptions of mathematics, but that you claim for your conception a value of its own. Is that right?

INT. Indeed, the only positive contention in the foundation of mathematics which I oppose is that classical mathematics has a clear sense; I must confess that I do not understand that. But even those who maintain that they do understand it might still be able to grasp our point of view and to value our work.

LETTER. It is shown by the paradoxes that classical mathematics is not perfectly clear.

FORM. Yes, but intuitionistic criticism goes much farther than is necessary to avoid the paradoxes; Mr. Int has not even mentioned them as an argument for his conception, and no doubt in his eyes consistency is but a welcome by-product of intuitionism.

SIGN. You describe your activity as mental construction, Mr. Int, but mental processes are only observable through the acts to which they lead, in your case through the words you speak and the formulas you write. Does not this mean that the only way to study intuitionism is to study the formal system which it constructs?

INT. When looking at the tree over there, I am convinced I see a tree, and it costs considerable training to replace this conviction by the knowledge that in reality lightwaves reach my eyes, leading me to the construction of an image of the tree. In the same way, in speaking to you I am convinced that I press my opinions upon you, but you instruct me that in reality I produce vibrations in the air, which cause you to perform some action, e.g. to produce other vibrations. In both cases the first view is the natural one, the second is a theoretical construction. It is too often forgotten that the truth of such constructions depends upon the present state of

science and that the words "in reality" ought to be translated into "according to the contemporary view of scientists". Therefore I prefer to adhere to the idea that, when describing intuitionistic mathematics, I convey thoughts to my hearers; these words ought to be taken not in the sense of some philosophical system, but in the sense of every-day life.

SIGN. Then intuitionism, as a form of interaction between men, is a social phenomenon and its study belongs to the history of civilization.

INT. Its study, not its practice. Here I agree with Mr. Prag: primum vivere, deinde philosophari, and if we like we can leave the latter to others. Let those who come after me wonder why I built up these mental constructions and how they can be interpreted in some philosophy; I am content to build them in the conviction that in some way they will contribute to the clarification of human thought.

PRAG. It is a common fault of philosphers to speak about things they know but imperfectly and we are near to being caught in that trap. Is Mr. Int willing to give us some samples of intuitionistic reasoning, in order that we may better be able to judge the quality of the stuff?

INT. Certainly, and even I am convinced that a few lessons will give you a better insight into it than lengthy discussions. May I beg those gentlemen who are interested in my explanations, to follow me to my classroom?

II

ARITHMETIC

2.1. Natural numbers

INT. We start with the notion of the natural numbers 1, 2, 3, etc. They are so familiar to us, that it is difficult to reduce this notion to simpler ones. Yet I shall try to describe their sense in plain words. In the perception of an object we conceive the notion of an entity by a process of abstracting from the particular qualities of the object. We also recognize the possibility of an indefinite repetition of the conception of entities. In these notions lies the source of the concept of natural numbers [L. E. J. Brouwer 1907, p. 3; 1948, p. 1237].

CLASS. Are these considerations not metaphysical in nature?

INT. They become so if one tries to build up a theory about them, e.g., to answer the question whether we form the notion of an entity by abstraction from actual perceptions of objects, or if, on the contrary, the notion of an entity must be present in our mind in order to enable us to perceive an object apart from the rest of the world. But such questions have nothing to do with mathematics. We simply state the fact that the concepts of an abstract entity and of a sequence of such entities are clear to every normal human being, even to young children.

CLASS. Let us admit that you have at your disposal the natural numbers. Now you must have some startingpoint for your deductions. Do you accept Peano's axioms?

INT. While you think in terms of axioms and deductions, we think in terms of evidence; that makes all the difference. I do not accept any axioms which I might reject if I chose to do so. The notion of natural numbers does not come to us as a bare notion, but from the beginning it is clothed in properties which I can detect by simple examination. Those properties which you describe by Peano's axioms are among them, as I shall show you. Let "N"

be an abbreviation for "natural number". The first two properties (1 is an N and if x is an N, then the successor of x is an N) can immediately be seen to be true by carrying out the generating construction. The same applies to the third and fourth axiom (If x and y are N and the successors of x and y are equal, then $x=y$; the successor of an N is not equal to 1). As to the so-called axiom of complete induction, it must be seen as a general theorem on natural numbers. Some remarks will be useful in preparation for its proof.

Clearly the construction of a natural number n consists in building up successively certain natural numbers, called the numbers from 1 to n, in signs: $1 \to n$. At any step in the construction we can pause to investigate whether the number reached at that step possesses a certain property or not. For instance, we can ascertain whether a given number m, different from n, occurs in $1 \to n$ or not. In the first case we say that $m<n$, in the second case that $m>n$. Now it is a theorem that $m \neq n$ and $m>n$ implies $n<m$. For if m does not occur in $1 \to n$, this fact proves that at the step at which we reach n, the construction of m is not terminated; thus n occurs in $1 \to m$.

The theorem of complete induction admits a proof of the same kind. Suppose $E(x)$ is a predicate of natural numbers such that $E(1)$ is true and that, for every natural number n, $E(n)$ implies $E(n')$, where n' is the successor of n. Let p be any natural number. Running over $1 \to p$ we know that the property E, which is true for 1, will be preserved at every step in the construction of p; therefore $E(p)$ holds.

Analogous remarks apply to the usual recursive definitions of sum and product in the domain of natural numbers. By running over $1 \to p$ we see that indeed $a+p$ and $p \cdot a$ are defined for arbitrary natural numbers a and p. Once we possess the fundamental methods of induction and recursion, the arithmetic of natural numbers meets with no serious difficulties, nor does that of integers or even of rationals. Difficulties arise only where the totality of integers is involved in some way, as in our attempt II to define an integer in our discussion. But such problems do not belong to elementary arithmetic.

FORM. You spoke repeatedly of equal natural numbers. What

does that mean? Is not a definition of equality, based for instance on a one-to-one relation, necessary?

INT. Indeed this point needs some clarification; it forces me even to revise somewhat our notion of a natural number. If a natural number were nothing but the result of a mental construction, it would not subsist after the act of its construction and it would be impossible to compare it with another natural number, constructed at another time and place. It is clear that we cannot solve this problem if we cling to the idea that mathematics is purely mental. In reality we fix a natural number, x say, by means of a material representation; to every entity in the construction of x we associate, e.g., a dot on paper. This enables us to compare by simple inspection natural numbers which were constructed at different times.

FORM. That amounts to the application of one-to-one relations.

INT. We may express it in that way, provided we are well aware that the process of comparison is staged at the pre-mathematical level. Mathematics begins after the concepts of natural numbers and of equality between natural numbers have been formed. Of course the dichotomy between mathematics and pre-mathematics is artificial, just as is every splitting up of human thought, but this dichotomy corresponds to an important difference in methods.

LETTER. One would expect that the basic notions of mathematics were simple and clear, but your notion of a natural number turns out to be pretty complicated.

INT. As far as I know, psychology has not discovered mental atoms. Every notion may be analysed, none is comprehensible by itself; any notion depends for its explanation upon its relations to other notions. The notion of a natural number is no exception to this rule. Yet it is suitable to serve as one of the main basic concepts of mathematics, mainly for the following three reasons:

1. It is easily understood by any person who has a minimum of education,

2. It is universally applicable in the process of counting,

3. It underlies the construction of analysis.

CLASS. Apart from these philosophical questions, your interpretation of the arithmetic of rationals is identical with ours.

2.2 Real number-generators

2.2.1. *Definition; relation of coincidence*

INT. Yes, but at the next station, that of real numbers, we enter a totally different landscape. As in the classical mathematics, so in intuitionism different equivalent theories of real numbers are possible [L. E. J. Brouwer 1919A, p. 3; A. Heyting 1935]. I shall briefly expound Cantor's theory, which has some advantages for our purpose.

Let us suppose that the theory of rationals, including their order relations, has been developed. A sequence $\{a_n\}$ of rational numbers is called a *Cauchy sequence*, if for every natural number k we can find a natural number $n = n(k)$, such that $|a_{n+p} - a_n| < 1/k$ for every natural number p. This must be so understood, that, given k, we are able to determine effectively $n(k)$.

Example. The sequence $a \equiv \{2^{-n}\}$ is a Cauchy sequence. Let the sequence $b \equiv \{b_n\}$ be defined as follows: If the nth digit after the decimal point in the decimal expansion of π is the 9 of the first sequence 0123456789 in this expansion, $b_n = 1$, in every other case $b_n = 2^{-n}$. b differs from a in at most one term, so b is classically a Cauchy sequence, but as long as we do not know whether a sequence 0123456789 occurs in π, we are not able to find n such that $|b_{n+p} - b_n| < 1/2$ for every p; we have no right to assert that b is a Cauchy sequence in our sense.

Definition 1. A Cauchy sequence of rational numbers is a *real number-generator*. Where no confusion is possible, we shall speak briefly of a number-generator.

Two number-generators $a \equiv \{a_n\}$ and $b \equiv \{b_n\}$ are *identical*, if $a_n = b_n$ for every n. We express this relation by $a \equiv b$. The following notion of coincidence is more important.

Definition 2. The number-generators $a \equiv \{a_n\}$ and $b \equiv \{b_n\}$ *coincide*, if for every k we can find $n = n(k)$ such that $|a_{n+p} - b_{n+p}| < 1/k$ for every p. This relation is denoted by $a = b$.

Theorem. The relation of coincidence between number-generators is reflexive, symmetrical and transitive. The easy proof is well-known.

Remark. Given any number-generator $a \equiv \{a_n\}$, a number-

generator $b \equiv \{b_n\}$ can be found such that $a=b$ and that the sequence $\{b_n\}$ converges as rapidly as we wish. For instance, in order that $|b_{n+p}-b_n| < 1/n$ for every n and p, it suffices to take $b_k = a_{n(k)}$ for every k.

If, in the following, a number-generator is denoted by one letter, v say, it will be silently understood that it can also be denoted by $\{v_n\}$, so that v_n is the nth component of the sequence v.

As the notion of a real number presupposes the fundamental notions of set theory, I postpone the definition of a real number (as a set of coincident number-generators) till chapter III.

2.2.2. *Inequality relation between number-generators*

If $a=b$ is contradictory (that means: if the supposition that $a=b$ leads to a contradiction), we write $a \neq b$.

Theorem 1. If $a \neq b$ is contradictory, then $a=b$ [L. E. J. Brouwer 1925, p. 254].

Proof. Determine n so that $|a_{n+p}-a_n| < 1/4k$ and $|b_{n+p}-b_n| < 1/4k$ for every p. Suppose $|a_n - b_n| \geqq 1/k$; then we would have $|a_{n+p}-b_{n+p}| > 1/2k$ for every p, which entails $a \neq b$. Thus $|a_n-b_n| < 1/k$ and $|a_{n+p}-b_{n+p}| < 2/k$ for every p, and as for every k we can find n so that this inequality is valid for every p, we have $a=b$.

CLASS. We must get used to the fact that such a theorem needs a proof.

INT. A proof of the impossibility of the impossibility of a property is not in every case a proof of the property itself. It will be instructive to illustrate this by an example [L. E. J. Brouwer 1925, p. 252]. I write the decimal expansion of π and under it the decimal fraction $\varrho = 0.333\ldots$, which I break off as soon as a sequence of digits 0123456789 has appeared in π. If the 9 of the first sequence 0123456789 in π is the kth digit after the decimal point, $\varrho = \frac{10^k-1}{3.10^k}$. Now suppose that ϱ could not be rational;
then $\varrho = \frac{10^k-1}{3.10^k}$ would be impossible and no sequence could appear in π; but then $\varrho = 1/3$, which is also impossible. The assumption that ϱ cannot be rational has lead to a contradiction; yet we have no right to assert that ϱ is rational, for this would mean that we

could calculate integers p and q so that $\varrho = p/q$; this evidently requires that we can either indicate a sequence 0123456789 in π or demonstrate that no such sequence can appear.

CLASS. And you reject my argument that ϱ is equal to one of the rational numbers 1/3, 0.3, 0.33, etc., though we do not know to which of them.

INT. Exactly, and I think that the real state of affairs is better expressed by saying that ϱ cannot be different from each of these numbers.

CLASS. It seems to me that the difficulty is caused by your interpretation of negation, which diverges from the usual one. To you "ϱ is not rational" means the same as "the supposition that ϱ is rational, leads to a contradiction". Thus you only speak of falsity "de jure", whereas usually negation refers to falsity "de facto". This may account for the peculiar behaviour of your negation.

INT. I can adhere to this view, if we agree that in intuitionistic mathematics only falsity "de jure" can play a part; the introduction of mere "de facto" falsity would conflict with the principle of constructivity.

Strictly speaking, we must well distinguish the use of "not" in mathematics from that in explanations which are not mathematical, but are expressed in ordinary language. In mathematical assertions no ambiguity can arise: "not" has always the strict meaning. "The proposition p is not true", or "the proposition p is false" means "If we suppose the truth of p, we are led to a contradiction". But if we say that the number-generator ϱ which I defined a few moments ago is not rational, this is not meant as a mathematical assertion, but as a statement about a matter of facts; I mean by it that as yet no proof for the rationality of ϱ has been given. As it is not always easy to see whether a sentence is meant as a mathematical assertion or as a statement about the present state of our knowledge, it is necessary to be careful about the formulation of such sentences. Where there is some danger of ambiguity, we express the mathematical negation by such expressions as "it is impossible that", "it is false that", "it cannot be", etc., while the factual negation is expressed by "we have no right to assert that", "nobody knows that", etc.

There is a criterion by which we are able to recognize mathe-

matical assertions as such. Every mathematical assertion can be expressed in the form: "I have effected the construction A in my mind". The mathematical negation of this assertion can be expressed as "I have effected in my mind a construction B, which deduces a contradiction from the supposition that the construction A were brought to an end", which is again of the same form. On the contrary, the factual negation of the first assertion is: "I have not effected the construction A in my mind"; this statement has not the form of a mathematical assertion.

PRAG. You seem to be very much interested in such examples as that of the number ϱ. I find them in almost every one of your papers. To an outsider the construction of such far-fetched pathological cases seems a somewhat futile occupation.

INT. We are forced to construct such examples in order to convince others of the necessity of a proof for certain propositions. But it would be wrong to consider them as an essential part of intuitionistic mathematics, just as it would be wrong to contend that the continuous non-differentiable function of Weierstrass is an essential part of the classical differential calculus.

2.2.3. *Apartness-relation between number-generators*

But we have already insisted too much on the negative notion of inequality; negative concepts are for us even less important than in classical mathematics; whenever possible we replace them by positive concepts. In the case of inequality between real number-generators we do this by the

Definition 1: For real number-generators a and b, *a lies apart from b*, $a \# b$, means that n and k can be found such that $|a_{n+p} - b_{n+p}| > 1/k$ for every p [L. E. J. Brouwer 1919A, p. 3].

$a \# b$ entails $a \neq b$, but the converse assertion is not correct. This can be shown by an example which is still more sophisticated than the previous one. As it raises some rather delicate questions, I shall postpone it till a later lecture (8.1.1). I hope that for the moment it is sufficiently clear that $a \# b$ is a stronger condition than $a \neq b$, because the former demands the actual indication of the numbers n and k, whereas the latter contents itself with a mere proof of impossibility.

I shall now deduce the main properties of the $\neq$ relation.

Theorem 1. If $a \neq b$, then $b \neq a$.

Theorem 2. If $a \neq b$ and $a = a'$, then $a' \neq b$.

Proof. We can find n and k so that

$$|a_{n+p} - b_{n+p}| > 1/k \text{ for every } p.$$

Now we determine m so that

$$|a_{m+p} - a'_{m+p}| < 1/2k \text{ for every } p.$$

Then, if h is the largest of m, n:

$$|a'_{h+p} - b_{h+p}| > 1/2k \text{ for every } p.$$

Theorem 3. If $a \neq b$ is impossible, $a = b$ [L. E. J. Brouwer 1925, p. 254].

The proof of 2.2.2. Th. 1 has been so arranged as to establish this stronger theorem.

Theorem 4. If $a \neq b$, then for any real number-generator c either $a \neq c$ or $b \neq c$ holds.

Proof. Similarly as in the proof of Th. 2 we find k and h so that

(i) $$|a_{h+p} - b_{h+p}| > 1/k \text{ for every } p;$$

(ii) $$|a_{h+1} - a_{h+p}| < 1/8k \text{ for every } p;$$

(iii) $$|b_{h+1} - b_{h+p}| < 1/8k \text{ for every } p;$$

(iv) $$|c_{h+1} - c_{h+p}| < 1/8k \text{ for every } p.$$

Then by (i) with $p = 1$, either $|a_{h+1} - c_{h+1}| > 1/2k$ or $|b_{h+1} - c_{h+1}| > 1/2k$. In the first case we derive from (ii) and (iv) that

$$|a_{h+p} - c_{h+p}| > 1/4k \text{ for every } p,$$

hence $a \neq c$; in the second case

$$|b_{h+p} - c_{h+p}| > 1/4k \text{ for every } p,$$

i.e. $b \neq c$.

2.2.4. *Fundamental operations with number-generators*

Definition 1. If a and b are real number-generators defined by the sequences of rational numbers $\{a_n\}$ and $\{b_n\}$ respectively, then

i. $a+b$ is the sequence $\{a_n+b_n\}$;

ii. ab is the sequence $\{a_n b_n\}$;

iii. $-a$ is the sequence $\{-a_n\}$;

iv. If $a \not\equiv 0$, then a^{-1} is the sequence $\{c_n\}$, where $c_n = a_n^{-1}$ if $a_n \neq 0$ and $c_n = 0$ if $a_n = 0$.

Theorem 1. $a+b$, ab, $-a$ and a^{-1} are again real number-generators.

Proof. It is easily proved by known methods that the defining sequences are Cauchy sequences.

Remarks. (1). Note that a^{-1} can only be defined if $a \not\equiv 0$; this condition is necessary and sufficient to show that the defining sequence of a^{-1} is bounded.

(2). The rational number r may be identified with the sequence every member of which is r. In this way the system of real number-generators appears as an extension of that of rationals. This remark has been applied in def. 1. iv.

(3). For rational numbers no difference need be made between $\neq$ and $\not\equiv$, for if $a \neq b$, then $a-b$ is a rational number which is not 0, $a-b = p/q > 1/2q$.

Theorem 2. If $a = a'$, $b = b'$, then $a+b = a'+b'$ and $ab = a'b'$.
If $a = a'$, then $-a = -a'$.
If $a \not\equiv 0$, and $a = a'$, then $a^{-1} = (a')^{-1}$.

Proof. It will suffice to prove the last assertion. As $a \not\equiv 0$, we can find n and j so that $|a_{n+p}| > 1/j$ for every p; similarly we find n' and j' so that $|a'_{n'+p}| > 1/j'$, for every p.
Given any natural number k, we can determine l so that

$$|a_{l+p} - a'_{l+p}| < \frac{1}{jj'k} \text{ for every } p.$$

Set $a^{-1} = c$, $(a')^{-1} = c'$, then, if m is the greatest of n, n', l,

$$|c_{m+p} - c'_{m+p}| = \frac{|a_{m+p} - a'_{m+p}|}{|a_{m+p} a'_{m+p}|} < 1/k \text{ for every } p.$$

Hence $c = c'$.

2.2.5. *Fundamental identities*

For a concise formulation of the fundamental identities I introduce the notion of a rational function.

Definition 1. A *rational function* $f(a, b, c, \ldots)$ is defined by a finite number of applications of the four fundamental operations. It is defined for such real number-generators $a, b, c, \ldots$, that, whenever in the calculation of f the inverse of a number-generator must be taken, this number-generator is $\neq 0$. E.g.

$$f(a, b) = a^{-1}\,(a+b^{-1})^{-1}$$

is defined for such real number-generators a, b, that a, b and $a+b^{-1}$ are all $\neq 0$.

Lemma. If $f(a, b, c, \ldots)$ is any rational function, a, b, c are given real number-generators for which f is defined and

$$f(a, b, c, \ldots) \equiv x,$$

then we can find a natural number n so that

$$x_{n+p} = f(a_{n+p}, b_{n+p}, c_{n+p}, \ldots)$$

for every p.

Proof. Let $\varphi_1(a, b, c, \ldots)$, $\varphi_2(a, b, c, \ldots)$, ... be the functions for which successively the inverse must be taken in the calculation of f. Then $\varphi_1(a, b, c, \ldots) \neq 0$ and we can find an index k_1 so that $\varphi_1(a_n, b_n, c_n, \ldots) \neq 0$ for $n > k_1$; thereupon we find $k_2 > k_1$ so that $\varphi_2(a_n, b_n, c_n, \ldots) \neq 0$ for $n > k_2$, etc. If k_r is the last index found in this way, it follows from 2.2.4. def. 1, that $x_n = f(a_n, b_n, c_n, \ldots)$ for $n > k_r$.

Theorem 1. Every rational identity that is valid for rational numbers holds also in the following sense for real number-generators:

Let $f(p, q, r, \ldots, x, y, z, \ldots)$ and $g(p, q, r, \ldots, x, y, z, \ldots)$ be rational functions such that $f = g$ if for $p, q, r, \ldots$ are substituted given rational numbers $p_0, q_0, r_0, \ldots$ and for $x, y, z, \ldots$ arbitrary rational numbers, for which f and g are defined. Then

$$f(p_0, q_0, r_0, \ldots, a, b, c, \ldots) = g(p_0, q_0, r_0, \ldots, a, b, c, \ldots)$$

for any real number-generators $a, b, c, \ldots$, for which f and g are defined.

Proof. Set

$$f(p_0, q_0, r_0, \ldots, a, b, c, \ldots) \equiv v$$

and

$$g(p_0, q_0, r_0, \ldots, a, b, c, \ldots) \equiv w.$$

As shown in the lemma, we can find an index k so that

$$f(p_0, q_0, r_0, \ldots, a_n, b_n, c_n, \ldots) = v_n$$

and

$$g(p_0, q_0, r_0, \ldots, a_n, b_n, c_n, \ldots) = w_n$$

for $n > k$. By supposition $v_n = w_n$; hence $v = w$.

This proves at one blow all the fundamental identities of arithmetic. We complete them by the following laws, which are of particular importance in analysis.

Theorem 2. $a \neq b$ implies $a + c \neq b + c$.

Proof. If $|a_{n+p} - b_{n+p}| > 1/k$ for every p, then

$$|(a_{n+p} + c_{n+p}) - (b_{n+p} + c_{n+p})| > 1/k \text{ for every } p.$$

Theorem 3. $a \neq 0$ and $b \neq 0$ imply $ab \neq 0$.

Proof. If $|a_{n+p}| > 1/k$ and $|b_{n+p}| > 1/k$ for every p, then

$$|a_{n+p} b_{n+p}| > 1/k^2 \text{ for every } p.$$

Theorem 4. $ab \neq 0$ implies $a \neq 0$ and $b \neq 0$.

Proof. We can find k and n so that

$$|a_{n+p} b_{n+p}| > 1/k, \quad |a_{n+p} - a_n| < 1, \quad |b_{n+p} - b_n| < 1$$

for every p; then

$$|a_{n+p}| > \frac{1}{k(|b_n| + 1)} \text{ and } |b_{n+p}| > \frac{1}{k(|a_n| + 1)}$$

for every p.

Theorem 5. $a + b \neq 0$ implies that either $a \neq 0$ or $b \neq 0$.

Proof. Let $a + b \neq 0$. By Th. 2, $-a + a + b \neq -a$, $b \neq -a$. By 2.2.3, Th. 4, either $b \neq 0$ or $-a \neq 0$. In the latter case, add a to both members and apply Th. 2.

Class. Classically, we read these theorems as negative properties and we give indirect proofs for them. For instance: If $a + c = b + c$,

then $a=b$; hence, if $a\neq b$, then $a+c\neq b+c$. But I see that you need direct proofs because you consider the positive relation $\#$ instead of the negative relation $\neq$.

INT. The theorems about $\#$ as well as their proofs are simpler than those involving $\neq$ and $=$; the latter must be handled with great caution, as I will illustrate by the following example.

I define two real number-generators a and b by the following laws: If in the first n decimals of π no sequence 0123456789 occurs, $a_n=b_n=2^{-n}$; if a sequence does occur in the first n decimals, let the 9 in the first sequence be the kth digit; now if k is odd, $a_n=2^{-k}$, $b_n=2^{-n}$ but if k is even, $a_n=2^{-n}$, $b_n=2^{-k}$. Neither for a nor for b are we able to decide whether it is 0 or not. But $ab=0$! In the first case $a_nb_n=2^{-2n}$, in the second $a_nb_n=2^{-k-n}$; in either $|a_nb_n|<1/m$ for $n>m$. Consequently the proposition "If $ab=0$, either $a=0$ or $b=0$" cannot be proved as long as there exist unsolved mathematical problems of the sort we used in the example.

CLASS. Yet, if $ab=0$, it must be impossible that neither a nor b is 0.

INT. That is right, for $a\neq 0$ and $b\neq 0$ implies $ab\neq 0$, as a negative counterpart of Th. 3. I prove this in different steps.

(i) If $ab=0$ and $a\#0$, then $b=0$, for $b\#0$ would give $ab\#0$ (Th. 3) and if $b\#0$ is impossible, then $b=0$ (2.2.3, Th. 3).

(ii) If $ab=0$ and $b\neq 0$, then $a=0$, for by (i) $a\#0$ would give $b=0$.

(iii) If $a\neq 0$ and $b\neq 0$, then $ab\neq 0$, for by (ii) $ab=0$ in combination with $b\neq 0$ would give $a=0$.

FORM. Could we not say, that if $ab=0$, either $a=0$ or $b=0$, but we need not know which of the two relations is true?

INT. It would be dangerous to adopt such a slipshod way of expression for such a subtle question. We only assert a proposition if we can prove it; so we only assert that either $a=0$ or $b=0$ if we can prove one of these propositions. By asserting that it is impossible that $a\neq 0$ and $b\neq 0$ we indicate exactly what we have proved, and this expression is scarcely more complicated than that which you propose. If we used the latter, we ought always to remember that it has another meaning than that which the words suggest.

2.2.6. *Order relations between number-generators*

I shall be brief about order relations.

Definition 1. $a<b$, if n and k can be found, so that $b_{n+p}-a_{n+p}>1/k$ for every p. $a>b$ means the same as $b<a$.

Theorem 1. If $a \# b$, either $a<b$ or $b<a$.

Proof. Find n and k so that $|a_{n+p}-b_{n+p}|>1/k$ for every p. Now determine $m>n$ so that $|a_m-a_{m+p}|<1/4k$ and $|b_m-b_{m+p}|<1/4k$ for every p. Either $a_m-b_m>1/k$ or $b_m-a_m>1/k$; in the first case $a_{m+p}-b_{m+p}>1/2k$ for every p, so $b<a$; in the second case we obtain $a<b$.

Theorem 2. If $a<b$, then $a \# b$.

This follows immediately from the definitions.

Theorems 1 and 2 show, that $a \# b$ is equivalent to ($a<b$ or $a>b$).

Theorem 3. If $a<b$ as well as $b<a$ are contradictory, then $a=b$.

Proof. We derive a contradiction from the supposition $a \# b$ by Th. 1; then 2.2.3, Th. 3 gives $a=b$.

Theorem 4. If $a<b$, then for every real number c holds either $a<c$ or $c<b$.

Proof similar to that of 2.2.3, Th. 4.

Theorem 5. If $a<b$, $b=c$, then $a<c$.

Proof as for 2.2.3, Th. 2.

Theorem 6. If $a<b$ and $b<c$, then $a<c$.

Proof easy from the definition.

Theorem 7. $a<b$ implies $a+c<b+c$.

Proof as for 2.2.5, Th. 2.

Theorem 8. If $a>0$ and $b>0$, then $ab>0$.

Proof as for 2.2.5, Th. 3.

Definition 2. We write $a \not> b$ if $a>b$ is impossible, and $a \not< b$ if $a<b$ is impossible.

Note that $a \not> b$ is not the same as ($a<b$ or $a=b$). For instance, in the example of 2.2.2, $\varrho \not> 1/3$, but we do not know whether $\varrho<1/3$ or $\varrho=1/3$.

Theorem 9. If $a \not< b$ and $b > c$, then $a > c$.

Proof. From $b > c$ it follows that $b > a$ or $a > c$. As the former is impossible, we have $a > c$.

Theorem 10. If $a > b$ and $b \not< c$, then $a > c$.

Proof analogous.

Theorem 11. If $a \not> b$ and $b \not> c$, then $a \not> c$.

Proof. Suppose $a > c$, then $a > b$ or $b > c$ by Th. 4.

2.2.7. *Maximum and minimum of two number-generators*

Definition 1. If $a \equiv \{a_n\}$, $b \equiv \{b_n\}$, then $\max(a,b) \equiv \{\max(a_n, b_n)\}$ and $\min(a, b) \equiv \{\min(a_n, b_n)\}$.

Theorem 1. If a and b are real number-generators, then $c \equiv \max(a, b)$ and $d \equiv \min(a, b)$ are real number-generators.

Proof. Find n such that for every p we have

$$|a_{n+p} - a_n| < 1/k, \quad |b_{n+p} - b_n| < 1/k.$$

Suppose $a_n \geqq b_n$, so that $c_n = a_n$.
Then $c_{n+p} \geqq a_{n+p} > a_n - 1/k = c_n - 1/k$,
and $a_{n+p} < a_n + 1/k$, $b_{n+p} < b_n + 1/k \leqq a_n + 1/k$, so $c_{n+p} < c_n + 1/k$.
It follows that the sequence $\{c_n\}$ is a Cauchy sequence.

Theorem 2. $\max(a,b) \not< a$. $\max(a,b) \not< b$. $\max(a,b) = \max(b,a)$. $\min(a,b)$ has analogous properties. $\max(a,b) \not< \min(a,b)$.

Theorem 3. If $x > \max(a, b)$, then $x > a$ and $x > b$.
Conversely, if $x > a$ and $x > b$, then $x > \max(a, b)$.

Proof. The first part of the theorem follows immediately from $\max(a, b) \not< a$ and 2.2.6, Th. 10. To prove the second part it suffices to remark that from $x_{n+p} - a_{n+p} > 1/k$ and $x_{n+p} - b_{n+p} > 1/k$ there follows

$$x_{n+p} - \max(a_{n+p}, b_{n+p}) > 1/k.$$

Theorem 4. $\max(ab, 0) = \max(a, 0) \max(b, 0) + \min(a, 0) \min(b, 0)$.

Proof. This equation is easily verified for rational numbers a, b by examining the different cases as to the signs of a and b. It

follows for real number-generators by the definitions of ab and of $\max(a, b)$, $\min(a, b)$.

Theorem 5. $\max(a, b) + \min(a, b) = a + b$.

2.2.8. *Absolute value of a number-generator*

Definition. If a is a real number-generator, then its *absolute value* $|a| = \max(a, -a)$, or what comes to the same thing, if $a \equiv \{a_n\}$, then $|a| \equiv \{|a_n|\}$.

Remark. The graph of the function $y = |x|$ is not the union of two half lines; it must be complemented by points for which it is unknown whether they correspond to a value of x which is > 0, < 0 or $= 0$. This is done by the definition above, which defines $|x|$ for every real number-generator x. In general, if we wish to consider a polygonal line as a continuous curve, we must complement it at every corner in an analogous way.

Theorem 1. If a and b are real number-generators, then $|a| + |b| \not< |a + b|$.

Proof. Suppose $|a| + |b| < |a + b|$; then we could find n and k so that

$$|a_{n+p} + b_{n+p}| - (|a_{n+p}| + |b_{n+p}|) > 1/k \text{ for every } p;$$

but this can be true for no value of p.

Theorem 2. $|a||b| = |ab|$.

Theorem 3. $|-a| = |a|$. If $a \neq 0$, $|a^{-1}| = |a|^{-1}$.

2.3. **Respectable real numbers**

FORM. Permit me to return to your example of two numbers whose product was zero without one of them being zero. In this and similar examples you use real numbers for which the order relations with respect to the rationals are not completely known. Could we not evade such complications by restricting arithmetic to such real numbers which I venture to call *respectable*, I mean those that are completely located with respect to the rationals?

INT. That seems a wise proposal, as most of the real numbers that occur in analysis are respectable in your sense. Yet I do not think that the difficulties to which you allude can really be solved in this way, but I shall not go into that question, because there

are even more serious objections. Before I discuss these, I think it is worth while to prove that some of the most important real numbers are respectable.

The proof for e is quite simple.

$e = \sum_0^\infty \frac{1}{k!}$ (the partial sums of the series form a number-generator).

Suppose $e = m/n$.

Set $s_n = \sum_0^n \frac{1}{k!}$, $r_n = \sum_{n+1}^\infty \frac{1}{k!}$.

$$r_n = \frac{1}{(n+1)!} \sum_1^\infty \frac{(n+1)!}{(n+k)!} < \frac{1}{(n+1)!} \sum_1^\infty \frac{1}{(n+2)^{k-1}} =$$

$$= \frac{1}{(n+1)!} \frac{n+2}{n+1} < \frac{1}{(n+1)!} \frac{n+1}{n} = \frac{1}{n!\,n}.$$

$$\frac{m}{n} = s_n + r_n. \qquad m(n-1)! = n!\,s_n + n!\,r_n.$$

This is impossible, for $n!\,r_n < 1/n$ is not an integer.

CLASS. This is a well-known proof for the irrationality of e, but does it settle the order relation between e and m/n?

INT. To do that, we must convert the last part into a positive reasoning. We note that

$$\frac{1}{(n+1)!} < e - s_n < \frac{1}{n!\,n}.$$

$$\frac{1}{n+1} < n!\,(e - s_n) < \frac{1}{n}.$$

As $n!\left(\frac{m}{n} - s_n\right)$ is an integer, $\left|n!\left(e - \frac{m}{n}\right)\right| > \frac{1}{n+1}$.

This gives $\left|e - \frac{m}{n}\right| > \frac{1}{(n+1)!}$; consequently, by calculating e with a sufficient degree of accuracy, we can decide of its order relation with respect to m/n.

As you see, in order to prove that a real number a is respectable, it suffices to give a proof for the irrationality of a, that establishes at the same time a measure for the degree of irrationality of a, that is, an arithmetical function $\varphi(n)$ such that $\left|a - \frac{m}{n}\right| > \frac{1}{\varphi(n)}$ for all values of m and n.

In order to find a measure of irrationality for the algebraic

numbers [L. E. J. Brouwer 1920, p. 960], we consider two algebraic numbers a and b; by a calculation involving only operations on rational numbers we find an algebraic equation

$$f(x) \equiv c_0 x^n + \ldots + c_n = 0 \quad (c_0 > 0)$$

with integer coefficients, that has a and b among its roots and whose discriminant $d \neq 0$ (d is an integer). Now

$$d = c_0^{2n-2} \prod_{i<k} (w_i - w_k)^2,$$

where $w_1 = a$, $w_2 = b$, $w_3, \ldots, w_n$ are the roots of $f(x) = 0$. Determine the integer h so that

$$|w_i - w_k| < h \quad (i, k = 1, \ldots, n);$$

then $|a-b|^2 > \frac{|d|}{c_0^{2n-2} h^{n^2-n-2}}$.

If for b we take a rational number, we find a measure of irrationality for a.

All this is rather satisfactory, but on the other hand we do not know if Euler's constant C is rational, so at the present state of our knowledge it does not satisfy your condition. Even more grave is the fact that the sum of two respectable numbers need not be respectable. To give an example, I define a real number-generator c by writing the decimal expansion of π, stopping after the first sequence 0123456789 or 9876543210, but if the latter occurs first, I change the 0 at the end into 1. π as well as c are respectable numbers. In the case of π this was shown by Brouwer [L. E. J. Brouwer 1920, p. 961]; his proof shows that for any rational number r we can find a natural number n so that $|\pi - r| > 10^{-n}$. By calculating π to n digits after the decimal point we find the order relation between c and r. But $c - \pi$ is not respectable (at least we do not know if it is), for at this moment nobody is able to decide whether $c - \pi$ is equal to, greater than or less than 0.

For these reasons it seems unpractical to restrict arithmetic to respectable numbers. Therefore I drop this notion and stick to the definition 2.2.1, Def. 1.

2.4. Limits of sequences of real number-generators

For the sake of completeness I formulate the definition of positive convergence:

Definition 1. The sequence $\{a_n\}$ of real number-generators is (*positively*) *convergent* to the *limit* a, if, given any natural number k, a natural number n can be found such that for every natural number p,

$$|a - a_{n+p}| < 2^{-k}. \tag{1}$$

CLASS. In such definitions as this, it is not the words that are important, but the way in which they must be understood. You emphasize this by using the words "can be found", which express that the number n must not only exist, but that it must be effectively known. I have still a question to ask. Why do you use the expression "positively convergent"? Is there also a notion of negative convergence?

INT. In fact, as is the case for many other concepts, besides the positive notion of convergence, which results in a natural way if we take the usual definition in a constructive sense, there can be defined a weaker negative notion which is classically equivalent to the positive one. However, as the use of these negative notions involves logical difficulties connected with the properties of the intuitionistic negation, I think it better to treat them in a chapter on logic (7.3.2).

Cauchy's general principle of convergence is valid. I leave it to you to give the positive formulation of the principle and to prove that it gives a necessary and sufficient condition for convergence. Also the theorems about the convergence of a sub-sequence, those about the limit of the sum-sequence and the product-sequence of two sequences, and similar theorems subsist intuitionistically. However, many other classical theorems are no longer valid. I state as an example that a bounded monotone sequence need not be convergent. A simple counterexample is the sequence $\{a_n\}$ which is defined as follows: $a_n = 1 - 2^{-n}$ if among the first n digits in the decimal expansion of π no sequence 0123456789 occurs; $a_n = 2 - 2^{-n}$ if among these n digits such a sequence does occur. Nobody knows if the limit of this sequence, if it exists, will be 1

or 2; so we are not allowed to say that this limit exists as a well defined real number-generator.

I shall not stop to give definitions of such notions as the upper limit of a sequence, or the least upper bound of a sequence. They result from the usual definitions by wording them positively and by taking every existential assertion positively.

Note that $a \equiv \{a_n\}$, where the a_n are rational, implies $\lim a_n = a$.

III

SPREADS AND SPECIES

3.1. Spreads

3.1.1. *Infinitely proceeding sequences*

Until now we have supposed that a real number-generator is determined by a law which, if n is any natural number, gives full prescriptions for the calculation of the n-th number of the sequence. This point of view suffices as long as we consider only isolated real number-generators, but it becomes unsatisfactory in the theory of the continuum, considered as the set of all real number-generators. The notion of an arbitrary law is unnatural and unmanageable; it may be useful on the basis of some formal system, but then it does not lead to a theory which corresponds sufficiently to our intuitive idea of the continuum. Brouwer was the first to show how a satisfactory theory of the continuum on an intuitive basis can be obtained without using the notion of an arbitrary law.

A real number-generator is never ready at hand; we never possess more than a finite part of its defining sequence. This leads us to think of a real number-generator as in a constant state of growth. The notion of a law of progression is essential here only in so far as it warrants the possibility of unlimited continuation of the sequence, hence we may eliminate it by postulating directly this possibility [L. E. J. Brouwer 1919A, p. 3; 1920, p. 956; 1924, p. 245]. Thus by an *infinitely proceeding sequence* (abbreviated: ips) we mean exactly what the words express, i.e. a sequence that can be continued ad infinitum. The question how the components of the sequence are successively determined, whether by a law, by free choices, by throwing a die, or by some other means, is entirely irrelevant. Of course two questions arise here: Is the introduction of the concept of an ips legitimate and is it expedient? The former question can be answered in the affirmative if the concept is sufficiently clear to be acceptable as a fundamental

notion in mathematics. Now I presume that you have serious doubts about its clearness.

FORM. Indeed I have. By admitting this concept you introduce into mathematics the notion of time and a subjective element that do not belong there. An infinitely proceeding sequence proceeds in time and the way in which it proceeds may depend upon choices, that is upon voluntary acts of the choosing subject.

INT. I agree to that; yet if we examine the proofs of the theorems on real number arithmetics in sections 2.2.4 and 2.2.5, we see that they only depend upon the possibility of indefinitely continuing the sequences; we never used the fact that their continuation was governed by a law; hence it must be possible to consider the continuation without demanding that a law governs it. For instance, in the definition of the sum of two real number-generators (2.2.4), the n-th approximation of $a+b$ is known as soon as the n-th approximations of a and b are given. Hence, if a and b are infinitely proceeding sequences, $a+b$ is an infinitely proceeding sequence. To arrive at the notion of an ips, we need not introduce new ideas, in particular not the notion of choice; the word "choice" is used here as a short expression for the generation of a component of the sequence. The idea of a law governing the production of the sequence is not necessary here and can be removed by a process of abstraction. On these grounds I beg you to admit that the notion of an ips is sufficiently clear.

With regard to the dependence of an ips upon the concept of time and its subjectivity a lengthier discussion is necessary. Let us, extending 2.2.1, Def. 1, define a real number-generator as an ips which is a Cauchy sequence of rational numbers. Here the condition of being a Cauchy sequence deserves special attention. If the numbers of the sequence are freely chosen, how can we know beforehand that the sequence will be a Cauchy sequence? Evidently the only way is to restrict the freedom of choice by rules that warrant the Cauchy property *before the choices are made*, e.g., by the condition that $|a_n - a_{n+p}| < 1/n$ for every n and p. An ips subjected only to this condition is certainly a Cauchy sequence.

In the same way the whole theory of real number-generators as explained in sections 2.2 and 2.4 can be extended to real number-

generators in the wider sense as defined here: accordingly, from this moment on we shall adopt the latter definition.

Generally speaking, we eliminate the subjective and temporal aspects in the notion of an ips by admitting only such reasonings as apply to a sequence independently of the choices that must still be made.

The choices by which an ips is generated need not be entirely free; their freedom can be restricted in various ways, provided at every stage we are able to decide which components may be taken for the next choice and which may not. E.g., the condition that $|a_n - a_{n+p}| < 1/n$ for every n and every p meets this requirement, for after the choices of $a_1, \ldots, a_n$ we know which rational numbers can be chosen as a_{n+1} and which may not. Brouwer allows that, after a certain number of choices, new restrictions for the further choices are added by a free decision [L. E. J. Brouwer 1924, p. 245].

3.1.2. *Definition of a Spread*

The main interest of the notion of an ips lies in the mode of generality it conveys. A free Cauchy sequence of rational numbers represents the continuum of real number-generators much better than a sequence determined by an unspecified law; it corresponds to the intuitive concept of the continuum as a possibility of a gradual determination of points. Generalization of this idea leads to Brouwer's definition of a spread [L. E. J. Brouwer 1918, p. 3; 1924, p. 244; 1954, p. 8].

A *spread* M is defined by two laws; the first, which I shall call the *spread-law* Λ_M, regulates the choices of natural numbers, while the effect of the second or *complementary law* Γ_M is to assign a sequence of mathematical entities to any ips of natural numbers which is generated according to the first law.

It is convenient to introduce the following expressions concerning finite sequences. $a_1, \ldots, a_n, a_{n+1}$ is an *immediate descendant* of $a_1, \ldots, a_n$. Conversely, $a_1, \ldots, a_n$ is the *immediate ascendant* of $a_1, \ldots, a_n, a_{n+1}$.

Definition 1. A *spread-law* is a rule Λ which divides the finite sequences of natural numbers into admissible and inadmissible sequences, according to the following prescriptions:

(1) It can be decided by A for every natural number k whether it is a one-member admissible sequence or not;
(2) Every admissible sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ is an immediate descendant of an admissible sequence $a_1, a_2, \ldots, a_n$;
(3) If an admissible sequence $a_1, \ldots, a_n$ is given, A allows us to decide for every natural number k whether $a_1, \ldots, a_n, k$ is an admissible sequence or not.
(4) To any admissible sequence $a_1, \ldots, a_n$ at least one natural number k can be found such that $a_1, \ldots, a_n, k$ is an admissible sequence.

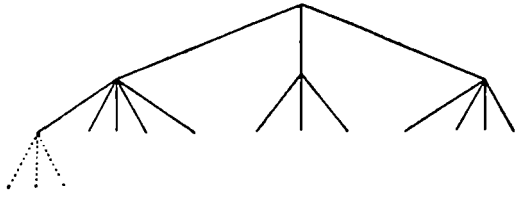


Fig. 1

Thus the admissible sequences can be represented as spreading out fanwise as in Fig. 1; it must be remembered that an infinite number of branches can start from any vertex of the diagram.

Definition 2. The *complementary law* Γ_M of a spread M assigns a definite mathematical entity to any finite sequence which is admissible according to the spread-law of M.

Form. What sort of entities does Γ_M assign to the admissible sequences?

Int. In the theory of real number-generators these are rational numbers; in general they can be any previously introduced mathematical entities.

Definition 3. An ips $\{a_n\}$, subjected to the condition that, for every n, $a_1, \ldots, a_n$ must be an admissible sequence according to the spread-law A_M, is called an *admissible* ips. Then the ips of the entities which the complementary law assigns to the sequences a_1; a_1, a_2; $\ldots$; $a_1, \ldots, a_n$; $\ldots$ is an *element* of the spread M.

Thus an element b of a spread is an ips $\{b_n\}$; we refer to b_n as the nth *component* of b.

Definition 4. Two elements of spreads are *equal* if their nth components are equal for every n.

Definition 5. Two spreads are *equal* if to every element of either of them an equal element of the other can be found.

It is easily seen that the relations of equality between elements of spreads and between spreads satisfy the usual conditions of reflexivity, symmetry and transitivity.

3.1.3. *Examples of spreads*

(i) Let $r_1, r_2, \ldots$ designate an enumeration of the rational numbers. Λ_M: Every natural number forms an admissible one-member sequence; if $a_1, \ldots, a_n$ is an admissible sequence, then $a_1, \ldots, a_n, a_{n+1}$ is an admissible sequence if and only if $|r_{a_n} - r_{a_{n+1}}| < 2^{-n}$.
Γ_M: To the sequence $a_1, \ldots, a_n$ (if admissible) is assigned the rational number r_{a_n}.

The elements of M are real number-generators $r_{a_1}, r_{a_2}, \ldots$. To any real number-generator c a member m of M can be found so that $c = m$; in this sense M represents the continuum of real number-generators.

(ii) In example i, add to Λ_M the condition that $0 < r_{a_n} < 1$ for every n. Now M represents the real number-generators x such that $0 \not> x \not> 1$.

(iii) In example ii, add to Λ_M the condition that

$$|1/2 - r_{a_n}| \leqslant |1/2 - r_{a_1}| \text{ for every } n > 1.$$

Now M represents the real number-generators y such that $0 < y < 1$.

(iv) Λ_M: Any sequence consisting only of 0 and 1, and no other, is admissible.
Γ_M: To the sequence $a_1, \ldots, a_n$ is assigned the rational number

$$\sum_{k=1}^{n} a_k 2^{-k}.$$

M represents the dually developable real number-generators x with $0 \not> x \not> 1$, where x is called dually developable if for any number of the form $a/2^n = b$, either $x \not> b$ or $x \not< b$ can be proved.

(v) To Λ_M in example iv add the restriction that $a_n = 0$ if n is odd.

M is a spread of real number-generators which is similar to Cantor's discontinuum.

It is clear that by varying the restrictions contained in A_M we can define various spreads of real number-generators.

Note on the terminology. In his earlier publications, Brouwer used "Menge" for what is here called a spread. Later on he avoided terms like Menge, set, class, ensemble, which suggest the conception, not uncommon in the classical theory, of a set as the totality of its elements, and introduced the word "spread" (Dutch: "spreiding", French: "déploiement"). Another notion, also analogous to the classical notion of set, is introduced in the next section under the name of "species". I follow Brouwer's terminology, using the word "set" in an informal way, in explanatory paragraphs, where I compare the intuitionistic with the classical theory.

3.2. Species

3.2.1. *Definition of a species*

Intuitionistically, there are two ways of defining a set: (i) by a common mode of generation for its elements; this case is realized in the spreads; (ii) by a characteristic property of its elements; sets of this sort are called species.

Definition 1. A *species* is a property which mathematical entities can be supposed to possess [L. E. J. Brouwer 1918, p. 4; 1924, p. 245; 1952, p. 142].

Definition 2. After a species S has been defined, any mathematical entity which has been or might have been defined before S and which satisfies the condition S, is a *member* of the species S.

3.2.2. *Examples of species*

(i) The real number-generators which coincide with a given real number-generator form a species (more exactly: the property of coinciding with a given number-generator is a species; every definition of a species, given in the first form, ought to be so transposed); this species is called a *real number*. If x is a real number and if the number-generator ξ is one of its members, then we say that ξ *represents* x and also that ξ *coincides with* x.

(ii) The ipss which are equal to the elements of a spread M form a species, the corresponding *spread-species* S_M.

(iii) All real numbers form a species, which is not defined as a spread-species. This species is the (one-dimensional) *continuum* (of real numbers).

(iv) The components of an ips ξ of natural numbers form a species, which, for brevity, we identify with ξ.

3.2.3. *Type of a species*

CLASS. In order to avoid circular definitions, is it not necessary to introduce a hierarchy of types of species, analogous to that in *Principia Mathematica*?

INT. Circular definitions are excluded by the condition that the members of a species S must be definable independently of the definition of S; this condition is obvious from the constructive point of view. It suggests indeed an ordination of species which resembles the hierarchy of types. Infinitely proceeding sequences and spread-species are also called species of *zero type*. A species that has as its members species of type zero is of *type one*.

Definition 1. A species is of *type* n if all its members have type less than n and at least one of its members has type $n-1$.

Examples i and ii and iv are of type 0; iii is of type 1.

3.2.4. *Subspecies*

The notion of a subspecies of a given species gives no difficulty. I use the signs $\in$, $\cup$, $\cap$ with the usual meanings. $S \subseteq T$ means that every member of S is also a member of T; $S = T$ (S is *equal* to T) if $S \subseteq T$ and $T \subseteq S$. $a \notin S$ means that it is impossible that a is a member of S. If T is a subspecies of S, then $S - T$ is the species of those elements of S which cannot belong to T.

If $T \subseteq S$, then $S' = T \cup (S - T)$ is not always identical with S, for S' contains only those elements of S for which it can be decided whether they belong to T or not. For instance, if S is the species of real numbers and T that of rational numbers, then $S - T$ is the species of negatively irrational numbers. A real number such as Euler's constant C, for which it is unknown whether it is rational or not, cannot be said to belong to $T \cup (S - T)$.

Definition 1. If the species S and T have the property that S can contain no element which does not belong to T, and that T can contain no element which does not belong to S, then S and T are *congruent* species [L. E. J. Brouwer 1924, p. 246].

Theorem. If T is a subspecies of S, then $S' = T \cup (S - T)$ is congruent with S.

Proof. As $S' \subseteq S$, we need only prove that S contains no element which does not belong to S'. Suppose that $a \in S$, but $a \notin S'$; then $a \notin T$, so $a \in S - T$, so $a \in S'$, which contradicts the hypothesis $a \notin S'$. Hence $a \notin S'$ is impossible for every element of S.

Definition 2. If $T \subseteq S$ and $T \cup (S - T)$ is equal to S, then T is a *detachable* subspecies of S, and S is *split up* into T and $S - T$ [L. E. J. Brouwer 1924, p. 247].

CLASS. So this amounts to saying that we can decide for every element of S whether it belongs to T or not. Clearly, if T is a detachable subspecies of S, $S - T$ is also a detachable subspecies of S.

INT. Let me give some examples.

The species of positive even numbers is a detachable subspecies of the species N of natural numbers.

The species of exponents n for which the equation $x^n + y^n = z^n$ has non-trivial integral solutions, is not known to be a detachable subspecies of N.

We shall see later on (3.4.3, Th. 2) that the continuum has no other detachable subspecies than itself and the null species.

3.2.5. *The relation of equivalence between species*

As usual, two species, between which a one-to-one correspondence has been established, are called *equivalent* species. As we remarked in 2.1, the construction of a natural number n consists in building up successively the numbers from 1 to n; these numbers form the species $1 \to n$. A *finite* species is a species that is equivalent to $1 \to n$ for some natural number n.

Definition 1. A species that is equivalent to the species N of all natural numbers, is *denumerably infinite*.

Definition 2. A species that contains a denumerably infinite subspecies is called *infinite*.

Thus a species that cannot be finite, is not necessarily infinite.

Definition 3. A species that is equivalent to a detachable subspecies of N is called *numerable* [L. E. J. Brouwer 1918, p. 7; 1924, p. 248], [A. Heyting 1929, p. 51].

Example. The species of twin primes $(p, p+2)$ is numerable, though nobody knows whether it is finite or infinite.

I shall not go into the theory of cardinal numbers, which differs much from the classical theory [L. E. J. Brouwer 1924], in this among other respects, that two species need not at all be comparable as to their cardinal numbers. It is easy to give an example of a species for which it is unknown whether it is the null species, a finite species or an infinite species.

CLASS. The species of numbers n such that the nth to $(n+9)$th digits in π form a sequence 0123456789, provides such an example.

INT. That the continuum is not denumerably infinite, is an immediate consequence of 3.4.3, Th. 2.

3.3. Arithmetic of real numbers

3.3.1. *Relations and operations for real numbers*

In 2.2 we treated the arithmetic of number-generators. You will readily supply the definitions of equality and inequality between real numbers, and the proofs of the main properties of these relations. Also, after the definitions of the arithmetical operations have been given in an obvious way, the theorems in the arithmetic of real numbers are immediate consequences of those about real number-generators. I leave it to you to work this out.

3.3.2. *Intervals*

Definition 1. If a and b are real numbers, the *closed interval* $[a, b]$ is the species of real numbers x such that it is impossible that $x>a$ and $x>b$, and also impossible that $x<a$ and $x<b$.

Note. The definition must be given in this complicated form because it may happen that we do not know which of a, b is the greater.

Theorem 1. If $\max(a, b)=c$, $\min(a, b)=d$, then $[a, b]=[d, c]$.

Proof. We saw in 2.2.7, Th. 3 that ($x>a$ and $x>b$) is equivalent

to $x>c$, and this is equivalent to ($x>c$ and $x>d$); thus the impossibility of ($x>a$ and $x>b$) is equivalent to the impossibility of ($x>c$ and $x>d$). By an analogous argument with d in place of c, we see that $x \in [a, b]$ is equivalent to $x \in [d, c]$.

Theorem 2. If $a \not> b$, then $[a, b]$ is the species of the real numbers x which satisfy ($x \not< a$ and $x \not> b$).

Proof. Though this proof is very simple, it may seem difficult to those who are not accustomed to intuitionistic reasoning; for this reason I shall give it in some detail. First suppose $x \not< a$ and $x \not> b$. From $x \not< a$ it follows that a fortiori ($x<a$ and $x<b$) is impossible; similarly, from $x \not> b$ it follows that ($x>a$ and $x>b$) is impossible. Thus $x \in [a, b]$.

Now suppose $x \in [a, b]$, and let us admit for a moment that $x<a$; then, since $a \not> b$, we have $x<b$, so that ($x<a$ and $x<b$), which is impossible by hypothesis. We have now proved that $x \not< a$, and we can prove analogously that $x \not> b$.

Corollary. If $\max(a, b)=c$, $\min(a, b)=d$, then $[a, b]$ is the species of the real numbers x which satisfy $x \not> c$ and $x \not< d$.

3.3.3. *Canonical number-generators*

It is often convenient to represent a real number by a number-generator of a simple form. Let the real number x be given by the number-generator $\{r_n\}$. We can find k so that $|x-r_k|<2^{-n-3}$, and after that we can determine an integer x_n so that $|r_k-x_n 2^{-n}| \leqslant 2^{-n-1}$, so that

$$|x-x_n 2^{-n}|<{}^5/_8\, 2^{-n}. \tag{1}$$

If we do this for every n, we obtain a number-generator $\{x_n 2^{-n}\}$ which coincides with x and which has the property that

$$|x_n 2^{-n} - x_{n+1} 2^{-n-1}|<{}^5/_8\, 2^{-n}+{}^5/_{16}\, 2^{-n}={}^{15}/_{16}\, 2^{-n};$$

this implies that

$$|x_n 2^{-n}-x_{n+1}2^{-n-1}| \leqslant 2^{-n-1}. \tag{2}$$

Definition. A number-generator of the form $\{x_n 2^{-n}\}$, where every x_n is an integer, and which satisfies (2), will be called a *canonical* number-generator.

We have proved:

Theorem 1. Every real number x coincides with a canonical number-generator $\{x_n 2^{-n}\}$ which satisfies (1).

It is clear from the proof, that in (1) the factor 5/8 can be replaced by $1/2+\varepsilon_n$, where $\varepsilon_n > 0$.

3.4. Finitary spreads (Fans)

3.4.1. *Definition*

A spread M is *finitary* (is a *fan*) if the spreadlaw Λ_M is such that only a finite number of one-member sequences are admissible and that for every admissible sequence $a_1, \ldots, a_n$ there is only a finite number of values of k such that $a_1, \ldots, a_n, k$ is an admissible sequence.

LETTER. I wonder whether "finitary" is a new word in English.

INT. The word has been used by Kleene for a similar notion. As to "fan", I think it wise to introduce sparingly such new words in mathematics, but the notion of a finitary spread will prove so important that it is convenient to have a short word for it.

Theorem 1. Every closed interval of the continuum coincides with a finitary spread [L. E. J. Brouwer 1919A, p. 14; 1924B, p. 192].

Proof. Let two real numbers a, b be given and set $\max(a, b) = c$, $\min(a, b) = d$; then $[a, b] = [d, c]$. As in 3.3.3 we construct the canonical number-generators $\{d_n 2^{-n}\}$, $\{c_n 2^{-n}\}$, which coincide with d and c respectively. We may suppose that $d_n \leqq c_n$, for if $d_n > c_n$ we easily see that also $|d - c_n 2^{-n}| < {}^5/_8\, 2^{-n}$.

Consider the spread S of the canonical ipss $\{x_n 2^{-n}\}$ where x_n satisfies

$$d_n \leqq x_n \leqq c_n. \tag{1}$$

After x_n has been chosen, at least one and at most three values are admissible for x_{n+1}, so S is finitary. I shall show that S coincides with $[a, b]$. (1) shows that every element of S coincides with an element of $[d, c]$. Conversely, let x be any element of $[d, c]$, and let $\{x_n 2^{-n}\}$ be a canonical ips coinciding with x. As before, we may suppose that $d_n \leqq x_n \leqq c_n$, so that x coincides with an element of S.

3.4.2. *The fan theorem*

Theorem. If an integer-valued function $\varphi(\delta)$ is defined for every element δ of a finitary spread S, then a natural number can

be computed from the definition of φ, such that $\varphi(\delta)$ is determined by the first N components of δ; that is, if δ_1 and δ_2 are such elements of S that the first N components of δ_1 are equal to the first N components of δ_2, then $\varphi(\delta_1)=\varphi(\delta_2)$ [L. E. J. Brouwer 1923, p. 4; 1924B, p.192; 1924D, p.646; 1926A, p. 66; 1952, p. 143; 1954, p. 15].

Proof. Let A be the spread-law of S and F the species of admissible finite sequences by A, to which the null-sequence is added. Let K be the finitary spread of admissible ipss by A. It is convenient to introduce the following expressions (see also 3.1.2).

$a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k}$ $(k \geqq 1)$ is a *descendant* of $a_1, \ldots, a_n$; the latter sequence is an *ascendant* of the former.

The ips $\sigma = a_1, \ldots, a_n, \ldots$ is a *continuation* of the finite sequence $a_1, \ldots, a_n$, and this sequence is a *segment* of σ. If σ is an element of K, it is a K-continuation of $a_1, \ldots, a_n$. A sequence in F is also called an F-sequence.

If d is the element of K to which δ is connected by Γ_S, we set $\varphi(\delta)=f(d)$. f is defined for every element of K. As $f(d)$ must be calculable, its value must be determined by a finite number of the components of d; that is to say, by a sequence $a(d)$ in F. Let C be the species of the $a(d)$ which correspond in this way to the elements of K; then every element of K has a segment $a(d)$ in C. Also, if b is a sequence in F, every K-continuation of b has a segment in C; we shall express this property by saying that b is *K-barred* by C.

Obviously it is only by means of a proof $\mathfrak{R}$, based on the data of the theorem, that we can become aware of the fact that every F-sequence is K-barred by C.

Now these data are of two sorts, to wit:

(i) the species C,

(ii) the relations between a sequence in F and its immediate descendants in F.

Thus $\mathfrak{R}$, if expounded without abbreviations, consists of a finite number of inferences, each of which is either a ζ-inference or a F-inference, where the latter are defined as follows:

F-inference: for a certain F-sequence a, every immediate descendant of a in F is K-barred by C, so a is K-barred by C.

ζ-inference: for a certain F-sequence a, the immediate ascendant of a is K-barred by C, so a is K-barred by C.

The last inference of $\mathfrak{R}$, affirming that the null-sequence is K-barred by C, must be a F-inference, so the barred condition (that is, the property of being K-barred by C) of every 1-sequence must be proved before that of the null-sequence. Thus, if the barred condition of a 1-sequence is proved by a ζ-inference, it has been proved previously in $\mathfrak{R}$; for the first time it has been proved by a F-inference. It follows that every ζ-inference which proves the barred condition of a 1-sequence is superfluous. Moreover, the proof of the barred condition of a 1-sequence must be preceded by the proof of the barred conditions of its immediate descendants. Repeating the same argument for 2-sequences, and so on, we prove by induction that all ζ-inferences can be eliminated from $\mathfrak{R}$; by doing this we obtain a proof $\mathfrak{R}'$.

Furthermore from $\mathfrak{R}'$ we may omit every F-inference, which proves the barred condition of a sequence that belongs to C, or has an ascendant in C, or that has been proved previously in $\mathfrak{R}'$ to be barred by C. After all these simplifications we obtain a proof $\overline{\mathfrak{R}}$.

Let C_0 be the subspecies of C, whose elements occur in inferences in $\overline{\mathfrak{R}}$. The first inference of $\overline{\mathfrak{R}}$ must have the form: "Every immediate F-descendant of the F-sequence a belongs to C, so a is K-barred by C." Obviously in this case every immediate F-descendant of a belongs to C_0, so a is K-barred by C_0. Any other inference in $\overline{\mathfrak{R}}$ is of the form: "Every immediate F-descendant of the F-sequence a belongs to C or has previously been proved to be K-barred by C, so a is K-barred by C". From this it is easily seen by induction, that if it is proved in the course of $\overline{\mathfrak{R}}$ that a sequence a is K-barred by C, then a is also K-barred by C_0. In particular the null-sequence is K-barred by C_0.

The number of steps in $\overline{\mathfrak{R}}$ is finite; every step is a F-inference which uses but a finite number of previous F-inferences. It follows that the total number of F-inferences in $\overline{\mathfrak{R}}$ is finite. The number of elements of C_0 which occur in any inference of $\overline{\mathfrak{R}}$ is finite; it follows that C_0 is finite. Thus there is a finite maximum N for the length of a sequence in C_0. This proves the theorem.

CLASS. This is a remarkably simple proof for a theorem with such far-reaching consequences.

INT. It is Brouwer's original proof [L. E. J. Brouwer 1926A]; only Brouwer deduces the fan theorem as a special case of a general theorem on spreads. The above proof is obtained by selecting from his argument what is necessary for the proof of the fan theorem.
FORM. What strikes me most in this proof is the sentence: "Evidently we must be aware of the fact that every F-sequence is K-barred by C, by means of a proof $\mathfrak{R}$, based on the data of the theorem". It seems here that a mathematical result is deduced by methods which, in the terminology of formalism, would be called metamathematical. Now the use of metamathematics for the deduction of mathematical results is not new; the simplest example is the principle of duality in projective geometry, and important applications of this method to algebra have recently been made by A. Tarski, Abraham Robinson and Leon Henkin (see e.g. [A. Tarski 1950], [L. Henkin 1953], [A. Robinson 1951]). But it is noteworthy that similar methods are used by the intuitionists, who are opposed to strict formalization and therefore cannot construct metamathematics in the proper sense.
INT. This is an important question which concerns the nature of intuitionistic proofs in general. You are right that the distinction between mathematics and metamathematics cannot be maintained if no strict formalization of mathematics is performed. In order to clarify in which respect the proof of the fan theorem differs from other proofs, the following remark may be useful. In every mathematical theorem there occurs a reference to previous constructions. To take an example at random, consider 2.2.5, Th. 2: For real numbers a, b and c, $a \neq b$ implies $a+c \neq b+c$. Here the hypothesis $a \neq b$ does not refer to a supposed fact, but to a supposed construction of natural numbers n and k, satisfying 2.2.3, Def. 1, and the theorem affirms that this construction can be so completed as to produce natural numbers satisfying 2.2.3, Def. 1 with $a+c$ and $b+c$ instead of a and b. But in almost every case it is not the supposed construction itself that plays a part in the proof, but only its result. The new feature in the proof of the fan theorem is, that the possible form of the supposed construction is explicitly involved in it. If we are well aware that the hypothesis of a theorem consists always in the assumption of a previous execution of some construction, we can offer no objection against the use of con-

siderations about the way in which such a construction can be performed as a means of proof.

Kreisel [1958 A], Spector [1962] and Kleene [1965] have undertaken a formalization of intuitionistic analysis including the theory of ipss, with special attention to the methods which Brouwer applied in his proof of the fan theorem.

Applications of the fan theorem

3.4.3. *Continuity of functions*

The most conspicuous application of the fan theorem is

Theorem 1. A real-valued function $f(x)$ which is defined everywhere on a closed interval of the continuum is uniformly continuous on that interval [L. E. J. Brouwer 1923, p. 5; 1924B, D; 1926A, p. 67; 1954, p. 17].

Proof. By 3.4.1, Th. 1, the interval $[a, b]$ coincides with a fan S. To every element ξ of S there is associated a real number $y = f(\xi)$; y coincides with a canonical number-generator $\eta = \{\eta_n 2^{-n}\}$. For a fixed value of n we associate η_n to ξ; in this way we obtain an integer-valued function on S; thus, by the fan-theorem, a number $N(n)$ can be found so that, for every ξ in S, η_n is determined by the first N components of ξ.

Now let x_1 and x_2 be real numbers in $[a, b]$ such that $|x_1 - x_2| < 2^{-N-2}$; then x_1 and x_2 coincide with canonical number-generators ξ_1 and ξ_2 in which the first N components are the same. It follows that η_n is the same for ξ_1 and ξ_2, consequently

$$|f(\xi_1) - f(\xi_2)| < {}^5/_4\, 2^{-n}.$$

We have now proved: If $|x_1 - x_2| < 2^{-N}$, then $|f(x_1) - f(x_2)| < {}^5/_4\, 2^{-n}$. That is to say, that $f(x)$ is uniformly continuous in $[a, b]$.

Theorem 2. A pointspecies which is a detachable subspecies of a closed interval is either the null species or the whole interval [L. E. J. Brouwer 1926A, p. 66].

Proof. Let the pointspecies Q be a detachable subspecies of the interval E. The function $f(x)$, which is 1 if x belongs to Q and 0 if x belongs to $E - Q$, is defined in every point of E and thus must be continuous on E; this means that $f(x)$ is constant.

Theorem 3. A function that is defined everywhere on a closed interval E, has on E a least upper bound and a greatest lower bound.

Proof. From the proof of theorem 1 it follows that there is only a finite number of values for η_n; let ζ_n be the smallest of them. Set $\zeta_n 2^{-n} = z_n$. It is easy to see that $|z_n - z_{n+1}| \leqq 2^{-n-1}$, so $\lim_{n\to\infty} z_n$ exists and is the g.l.b. of $f(x)$.

Theorem 4. If a function is defined and positive everywhere on a closed interval E, its g.l.b. is positive.

Proof. As in the proof of Th. 1, to every element ξ of S there is associated a canonical number-generator $\eta = \eta_n 2^{-n}$; $\eta > 0$, so for some value n_1 of n, $\eta_{n_1} > 0$. n_1 is a function of ξ; by the fan theorem, a number M can be found so that n_1 is determined by the first M components of ξ, so there is only a finite number of values for n_1; let n_0 be the maximum of these values. Then $z_{n_0} \geqq 2^{-n_0}$. Moreover, for every x, $f(x) > z_{n_0} - {}^5/_8\, 2^{-n_0}$, so the g.l.b. of $f(x)$ is positive.

Remark. We may not assert that $f(x)$ attains its g.l.b. for a definite value of x. This is illustrated by the following example.
$f(x) = -3x^4 + 4cx^3 + 6x^2 - 12cx$,
where c is a real number for which it is unknown whether $c > 0$, $c = 0$ or $c < 0$.
$f'(x) = -12(x+1)(x-1)(x-c)$.
$f(-1) = 3 + 8c$, $f(1) = 3 - 8c$, $f(c) = c^4 - 6c^2$.
The least upper bound of $f(x)$ is $3 + 8|c|$, but it is unknown whether $f(x)$ takes this value for $x = -1$ or for $x = 1$.

3.4.4. *The Bolzano–Weierstrass theorem*

Brouwer investigated this theorem (L. E. J. Brouwer 1952B].

We consider the following special case

(A) To every bounded infinite species of real numbers a point of accumulation can be found.

This is classically equivalent to:

(B) Every bounded species of real numbers without a point of accumulation is finite.

Let the sequence $\{a_n\}$ be defined as follows. If among the first n digits in the decimal expansion of π no sequence 0123456789 occurs, then $a_n = 2^{-n}$; if such a sequence does occur in the first n

digits of π, then $a_n = 1 - 2^{-n}$. The species of the real numbers a_n is infinite, but nobody knows whether a point of accumulation, if it exists, would be 0 or 1. So at present we are not able to prove (A) intuitionistically.

As to (B), Brouwer showed that there is no hope of proving even the weaker proposition (C), given below. I shall repeat his argument in 8.1.3.

Definition. A species S is *bounded in number* if a natural number n is known such that S can contain no subspecies of n elements.

Remark. Every finite species is bounded in number but the converse need not be true.

(C) Every bounded species of real numbers without a point of accumulation is bounded in number.

However, somewhat weakened versions of (A) and of (C) are true.

Theorem 1. Let Q be a bounded infinite species of real numbers and let m, n be natural numbers. Then there exists an interval of length 2^{-n} which contains at least m elements of Q.

Proof. Let h, k be integers such that Q is contained in the interval (h, k). Set $r = (k-h+1)2^{n+1}$. Let R be a subspecies of Q with rm elements. For every element x of R we determine an integer x_{n+1} so that $(x_{n+1}-1)2^{-n-1} < x < (x_{n+1}+1)2^{-n-1}$. Since the number of these intervals which overlap (h, k) is r, at least one of them contains m or more elements of R.

Theorem 2. If Q is a bounded species of real numbers with the property that for every real number x a natural number $r(x)$ can be found such that the interval $(x-2^{-r}, x+2^{-r})$ cannot contain two different members of Q, then Q is bounded in number.

Proof. Let Q be contained in the interval (h, k). The interval $(x-2^{-r}, x+2^{-r})$, where $x \in [h, k]$, contains an interval $i(a, r) = (a \cdot 2^{-r-1}, (a+2)2^{-r-1})$, where a is an integer, and where $i(a, r)$ contains x.

Let J be the canonical point fan which coincides with $[h, k]$. Since to every element ξ of J there are associated integers $r(\xi)$ and $a(\xi)$, the fan theorem allows us to find m so that $r(\xi)$ and $a(\xi)$ depend only upon the first m choices of ξ. Consequently, there is only a finite number, say s, of different intervals $i(a, r)$, which together cover $[h, k]$. Since no $i(a, r)$ contains two different elements of Q, a subspecies of Q with $s+1$ elements cannot exist.

IV

ALGEBRA

4.1. Algebraic fields

I shall not give here a connected treatment of intuitionistic algebra [A. Heyting 1941]; the following fragments are mainly intended as applications of the theory of real numbers, but it is easy to formulate them for the case of an abstract algebraic field.

4.1.1. *Apartness relations*

In a field, division must be defined, and, as we have seen, for real numbers division is only possible if the divisor lies apart from 0; it follows that an apartness relation will be essential in the definition of a field.

Definition. A symmetric relation $\#$ between the elements of a species S will be called an *apartness relation* if it has the following properties (i)–(iii) (Compare 2.2.3). $a, b, \ldots$ are elements of S

(i) If $a \# b$, $a = b$ is impossible.

(ii) If $a \# b$ is impossible, $a = b$.

(iii) If $a \# b$, then for any element c of S, either $a \# c$ or $b \# c$.

4.1.2. *Definition of a field*

A mathematical species F is a *field* if it has the following properties R, A1, A2, M1, M2, M3.

R. In F an apartness relation $\#$ is defined.

A1. In F a commutative and associative addition is defined; F contains a zero element and the negative of any of its elements.

A2. If $a \# b$, then for any element c of F, $a + c \# b + c$.

M1. In F a commutative and associative multiplication is defined,

which is distributive with respect to addition; F contains a unit element 1 and $1 \neq 0$.

M2. If $a \neq 0$, the reciprocal a^{-1} exists and $a^{-1} \neq 0$.

M3. If $a \neq b$ and $c \neq 0$, then $ac \neq bc$.

4.1.3. *Properties of the apartness relation in a field*

Theorem 1. $ab \neq 0$ entails $a \neq 0$ and $b \neq 0$.

Proof. If $ab \neq 0$, either $a \neq 0$ or $ab \neq a$. In case $a \neq 0$, also $a^{-1} \neq 0$ and $(ab)a^{-1} \neq 0$, that is $b \neq 0$. Now suppose $ab \neq a$, that is $a(b-1) \neq 0$. Now $1 \neq 0$ gives that either $b \neq 0$ or $b-1 \neq 0$. If $b \neq 0$, also $b^{-1} \neq 0$ and $(ab)b^{-1} \neq 0$, $a \neq 0$. In the same way we treat the case $b-1 \neq 0$.

Theorem 2. $a+b \neq 0$ implies that either $a \neq 0$ or $b \neq 0$.

Proof. As for 2.2.5, Th. 5.

Theorem 3. If $ab \neq cd$, then either $a \neq c$ or $b \neq d$.

Proof. $ab-cd \neq 0$; $a(b-d)+d(a-c) \neq 0$; by Th. 2 and Th. 1 either $b-d \neq 0$ or $a-c \neq 0$.

Theorem 4. If $f(x_1, \ldots, x_n)$ is a polynomium with coefficients in the field F, and if $p_1, \ldots, p_n, q_1, \ldots, q_n$ are elements of F such that $f(p_1, \ldots, p_n) \neq f(q_1, \ldots, q_n)$, then for at least one value of the subscript i, $p_i \neq q_i$.

Proof. By repeated application of Th. 2 we find a term $cx_1^{a_1} \ldots x_n^{a_n}$ in $f(x_1, \ldots, x_n)$ such that $cp_1^{a_1} \ldots p_n^{a_n} \neq cq_1^{a_1} \ldots q_n^{a_n}$. Then we establish the theorem by repeated application of Th. 3.

FORM. This theory of algebraic fields is essentially an axiomatic theory.

INT. It illustrates how the axiomatic method may be applied in intuitionistic mathematics. But we must keep in mind that it plays no part in the foundations of mathematics; it is but a convenient way of presenting a theory in which many theorems have the same complicated system of suppositions.

4.2. Linear equations

The theory of linear equations illustrates clearly in what way classical theories may be made more precise.

4.2.1. *Cramer's rule*

Let d be the determinant of the coefficients in the left members of the equations

$$\sum_{k=1}^{n} a_{ik}x_k = b_i \qquad (i = 1, \ldots, n) \tag{1}$$

If $d \neq 0$, (1) can be solved by Cramer's rule:

$$x_k = \frac{d_k}{d} \qquad (k = 1, \ldots, n) .$$

This solution is unique in the following sharp sense:

Theorem 1. If $p_1, \ldots, p_n$ are numbers such that for some value of r, $p_r \neq d_r/d$, then a subscript i can be found such that

$$\sum_{k=1}^{n} a_{ik}p_k \neq b_i .$$

Proof. Let m_{ik} denote the minor of a_{ik} in d. We have

$$\sum_{i=1}^{n} m_{ir} \sum_{k=1}^{n} a_{ik}p_k = dp_r .$$

$$\sum_{i=1}^{n} m_{ir} b_i = d_r .$$

$$\sum_{i=1}^{n} m_{ir} \sum_{k=1}^{n} a_{ik}p_k \neq \sum_{i=1}^{n} m_{ir}b_i .$$

Then by 4.1.3, Th. 2, for at least one value of i,

$$m_{ir} \sum_{k=1}^{n} a_{ik}p_k \neq m_{ir}b_i ,$$

which proves the theorem.

4.2.2. *m Equations in n variables, with known rank*

Let us now consider a system of m equations in n variables:

$$L_i = \sum_{k=1}^{n} a_{ik}x_k = b_i \qquad (i = 1, \ldots, m) . \tag{2}$$

In order to solve these equations by one of the usual methods, it is necessary to know the rank r of the matrix $A = (a_{ik})$; moreover, division must be possible by some r-rowed minor. Therefore we define the notion of rank more precisely:

Definition 1. The matrix A is of *rank* r, if at least one r-rowed

minor in A is apart from 0, while all $r+1$-rowed minors are 0. If d is an r-rowed minor and $d \neq 0$, d is called a *principal minor* of A.

Definition 2. A *characteristic determinant* c_s of (2) is obtained from a principal minor d of A by adding a row containing coefficients of the sth equation (2) and the column of the right members of (2).

A necessary and sufficient condition that (2) has a solution is that every characteristic determinant is 0. The necessity of this condition can be stated in a more precise form:

Theorem 1. If some characteristic determinant $c_s \neq 0$, then for arbitrary values $x_k = p_k$ $(k = 1, \ldots, n)$, there is a value of i such that $L_i(p_1, \ldots, p_n) \neq b_i$.

Proof. Suppose the principal minor is

$$d = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix} \text{ and } c_s = \left|\begin{array}{ccc} & d & \\ & & \\ \hline a_{s1} & \cdots & a_{sr} \end{array}\right| \begin{array}{c} b_1 \\ \vdots \\ b_r \\ \hline b_s \end{array} \neq 0.$$

Take arbitrary values $x_1 = p_1, \ldots, x_n = p_n$.
Substituting $x_{r+1} = p_{r+1}, \ldots, x_n = p_n$ in the first r equations and solving for $x_1, \ldots, x_r$, we obtain $x_1 = q_1, \ldots, x_r = q_r$.

$$L_s(q_1, \ldots, q_r, p_{r+1}, \ldots, p_n) - b_s = -c_s/d \neq 0.$$

Thus, either

$$L_s(p_1, \ldots, p_r, p_{r+1}, \ldots, p_n) \neq b_s,$$

or, by 4.1.3, Th. 4, for at least one value of k $(1 \leqq k \leqq r)$, $p_k \neq q_k$. In the latter case, by Th. 1, for at least one value of i $(1 \leqq i \leqq r)$,

$$L_i(p_1, \ldots, p_r, p_{r+1}, \ldots, p_n) \neq b_i.$$

If $c_s = 0$ $(s = r+1, \ldots, m)$, the system (2) has a solution of the form

$$x_k = f_k(x_{r+1}, \ldots, x_n) \qquad (k = 1, \ldots, r). \tag{3}$$

This solution is complete in the precise sense; that is, if $(p_1, \ldots, p_n)$ is a vector such that for every vector $(q_1, \ldots, q_n)$ contained in (3), $p_k \neq q_k$ for at least one value of k, then $L_i(p_1, \ldots, p_n) \neq b_i$ for at least one value of i. The proof of this theorem is analogous to that of 4.2.1., Th. 1.

4.2.3. *Rank unknown*

If the equations (2) are homogeneous and of rank r, it follows that they have a $(n-r)$-parameter solution which is complete in the precise sense. However, if the rank is not known, it may occur that we can find no solution apart from the null-solution, even if all n-rowed minors are 0. Take, for instance, the equation

$$ax+by=0,$$

where a, b are real numbers such that neither $a=0$ nor $a \neq 0$ nor $b=0$ nor $b \neq 0$ is known, while the proportion of a and b is also unknown. (Example: $a=\{a_n\}$, where $a_n=2^{-n}$ if in the first n digits of π no sequence 012345678[9] occurs, $a_n=2^{-k}$ if such a sequence occurs and its 9 is the kth digit in π; b is defined analogously with e instead of π). If $a \neq 0$ or $b \neq 0$, $(x=b,\ y=-a)$ is a solution apart from (0, 0); if $a=b=0$, arbitrary numbers $\neq 0$ can be chosen for x and y. But the first solution does not apply in the second case, nor does the second solution in the first case, and as long as we do not know which case is realized, we are unable to give a solution.

4.2.4. *Homogeneous linear equations*

As a special case of 4.2.1, Th. 1 we have for the equations

$$\sum_{k=1}^{n} a_{ik}x_k=0 \qquad (i=1,\ldots,m)\,, \tag{4}$$

Theorem 1. If the rank of the matrix $A=(a_{ik})$ is n, then for any values $u_1, \ldots, u_n$ such that $u_k \neq 0$ for at least one value of k, there is at least one value of i such that

$$\sum_{k=1}^{n} a_{ik}u_k \neq 0\,.$$

Also the converse is true:

Theorem 2. If for any values $u_1, \ldots, u_n$ such that $u_k \neq 0$ for at least one value of k, at least one of the left members in (4) is $\neq 0$, then the rank of the matrix A is n.

Proof by induction with respect to n. For one variable the result is trivial. Let it be proved for equations in $n-1$ variables. Putting $x_n=0$ in (4) we obtain equations in $x_1, \ldots, x_{n-1}$ satisfying the condition of the theorem; thus by hypothesis the matrix of the first $n-1$ columns in A has rank $n-1$. Suppose the determinant

formed by its first $n-1$ rows is $d \neq 0$. Solve the first $n-1$ equations taking $x_n = 1$ and substitute the result in the other equations (4). The left member of at least one, say the tth equation will be $\neq 0$. The left member is equal to an n-rowed determinant of A, divided by d.

4.3. Linear dependence

4.3.1. *Definitions*

As usual, the n-dimensional vector space F^n over F is the species of sequences $(a_1, \ldots, a_n)$ of elements of F. We denote elements of F^n by bold type: $\mathbf{a} = (a_1, \ldots, a_n)$. Addition of vectors and multiplication of a vector by an element of F are defined in the usual way. As to the notion of linear dependence, it can be defined in two ways; in

$$\lambda_1 \mathbf{a}_1 + \ldots + \lambda_n \mathbf{a}_n = \mathbf{0} \tag{1}$$

we can require the coefficients λ to be $\neq 0$ or to be $\neq 0$; this gives respectively the notions of *strong* and of *weak dependence*. As the former is by far the most important, *dependence* without adjective will mean strong dependence. A system of vectors that cannot be dependent will be called *independent*. Just as in many other cases, besides this negative notion we can define a positive one, classically equivalent to it. I think you can now give this definition yourselves.

FORM. If at least one λ_i is $\neq 0$, then

$$\lambda_1 \mathbf{a}_1 + \ldots + \lambda_n \mathbf{a}_n \neq \mathbf{0}.$$

INT. In this case we say that the vectors are (mutually) *free*.

Theorem 1. A necessary and sufficient condition that the vectors

$$\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \qquad (i = 1, \ldots, p)$$

are free, is that their matrix has rank p. This is an immediate consequence of 4.2.4, Th. 1 and 2.

4.3.2. *A theorem and a counter-example*

The following theorem requires for its validity an extra condition.

Theorem 1. If the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r$ are free and the vectors

$\mathbf{b}_1, \ldots, \mathbf{b}_{r+1}$ are free, then at least one vector $\mathbf{b}_i$ is free from $\mathbf{a}_1, \ldots, \mathbf{a}_r$ (that is, $\mathbf{a}_1, \ldots, \mathbf{a}_r, \mathbf{b}_i$ are free).

Proof. The matrix of the $\mathbf{a}_j$ has rank r; we may suppose that the determinant d formed out of its first r columns is $\neq 0$. Determine numbers λ_{sk} by the equations

$$b_{si} = \sum_{k=1}^{r} \lambda_{sk} a_{ki} \qquad \begin{matrix}(i = 1, \ldots, r;\\ s = 1, \ldots, r+1)\end{matrix},$$

and put

$$\mathbf{c}_s = \sum_{k=1}^{r} \lambda_{sk} \mathbf{a}_k \qquad (s = 1, \ldots, r+1)$$

As the $\mathbf{b}_j$ are free, there is a determinant, formed out of their components with subscripts $j_1, \ldots, j_{r+1}$, which is $\neq 0$. The corresponding determinant formed out of the c_{ik} is 0, so we find at least one pair of subscripts t, u with $b_{tu} \neq c_{tu}$. Now form a $r+1$-rowed determinant by extending d with the column $a_{1u}, \ldots, a_{ru}, c_{tu}$ and the row $c_{t1}, \ldots, c_{tr}, c_{tu}$. This determinant is 0, so, as $d \neq 0$, the corresponding determinant with b instead of c is $\neq 0$. This proves the theorem.

CLASS. Classically, the condition that $\mathbf{a}_1, \ldots, \mathbf{a}_r$ must be free, is superfluous.

INT. That is why I drew your attention to this theorem; I shall show by a counter-example that for arbitrary vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r$ we have no hope of proving it. Take for F the field of reals, $n = 3$, $r = 2$, $\mathbf{a}_1 = (0, 0, 1)$, $\mathbf{a}_2 = (a, b, 1)$, where a and b are real numbers such that neither for a nor for b it is known whether it is 0, and let nothing be known about the value of a/b (see 4.2.3). Then for none of the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ can we assert that it is free from $\mathbf{a}_1, \mathbf{a}_2$.

4.3.3. *Systems of unknown rank*

If the rank of a system of equations is unknown, in general no solution can be found. Yet in some cases we can derive a negative result. As an example we prove the

Theorem 1. If every n-rowed determinant from the matrix A of the equations

$$\sum_{k=1}^{n} a_{ik} x_k = 0 \qquad (i = 1, \ldots, m)$$

is 0, then it is impossible that the equations have no solution $\neq \mathbf{0}$.

Proof. Suppose that there could be no solution $\neq \mathbf{0}$; we have to deduce a contradiction from this supposition.

The rank of A is not n. If it were $n-1$, there would be a solution $\neq \mathbf{0}$; it follows that no $(n-1)$-rowed determinant in A can be $\neq 0$, thus they are all 0. If the rank were $n-2$, we could deduce a contradiction in the same way, and so on. After n steps we find that all the coefficients are 0, but then there certainly is a solution $\neq \mathbf{0}$. This is the desired contradiction.

V

PLANE POINTSPECIES

5.1. General notions

We shall develop the theory of plane point-species [L. E. J. Brouwer 1919A]. An analogous theory can be developed for any number n of dimensions. For $n=1$ it is identical with the theory of species of real numbers. Though it may seem somewhat tedious to give the definitions of all the fundamental concepts, including those which are identical with the usual definitions, it is nevertheless necessary to do so, because for almost every notion several definitions occur in the literature, which are equivalent classically, but not intuitionistically.

5.1.1. *Point-generators and points*

Definition 1. *A point-generator* ξ of the plane (abbreviated: *p–g*) is an ordered pair (ξ_1, ξ_2) of real number-generators.

Definition 2. A *point* x of the plane is an ordered pair (x_1, x_2) of real numbers.

I leave it to the reader to supply the definitions of coincidence between two point-generators between two points and between a point-generator and a point.

Theorem 1. Every point-generator determines one and only one point with which it coincides.

Definition 3. A *canonical point-generator* is an ordered pair of canonical number-generators.

By the proof of 3.3.3, Th. 1 every point coincides with a canonical p–g.

5.1.2. *Species and spreads*

Definition 1. A *p–g-species* (*pointspecies*) is a species each of whose members is a p–g (point).

Definition 2. A *p–g-spread* is a spread each of whose elements is a p–g. A p–g-spread is *canonical* if every one of its elements is a canonical p–g.

Definition 3. Two p–g-species *coincide* if every member of either of them coincides with some member of the other.

Analogous definitions for the relation of coincidence between A and B, where each of A and B is a p–g-species, or a pointspecies or a p–g-spread, are easily given.

Theorem 1. Every p–g-species or p–g-spread determines one and only one pointspecies with which it coincides.

Definition 4. Two p–g-species are *geometrically congruent* if neither of them can contain a member that cannot coincide with a member of the other.

Here also analogous definitions must be given, as in the case of def. 3.

5.1.3. *Distance and topology*

Definition 1. The *distance* $|x-y|$ of two points $x=(x_1, x_2)$ and $y=(y_1, y_2)$ is $\max(|x_1-y_1|, |x_2-y_2|)$.

Remark. The theory could be developed as well with

$$\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2\}}$$

as the distance of x and y. The above definition is chosen for the sake of simplicity of formulas.

The notions of an ε-neighbourhood and of a neighbourhood of a point p can be introduced in the usual way by means of this notion of distance.

Definition 2. The points x and y are *apart* from each other (abbreviation $x \# y$) if either $x_1 \# y_1$ or $x_2 \# y_2$.

Theorem 1. (i) If $p \# q$, then $q \# p$.

(ii) If $p \# q$ is impossible, then p coincides with q.

(iii) If p coincides with q, then $p \# q$ is impossible.

(iv) If $p \# q$ and q coincides with r then $p \# r$.

(v) If $p \# q$, then for every point r either $p \# r$ or $q \# r$.

The proofs are easy, using the properties of the relation $\#$ between real numbers (2.2.3).

Analogous definitions for the relation of apartness between two p–gs and that between a point and a p–g are easily formulated.

Definition 3. The point p is a *closurepoint* of the pointspecies Q if for every n there can be found a point q_n of Q such that $|p-q_n|<2^{-n}$.

Definition 4. The point p is a *limit-point* of the pointspecies Q if for every n we can find two points q_n and r_n of Q such that $q_n \# r_n$, $|p-q_n|<2^{-n}$ and $|p-r_n|<2^{-n}$.

CLASS. Is it true, that a closurepoint of Q is either a point of Q or a limit-point of Q?

INT. The following example shows that this disjunction need not be valid. Let the sequence $\{a_n\}$ be defined as follows. If among the first n digits in π no sequence 0123456789 occurs, $a_n=2^{-n}$; if among these n digits a sequence does occur, then $a_n=0$. Let S be the species of the components of the sequence $\{a_n\}$. 0 is a closure-point of S, but it is unknown whether 0 is a point of S or a limitpoint of S.

Definition 5. The *closure* $\bar{Q}$ of a pointspecies Q is the species of the closurepoints of Q.

Definition 6. The *derived species* of a pointspecies Q is the species of the limit-points of Q.

Definition 7. A pointspecies is *closed* if it coincides with its closure.

Theorem 2. The closure and the derived species of a pointspecies are closed.
The proofs are simple.

Theorem 3. Every limit-point of the closure of Q belongs to the derived species of Q.

Proof. Let p be a limit-point of $\bar{Q}$, then for every n we can find points q_n and r_n in $\bar{Q}$ such that $q_n \# r_n$, $|p-q_n|<2^{-n-1}$ and $|p-r_n|<2^{-n-1}$. Further we can find m so that $|q_n-r_n|>2^{-m}$, $m>n$, and points q_n', r_n' in Q such that $|q_n'-q_n|<2^{-m-2}$ and $|r_n'-r_n|<2^{-m-2}$. Then $|q_n'-r_n'|>2^{-m-1}$, so $q_n' \# r_n'$, $|p-q_n'|<2^{-n}$ and $|p-r_n'|<2^{-n}$. This proves that p is a limit-point of Q.

5.1.4. *Open species, regions and regioncomplements*

An open pointspecies can be defined as a pointspecies with only interior points. Then the complement of an open pointspecies is a closed pointspecies, but the complement of a closed pointspecies is not necessarily open; in one dimension take for the closed pointspecies the point 0, then the complement is the species of points p such that $p \neq 0$, but the points for which $p \# 0$ form an open species. However, this notion of an open species includes pathological cases such as the following. If Euler's constant is rational, S denotes the square $|p-x|<1$; if not, S denotes the square $|q-x|<1/2$, where p and q are different points. In order to avoid such cases, we replace the notion of an open species by the more constructive notion of a *region* [L. E. J. Brouwer 1918, p. 8; 1919A, p. 20]. This notion is quite simple, but the precise formulation of its definition is most conveniently obtained after some preparations.

In the following, "E" denotes either the whole plane or a rectangle with rational vertices and with sides parallel to the axes of coordinates. If E is a rectangle, only points of E, where E is considered as a closed pointspecies, are considered, even if this restriction is not mentioned. It is supposed that E has been chosen once and for all and remains fixed throughout the chapter.

Definition 1. An *elementary set of rectangles* is a finite set of rectangles with rational vertices and with sides parallel to the axes of coordinates. No rectangles which degenerate into a line segment are admitted. The letters V, W, X, Y, Z will always denote such sets.

Definition 2. The *rational elementary domain* $R(V)$ is the species of the rational points which are interior to or lie on the frontier of at least one of the rectangles of V. The *exterior rational elementary domain* $R^*(V)$ is the species of the rational points which are not interior to $R(V)$ ("interior" being taken relatively to the species of the rational points).

It is clear what it means to say that V is interior to W; as only rational points occur, no intuitionistic difficulties appear.

Definition 3. The *elementary domain* αV is the same as $\overline{R(V)}$; the *exterior elementary domain* $\alpha^*(V)$ is the same as $\overline{R^*(V)}$.

Theorem 1. $\alpha^* V = \overline{E - \alpha V}$.

Proof. (i) If $p \in \alpha^* V$, then p is the limitpoint of a sequence of rational points in $E - \alpha V$, so $p \in \overline{E - \alpha V}$.

(ii) If $p \in \overline{E - \alpha V}$, then we can find for every n a point p_n in $E - \alpha V$ such that $|p - p_n| < 2^{-n-1}$, and furthermore a rational point q_n such that $|p_n - q_n| < 2^{-n-2}$. p_n is contained in the square $|q_n - x| < 2^{-n-2}$; let σ be this square. If σ were contained in V, there would follow $p_n \in \alpha V$, which is false, so that a part of σ lies outside V (observe in this part of the proof that only rational points occur!). Let r_n be a rational point in σ outside V, then $r_n \in R^*(V)$ and $|p_n - r_n| < 2^{-n-1}$, so $|p - r_n| < 2^{-n}$. The sequence $\{r_n\}$ shows that $p \in \alpha^* V$.

Remark. If E is a rectangle, $\alpha^* V = \alpha W$, where V and W fill up E without having interior points in common. We say that V and W *simply cover* E. Where no ambiguity need be feared, we often omit the α and denote by V the elementary domain αV. Yet some caution is necessary here, as the following example shows. Let V consist of the square with opposite vertices $(0, 0) - (1, 1)$ and W of the square $(0, 1) - (1, 2)$, and let X be the set of these two squares; then αX is not the same as $\alpha V \cup \alpha W$; for, a point (p_1, p_2) with $0 < p_2 < 1$, but for which neither $p_1 \gtrdot 1$ nor $p_1 \lessdot 1$ is known, belongs to αX, but cannot be said to belong to $\alpha V \cup \alpha W$.

This example shows at the same time that the formula $\overline{Q \cup R} = \bar{Q} \cup \bar{R}$ does not hold. $\overline{Q \cup R} \subseteq \bar{Q} \cup \bar{R}$ is always true.

Definition 4. A *region* is the union of a sequence $\{V_n\}$ of elementary domains such that V_n lies inside V_{n+1} for every n. (If E is a rectangle and if a part λ of the frontier of V_n falls in the frontier of E, then it is allowed that λ belongs to the frontier of V_{n+1}.)

Every region is an open species; the complement of every region is a closed species. For many purposes the notion of a region-complement can replace that of a closed species.

Remark on notation. We use the letters A, B, C, with or without subscripts, for regions; V, W, and sometimes X, Y, Z for elementary sets of rectangles or for elementary domains; M, N for region-complements; Q, R for unspecified pointspecies. Thus if the letter A is used, the reader is supposed to know that it designates

a region. $A=\{V_n\}$ means that the sequence $\{V_n\}$ satisfies the condition of def. 4 and that $A=\bigcup_n V_n$.

Theorem 2. If $A=\{V_n\}$, then

$$E-A=\bigcap_{m=1}^{\infty}(E-V_m)=\bigcap_{m=1}^{\infty}\overline{(E-V_m)}=\bigcap_{m=1}^{\infty}\alpha^*V_m.$$

Proof. (i). If $p\in E-A$, then $p\notin V_m$ for every m, so

$$p\in\bigcap_{m=1}^{\infty}(E-V_m)\text{, so }p\in\bigcap_{m=1}^{\infty}\overline{(E-V_m)}.$$

(ii). Now suppose $p\in\bigcap_{m=1}^{\infty}\alpha^*V_m$. For a fixed value n of m, $p\in\alpha^*V_{n+1}$. Let d be the minimum distance between the frontiers of V_n and V_{n+1} (apart from parts of the frontier of E). We can find a rational point q outside V_{n+1} such that $|p-q|<{}^1/_2d$; then we have for every rational point r in V_n, $|p-r|>{}^1/_2d$, so that $p\notin V_n$. As n is arbitrary, p belongs to no V_n, so $p\in E-A$.

This theorem shows that the notion of a region-complement, though its definition was negative, can be considered as constructive.

Theorem 3. If it is impossible that p does not belong to $E-A$, then p belongs to $E-A$.

Proof. This follows immediately from the fact that $E-A$ is defined by a negation.

LETTER. What strange logic are you applying here?

INT. In fact I do not apply logic; I could not, for I have not yet developed it. But my argument may indeed be called logical, as it refers to the structure of the proposition. I shall expose it more elaborately.

Let "ϱ" be an abbreviation for "$p\in A$"; then the negation "$\neg\,\varrho$" is "$p\in(E-A)$". The above theorem says that $\neg\neg\neg\,\varrho$ implies $\neg\,\varrho$. This is obvious; for, if $\neg\neg\neg\,\varrho$ is given, the supposition that ϱ is true leads to $\neg\neg\,\varrho$, which contradicts $\neg\neg\neg\,\varrho$. (Compare 7.1.2, (5).)

5.1.5. *Union and intersection*

Theorem 1. The union of a finite or infinite sequence of regions is a region [L. E. J. Brouwer 1919A, p. 22].

Proof. Let $\{A_n\}$ be a sequence of regions; $A_n = \{V_{nk}\}$.

Set $W_r = \bigcup_{m=1}^{r} V_{mr}$; then the region B, defined by $\{W_r\}$, is $\bigcup_{m=1}^{\infty} A_m$.

Theorem 2. The intersection of two regions $A = \{V_n\}$ and $B = \{W_n\}$ is a region, provided for some value of n, V_n and W_n have a rectangle in common.

Proof. Let X be the rectangle that is contained in V_n as well as in W_n. Set $V_{n+p} \cap W_{n+p} = Y_p$ for every p; then the sequence $\{Y_p\}$ defines a region C that coincides with $A \cap B$.

Theorem 3. The intersection of a finite or infinite sequence of region-complements is a region-complement; explicitly:

$$\bigcap_n (E - A_n) = E - \bigcup_n A_n .$$

Proof. "$x \in \bigcap_n (E - A_n)$" means: "for any n, x cannot belong to A_n".

"$x \in (E - \bigcup_n A_n)$" means: "for no n, x belongs to A_n". The two meanings are the same.

The union of the two complements of two regions is not always a region-complement; it need not even be closed, as is illustrated by the example following 5.1.4, Th. 1. The elementary domains V and W, defined there, are region-complements; the point p, which I considered there, is a closurepoint of $V \cup W$.

In the same way that the elementary domain X was formed from V and W in the example cited just now, we can associate a region-complement to any pair (or finite set) of region-complements. The definition is the following.

Definition. If $M = \bigcap_n \alpha^* V_n$ and $N = \bigcap_n \alpha^* W_n$, then

$$M \overset{*}{\cup} N = \bigcap_n \alpha^* (V_n \cap W_n)$$

$M \overset{*}{\cup} N$ is a region-complement, provided for some value of n, $V_n \cap W_n$ contains a square.

Theorem. If A and B are regions such that $A \cap B$ is a region (Th. 2), then $(E - A) \overset{*}{\cup} (E - B) = E - (A \cap B)$.

Proof. Clear from the definitions.

5.2. Located pointspecies

5.2.1. *Located pointspecies and pointfans*

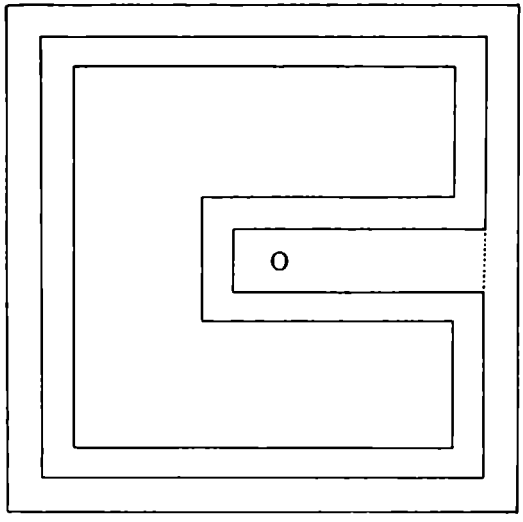
Definition 1. The *distance* of the point p to the pointspecies Q, written $\varrho(p, Q)$, is the greatest lower bound of the distances of p to the points of Q. Thus $\varrho(p, Q)$ satisfies the conditions:

(i) for every point q in Q, $|p-q| \not< \varrho$.

(ii) for every natural number n a point q_n in Q can be found such that $|p-q_n| < \varrho + 2^{-n}$.

Definition 2. A pointspecies Q is *located* [L. E. J. Brouwer 1919A, p. 13], if $\varrho(p, Q)$ can be calculated for every point p.

Remark. It is clear that $\varrho(p, Q)$ is a continuous function of p; thus, if the distance of every rational point to Q can be calculated, Q is located.



Example. Let E be the square with vertices $(\pm 1, \pm 1)$, and let A be the region which is defined by the sequence $\{V_n\}$ which is obtained as follows:
If in the first n digits of π no sequence 0123456789 occurs, V_n is a set of three rectangles, with the opposite vertices

$(-1+2^{-n}, -1+2^{-n})$ and $(1-2^{-n}, -2^{-n})$,
$(-1+2^{-n}, -2^{-n})$ and $(-2^{-n}, +2^{-n})$,
$(-1+2^{-n}, +2^{-n})$ and $(1-2^{-n}, 1-2^{-n})$

respectively; if in the first n digits of π a sequence 0123456789 does occur, V_n is the square with vertices $\pm(1-2^{-n})$.

Now let p be any rational point. Either $p \in E$, so that $\varrho(p, A)=0$, or p lies outside E; then $\varrho(p, A)=\varrho(p, E)$.

Thus A is located. However, $M=E-A$ is not located; for, O being the point (0, 0), $\varrho(O, M)=0$ if no sequence 0123456789 occurs in π, but $\varrho(O, M)=1$ if such a sequence does occur.

Theorem 1. Every bounded closed located pointspecies coincides with a canonical finitary p–g-spread (with a p–g-fan) [L. E. J. Brouwer 1919A, p. 14].

Proof. Let Q be a bounded closed located pointspecies. For convenience, let us call a point $(a\cdot 2^{-n}, b\cdot 2^{-n})$, a and b integers, an nth *lattice-point*. We divide the nth lattice-points, for $n=1, 2, \ldots$ successively, into *admissible* and *inadmissible* points, in such a way that for every admissible nth lattice-point p we have $\varrho(p, Q)<{}^5/_8\cdot 2^{-n}$, and for every inadmissible nth lattice-point q we have $\varrho(q, Q)>2^{-n-1}$; this can be done in view of 2.2.3, Th. 4. If p is an admissible nth lattice-point, a point q in Q can be found so that $|p-q|<{}^5/_8\cdot 2^{-n}$, and after that an $(n+1)$th lattice-point r so that $|q-r|<{}^5/_8\cdot 2^{-n-1}$. $\varrho(r, Q)<{}^5/_8\cdot 2^{-n-1}$, so r can be taken as admissible, and $|p-r|<2^{-n}$. Consider the spread of all canonical point-generators $\{p_n\}$, where p_n is an admissible nth lattice-point. After p_n has been chosen, at least one and at most nine choices for p_{n+1} are admissible, so S is a p–g-fan. I wish to prove that S coincides with Q.

(i) If $p \in Q$, then a canonical p–g of p belongs to S.

(ii) Let s be a point which coincides with a p–g s_0 in S. $s_0=\{p_n\}$, where p_n is an admissible nth lattice-point; $|s_0-p_n| \not> 2^{-n}$ and we can find a point q_n in Q so that $|p_n-q_n|<2^{-n}$; consequently $|s_0-q_n|<2^{-n+1}$. As every q_n is in Q, s_0 is a closurepoint of Q, and as Q is closed, $s_0 \in Q$.

Corollary. Let q and r be points of the closed located point-

species Q, such that $|q-r|<2^{-n-3}$ and let s be the middle of qr. We can find an nth lattice-point p_n such that $|s-p_n|<9\cdot 2^{-n-4}$; then $|q-p_n|<{}^5/_8\cdot 2^{-n}$ and $|r-p_n|<{}^5/_8\cdot 2^{-n}$. It follows that p_n is admissible. We have proved the lemma:

If q and r are points of a closed located pointspecies Q, such that $|q-r|<2^{-n-3}$, then the p–g-fan S that coincides with Q contains two elements which coincide with q and r respectively and in which the first n components $p_1, \ldots, p_n$ are the same.

Theorem 2. The closure of every canonical pointfan S coincides with a located region-complement.

Proof. Let H be the set of the n-th components in the elements of S; H is a finite set of n-th lattice-points. With every point of H as centre, describe the square with sidelength $3\cdot 2^{-n}$; let V_n be the set of these squares. Set $M=\bigcap_{n=1}^{\infty} V_n$, then M is a region-complement. It is clear that S is geometrically contained in M; as M is closed, also $\bar{S}$ is geometrically contained in M. I shall prove that conversely M is geometrically contained in $\bar{S}$.

Let q be a point of M, then $q\in V_n$, so we can find a point p_n of H so that $|q-p_n|<2^{-n+1}$. p_n is a component of at least one p–g of S, say of s_n; $|s_n-p_n|\not> 2^{-n}$. In this way we find for every n, a p–g s_n in S such that $|q-s_n|<2^{-n+2}$; thus q is a closure point of S.

It must still be proved that M is located. Let p be any point of the plane and set $\varrho(p, V_n)=\varrho_n$. It is clear from the construction of V_n and V_{n+1} that $\varrho_n \not> \varrho_{n+1} \not> \varrho_n+{}^5/_4\cdot 2^{-n}$. Thus $\lim \varrho_n=\varrho_0$ exists and $\varrho_0-\varrho_n \not> {}^5/_2\cdot 2^{-n}$.

I wish to prove that $\varrho(p, M)=\varrho_0$, that is to say that

(i) if $q\in M$, then $|p-q| \not< \varrho_0$;

(ii) for every n a point q_n in M can be found so that

$$|p-q_n|<\varrho_0+2^{-n}.$$

Proof of (i). $q\in V_n$, so $|q-p| \not< \varrho_n \not< \varrho_0+{}^5/_4\cdot 2^{-n}$ for every n, that is $|q-p| \not< \varrho_0$.

Proof of (ii). We can find successively a square a of V_n so that $\varrho(p, a)<\varrho_n+2^{-n}$, and a point r in a so that $|p-r|<\varrho_n+2\cdot 2^{-n}$; then $|p-r|<\varrho_0+{}^9/_2\cdot 2^{-n}$.

Corollary 1. It follows from the two preceding theorems that

every bounded closed located pointspecies coincides with a region-complement.

Theorem 3. A real-valued function which is defined everywhere on a bounded closed located pointspecies is uniformly continuous on that species.

Theorem 4. A real-valued function which is defined everywhere on a bounded closed located pointspecies has a g.l.b. and a l.u.b. on that species.

Theorem 5. If a real-valued function is defined and everywhere positive on a bounded closed located pointspecies, its g.l.b. is positive.

The proofs of these theorems are exactly analogous to those in 3.4.3.

5.2.2. *The intuitionistic form of the Heine–Borel theorem*

Theorem. Let Q be a bounded closed located pointspecies. If to every point p of Q there is associated a neighbourhood $U(p)$ in E, then we can find a finite set $U(q_1), \ldots, U(q_m)$ of these neighbourhoods so that Q is contained in $\bigcup_{k=1}^{m} U(q_k)$.

Proof. By definition a neighbourhood $U(p)$ contains a square $|p-x|<2^{-h}$, where h is a natural number, associated to p. Let S be a canonical pointfan that coincides with Q; if the point p of Q coincides with the p-g $p'=\{p'_n\}$ of S, then h is associated to p'. By the fan-theorem we can find a maximal value h_0 for h, so that, for every p, $U(p)$ contains the square $|p-x|<2^{-h_0}$. The (h_0+1)th lattice-points which occur as components of elements of S can be arranged in a finite sequence $p'_{h_0+1,i}$ $(i=1, \ldots, m_0)$. For each of these values of i we determine a fixed element of S, say q'_i, in which $p'_{h_0+1,i}$ occurs. Let q_i be the point of Q that coincides with q'_i; $|q_i-p'_{h_0+1,i}|<{}^5/_8\cdot 2^{-h_0-1}$.

An arbitrary point r of Q coincides with an element $r=\{r'_n\}$ of S; $r'_{h_0+1}=p'_{h_0+1,i}$ for some i, so $|r-p'_{h_0+1,i}|<{}^5/_8\cdot 2^{-h_0-1}$. It follows that $|q_i-r|<{}^5/_8\cdot 2^{-h_0}$, that is $r\in U(q_i)$. This means that $U(q_1)\cup\ldots\cup U(q_{m_0})$ contains Q.

Remark. L. E. J. Brouwer [1926C, p. 867] proved this theorem in a more general form (for what he calls located-compact species).

VI

MEASURE AND INTEGRATION

In this chapter E is the unit-square $|x-O| \not> 1/2$.

6.1. Measurable regions and region-complements

6.1.1. *Measurable regions*

Definition. The *measure*, to be denoted by mV, of an elementary domain V is its area in the ordinary sense. If a region A is defined by the sequence $\{V_n\}$ of elementary domains, and if $\lim_{n\to\infty} mV_n$ exists, then A is *measurable* and its *measure* is

$$mA = \lim_{n\to\infty} mV_n \text{ [L. E. J. Brouwer 1919A, p. 26].}$$

Class. By this definition, not even every bounded region is measurable.

INT. That is an immediate consequence of the fact that a bounded monotone sequence of real numbers need not be convergent. The proof of the following theorem is less simple than might be expected.

Theorem 1. If A and B are measurable regions, and if $A \supseteq B$, then $mA \not< mB$.

The proof rests on the following Lemma:

Lemma 1. If the elementary domain W is contained in the region $A=\{V_n\}$, then a number m can be found so that W is covered by V_m.

Proof. To every point p of W there is associated a natural number $h(p)$ so that p belongs to $V_{h(p)}$. As W is bounded, closed and located, it follows from 5.2.1, Th. 1 and the fan-theorem that there is a maximum m for $h(p)$; every point of W belongs to V_m.

Proof of theorem 1. Let $A=\{V_n\}$, $B=\{W_n\}$. Given k, we first determine n so that $mB-mW_n<2^{-k}$, and then (by the lemma)

m so that V_m covers W_n. Then $mA \geqslant mV_m \geqslant mW_n \geqslant mB - 2^{-k}$; as this is true for every k, $mA \geqslant mB$.

Theorem 2. If the region A coincides with the region B, and if A is measurable, then B is measurable and $mA = mB$.

Proof. $A = \{V_n\}$, $B = \{W_n\}$. By lemma 1, for every n, $mW_n < mA$. On the other hand, as A is measurable, given k, we can find i so that $mV_i > mA - 2^{-k}$, and then h so that W_h covers V_i (using the lemma), so that $mW_h > mA - 2^{-k}$. Hence $\lim\limits_{n\to\infty} mW_n = mA$.

Theorem 3. If A and B are measurable regions, the region $C = A \cup B$ is measurable and $mC \leqslant mA + mB$.
The proof follows easily from the definition.

Theorem 4. Let $\{A_n\}$ be a sequence of measurable regions such that $\lim\limits_{r\to\infty} m \bigcup\limits_{n=1}^{r} A_n = \mu$ exists; then $A = \bigcup\limits_{n=1}^{\infty} A_n$ is a measurable region and $mA = \mu$.

Proof. Put $\bigcup\limits_{n=1}^{r} A_n = B_r$, $\bigcup\limits_{n=1}^{\min.\, r,\, s} V_{ns} = W_{rs}$, $W_{ss} = U_s$; then
$B_r = \{W_{rs}\}$, $A = \{U_s\}$.
$mU_s = mW_{ss} < mB_s \leqslant \mu$.
Determine n_1 such that $\mu - mB_{n_1} < 2^{-k-1}$ and $n_2 > n_1$ such that $mB_{n_1} - mW_{n_1 n_2} < 2^{-k-1}$. Then $\mu - mU_{n_2} < \mu - mW_{n_1 n_2} < 2^{-k}$.
It follows that $\lim\limits_{s\to\infty} mU_s = \mu$, thus $mA = \mu$.

6.1.2. *Measurable region-complements*

Definition 1. If the region-complement M is the complement of the measurable region A, then M is *measurable* and its *measure* mM is $1 - mA$.

Theorem 1. If M_1 and M_2 are measurable region-complements, $mM_1 = m_1$, $mM_2 = m_2$, $m(M_1 \cap M_2) = m$, then $m_1 + m_2 \geqslant 1 + m$.

Proof. $M_1 = E - A_1$, $M_2 = E - A_2$, $M_1 \cap M_2 = E - (A_1 \cup A_2)$.
In view of 6.1.1, Th. 3 we have

$$m(A_1 \cup A_2) \leqslant mA_1 + mA_2$$
$$1 - m \leqslant 1 - m_1 + 1 - m_2$$
$$m_1 + m_2 \geqslant 1 + m.$$

Theorem 2. [L. E. J. Brouwer 1919A, p. 26]. Every measurable region-complement of positive measure contains a located measurable region-complement L such that $mL > mM - 2^{-p}$, where p is an arbitrary natural number.

Proof. There is a sequence $\{V_n\}$ of elementary domains such that $M = \bigcap_{n=1}^{\infty} V_n$. From this sequence we select a subsequence $\{W_n\}$ such that $mW_n - mM < 2^{-4n-p}$ for every n. Now L is obtained by the following construction. Divide E into subsquares of side-length 2^{-n}; these squares are called $\varkappa_n$-squares. From every W_n we remove its intersection with those $\varkappa_1$-squares which have with W_1 an intersection $< 2^{-p-3}$; the rest is W_n'. From every W_n' we remove the intersection with those $\varkappa_2$-squares with which W_2' has an intersection $< 2^{-p-7}$, and so on. In general, $W_n^{(k+1)}$ $(n > k)$ is obtained from $W_n^{(k)}$ by removing the intersections with those $\varkappa_{k+1}$-squares which have with $W_{k+1}^{(k)}$ an intersection $< 2^{-p-4k-3}$.

$$L = \bigcap_{n=1}^{\infty} W_n^{(n)}.$$

In view of 5.1.5, Th. 3, L is a region-complement.

By hypothesis, $mW_n - mW_{n+1} < 2^{-4n-p}$. A fortiori,

$$mW_n^{(n)} - mW_{n+1}^{(n)} < 2^{-4n-p}.$$

In passing from $W_{n+1}^{(n)}$ to $W_{n+1}^{(n+1)}$, the removed area is at most $2^{2n+2} \cdot 2^{-p-4n-3} = 2^{-p-2n-1}$, so $mW_n^{(n)} - mW_{n+1}^{(n+1)} < 2^{-4n-p} + 2^{-p-2n-1} < 2^{-p-2n}$.

It follows that $\lim mW_n^{(n)}$ exists.

One is not allowed to infer at once that $\lim mW_n^{(n)} = mL$, because $W_{n+1}^{(n+1)}$ need not be strictly interior to $W_n^{(n)}$. However, by a slight extension of every $W_n^{(n)}$ we easily construct a sequence $\{U_n\}$ of elementary domains such that U_{n+1} is always strictly interior to U_n, $L = \bigcap U_n$, and $\lim mU_n = \lim mW_n^{(n)}$.

Thus L is measurable and $mL = \lim mW_n^{(n)}$.

From the construction of $W_n^{(n)}$ we see that

$$mW_n - mW_n^{(n)} < \sum_{k=0}^{n-1} 2^{2k+2} \cdot 2^{-p-4k-3} < 2/3 \cdot 2^{-p}.$$

By a passage to the limit we have

$$mM - mL < 2^{-p}.$$

It remains to be shown that L is located. Let $\varkappa'$ be one of the $\varkappa_n$-squares which have a positive intersection with $W_n^{(n)}$, then, by the above construction, this intersection is $\geqq 2^{-p-4n+1}$. As $mW_n^{(n)} - mW_{n+1}^{(n)} < 2^{-p-4n}$, $m(W_{n+1}^{(n)} \cap \varkappa') > 2^{-p-4n+1} - 2^{-p-4n} = 2^{-p-4n}$. Passing from $W_{n+1}^{(n)}$ to $W_{n+1}^{(n+1)}$, we remove from $\varkappa'$ at most an area $4.2^{-p-4n-3}$, so $W_{n+1}^{(n+1)}$ has a part in common with $\varkappa'$; hence $W_{n+1}^{(n+1)}$ has a part in common with at least one of the $\varkappa_{n+1}$-squares which are parts of $\varkappa'$. Continuing this process, we find a sequence $\{q_h\}$ in which every q_h is a $\varkappa_{n+h}$-square, so that for every h, $W_{n+h}^{(n+h)}$ has a part in common with q_h. Let $x^{(h)}$ be a point in $W_{n+h}^{(n+h)} \cap q_h$; the sequence $\{x^{(h)}\}$ converges to a point x_0. As $x^{(j)} \in W_{n+h}^{(n+h)}$ for $j \geqq h$, $x_0 \in W_{n+h}^{(n+h)}$; as this is true for every h, $x_0 \in L$. Thus every $\varkappa_n$-square which has a positive intersection with $W_n^{(n)}$, has at least one point in common with L.

Now let H be the set of $\varkappa_n$-squares which have a positive intersection with $W_n^{(n)}$. L is contained in H, while every square of H has at least one point in common with L. Therefore for any rational point p, $\varrho(p, H) \not> \varrho(p, L) \not> \varrho(p, H) + 2^{-n}$.
This proves that L is located.

Corollary. It follows from the proof just given that every bounded region-complement of positive measure contains at least one point.

Theorem 3. If the measurable region-complement M is contained in the elementary domain V, then $mM \not> mV$.

Proof. $M = \cap W_n$. Suppose that $mM > mV$. We can find q so that $mM > mV + 2^{-q}$; then for every n, $mW_n > mV + 2^{-q}$. Let U_n be the part of W_n outside V. $mU_n > 2^{-q}$. By cutting from every U_n a small strip along the boundary of V, we obtain T_n. We can do this in such a manner that T_{n+1} is interior to T_n and that $mT_n > 2^{-q}$ for every n. Set $\cap T_n = N$. N is a measurable region-complement and $mN \not< 2^{-q}$. Hence, in view of the last corollary, N contains at least one point p. It is clear that p cannot belong to V, so $mM > mV$ has led to a contradiction.

6.1.3. *Negligeable and almost full pointspecies*

Definition 1. A pointspecies which can be enclosed in a measurable region of arbitrarily small measure is called *negligeable*.

Definition 2. A pointspecies which contains for every n a region-complement of measure greater than $1-2^{-n}$, is called *almost full*. A property which is true on an almost full species will be said to be true *almost everywhere*.

The complement of a negligeable pointspecies Q is almost full, for $Q \subseteq A$ implies $E-A \subseteq E-Q$. The converse is not always true, for from $E-A \subseteq R$ it follows that $E-R \subseteq E-(E-A)$, but this last species does not necessarily coincide with A, and we are not sure that it is a region.

Theorem 1. The meet of two almost full species is an almost full species.

Proof. Let Q and R be almost full species, and M, N region complements of measure $>1-2^{-n}$, contained in Q and R respectively. Then $M \cap N \subseteq Q \cap R$; set $m(M \cap N)=m$. By 6.1.2, Th. 1, $1-2^{-n}+1-2^{-n}<1+m$; $m>1-2^{-n+1}$.

Remarks. If V is an elementary domain, V is a region-complement and the interior of V is a region of the same measure. The boundary of V is a region-complement of measure 0. It can be enclosed in a measurable region A of measure $<2^{-n}$, which can be so chosen that, for some number s, every point at a distance less than 2^{-s} from the boundary belongs to A.

Let W be the elementary set of rectangles which together with V simply covers E. For an arbitrary point p in $E-A$ we can decide, by approximating it to less than 2^{-s}, whether it belongs to V or to W.

THE BROUWER INTEGRAL

I shall develop the theory of integration for the case of functions, defined on a subspecies Q of the unit-square E. Various extensions are possible, but their place is not in this introduction.

6.2. Bounded measurable functions

6.2.1. *Definition of the integral*

Definition 1. A bounded function $f(x)$, defined on a subspecies Q of E, is *measurable* [L. E. J. Brouwer 1923, p. 6], if for every natural number n the following conditions are fulfilled.

(i) a measurable plane region A_n with $mA_n<2^{-n}$ is given;

(ii) E is simply covered by elementary domains

$$V_{nh}\ (h = -l_n, \ldots, -1, 0, 1, \ldots, k_n),$$

(iii) If $M_n = E - A_n$, then

(a) for $x \in V_{nh} \cap M_n$, $h \neq 0$, we have

$$(h-1)2^{-n-1} \not> f(x) \not> (h+1)2^{-n-1},$$

(b) for $x \in Q \cap V_{n0} \cap M_n$ we have

$$-2^{-n-1} \not> f(x) \not> 2^{-n-1}.$$

The set of threedimensional intervals

$$x \in V_{nh},\ (h-1)2^{-n-1} \not> z \not> (h+1)2^{-n-1}$$
$$(h = -l_n, \ldots, 0, \ldots, k_n)$$

will be called the n-th *approximating strip* of $f(x)$. $f(x)$ is said to be measurable by $\{V_{nh}\}$ and $\{A_n\}$.

Definition 2. The *integral* of the measurable function $f(x)$ is defined by

$$\int_E f(x)dx = \lim_{n\to\infty} 2^{-n-1} \sum_{h=-l_n}^{k_n} h\, m V_{nh}\,. \tag{1}$$

Remarks. 1. It imposes no restriction to suppose $A_{n+1} \subseteq A_n$ for every n, for if we set $\bigcup_{k=n}^{\infty} A_k = B_n$ and $W_{n-1,h} = V_{n,2h}$, then $f(x)$ is also measurable by $\{B_n\}$ and $\{W_{nk}\}$, and the sequence $\{W_{nk}\}$ gives the same integral. Similarly we may suppose that the boundaries of all the V_{nh} are contained in A_n; for if $mA_n < 2^{-n} - 2^{-n'}$, we may enclose these boundaries in a measurable region C_n with measure less than $2^{-n'}$ and then substitute $A_n \cup C_n$ for A_n. This has the advantage that for every point of $E - (A_n \cup C_n)$ we know to which of the V_{nh} it belongs.

2. It is often convenient to modify clause (ii) of def. 1 as follows: (ii bis). E is simply covered by elementary domains $V_{nh}^{(p)}$; every $V_{nh}^{(p)}$ is contained in one $V_{n-1,j}^{(q)}$, and to read in clause (iii) $\bigcup_p (V_{nh}^{(p)} \cap M_n)$ in place of $V_{nh} \cap M_n$.

Class. That is a well-known definition of a Lebesgue integral.

Int. Yes, but it was necessary to select a constructive definition from among those which are classically possible. For instance, if $f(x)$ is defined in E and has the constant value a, where a is a real number for which neither $a < 1/2$ nor $a = 1/2$ nor $a > 1/2$ is known, then the species of points for which $f(x) < 1/2$ is not measurable;

I can say that now, though I have not yet given a general definition of measurable pointspecies. Of course the same remark applies to $f(x)=1/2$ and to $f(x)>1/2$.

An important difference is that a classical mathematician can always suppose his functions to be defined over all of E, by assigning to them the value 0 at every point where they were undefined. This does not work intuitionistically, since for some points it will be unknown whether the function is defined or not. Brouwer's definition of a measurable function is so formulated, that for such a function we know almost everywhere either that it is defined or that, if it is defined, it is smaller than 2^{-n}, where n can be chosen arbitrarily.

The pointspecies on which $f(x)$ is defined, will be called the *domain* of $f(x)$.

Theorem 1. The limit (1) exists for every measurable function.

Proof. Let $f(x)$ be measurable by $\{V_{nh}\}$ and $\{A_n\}$. We suppose that the A_n fulfill the conditions of remark 1.

Let p be any number greater than n, and let $V_{nh} \cap V_{pk} = W_{hk}$. Each W_{hk} is either an X_{hk} or a Y_{hk}, as follows:

$$W_{hk} = X_{hk} \text{ if } |h2^{-n-1} - k2^{-p-1}| < 2^{-n};$$
$$W_{hk} = Y_{hk} \text{ if } |h2^{-n-1} - k2^{-p-1}| \geqq 2^{-n}.$$

It is clear that

$$(2) \quad 2^{-p-1} \sum_k k\, m V_{pk} - 2^{-n-1} \sum_h h\, m V_{nh} = \sum_{h,k} (k2^{-p-1} - h2^{-n-1})\, m W_{hk}.$$

From the definitions we have for $x \in V_{nh} \cap M_n$ $(h \neq 0)$,

$$(3) \quad (h-1)2^{-n-1} \not> f(x) \not> (h+1)2^{-n-1},$$

and for $x \in V_{pk} \cap M_p$ $(k \neq 0)$,

$$(4) \quad (k-1)2^{-p-1} \not> f(x) \not> (k+1)2^{-p-1}.$$

Consequently for $x \in W_{hk} \cap M_n \cap M_p$,

$$(5) \quad |h2^{-n-1} - k2^{-p-1}| \not> 2^{-n-1} + 2^{-p-1} < 2^{-n}.$$

This is also true for $h=0$, $k \neq 0$ or $h \neq 0$, $k=0$, for, in the first case, $V_{pk} \cap M_p \subseteq Q$, so $W_{0k} \cap M_p \cap M_n \subseteq Q$, and (3) is valid on $W_{0k} \cap M_p \cap M_n$. For $h=k=0$, (5) is trivial.

Every point of $M_n \cap M_p$ belongs to some W_{hk}; in view of (5) this can be no Y_{hk}, so $M_n \cap M_p \subseteq \cup X_{hk}$.

Now, by 6.1.2, Th. 1, $m(M_n \cap M_p) \not< 1 - 2^{-n} - 2^{-p} > 1 - 2^{-n+1}$.

So $m \cup X_{hk} > 1 - 2^{-n+1}$ and $m \cup Y_{hk} < 2^{-n+1}$.
Thus if $|f(x)| < M$,

(6) $$\sum_{h,k} |k2^{-p-1} - h2^{-n-1}| m Y_{hk} < (2M+2)2^{-n+1}.$$

(7) $$\sum_{h,k} |k2^{-p-1} - h2^{-n-1}| m X_{hk} < 2^{-n}\, m \cup X_{hk} \not> 2^{-n}.$$

By (2),

(8) $$|2^{-p-1} \sum k\, m V_{pk} - 2^{-n-1} \sum h\, m V_{nh}| < (4M+5)2^{-n}.$$

Applying Cauchy's principle (2.4) we see that the limit in (1) exists.

Theorem 2. If the bounded function $f(x)$ is measurable by $\{V_{nh}\}$ and $\{A_n\}$ and also by $\{W_{nk}\}$ and $\{B_n\}$, then

(9) $$\lim_{n\to\infty} 2^{-n-1} \sum h\, m V_{nh} = \lim_{n\to\infty} 2^{-n-1} \sum k\, m W_{nk}.$$

Let $A_n \cup B_n = C_n$, $E - C_n = M_n$. We may suppose that C_n contains the boundaries of V_{nh} and W_{nk}.
Set $V_{nh} \cap W_{nk} = X_{hk}$ if $|h-k| \leqq 1$.
$V_{nh} \cap W_{nk} = Y_{hk}$ if $|h-k| > 1$.

A point of Y_{hk} cannot belong to M_n; so $M_n \subseteq \cup X_{hk}$. Hence (6.1.2, Th. 3) $mM_n \not> m \cup X_{hk} . m \cup X_{hk} \geqq 1 - 2^{-n+1} . m \cup Y_{hk} \leqq 2^{-n+1}$.

$$|2^{-n-1} \sum h\, m V_{nh} - 2^{-n-1} \sum k\, m V_{nk}| =$$
$$= 2^{-n-1} |\sum (h-k) m X_{hk} + \sum (h-k) m Y_{hk}| \leqq$$
$$\leqq 2^{-n-1} \sum m X_{hk} + (2M + 2^{-n}) \sum m Y_{hk} \leqq 2^{-n-1} + 2M \cdot 2^{-n+1} +$$
$$+ 2^{-2n+1} < (8M+2)2^{-n-1}.$$ This proves (9).

6.2.2. *Conditions of measurability*

Theorem 1. A bounded function which is defined almost everywhere is measurable [B. van Rootselaar 1954. p. 7].

Proof. By hypothesis $f(x)$ is defined on the species Q, and Q contains for every n a measurable region-complement M_n such that $mM_n > 1 - 2^{-n-3}$.

By 6.1.2, Th. 2, M_n contains a measurable located region-complement L_n such that $mL_n > 1 - 2^{-n-2}$. $L_n = E - A_n'$. Applying 5.2.1, Th. 3, we see that $f(x)$ is uniformly continuous on L_n, so that for every point x of L_n there is given a neighbourhood $U_n(x)$ such that $|f(x) - f(x')| < 2^{-n-2}$, if x' lies in $U_n(x) \cap L_n$. $U_n(x)$ contains an elementary domain $V_n(x)$ which contains x in its interior. The

Heine–Borel-theorem (5.2.2) can be applied to L_n, so there is a finite number of domains V_{ni} $(i=1, \ldots, m)$ which cover L_n. We define disjoint elementary domains W_{ni} by

$$W_{n1} = V_{n1}\,, \quad W_{ni} = V_{ni} - \bigcup_{k=1}^{i-1} (V_{ni} \cap W_{nk})\,.$$

The total area covered by the W_{ni} is at least $1-2^{-n-2}$; the rest of E can be enclosed in a measurable region A_n of measure $<2^{-n-1}$. Let us enclose the boundaries of the W_{ni} in a measurable region B_n with $mB_n<2^{-n-2}$, and set $C_n=A_n \cup B_n \cup A_n'$, so that $mC_n<2^{-n}$. Then $f(x)$ is measurable by $\{W_{ni}\}$ and $\{C_n\}$.

Conversely we shall show that every measurable function can be completed to an almost everywhere defined measurable function with the same integral.

Definition 1. $g(x)$ is a *completion* of $f(x)$ if it satisfies the following conditions:

(i) $g(x)$ is defined almost everywhere,

(ii) $g(x)=f(x)$ in every point where $f(x)$ and $g(x)$ are defined,

(iii) $f(x)$ is defined in every point where $g(x) \neq 0$.

Theorem 2. If $f(x)$ is measurable by $\{V_{nh}\}$ and $\{A_n\}$, then the function $g(x)$ defined below is a completion of $f(x)$.
$g_n(x)=h2^{-n-1}$ in $V_{nh} \cap (E-A_n)$; $g(x)=\lim g_n(x)$.

Theorem 3. If $g(x)$ is a completion of the bounded measurable function $f(x)$, then $g(x)$ is measurable and

$$\int g(x)dx = \int f(x)dx.$$

Theorem 4. If a completion of $f(x)$ exists, then $f(x)$ is measurable.

The proofs of these theorems are immediate from the definitions. In the proof of Th. 4 we use Th. 1 to prove that the completion of $f(x)$ is measurable.

Theorem 5. If $f(x)$ is measurable, $f(x)$ and $g(x)$ are bounded and $f(x)=g(x)$ almost everywhere, then $g(x)$ is measurable and $\int f(x)dx = \int g(x)dx$.

Proof. Let $f(x)$ be measurable by $\{V_{nh}\}$ and $\{A_n\}$, and let B_n be a region with measure less than 2^{-n} such that $f(x)=g(x)$ on $E-B_n$. Then $f(x)$ is also measurable by $\{V_{n-1,h}\}$ and $\{A_n \cup B_n\}$, and from the definition the statement in the theorem is clear.

6.2.3. *Sums and products of measurable functions*

Theorem 1. If the bounded measurable functions $f(x)$ and $g(x)$ have the same domain Q, or if $f(x)$ and $g(x)$ are defined a.e., then $f(x)+g(x)$ is measurable and $\int f(x)dx+\int g(x)dx=\int (f(x)+g(x))dx$.

Proof. Let $f_0(x)$ and $g_0(x)$ be completions of $f(x)$, $g(x)$, respectively. Then $f_0(x)+g_0(x)$ is a completion of $f(x)+g(x)$. The equality of the integrals is easily inferred from the definition (6.2.1, Def. 1).

Theorem 2. If $f(x)$ and $g(x)$ are bounded measurable functions, then $f(x)\,g(x)$ is measurable.

Proof. Here also, if $f_0(x)$ and $g_0(x)$ are completions of $f(x)$, $g(x)$, respectively, then $f_0(x)\,g_0(x)$ is a completion of $f(x)\,g(x)$.

Remark. Note that in Th. 2 it is not necessary to suppose, as in theorem 1, that $f(x)$ and $g(x)$ have the same domain. The reason is, that $f_0(x)\,g_0(x)\not\equiv 0$ implies $f_0(x)\not\equiv 0$ and $g_0(x)\not\equiv 0$, while $f_0(x)+g_0(x)\not\equiv 0$ implies only that $f_0(x)\not\equiv 0$ or $g_0(x)\not\equiv 0$. The domain of $f(x)\,g(x)$ is, of course, the intersection of the domains of $f(x)$ and of $g(x)$.

Definition 1. The *non-negative part* $f^+(x)$ and the *non-positive part* $f^-(x)$ of $f(x)$ are defined as follows:

$$f^+(x)=\max(f(x),0);\quad f^-(x)=\min(f(x),0).$$

Obviously, $f^+(x)$ and $f^-(x)$ have the same domain as $f(x)$, and $f(x)=f^+(x)+f^-(x)$.

Theorem 3. If $f(x)$ is bounded and measurable, then $f^+(x)$ and $f^-(x)$ are measurable, and

$$\int f^+(x)dx+\int f^-(x)dx=\int f(x)dx.$$

Proof. If $f_0(x)$ is a completion of $f(x)$, then $\max(f_0(x),0)$ is a completion of $f^+(x)$; the analogous result holds for $f^-(x)$. $f^+(x)$ and $f^-(x)$ have the same domain as $f(x)$, so we may complete the proof by applying Th. 1.

Theorem 4. If $f(x)$ is bounded and measurable, then $|f(x)|$ is measurable.

Proof. $|f(x)|=f^+(x)-f^-(x)$.

6.2.4. *A limit theorem*

Definition 1. The sequence of functions $\{f_n(x)\}$ is *convergent at the point* x_0, if a number N is known so that $f_n(x_0)$ is defined for $n>N$ and if the sequence $f_{N+n}(x_0)$ converges in the ordinary sense.

Theorem 1. If the functions of the sequence $\{f_n(x)\}$ are measurable and uniformly bounded, and if the sequence converges a.e. to $f(x)$, then $f(x)$ is measurable and

$$\lim_{n\to\infty} \int f_n(x)dx = \int f(x)dx.$$

Proof. For every m we can find a region-complement M_m such that $mM_m > 1-2^{-m-1}$ and that $\lim_{n\to\infty} f_n(x)$ exists at every point of M_m. M_m contains a located region-complement L_m of measure $>1-2^{-m}$. For every point x of L_m we can find a number $N(x)$, such that $f_n(x)$ is defined for $n>N(x)$ and that $|f(x)-f_n(x)|<2^{-m}$ for $n>N(x)$. In view of 5.2.1, Th. 1 and the fan-theorem we can find a maximum value of $N(x)$, say N_1. Hence if $n>N_1$, then $f_n(x)$ is defined and $|f(x)-f_n(x)|<2^{-m}$ for every x in L_m. In the following I suppose that $n>N_1$.

L_m is the complement of a region B_m. Let $f_n(x)$ be measurable by $\{V_{npk}\}$ and $\{A_{np}\}$. Set $C_{nm}=A_{nm}\cup B_m$, $K_{nm}=E-C_{nm}$. We have $mC_{nm}<2^{-m+1}$.

For $x\in V_{nmh}\cap K_{nm}$ we have $|f(x)-f_n(x)|<2^{-m}$ and further

(1) $(h-1)2^{-m-1}-2^{-m} \not> f(x) \not< (h+1)2^{-m-1}+2^{-m}$.

Then we can find a number k such that

(2) $(k-1)2^{-m+2} \not> f(x) \not< (k+1)2^{-m+2}$; here

(3) $|k2^{-m+2}-h2^{-m-1}| \leqq 2^{-m+1}$.

This proves that $f(x)$ is measurable by $\{V_{nmh}\}$ and $\{C_{nm}\}$, where n is a function of m.

From inequality (8) in the proof of 6.2.1, Th. 1 we infer that

(4) $|\int f(x)dx - 2^{-m+2}\sum k\, mV_{nmh}| < (4M+5)2^{-m+3}$;

(5) $|\int f_n(x)dx - 2^{-m-1}\sum h\, mV_{nmh}| < (4M+5)2^{-m}$.

From (3) we deduce

(6) $|2^{-m+2}\sum k\, mV_{nmh} - 2^{-m-1}\sum h\, mV_{nmh}| \not> \sum |2^{-m+2}k - 2^{-m-1}h| mV_{nmh} \not> 2^{-m+1}\sum mV_{nmh} \not> 2^{-m+1}$.

By (4), (5) and (6) we see that

(7) $|\int f(x)dx - \int f_n(x)dx| < (36M+47)2^{-m}$.

Since this is true for every $n > N(m)$, we have proved the theorem.

6.3. Measurable pointspecies

6.3.1. *Fundamental notions*

Definition 1. The *characteristic function* $f_Q(x)$ of a pointspecies Q is defined as follows:

$$f_Q(x) = 1 \text{ if } x \in Q\,;\ f_Q(x) = 0 \text{ if } x \notin Q.$$

Definition 2. Q is *measurable* if $f_Q(x)$ is measurable and the *measure* of Q is $mQ = \int f_Q(x)dx$.

Theorem 1. Definition 2 is equivalent with the following one: Q is measurable if, for every n, an elementary domain V_n and a measurable region A_n can be found such that $mA_n < 2^{-n}$ and $Q \cap (E - A_n) = V_n \cap (E - A_n)$. Then $mQ = \lim_{n\to\infty} mV_n$ [L. E. J. Brouwer 1919A, p. 29]. We say in this case that Q is measurable by $\{V_n\}$ and $\{A_n\}$.

Proof. (i) If $f_Q(x)$ is measurable by $\{W_{nh}\}$ and $\{A_n\}$, then Q is measurable by $\{W_{n,2^{n+1}}\}$ and $\{A_n\}$.
(ii) If Q is measurable by $\{V_n\}$ and $\{A_n\}$, then $f_Q(x)$ is measurable by $\{W_{nh}\}$ and $\{A_n\}$, where $W_{n,2^{n+1}} = V_n$, $W_{n0} = E - V_n$ and the other W_{nh} are empty.

Theorem 2. If Q is measurable, then $Q \cup (E-Q)$ is an almost full species.

Proof. Let $f_Q(x)$ be measurable by $\{W_{nh}\}$ and $\{A_n\}$. For abbreviation, set $X_n = W_{n,2^{n+1}}$. We suppose that A_n contains the frontier of X_n and of W_{n0}, so that every point of $E - A_n$ belongs either to X_n or to W_{n0}. $E - A_n = M_n$. $Q \cap M_n = X_n \cap M_n$.
If $x \in W_{n0} \cap M_n$, $f_Q(x) \neq 1$, so $x \notin Q$, $x \in E - Q$; consequently $W_{n0} \cap M_n = (E-Q) \cap M_n$.

$(Q \cup (E-Q)) \cap M_n = (X_n \cup W_{n0}) \cap M_n = M_n$. $Q \cup (E-Q) \supseteq M_n$.

6.3.2. *Conditions of measurability*

Theorem 1. If Q is measurable, then $E-Q$ is measurable and $m(E-Q)=1-mQ$ [L. E. J. Brouwer 1919A, p. 30].

Proof. As in the preceding proof, we find M_n so that $Q \cup (E-Q) \supseteq M_n$ and $mM_n > 1-2^{-n}$. Then for every n, $f_Q(x)+f_{E-Q}(x)=1$ on M_n. Hence $f_Q(x)+f_{E-Q}(x)=1$ almost everywhere. Consequently

$$\int f_Q(x)dx+\int f_{E-Q}(x)dx=\int dx=1.$$

Corollary. If Q is measurable, then $E-(E-Q)$ is measurable and has the same measure. This is very convenient, as it often permits us in the theory of measure to neglect double negations.

Theorem 2. Every measurable species Q contains for every n a located measurable region-complement L_n of measure greater than $mQ-2^{-n}$.

Proof. Let Q be measurable by $\{V_n\}$ and $\{A_n\}$, $E-A_n=M_n$. We can find $k=k(n)>n+1$ so that $|mV_k-mQ|<2^{-n-2}$. Also $mM_k \not< 1-2^{-n-2}$. By 6.1.2, Th. 1,

$$mV_k+mM_k \not> 1+m(M_k \cap V_k);$$
$$m(M_k \cap V_k) \not< mV_k+mM_k-1>mQ-2^{-n-2}+1-2^{-n-2}-1=$$
$$=mQ-2^{-n-1}.$$

The region-complement $M_k \cap V_k$ contains a located region-complement L_n such that $mL_n > m(M_k \cap V_k)-2^{-n-1}$. Thus $mL_n > mQ-2^{-n}$.
As $M_k \cap V_k = Q \cap V_k \subseteq Q$, the theorem is proved.

Theorem 3. A necessary and sufficient condition that the point-species Q is almost full, is that Q is measurable and $mQ=1$.

Proof. (i) The condition is necessary, for if Q is almost full, then $f_Q(x)$ is defined a.e., so by 6.2.2, Th. 1, $f_Q(x)$ is measurable, and $f_Q(x)=1$ a.e. so $mQ=1$.

(ii) The condition is sufficient by the preceding theorem.

Theorem 4. A necessary and sufficient condition that the pointspecies Q is measurable, is that $Q \cup (E-Q)$ is an almost full pointspecies.

Proof. (i) The condition is necessary by 6.3.1, Th. 2.

(ii) The condition is sufficient: as (i) in the preceding proof.

Theorem 5. Every detachable subspecies R of a measurable species Q is measurable [B. van Rootselaar 1954, p. 5].

Proof. $Q \cup (E-Q) \subseteq R \cup (E-R)$, for every element of Q belongs either to R or to $E-R$, and $E-Q \subseteq E-R$. Thus, as $Q \cup (E-Q)$ is almost full, $R \cup (E-R)$ is almost full, and R is measurable.

Corollary. If $Q \cup (E-Q)$ is measurable, then Q is measurable.

Theorem 6. If Q and R are measurable and $R \subseteq Q$, then $mQ = mR + m(Q-R)$.

Proof. The characteristic functions $f_Q(x)$, $f_R(x)$ and $f_{Q-R}(x)$ are all defined on the almost full species

$$(Q \cup (E-Q)) \cap (R \cup (E-R)),$$

and on this species $f_Q(x) = f_R(x) + f_{Q-R}(x)$.
This proves the theorem.

Theorem 7. If Q and R are geometrically congruent measurable pointspecies, then $mQ = mR$.

Proof. $E-Q = E-R$, so $1-mQ = 1-mR$.

6.3.3. *Union and intersection of measurable pointspecies*

Theorem 1. If Q and R are measurable pointspecies, then $Q \cap R$ and $Q \cup R$ are also measurable. $m(Q \cup R) \not< mQ + mR$; if Q and R are disjoint, then $m(Q \cup R) = mQ + mR$ [L. E. J. Brouwer 1919A, p. 32].

Proof. $f_{Q \cap R}(x)$ and $f_{Q \cup R}(x)$ are defined on the almost full species $(Q \cup (E-Q)) \cap (R \cup (E-R))$, and on this species we have $f_{Q \cap R}(x) = f_Q(x)\, f_R(x)$ and $f_{Q \cup R}(x) \not< f_Q(x) + f_R(x)$; if Q and R are disjoint, then $f_{Q \cup R}(x) = f_Q(x) + f_R(x)$.

Theorem 2. If $\{Q_n\}$ is a sequence of measurable pointspecies, $S_n = \bigcup_{h=1}^{n} Q_h$, $m_n = mS_n$, and if $\lim_{n \to \infty} m_n = m$ exists, then the species $S_\omega = \bigcup_{h=1}^{\infty} Q_h$ is measurable and $mS_\omega = m$.
[L. E. J. Brouwer 1919 A, p. 33].

Proof. By 6.3.1, Th. 2, we can find for every Q_n a measurable region B_n such that $mB_n < 2^{-h-n}$ and that $Q_n \cup (E-Q_n) \supseteq E-B_n$. Set $B = \bigcup_{n=1}^{\infty} B_n$, then $mB < 2^{-h}$.

Next we determine the numbers $n_1, n_2, \ldots$ so that $m - m_{n_1} < 2^{-h-2}$, $m - m_{n_2} < 2^{-h-3}$, etc. After that we find the measurable regions $C_1, C_2, \ldots$ such that $mC_i < 2^{-h-i}$ and $E - C_i \subseteq E - (S_{n_{i+1}} - S_{n_i})$. Set $C = \bigcup_{i=1}^{\infty} C_i$, so that $mC < 2^{-h}$, and $D = B \cup C$, so that $mD < 2^{-h+1}$.

Let x be any point of $E-D$. $x \in E-B_n$ for every n, so $x \in Q_n$ or $x \in E-Q_n$; consequently, $x \in S_{n_1}$ or $x \in E-S_{n_1}$. If $x \in S_{n_1}$, then $x \in S_\omega$.

Let us now suppose that $x \in E-S_{n_1}$. As $x \in E-C_1$, x does not belong to $S_{n_2}-S_{n_1}$, so $x \in E-S_{n_2}$.

Similarly, because $x \in E-C_2$, we have $x \in E-S_{n_3}$, and so on. It follows that $x \in E-S_\omega$. We have proved that any point of $E-D$ belongs either to S_ω or to $E-S_\omega$; thus f_{S_ω} is defined on $E-D$. Moreover, we have on $E-D$, $\lim_{n\to\infty} f_{S_n} = f_{S_\omega}$. Now the theorem follows by 6.2.4.

Theorem 3. If $\{Q_n\}$ is a sequence of measurable pointspecies, $R_n = \bigcap_{h=1}^{n} Q_h$, $mR_n = m_n$, and if $\lim_{n\to\infty} m_n = m$ exists, then the species $R_\omega = \bigcap_{h=1}^{\infty} Q_h$ is measurable and $mR_\omega = m$.

[L. E. J. Brouwer 1919A, p. 33].

The proof is almost word for word the same as that for the preceding theorem.

Theorem 4. Every measurable region or region-complement (6.1.1 or 6.1.2) is also a measurable point species (as defined in 6.3.1) and the two measures are equal [L. E. J. Brouwer 1919A, p. 30].

Proof. For a region this is an immediate consequence of theorem 2; for a region-complement it follows by 6.3.2, Th. 1.

Theorem 5. If $f(x)$ is a bounded measurable function, defined on the pointspecies Q, and if p is any natural number, then we can find disjoint measurable species Q_{ph} ($h = -l_p, \ldots, 0, \ldots, k_p$), such that $\sum_h mQ_{ph} = 1$ and

$$(h-1)2^{-p-1} < f(x) < (h+1)2^{-p-1} \quad \text{on } Q_{ph}\ (h \neq 0),$$
$$-2^{-p-1} < f(x) < 2^{-p-1} \quad \text{on } Q_{p0} \cap Q.$$

Proof. Let $f(x)$ be measurable by $\{V_{nh}\}$ and $\{A_n\}$. Set $M_n = E - A_n$ and $N_{nh} = M_n \cap V_{nh}$. Order the N_{nh} $(n \geqq p)$ by the following rule: N_{nh} precedes N_{mi}, if either $n<m$ or $n=m$, $h<i$. Let R_{nh} be the species of the points of N_{nh} which belong to none of the region-complements that precede N_{nh} in this sequence. Now divide the sequence of the R_{nh} into partial sequences, beginning with $N_{p,-l_p}, \ldots, N_{p,k_p}$ respectively, as follows. R_{ni} belongs to the partial sequence beginning with N_{ph}, if h is the smallest number such that

$$(h-1)2^{-p-1} \leqq (i-1)2^{-n-1} < (i+1)2^{-n-1} \leqq (h+1)2^{-p-1}.$$

Let Q_{ph} be the union of the species in the sequence beginning with N_{ph}; the $Q_{p,-l_p}, \ldots, Q_{p,k_p}$ have the desired properties.

Remark. Brouwer [L. E. J. Brouwer 1923, p. 9] proves that the Q_{ph} can be taken as outer limiting species. I do not introduce this notion.

6.4. The integral as the measure of a pointspecies

In the following three-dimensional as well as plane pointspecies occur. In order to prevent confusion, three-dimensional species will be denoted by german capitals.

Theorem 1. Let $f(x)$ be a bounded function, defined on a subspecies S of E. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the species of the points in space with coordinates $(p_1\, p_2, p_3)$, such that $(p_1, p_2, 0) \in S$ and $0 \not> p_3 \not> f(p_1, p_2)$, respectively $0 \not< p_3 \not< f(p_1, p_2)$; then
(a) $f(x)$ is measurable if and only if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are measurable three-dimensional pointspecies;
(b) if $f(x)$ is measurable, then $\int f(x)dx = m\mathfrak{T}_1 - m\mathfrak{T}_2$.

Proof of (a), if. Let $|f(x)| < M$ and let $\mathfrak{E}$ be the elementary domain in space defined by $(p_1, p_2, 0) \in E$, $|p_3| < M$. Let $\mathfrak{T}_1$ be measurable by $\{\mathfrak{V}_{1n}\}$ and $\{\mathfrak{A}_{1n}\}$. Suppose the notation is so chosen that $m\mathfrak{A}_{1n} < 2^{-2n}$. $\mathfrak{A}_{1n}$ is defined by a sequence of elementary domains $\{\mathfrak{U}_{1n,s}\}$; let i_q be chosen such that $m\mathfrak{U}_{1n,i_q} > (1-2^{-2q-5})m\mathfrak{A}_{1n}$. Then, if

$$\mathfrak{W}_{1n,q} = \mathfrak{U}_{1n,i_{q+1}} - \mathfrak{U}_{1n,i_q}, \quad m\mathfrak{W}_{1n,q} < 2^{-2q-5} m\mathfrak{A}_{1n} < 2^{-2n-2q-5}.$$

The parallel to the axis of p_3 drawn through a point $x = (p_1, p_2, 0)$, intersects $\mathfrak{W}_{1n,q}$ in a segment or segments of total length $l(x)$; let $X_{1n,q}$ be the species of the points x such that $l(x) \geqq 2^{-n-q-3}$. $mX_{1n,q} < 2^{-n-q-2}$ and we may enclose $X_{1n,q}$ in a region $C_{1n,q}$ with measure $< 2^{-n-q-2}$. Set $B_{1n} = \bigcup_{q=1}^{\infty} C_{1n,q}$, then $mB_{1n} < 2^{-n-1}$, and if $x \in E - B_{1n}$, then the parallel to the axis of Z drawn through x intersects $\mathfrak{A}_{1n}$ in a species D such that $mD < \sum_{q=1}^{\infty} 2^{-n-q-3} = 2^{-n-3}$. Set $E - B_{1n} = M_{1n}$.

Now we divide E into a finite set of elementary domains Y_{nh}, such that if $x \in Y_{nh}$, the parallel of the axis of p_3 drawn through x intersects $\mathfrak{B}_{1n}$ in a segment $(0, s)$ which satisfies the condition

$$(2h-1)2^{-n-2} \not> s \not> (2h+1)2^{-n-2}.$$

Combining this inequality with the earlier result: if $x \in M_{1n}$, then the parallel to the axis of p_3 drawn through x intersects $\mathfrak{A}_{1n}$ in a species of measure $< 2^{-n-3}$, and with the fact that $\mathfrak{B}_{1n} \cap (\mathfrak{E} - \mathfrak{A}_{1n}) = \mathfrak{T}_1 \cap (\mathfrak{E} - \mathfrak{A}_{1n})$, we see that for $x \in Y_{nh} \cap M_{1n}$,

$$(h-1)2^{-n-1} \not> f(x) \not> (h+1)2^{-n-1};$$

for $h = 0$ this inequality is valid at all those points where $f(x)$ is defined and $f(x) \not< 0$.

We have now proved that $f^+(x)$ is measurable by $\{Y_{nh}\}$ and $\{B_{1n}\}$; in the same way, by starting with $\mathfrak{T}_2$, we can prove that $f^-(x)$ is measurable.

Then $f(x)$ is measurable by 6.2.3, Th. 1.

Proof of (a), only if. Without loss of generality we may assume that $f(x)$ is bounded by 1. Let $f(x)$ be measurable by $\{V_{nh}\}$ and $\{A_n\}$, and let $\mathfrak{W}_n$ be the n-th approximation-strip, $\mathfrak{B}_n$ the three-dimensional region of points (p_1, p_2, p_3) such that $(p_1, p_2, 0) \in A_n$, and $\mathfrak{C}_n$ the species of points in space such that $-2^{-n} < p_3 < 2^{-n}$, then $\mathfrak{D}_n = \mathfrak{W}_n \cup \mathfrak{B}_n \cup \mathfrak{C}_n$ is measurable and $m\mathfrak{D}_n < 2^{-n+2}$; $\mathfrak{D}_n$ can easily be enclosed in a region $\mathfrak{F}_n$ such that $m\mathfrak{F}_n < 2^{-n+3}$. Consider the finite set of rectangular blocks $\mathfrak{U}_{1n}$, consisting of the points (p_1, p_2, p_3) such that for some h, $(p_1, p_2, 0) \in V_{nh}$ $(h > 0)$ and $0 \not> p_3 \not> h2^{-n-1}$, and the set $\mathfrak{U}_{2n}$, which contains the points such

that, for some h, we have $(p_1, p_2, 0) \in V_{nh}$ $(h<0)$ and $0 \not< p_3 \not< h2^{-n-1}$. $\mathfrak{T}_1$ is measurable by $\{\mathfrak{U}_{1n}\}$ and $\{\mathfrak{F}_n\}$; $\mathfrak{T}_2$ is measurable by $\{\mathfrak{U}_{2n}\}$ and $\{\mathfrak{F}_n\}$.

Proof of (b). This follows at once from

$$m\mathfrak{U}_{1n} - m\mathfrak{U}_{2n} = 2^{-n-1} \sum_h hmV_{nh}$$

by passing to the limit.

As an application we prove

Theorem 2. If $f(x)$ and $g(x)$ are bounded measurable functions, then $\max(f(x), g(x))$ and $\min(f(x), g(x))$ are measurable functions.

Proof. Set $\max(f(x), g(x)) = h(x)$, $\min(f(x), g(x)) = k(x)$. Let $\mathfrak{T}_i(F)$ $(i=1, 2;\ F=f, g, h, k)$ be defined in the same way as $\mathfrak{T}_1$ and $\mathfrak{T}_2$ in Th. 1.

$$\mathfrak{T}_1(h) = \mathfrak{T}_1(f) \cup \mathfrak{T}_1(g). \qquad \mathfrak{T}_2(h) = \mathfrak{T}_2(f) \cap \mathfrak{T}_2(g).$$
$$\mathfrak{T}_1(k) = \mathfrak{T}_1(f) \cap \mathfrak{T}_1(g). \qquad \mathfrak{T}_2(k) = \mathfrak{T}_2(f) \cup \mathfrak{T}_2(g).$$

$\mathfrak{T}_i(f)$ and $\mathfrak{T}_i(g)$ are measurable; by 6.3.3, Th. 1, so are $\mathfrak{T}_i(h)$ and $\mathfrak{T}_i(k)$. It follows that $h(x)$ and $k(x)$ are measurable.

Remark. This theorem can also be proved by the completion method (see 6.2.2).

6.5. Unbounded functions

6.5.1. *Unbounded measurable functions*

The definition and many properties of bounded measurable functions can be immediately extended to unbounded functions. I repeat them for the convenience of the reader.

Definition 1. A function f, defined on a subspecies Q of E, is *measurable*, if for every natural number n the following conditions are fulfilled:

(i) a measurable plane region A_n with $mA_n < 2^{-n}$ is given;

(ii) E is simply covered by elementary domains V_{nh} $(h = -l_n, \ldots, -1, 0, 1, \ldots, k_n)$;

(iii) if $M_n = E - A_n$, then

(a) for $x \in V_{nh} \cap M_n$, $h \neq 0$, we have $(h-1)2^{-n-1} \not> f(x) \not> (h+1)2^{-n-1}$,

(b) for $x \in Q \cap V_{n0} \cap M_n$ we have $-2^{-n-1} \not> f(x) \not> 2^{-n-1}$.

Theorem 1. A function which is defined almost everywhere is measurable [B. van Rootselaar 1954, p. 7].

Proof. See 6.2.2, Th. 1.

Theorem 2. The function f is measurable if and only if a completion of f exists.

Proof. See 6.2.2, Ths. 2 and 4.

Theorem 3. If the measurable functions f and g have the same domain, or if both are defined a.e., then $f+g$ is measurable.

Proof. See 6.2.3, Th. 1.

Theorem 4. If f and g are measurable functions, then their product is a measurable function.

Proof. See 6.2.3, Th. 2.

Another proof of Th. 3 and Th. 4 was given by van Rootselaar [1956].

Theorem 5. If f and g are measurable functions, defined on the same domain, then $\max(f, g)$ and $\min(f, g)$ are measurable functions.

Proof. If f_0 and g_0 are completions of f and g respectively, then $\max(f_0, g_0)$ is a completion of $\max(f, g)$ and $\min(f_0, g_0)$ is a completion of $\min(f, g)$.

Corollary. If f is a measurable function, then f^+ and f^- are measurable functions, and $|f|$ is a measurable function.

Theorem 6. If the functions of the sequence $\{f_n\}$ are measurable, and if the sequence $\{f_n(x)\}$ converges a.e. to $f(x)$, then f is a measurable function.

Proof. See the first part of the proof of 6.2.4, Th. 1.

Definition 2. For any function f we denote $\min(f(x), 2^k)$ by ${}_kf(x)$.

Theorem 7. A non-negative function f is measurable if and only if ${}_kf$ is measurable for every natural number k.

Proof. The only-if-part is an immediate consequence of Th. 5. To prove the if-part, suppose that ${}_kf$ is measurable by $\{V_{knh}\}$ and $\{A_{kn}\}$ and that each A_{kn} contains the boundaries of the corresponding V_{knh}. $E - A_{kn} = M_{kn}$. Then $W_{kn} = \bigcup_{h>0} (V_{knh} \cap M_{kn})$ is a region-complement. From here on, k is a fixed natural number. By 6.2.1, Th. 2, W_{kn} contains a located region-complement L_n such that $mL_n > mW_{kn} - 2^{-n}$. ${}_kf$ and also f is defined everywhere on L_n; by 5.2.1, Th. 4, f is bounded on L_n, say $f(x) < 2^l$ on L_n. If we denote

$L_n \cup (V_{kn0} \cap m_{kn})$ by K_n. then K_n is a region-complement with the properties

(i) $mK_n > 1 - 2^{-n+1}$,

(ii) $x \in K_n \to f(x) < 2^l$.

By hypothesis $_lf$ is measurable by $\{V_{lnh}\}$ and $\{A_{ln}\}$; this means that

$$x \in V_{lnh} \cap M_{ln} \;\&\; h > 0 \to (h-1)2^{-n} \not> {}_lf(x) \not> (h+1)2^{-n}.$$

$$x \in V_{ln0} \cap M_{ln} \cap Q \quad \to \quad 0 \not> {}_lf(x) \not> 2^{-n}.$$

Q is the domain of f.

These inequalities remain valid if M_{ln} is replaced by $M_{ln} \cap K_n = H_{ln}$. $mH_{ln} > 1 - 2^{-n+2}$. On H_{ln}, $f(x) = {}_lf(x)$, so

$$x \in V_{lnh} \cap H_{ln} \;\&\; h > 0 \to (h-1)2^{-n} \not> f(x) \not> (h+1)2^{-n},$$

$$x \in V_{ln0} \cap H_{ln} \cap Q \quad \to \quad 0 \not> f(x) \not> 2^{-n}.$$

Thus, considering l as a function of n, we see that f is measurable by $\{V_{lnh}\}$ and $\{D_{ln}\}$, where $H_{ln} = E - D_{ln}$.

6.5.2. *Summable functions*

Definition. A measurable function $f(x)$ is *summable* if

$$\lim_{k\to\infty} \int {}_kf^+(x)dx + \lim_{l\to\infty} \int {}_lf^-(x)dx$$

exists; this limit is $\int f(x)dx$.

Theorem 1. If f and g have the same domain, $f(x)$ is summable and $g(x)$ is measurable, and if $0 \not> g(x) \not> f(x)$, then $g(x)$ is summable.

Proof. Let $g(x) \not> f(x)$ be true on the almost full pointspecies Q. The inequality

(1) $\min(g(x), 2^{k+m}) - \min(g(x), 2^k) \not> \min(f(x), 2^{k+m}) - \min(f(x), 2^k)$

is obvious for such points x in Q where

(2) either $g(x) \not< 2^k$, or $g(x) < 2^k$.

Thus, if for some x in Q (1) were false, x could not satisfy condition (2); but this is a contradiction.

Finally, if (1) cannot be false, then (1) is true.

From (1) we infer that

$$\int_{k+m} g(x)dx - \int_k g(x)dx \not> \int_{k+m} f(x)dx - \int_k f(x)dx.$$

It follows that $\lim_{k\to\infty} \int_k g(x)dx$ exists.

Theorem 2. If $f(x)$ and $g(x)$ are summable functions with the same domain, then $f(x) + g(x)$ is summable and

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx.$$

The proof will be given by means of three lemmas.

Lemma 1. Th. 2 is true for non-negative functions $f(x)$ and $g(x)$.

Proof. Set $f(x)+g(x)=h(x)$, then $h(x)$ is measurable by 6.5.1, Th. 3

$$_{k-1}f(x)+_{k-1}g(x) \succcurlyeq {}_kh(x) \succcurlyeq {}_kf(x)+{}_kg(x).$$

Passing to the limit, we see that $\int h(x)dx$ exists and is equal to $\int f(x)dx + \int g(x)dx$.

Lemma 2. If $f(x)$ and $(g)x$ are summable functions, defined on the same domain, and such that $0 \succcurlyeq g(x) \succcurlyeq f(x)$, then $f(x)-g(x)$ is summable.

Proof. By 6.5.1, Th. 3, $f-g$ is measurable. Further $0 \succcurlyeq f(x)-g(x) \succcurlyeq f(x)$, so $f-g$ is summable by Th. 1.

Lemma 3. If $f(x)$ and $g(x)$ are non-negative summable functions with the same domain, then $f(x)-g(x)$ is summable.

Proof. If $f(x)-g(x)=h(x)$, then

$$h^+(x) = \max(f(x), g(x)) - g(x),$$
$$-h^-(x) = g(x) - \min(f(x), g(x)).$$

As $0 \succcurlyeq g(x) \succcurlyeq \max(f(x), g(x)) \succcurlyeq f(x)+g(x)$ and
$0 \succcurlyeq \min(f(x), g(x)) \succcurlyeq g(x)$, we see by 6.5.1, Th. 5, 6.5.2, Th. 1 and lemma 2, that $h(x)$ is summable.

Proof of theorem 2.

$$f(x)+g(x) = (f^+(x)+g^+(x)) - (-f^-(x)-g^-(x))$$

is summable by lemmas 1 and 3.

Theorem 3. If f and g have the same domain, $f(x)$ is measurable and $g(x)$ is summable, and if $|f(x)| < g(x)$, then $f(x)$ is summable.

Proof. $0 \succcurlyeq f^+(x) \succcurlyeq |f(x)|$ and $0 \succcurlyeq -f^-(x) \succcurlyeq |f(x)|$.
It follows that

$$0 \succcurlyeq f^+(x) < g(x) \text{ and } 0 \succcurlyeq -f^-(x) < g(x).$$

In view of Th. 1, $f^+(x)$ and $-f^-(x)$ are summable.
As $f(x)=f^+(x)-(-f^-(x))$, lemma 3 gives the desired result.

Theorem 4. If $f_n(x)$ is summable for every n and $\lim_{n\to\infty} f_n(x) = f(x)$

uniformly, then $f(x)$ is summable and $\lim_{n\to\infty} \int f_n(x)dx = \int f(x)dx$.

Proof. For sufficiently large values of n,

$$|f(x)-f_n(x)|<\varepsilon, \text{ so by Th. 3, } f(x)-f_n(x)$$

is summable. By Th. 2, also $f(x)$ is summable.
The second part of the theorem is easily proved.

Theorem 5. If $f(x)$ is summable and $g(x)$ is bounded and measurable, then $h(x)=f(x)g(x)$ is summable.

Proof. Suppose $|g(x)|<s$. ${}_kf(x)g(x)$ is measurable for every k, so $h(x)$ is measurable.

First let $f(x) \not< 0$ and $g(x) \not< 0$. If $f(x)$ is summable, then $sf(x)$ is summable, so we need only consider the case that $g(x)<1$. Then it is easily seen that, for $l>k$,

$$\min(fg, 2^l)-\min(fg, 2^k) \not> \min(f, 2^l)-\min(f, 2^k).$$
$$\int {}_lh(x)dx \; -\int {}_kh(x)dx \quad \not> \int {}_lf(x)dx \; -\int {}_kf(x)dx.$$

This proves that $h(x)$ is summable.

In the general case we have

$$h^+(x)=f^+(x)g^+(x)+f^-(x)g^-(x);$$
$$h^-(x)=f^+(x)g^-(x)+f^-(x)g^+(x);$$

so $h^+(x)$ and $h^-(x)$ are summable.

6.5.3. *Functions, summable on a pointspecies*

Definition 1. The function $f(x)$ is *measurable on the species* G, if $f(x)f_G(x)$ is measurable.
The function $f(x)$ is *summable on* G, if $f(x)f_G(x)$ is summable; in this case

$$\int_G f(x)dx=\int_E f(x)f_G(x)dx.$$

Theorem 1. A measurable function is measurable on every measurable species G. A summable function is summable on every measurable species G.

Proof. (i) If $f(x)$ is bounded, $f(x)f_G(x)$ is measurable by 6.2.3, Th. 2.

(ii) If $f(x)$ is not bounded, ${}_kf(x)f_G(x)$ is measurable for every k, by (i).

(iii) By the preceding theorem, $f(x)f_G(x)$ is summable.

Definition 2. $\int_a^b f(x)dx = \int_G f(x)dx$ where G is the interval (a, b).

Theorem 2. If $f(x)$ is summable, then

$$\lim_{mG \to 0} |\int_G f(x)dx| = 0 \text{ uniformly.}$$

Proof. (i) If $f(x)$ is bounded, $|\int_G f(x)dx| \not> (\max |f(x)|)mG$.

(ii) In the general case we determine k so that

$$\int_E (f^+(x) - {}_k f^+(x))dx < 2^{-n-2} \quad \text{and}$$
$$\int_E (f^-(x) - {}_k f^-(x))dx < 2^{-n-2}.$$

For $mG < 2^{-n-k-2}$ we have

$$|\int_G f(x)dx| \not> \int_G {}_k f^+(x)dx + \int_G {}_k f^-(x)dx +$$
$$+ \int_G (f^+(x) - {}_k f^+(x))dx + \int_G (f^-(x) - {}_k f^-(x))dx \not>$$
$$\not> 2^k mG + 2^k mG + 2^{-n-2} + 2^{-n-2} < 2^{-n}.$$

Remark. Van Rootselaar [B. van Rootselaar 1954, p. 16] has given another definition of $\int_G f(x)dx$, which is perhaps more general than the definition given here.

6.5.4. *Limit theorems*

Theorem 1. If the sequence of non-negative functions $\{f_n(x)\}$ satisfies the conditions

(i) every f_n is summable,

(ii) $f_{n+1}(x) \not< f_n(x)$,

(iii) the limit $\lim_{n \to \infty} f_n(x) = f(x)$, exists almost everywhere,

(iv) $\lim_{n \to \infty} \int_E f_n(x)dx$ exists;

then $f(x)$ is summable and

$$\int_E f(x)dx = \lim_{n \to \infty} \int_E f_n(x)dx.$$

Proof. $\lim_{n \to \infty} \int f_n(x)dx = \lim_{n \to \infty} \lim_{k \to \infty} \int {}_k f_n(x)dx.$

As $\int {}_k f_n(x)dx$ is non-decreasing for increasing n and k, we may invert the order of the limits, obtaining

$$\lim_{k \to \infty} \lim_{n \to \infty} \int {}_k f_n(x)dx.$$

The sequence $\{{}_k f_n(x)\}$ satisfies the conditions of 6.2.4, Th. 1, so

$$\lim_{n\to\infty} \int_k f_n(x)dx = \int_k f(x)dx.$$

This proves the theorem.

CLASS. Is that theorem about inverting the order of the limits in a double series intuitionistically valid?

INT. Yes, and I propose its proof as an exercise.

Theorem 2. (Egorof's theorem [B. van Rootselaar 1954, p. 10].) If the functions of the sequence $\{f_n\}$ are all defined on the measurable species Q, and if $\lim_{n\to\infty} f_n(x) = f(x)$ exists a.e. on Q, then, for every m, a measurable subspecies Q_m of Q can be found such that $mQ - mQ_m < 2^{-m}$, and that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on Q_m.

Proof. Let Q_0 be the species of measure $mQ_0 = mQ$, on which $\lim_{n\to\infty} f_n(x) = f(x)$. For Q_m we choose a measurable located region-complement of measure greater than $mQ - 2^{-m}$, contained in Q_0 (6.3.2, Th. 2, 6.1.2, Th. 2). Q_m coincides with a point-fan S (5.2.1, Th. 1); to every element x of S there is associated a natural number $N(x)$ such that

$$|f_n(x) - f(x)| < 2^{-p} \text{ for } n > N(x).$$

The fan-theorem assures us that a natural number B can be found such that $N(x)$ depends only upon the first B components of the ips x; it follows that a maximum value N_0 of $N(x)$ can be calculated. Then

$$|f_n(x) - f(x)| < 2^{-p}, \text{ for every } n > N_0 \text{ and every } x \text{ in } Q_m.$$

6.6. Hilbert space

I shall prove that the functions $f(x)$ which are defined a.e. and such that $f^2(x)$ is summable, form a Hilbert space B^2.

Real Hilbert space can be defined in different ways, of which the following two are relevant here:

(i) the constructive definition as the species of (positively) convergent sequences of real numbers, in which the inner product is defined as usual.

(ii) the axiomatic definition by von Neumann's axioms [J. von Neumann 1929, p. 64–66], [M. H. Stone 1932, p. 3].

It can be proved that the two definitions are intuitionistically equivalent if the axioms are slightly modified in the following manner.

For the sake of briefness the elements of Hilbert space H will be called *vectors*.

Axioms for Hilbert space.

Axioms I. The usual axioms for a real linear space.

Axioms II. The usual axioms for the inner product, including

IIa, $(x, x) \not< 0$;

IIb, If $(x, x) = 0$, then $x = 0$.

(The null-element of H is simply denoted by 0).

An apartness relation is introduced by

Definition 1. *x lies apart from 0* ($x \# 0$) if $(x,x) \# 0$.

x lies apart from y ($x \# y$) if $x - y \# 0$.

Definition 2. The vectors $x^1, \ldots, x^k$ are *mutually free*, if for every set of k real numbers $a_1, \ldots, a_k$ such that $a_i \# 0$ for at least one value of i, we have

$$a_1 x^1 + \ldots + a_k x^k \# 0.$$

Axiom III. For every k we can find k mutually free vectors.

Axiom IV. (Axiom of separability in the strong form.)

There exists a sequence $S = \{e^n\}$ of vectors such that every finite subsequence of S is free and that the species of finite linear combinations of vectors in S is dense in H.

Axiom V, the axiom of completeness in its usual form.

Remark. Axiom III is a consequence of axiom IV.

In order to formulate the announced theorem, I need the following definitions.

Definition 3. $f(x) \approx 0$ means that $f(x) = 0$ a.e. $f(x)$ is *equivalent* to $g(x)$, $f(x) \approx g(x)$, if $f(x) - g(x) \approx 0$.

The species of the functions which are equivalent to a given function will be called a *metafunction*.

In this section $f(x)$ means sometimes a function, sometimes the metafunction of which the function $f(x)$ is a member.

Theorem 1. The species B^2 of metafunctions $f(x)$ such that the function $f(x)$ is defined a.e. and that $f^2(x)$ is summable, becomes a

Hilbert space if $(f(x), g(x))$ is defined as denoting $\int f(x)\, g(x)dx$.

Proof. Of axioms I, II, only IIb needs a proof.
We must prove that $(f, f)=0$ implies $f \approx 0$.
Set $h(x)=f^2(x)$, then ${}_k h(x)$ is measurable and $\int {}_k h(x)dx=0$.
Let ${}_k h(x)$ be measurable by $\{V_{nh}\}$ and $\{A_n\}$.
Suppose $m(\bigcup_{h>1} V_{nh})>2^{-n}$. Set $M_n=\bigcup_{h>1} V_{nh}-A_n$, then $mM_n>0$.
Define $g(x)={}_k h(x)$ on M_n and $g(x)=0$ on $E-M_n$.

$$\int g(x)dx \not> \int f^2(x)dx=0 \quad \text{and}$$
$$\int g(x)dx > 2^{-n-1}\, mM_n > 0.$$

The supposition $m(\bigcup_{h>1} V_{nh})>2^{-n}$ has led to a contradiction, so

$$m(\bigcup_{h>1} V_{nh}) \not> 2^{-n}.$$

Enclose $\bigcup_{h>1} V_{nh}$ in a region B_n of measure $<2^{-n+1}$;
set $C_n=A_n \cup B_n$. $mC_n<2^{-n+2}$. On $E-C_n$ we have $f(x)<2^{-n}$.
Set $D_n=\bigcup_{k=n}^{\infty} C_k$. $mD_n<2^{-n+3}$. Let $N_n=E-D_n$. On N_n we have $f(x)<2^{-n-p}$ for every p, so $f(x)=0$. $\bigcup_{k=n}^{\infty} N_k$ is a species of measure 1 on which $f(x)=0$.
Axioms III and IV are satisfied. The proof of this fact is identical with the classical proof.

Proof of axiom V (the Riesz–Fischer theorem).

I formulate this as

Theorem 2. Let $\{f_n\}$ be a sequence of functions in B^2 such that $\int (f_m-f_n)^2dx$ tends to 0, if m and n tend to infinity; then there exists a function f in B^2 such that $\lim_{n\to\infty} \int (f-f_n)^2dx=0$ [A. Heyting 1951].
The proof follows closely von Neumann's proof.
First, we determine the integers N_p so that

$$\int (f_m-f_n)^2dx<2^{-3p} \text{ for } m, n \geqq N_p.$$

For the purpose of abbreviation set $g_p=|f_{N_{p+1}}-f_{N_p}|$ and $h_p(x)=$ $=\min(g_p(x), 1)$. By the method of 6.3.3, Th. 5, we determine the disjoint measurable species Q_{ph} $(h=0, \ldots, s)$ such that $\sum_h mQ_{ph}=1$ and

$$(h-1)2^{-p-1}<h_p(x)<(h+1)2^{-p-1} \text{ on } Q_{ph}\ (h\neq 0),$$

$$-2^{-p-1}<h_p(x)<2^{-p-1} \qquad \text{on } Q_{p0}\cap Q.$$

Define $Q_{p0}\cup Q_{p1}=R_p$, $\bigcup_{h>1} Q_{ph}=S_p$. Then

$$2^{-3p}>\int h_p^2dx=\int_{R_p} h_p^2dx+\int_{S_p} h_p^2dx \not< 2^{-2p-2}\, mS_p.$$

$mS_p<2^{-p+2}\,.\,mR_p>1-2^{-p+2}$. Set $\bigcap_{p\geqq q} R_p=T_q$, then $mT_q>1-2^{-q+3}$. If, finally, $\bigcup_q T_q=T_0$, then $mT_0=1$.

The sequence $\{f_{N_p}\}$ converges on T_0 to a function f, the convergence being uniform on any T_q.

$$\int_{T_q}(f_m-f_{N_p})^2dx \not> \int_E (f_m-f_{N_p})^2dx<\varepsilon \quad \text{for } m, N_p>N(\varepsilon).$$

(A) $\int_{T_q}(f_m-f)^2dx \not> \varepsilon \quad \text{for } m>N(\varepsilon).$

The desired result is obtained by passing to the limit, $q\to\infty$. Classically, this can be done by the simple remark that the left members of (A) for $q=1, 2, \ldots$ form a bounded monotone sequence. Intuitionistically, the existence of the limit must be proved by direct calculation. To do this we first prove a lemma.

Lemma. If

1. every function of the sequence $\{f_n(x)\}$ is measurable,
2. $\int f_n^2(x)dx$ exists for every n,
3. $f_n(x)$ converges uniformly to $f(x)$ when $n\to\infty$;

then $f^2(x)$ is summable and $\lim_{n\to\infty}\int f_n^2(x)dx=\int f^2(x)dx$.

Proof. By 6.5.2, Th. 4, $f(x)$ and also $|f(x)|$ is summable. Let ε be an arbitrary small positive number and $\eta=\frac{\varepsilon}{6\int(|f(x)|+1)dx}$. A number N can be found so that

$$|f(x)-f_n(x)|<\eta \quad \text{for } n>N.$$

Then

$$|f_m^2(x)-f_n^2(x)|<2\eta|f_m(x)+f_n(x)|<$$
$$<4\eta(|f(x)|+\eta) \qquad \text{for } m, n>N.$$
$$|\int(f_m^2(x)-f_n^2(x))dx|<4\eta\int(|f(x)|+1)dx<\varepsilon.$$

It follows that $\lim_{n\to\infty}\int f_n^2(x)dx=S$ exists.

We choose a fixed value of $n>N$ so that

(1) $$|\int f_n^2(x)dx - S| < \varepsilon/3.$$

In the same way as above we find

$$|f^2(x) - f_n^2(x)| < 2\eta\,(|f(x)| + \eta).$$

A fortiori:

$$|{}_k f^2(x) - {}_k f_n^2(x)| < 2\eta\,(|f(x)| + \eta).$$

(2) $$|\int {}_k f^2(x)dx - \int {}_k f_n^2(x)dx| < 2\eta \int (|f(x)| + 1)dx = \varepsilon/3.$$

By definition 6.5.2, we can determine K so that

(3) $$|\int f_n^2(x)dx - \int {}_k f_n^2(x)dx| < \varepsilon/3 \text{ for } k > K.$$

From (1), (2) and (3), we see that

$$|\int {}_k f^2(x)dx - S| < \varepsilon \quad \text{for } k > K.$$

Thus $\int f^2(x)dx = S$.

I now continue the proof of Th. 2.

For abbreviation, set $f_{N_p}(x) = F_p(x)$.

From Schwarz's inequality there follows $(q > p)$:

$$\sqrt{\int F_p^2(x)dx} \geqslant \sqrt{\int F_q^2(x)dx} + \sqrt{\int (F_p(x) - F_q(x))^2 dx}.$$

(1) $$\left|\sqrt{\int F_p^2(x)dx} - \sqrt{\int F_q^2(x)dx}\right| \geqslant \sqrt{\int (F_p(x) - F_q(x))^2 dx} < 2^{-p}.$$

Thus, $\lim\limits_{p\to\infty} \sqrt{\int F_p^2(x)dx}$ exists and as a consequence also $\lim\limits_{p\to\infty} \int F_p^2(x)dx$ exists; $\lim\limits_{p\to\infty} \int F_p^2(x)dx = S$.

The inequality (1) remains true when the integrals are taken over an arbitrary T_k. Now for $r > r_0$,

$$\int_{T_k} F_r^2(x)dx < S + 1; \quad \sqrt{\int_{T_k} F_r^2(x)dx} < \sqrt{(S+1)} = A.$$

$$\left|\int_{T_k} F_p^2(x)dx - \int_{T_k} F_q^2(x)dx\right| < 2A\,2^{-p} \quad \text{for } q > p > r_0.$$

As $F_q(x)$ converges uniformly to $f(x)$ in T_k for $q \to \infty$, we may pass to the limit (6.5.2, Th. 4):

(2) $$\left|\int_{T_k} F_p^2(x)dx - \int_{T_k} f^2(x)dx\right| < A\,2^{-p+1} \quad \text{for } p > r_0.$$

On the other hand, as $mT_0 = 1$ and $\lim\limits_{p\to\infty} \int F_p^2(x)dx = S$,

(3) $$\left|\int_{T_0} F_p^2(x)dx - S\right| < 2^{-n-2} \quad \text{for } p > r_1(n).$$

By 6.5.3, Th. 2 we have

$$\int_{T_0 - T_k} F_p^2(x)dx < 2^{-n-2} \quad \text{for } k > r_2(n, p). \tag{4}$$

n being given, I choose p so that $p > r_0$, $p > r_1(n)$ and $A2^{-p+1} < 2^{-n-2}$; then by (2), (3) and (4),

$$\left| \int_{T_k} f^2(x)dx - S \right| < 2^{-n} \quad \text{for } k > r_2(n).$$

$\lim_{k\to\infty} \int_{T_k} f^2(x)dx = S$, or, more explicitly,

$$\lim_{k\to\infty} \lim_{h\to\infty} \int_{T_k} {}_h f^2(x)dx = S. \tag{5}$$

As $\int_{T_k} {}_h f^2(x)dx$ is monotone non-diminishing with respect to k as well as to h, we may, using the property which was mentioned already in the proof of 6.5.4, Th. 1, invert the order of the limits.

$$S = \lim_{h\to\infty} \lim_{k\to\infty} \int_{T_k} {}_h f^2(x)dx = \lim_{h\to\infty} \int_{T_0} {}_h f^2(x)dx = \int_{T_0} f^2(x)dx.$$

The existence of the last integral having been proved,

$$\int_{T_0} (f(x) - f_n(x))^2 dx$$

also exists and by 6.5.3, Th. 2, this integral is equal to

$$\lim_{k\to\infty} \int_{T_k} (f(x) - f_n(x))^2 dx.$$

Thus, according to formula (A),

$$\int_{T_0} (f(x) - f_n(x))^2 dx \not> \varepsilon \quad \text{for } m > N(\varepsilon).$$

As $mT_0 = 1$, the theorem is proved.

6.7. Derivation

I shall not treat the theory of derivation here. The proofs of the main theorems have been adapted to intuitionistic requirements by van Rootselaar [B. van Rootselaar 1954, p. 33].

VII

LOGIC

7.1. The propositional calculus

The word "logic" has many different meanings. I shall not try to give a definition of intuitionistic logic, any more than I have begun this course by a definition of mathematics. Yet a preliminary remark will be useful. Our logic has only to do with mathematical propositions; the question whether it admits any applications outside mathematics does not concern us here. The letters p, q, r occur in this chapter as variables for mathematical propositions; German letters $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ will be used as abbreviations for mathematical propositions. It is not my purpose to give a complete formal treatment of intuitionistic logic; a formal system which codifies all the logical inferences of intuitionistic mathematics known at present, is easily accessible in Kleene's book [S. C. Kleene 1952], where the reader will also find an account of the metamathematical investigations of this system. Among later investigations I mention papers by E. W. Beth [1956, 1959], R. Harrop [1956, 1960], K. Schröter [1956, 1957], G. Kreisel and H. Putnam [1957], J. Porte [1958], H. Rasiowa and R. Sikorski [1959], R. Sikorski [1959], Th. Skolem [1958], S. C. Kleene [1962] and N. N. Vorob'ev [1958]. Kreisel [1958, 1962 A] formulated the completeness problem for intuitionistic logic and obtained important results on this subject. He also proposed a formal theory of constructions [1962] which is intended to serve as a basis for intuitionistic mathematics. Here I shall only call your attention to some formulas which express interesting methods of reasoning and show why these methods are intuitively clear within the realm of intuitionistic mathematics.

It will be necessary to fix, as firmly as possible, the meaning of the logical connectives; I do this by giving necessary and sufficient conditions under which a complex expression can be asserted.

7.1.1. *Interpretation of the signs*

The *conjunction* $\wedge$ gives no difficulty. $\mathfrak{p} \wedge \mathfrak{q}$ can be asserted if and only if both $\mathfrak{p}$ and $\mathfrak{q}$ can be asserted.

I have already spoken of the *disjunction* $\vee$ (2.2.5, at the end). $\mathfrak{p} \vee \mathfrak{q}$ can be asserted if and only if at least one of the propositions $\mathfrak{p}$ and $\mathfrak{q}$ can be asserted.

The *negation* $\neg$ is the strong mathematical negation which we have already discussed (2.2.2). In order to give a more explicit clarification, we remember that a mathematical proposition $\mathfrak{p}$ always demands a mathematical construction with certain given properties; it can be asserted as soon as such a construction has been carried out. We say in this case that the construction *proves* the proposition $\mathfrak{p}$ and call it a *proof* of $\mathfrak{p}$. We also, for the sake of brevity, denote by $\mathfrak{p}$ any construction which is intended by the proposition $\mathfrak{p}$. Then $\neg\mathfrak{p}$ can be asserted if and only if we possess a construction which from the supposition that a construction $\mathfrak{p}$ were carried out, leads to a contradiction.

SIGN. Is it not necessary to clarify the notion of a contradiction?

INT. I think that contradiction must be taken as a primitive notion. It seems very difficult to reduce it to simpler notions, and it is always easy to recognize a contradiction as such. In practically all cases it can be brought into the form $1 = 2$. As a simple example, let us consider the proposition $\mathfrak{p} \equiv (\sqrt{2}$ is rational). It demands the construction of integers a, b, such that $a^2 = 2b^2$. By a well-known argument we may suppose that a and b are relatively prime. On the other hand, a is even, so 4 divides a^2, hence 4 divides $2b^2$, and b is even; thus a and b have the common divisor 2. This contradicts the fact that a and b are relatively prime. The contradiction can be given the form: The GCD of a and b is at the same time 1 and 2.

Some mathematicians, and notably Griss, have raised objections against the use of contradiction in mathematical reasoning. I shall treat these objections in the next chapter; here I take the point of view that the notion of a contradiction is sufficiently clear and that the negation which is based on it can be used in mathematics.

The *implication* $\mathfrak{p} \to \mathfrak{q}$ can be asserted, if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$

(supposing that the latter be effected), would automatically effect a construction proving $\mathfrak{q}$. In other words, a proof of $\mathfrak{p}$, together with $\mathfrak{r}$, would form a proof of $\mathfrak{q}$.

Almost every proof in this book consists of such a construction as $\mathfrak{r}$ above. One of the first instances, and a very clear one, is the proof of 2.2.3, Th. 2.

A logical formula with proposition variables, say $\mathfrak{A}(p, q, \ldots)$, can be asserted, if and only if $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$ can be asserted for arbitrary propositions $\mathfrak{p}, \mathfrak{q}, \ldots$; that is, if we possess a method of construction which by specialization yields the construction demanded by $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$. For example consider

$$\mathfrak{A}(p, q) \equiv (p \wedge \cdot\ p \to q \cdot \to q).$$

$\mathfrak{A}(\mathfrak{p}, \mathfrak{q})$ demands a construction E, which from a proof C of $\mathfrak{p}$ and a proof D of $\mathfrak{p} \to \mathfrak{q}$ yields a proof of $\mathfrak{q}$. By the definition of implication, E consists simply in the juxtaposition of C and D. Thus $\mathfrak{A}(p, q)$ can be asserted.

In 2.2.2 I gave a criterion for mathematical propositions, namely that every mathematical proposition has the form "I have effected a construction with the following properties: ....." . This form is preserved by the four logical connectives. It is necessary to understand the word "construction" in the wider sense, so that it can also denote a general method of construction, as was meant in the last paragraph but one. If I do this – and I shall do it –, every logical formula expresses a mathematical proposition.

7.1.2. *Theorems of the propositional calculus*

In the formulas I use points and brackets in the usual way, assuming the convention that $\to$ binds less strongly that $\wedge$ and $\vee$. Asserted formulas are marked with $\vdash$.

Though the main differences between classical and intuitionistic logic are in the properties of the negation, they do not coincide completely in their negationless formulas. $p \to q \cdot \vee \cdot q \to p$ is a valid formula in classical logic, but it cannot be asserted in intuitionistic logic, as is clear from the definitions.

In the theory of negation the principle of the excluded middle fails. $p \vee \neg p$ demands a general method to solve every problem, or more explicitly, a general method which for any proposition $\mathfrak{p}$

yields by specialization either a proof of $\mathfrak{p}$ or a proof of $\neg\mathfrak{p}$. As we do not possess such a method of construction, we have no right to assert the principle.

Another form of the principle is $\neg\neg p \to p$. We have met many examples of propositions for which this fails: the first was "ϱ is rational" in 2.2.2. However,

(1) $\qquad \vdash p \to \neg\neg p.$

It is clear that from $\mathfrak{p}$ it follows that it is impossible that $\mathfrak{p}$ is impossible. I leave it to you to describe completely the method of construction which is demanded by (1), according to the definitions of $\to$ and $\neg$.

Another important formula is

(2) $\qquad \vdash p \to q \cdot \to \cdot \neg q \to \neg p.$

Of course, the inverse formula, $\neg q \to \neg p \cdot \to \cdot p \to q$, is not assertable. (Take $\mathfrak{q} \equiv a \neq b$, $\mathfrak{p} \equiv a = b$, where a and b are real numbers.)

Applying (2) twice, we find

(3) $\qquad \vdash p \to q \cdot \to \cdot \neg\neg p \to \neg\neg q.$

By substitution in (1) we find

(4) $\qquad \vdash \neg p \to \neg\neg\neg p.$

If we substitute $\neg\neg p$ for q in (2), we find, using (1),

(5) $\qquad \vdash \neg\neg\neg p \to \neg p.$

(4) and (5) show that we need never consider more than two consecutive negations.

From $\vdash p \to p \vee q$ follows, by (2), $\vdash \neg(p \vee q) \to \neg p$; in the same way we have $\vdash \neg(p \vee q) \to \neg q$, so

(6) $\qquad \vdash \neg(p \vee q) \to \neg p \wedge \neg q.$

The inverse formula is easily seen to be also true:

(7) $\qquad \vdash \neg p \wedge \neg q \to \neg(p \vee q).$

(6) and (7) form one of de Morgan's equivalences. The other one is only half true:

(8) $\qquad \vdash \neg p \vee \neg q \to \neg(p \wedge q).$

$\neg (p \wedge q) \to \neg p \vee \neg q$ cannot be asserted, as the following example shows. Let $\mathfrak{p}$ be $a \neq 0$ and $\mathfrak{q}$ be $b \neq 0$, where a and b are real numbers; then $\neg \mathfrak{p}$ is $a = 0$ and $\neg \mathfrak{q}$ is $b = 0$. I proved in 2.2.5 that $ab \neq 0$ is equivalent to $\mathfrak{p} \wedge \mathfrak{q}$, so $ab = 0$ is equivalent to $\neg (\mathfrak{p} \wedge \mathfrak{q})$: but just before the cited place in 2.2.5 I gave an example of real numbers a and b for which $ab = 0$, but neither $a = 0$ nor $b = 0$ is known.

(9) $\quad \vdash \neg\neg (p \vee \neg p).$

For $\neg (\mathfrak{p} \vee \neg \mathfrak{p})$ would imply, by (6), $\neg \mathfrak{p} \wedge \neg\neg \mathfrak{p}$, which is a contradiction. (8) gives by means of (2) and (6)

$$\vdash \neg\neg (p \wedge q) \to \neg (\neg p \vee \neg q) \to \neg\neg p \wedge \neg\neg q.$$

(10) $\quad \vdash \neg\neg (p \wedge q) \to \neg\neg p \wedge \neg\neg q.$

The inverse formula is also true:

(11) $\quad \vdash \neg\neg p \wedge \neg\neg q \to \neg\neg (p \wedge q).$

For it is clear from the above interpretation of the logical connectives that $\vdash \neg (p \wedge q) \wedge p \to \neg q$; then also $\vdash \neg (p \wedge q) \wedge \neg\neg q \to \neg p$. So, if $\neg\neg \mathfrak{p} \wedge \neg\neg \mathfrak{q}$ is given, the hypothesis $\neg (\mathfrak{p} \wedge \mathfrak{q})$ would lead to $\neg \mathfrak{p}$, which is contradictory with the given $\neg\neg \mathfrak{p}$.
It is easy to see that

(12) $\quad \vdash \neg\neg p \vee \neg\neg q \to \neg\neg (p \vee q),$

but the inverse implication does not hold because of the strong interpretation of $\vee$.

7.1.3. *A formal system*

The intuitionistic propositional calculus has been developed [A. Heyting 1930] as a formal system with $\wedge$, $\vee$, $\to$, $\neg$ as undefined constants, and on the basis of the following axioms

I. $\quad \vdash p \to (p \wedge p).$

II. $\quad \vdash (p \wedge q) \to (q \wedge p).$

III. $\quad \vdash (p \to q) \to ((p \wedge r) \to (q \wedge r)).$

IV. $\quad \vdash ((p \to q) \wedge (q \to r)) \to (p \to r).$

V. $\quad \vdash q \to (p \to q).$

VI. $\quad \vdash (p \wedge (p \to q)) \to q.$

VII. $\vdash p \to (p \vee q)$.

VIII. $\vdash (p \vee q) \to (q \vee p)$.

IX. $\vdash ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$.

X. $\vdash \neg p \to (p \to q)$.

XI. $\vdash ((p \to q) \wedge (p \to \neg q)) \to \neg p$.

The rules of deduction are the usual ones from the classical propositional calculus.

Axiom X may not seem intuitively clear. As a matter of fact, it adds to the precision of the definition of implication. You remember that $\mathfrak{p} \to \mathfrak{q}$ can be asserted if and only if we possess a construction which, joined to the construction $\mathfrak{p}$, would prove $\mathfrak{q}$. Now suppose that $\vdash \neg \mathfrak{p}$, that is, we have deduced a contradiction from the supposition that $\mathfrak{p}$ were carried out. Then, in a sense, this can be considered as a construction, which, joined to a proof of $\mathfrak{p}$ (which cannot exist) leads to a proof of $\mathfrak{q}$. I shall interpret the implication in this wider sense.

A system of intuitionistic logic in which $\to$ is interpreted in the narrower sense and in which, accordingly, X is rejected as an axiom, has been developed by Johansson in his "minimal calculus" [I. Johansson 1936].

It must be remembered that no formal system can be proved to represent adequately an intuitionistic theory. There always remains a residue of ambiguity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof.

7.2. The first order predicate calculus

7.2.1. *Interpretation of the quantifiers*

Let $\mathfrak{p}(x)$ be a predicate of one variable x, this variable ranging over a given mathematical species Q. Then $\vdash (\forall x)\mathfrak{p}(x)$ means that $\mathfrak{p}(x)$ is true for every x in Q; in other words, we possess a general method of construction which, if any element a of Q is chosen, yields by specialization the construction $\mathfrak{p}(a)$. In the case that Q is a spread-species, we must be able to effect the construction $\mathfrak{p}(x)$ for every ips x in Q; in the proof of the fan-theorem we saw that this is a very strong interpretation of the generalizing quantifier. The existential quantifier will also be interpreted in a

strong way. $(\exists x)\mathfrak{p}(x)$ will be true if and only if an element a of Q for which $\mathfrak{p}(a)$ is true has actually been constructed.

The introduction of predicates with more than one argument presents no difficulty. A formula of the first order predicate calculus, which contains propositional and predicate variables, can be asserted if it is true for every substitution of propositions and predicates for these variables. A simple formalization of the intuitionistic predicate calculus is obtained by adjoining to the intuitionistic propositional calculus the symbols, axioms and rules of the usual predicate calculus as stated by Hilbert and Ackermann ([D. Hilbert und W. Ackermann 1949, p. 59]; see also [A. Heyting 1946]). I shall not develop this formal system, but instead I shall prove some formulas by intuitive methods.

7.2.2. *Theorems of the predicate calculus*

The following theorems are clear.

(1) $\quad \vdash (\forall x)p(x) \to \neg\, (\exists x) \neg\, p(x).$

(2) $\quad \vdash (\exists x)p(x) \to \neg\, (\forall x) \neg\, p(x).$

The inverse implications do not hold. Counterexamples:

(i) Let x range over the real numbers and let $\mathfrak{p}(x)$ be "x is rational or x is irrational".

(ii) Let x range over the rational numbers and let $\mathfrak{p}(x)$ be "x is equal to the real number ϱ, defined in 2.2.2.".

(3) $\quad \vdash (\forall x) \neg\, p(x) \to \neg\, (\exists x)p(x).$

(4) $\quad \vdash \neg\, (\exists x)p(x) \to (\forall x) \neg\, p(x).$

(5) $\quad \vdash (\exists x) \neg\, p(x) \to \neg\, (\forall x)p(x).$

The inverse implication of (5) does not hold.
Counterexample:
Let x range over the rational numbers and let $\mathfrak{p}(x)$ be "x is not equal to the real number ϱ, defined in 2.2.2.".
By substitution in (3) we obtain

(6) $\quad \vdash (\forall x) \neg\neg\, p(x) \to \neg\, (\exists x) \neg\, p(x).$

Substitution in (4) gives

(7) $\quad \vdash \neg\, (\exists x) \neg\, p(x) \to (\forall x) \neg\neg\, p(x).$

Applying the formulas of the propositional calculus 7.1.2. (3) and (5) to (6) of this section we obtain

(8) $\vdash \neg\neg (\forall x) \neg\neg p(x) \to \neg (\exists x) \neg p(x)$.

This can be weakened to

(9) $\vdash \neg\neg (\forall x) p(x) \to \neg (\exists x) \neg p(x)$.

(9) and (7) yield the important result

(10) $\vdash \neg\neg (\forall x) p(x) \to (\forall x) \neg\neg p(x)$.

It is one of the most striking features of intuitionistic logic that the inverse implication does not hold, especially because the formula of the propositional calculus which results if we restrict x to a finite set, is true. In fact, if x takes only two values we obtain 7.1.2, (11). Brouwer [L. E. J. Brouwer 1924, p. 256; A. Heyting 1930A, p. 65; S. C. Kleene 1952, p. 491] gave the following counter-example:

A spread M is defined by the spreadlaw Λ_M and the complementary law Γ_M.

Λ_M. The first component of an admissible sequence can be 0 or 1. If $a_1, \ldots, a_n$ is an admissible sequence and $a_n = 0$, then a_{n+1} may be 0 or 1; if $a_n = 1$, then $a_{n+1} = 1$.

Γ_M assigns a_n to the admissible sequence $a_1, \ldots, a_n$.

In plain words, M consists of all ipss in which only 0 and 1 can be components, while 1 can only be followed by 1.

The following law Ξ assigns numbers to certain elements of M.

Ξ: The sequence, all of whose components are 0, has number 1. The sequence, all of whose components are 1, has number 2. The sequence which consists of n components 0 followed by components 1 has number $n+2$.

Let x range over the elements of M and let $\mathfrak{p}(x)$ be "Ξ assigns a number to x". Then $(\forall x) \neg\neg p(x)$ is true. I prove the equivalent proposition, $\neg (\exists x) \neg \mathfrak{p}(x)$, by deducing a contradiction from the supposition that Ξ assigns no number to a certain element a of M. Under this supposition, the first component of a cannot be 1, for then $\Xi(a)$ would be 2, so the first component is 0. The second component cannot be 1, for then $\Xi(a)$ would be 3, so the second component is 0. Continuing in this way we see that every component of a is 0; it follows that $\Xi(a) = 1$, a result which contradicts the supposition.

Yet, $\neg\neg(\forall x)\mathfrak{p}(x)$ does not hold; we can even prove that $\neg(\forall x)\mathfrak{p}(x)$. Supposing that $(\forall x)\mathfrak{p}(x)$, the fan-theorem allows us to infer that $\Xi(x)$ must be known after a finite number N of components of x are given, but this clearly contradicts the definition of Ξ.

CLASS. This is the first instance of a classical theorem which is intuitionistically not only unprovable, but even false.

INT. Of course, this divergence is brought about by the introduction of ipss; especially, the generalizing quantifier has quite a different meaning if applied to ipss.

It has been conjectured [S. Kuroda 1951, p. 46] that the formula $(\forall x)\neg\neg p(x) \to \neg\neg(\forall x)p(x)$ is always true if x ranges over a denumerable infinite species. In all the counterexamples that have been given, x ranges over a species that is not denumerably infinite (this holds for M above, though M is denumerable from the classical point of view!), but no way of proving the conjecture presents itself at present.

From (3) and (4) we infer, applying 7.1.2,(2):

(11) $\quad \vdash \neg\neg(\exists x)p(x) \to \neg(\forall x)\neg p(x).$

(12) $\quad \vdash \neg(\forall x)\neg p(x) \to \neg\neg(\exists x)p(x).$

By substitution in (5):

(13) $\quad \vdash (\exists x)\neg\neg p(x) \to \neg(\forall x)\neg p(x).$

From (13) and (12):

(14) $\quad \vdash (\exists x)\neg\neg p(x) \to \neg\neg(\exists x)p(x).$

Here also the inverse implication is not valid, but the fact is less surprising than for (10), because the corresponding formula of the propositional calculus, which would be $\neg\neg(p \vee q) \to \neg\neg p \vee \neg\neg q$, is also not provable.

7.3. Applications

In some cases a negative theory can be developed parallel to a positive theory. In the latter every notion is defined by a positive definition; in the former some notions are introduced by negative definitions, involving double negations, but in such a way that

definitions of corresponding notions in the two theories are classically equivalent.

7.3.1. *Order relations in the continuum*

Definition 1. If S is a mathematical species and if $<$ is a predicate, defined on a subspecies of $S \times S$, which satisfies rules (1)–(4) below, then $<$ is a *partial order relation* in S and S is *partially ordered* by $<$. ($a>b$ means the same as $b<a$.)

(1) $a<b \to \neg\,(a>b) \wedge \neg\,(a=b)$.

(2) $a=b \wedge b<c \to a<c$.

(3) $a<b \wedge b=c \to a<c$.

(4) $a<b \wedge b<c \to a<c$.

Definition 2. A partial order relation in a species S is an *order relation* in S and S is *ordered* by it, if it satisfies rule (5a).

(5a) $a=b \vee a<b \vee a>b$.

The relation $<$ between natural numbers or rational numbers is an order relation. The relation $<$ between real numbers, as defined in 2.2.6. in connection with 3.3, is not an order relation; this follows immediately from 3.4.3, Th. 2.

Instead of (5a), it satisfies (5b) and (6), which are both consequences of (1)–(4), (5a).

(5b) $\neg\,(a>b) \wedge \neg\,(a<b) \to a=b$ (Compare 2.2.6, Th. 3).

(6) $a<b \to (\forall c)(a<c \vee c<b)$ (Compare 2.2.6, Th. 4).

Definition 3. A partial order relation in a species S is a *pseudo-order relation* and S is *pseudo-ordered* by it, if it satisfies rules (5b) and (6). (Note that (2) and (3) follow from (1) and (6).)

In the following $<$ always denotes a pseudo-order relation in a species S.

Definition 4. $a \lesssim b$ means $\neg\,(a>b) \wedge \neg\,(a=b)$.

Theorem 1. $\neg\,(a \lesssim b)$ is equivalent to $\neg\,(a<b)$.

Proof. By (1), $a<b \to a \lesssim b$, so $\neg\,(a \lesssim b) \to \neg\,(a<b)$. (7.1.2, (2)). To prove the converse, I remark that $a \lesssim b \wedge \neg\,(a<b)$ means the same as $\neg\,(a>b) \wedge \neg\,(a<b) \wedge \neg\,(a=b)$, which contradicts (5b). So $a \lesssim b \wedge \neg\,(a<b)$ is impossible; consequently $\neg\,(a<b) \to \neg\,(a \lesssim b$.

Theorem 2. $a \lesssim b$ is equivalent to $\neg\neg\,(a<b)$.

Proof. By def. 4 and 7.1.2. (6), (7), $a \lessdot b$ is equivalent to $\neg\,(a>b \vee a=b)$, so
$\neg\neg\,(a \lessdot b) \to a \lessdot b$ (7.1.2, (5)).
$\neg\,(a \lessdot b) \to \neg\,(a<b)$ by Th. 1, so
$\neg\neg\,(a<b) \to \neg\neg\,(a \lessdot b)$ (7.1.2, (2)) $\to a \lessdot b$.
$\neg\,(a<b) \to \neg\,(a \lessdot b)$ by Th. 1, so
$a \lessdot b \to \neg\neg\,(a<b)$ (7.1.2, (1), (2)).

Theorem 3. The relation $\lessdot$ satisfies (1)–(4), (5b).

Proof. (1) follows from def. 4 and Th. 1.
(2). From (2) for $<$, by 7.1.2, (3) and Th. 2, as follows:

$$a=b \wedge \neg\neg\,(b<c) \to \neg\neg\,(a<c) \to a \lessdot c.$$

(3). Analogously.
(4). From (4) for $<$ by 7.1.2, (11), (3) and Th. 2.
(5b) follows from Th. 1 and (5b) for $<$.

In the case of the continuum $\lessdot$ does not satisfy (6) (see 8.1.1). Instead, it satisfies (5c):

(5c) $\neg\,(a \lessdot b) \wedge \neg\,(a=b) \to a \gtrdot b$.

Theorem 4. If $<$ is a pseudo-order relation in a species S, the relation $\lessdot$ which corresponds to it by def. 4, satisfies (5c).

Proof. Immediately from Th. 1 and def. 4.

Definition 5. A partial order relation in a species S which satisfies (1)–(4), (5b), (5c), is a *virtual order relation* in S and S is *virtually ordered* by it [L. E. J. Brouwer 1925A, p. 453].

Theorems 3 and 4 can now be expressed as follows. If $<$ is a pseudo-order relation in a species S. then $\lessdot$, as defined by def. 4, is a virtual order relation in S.

Remark on notation. My notation is different from Brouwer's. Here follows a small dictionary.

This book:	$\lessdot$	$\not<$	$\lessdot$
Brouwer :	$<_0$	$>$	$<$

The reason for changing the notation was, that in analysis the pseudo-order relation between real numbers is the most important; therefore, it should be denoted by the simplest sign. Also in Brouwer's notation there is danger of confusion between $a>b$ and $(a>b \vee a=b)$. I use $a \geqq b$ in the latter sense.

According to def. 5, the rules for a virtual order relation are:

(1) $a \prec b \to \neg (a \succ b) \wedge \neg (a = b)$.

(2) $a = b \wedge b \prec c \to a \prec c$.

(3) $a \prec b \wedge b = c \to a \prec c$.

(4) $a \prec b \wedge b \prec c \to a \prec c$.

(5b) $\neg (a \prec b) \wedge \neg (a \succ b) \to a = b$.

(5c) $\neg (a \prec b) \wedge \neg (a = b) \to a \succ b$.

Theorem 5. If $\prec$ is a virtual order relation in a species S, then $\neg\neg (a \prec b) \to a \prec b$ and $\neg\neg (a = b) \to a = b$.

Proof. By (1) and (5c), $a \prec b$ is equivalent to $\neg (a \succ b) \wedge \neg (a = b)$, which, by 7.1.2, (6), (7), is equivalent to $\neg (a \succ b \vee a = b)$. Thus, by 7.1.2, (5), $\neg\neg (a \prec b) \to a \prec b$.

From (1) and 7.1.2, (2), (1) we infer that $a = b \to \neg (a \prec b)$. Similarly, $a = b \to \neg (a \succ b)$. Thus, using (5b) and 7.1.2, (6), (7), we see that $a = b$ is equivalent to $\neg (a \prec b \vee a \succ b)$. This proves that $\neg\neg (a = b) \to a = b$.

Definition 6. Let ${}^{\circ}\!\prec$ be a partial order relation in the species S, and let $\sum$ be the species of formulas $a = b$ or $a \overset{\circ}{\prec} b$, which are valid for elements a, b of S. $\overset{\circ}{\prec}$ is called *unextensible* if it has the following properties: Whenever $x, y \in S$ and $\neg (x \overset{\circ}{\prec} y)$ cannot be deduced from $\sum$ by applications of (1)–(4), then $x \overset{\circ}{\prec} y \in \sum$; similarly, whenever $x, y \in S$ and $\neg (x = y)$ cannot be deduced from $\sum$ by applications of (1)–(4), then $x = y \in \sum$.

Theorem 6. Every virtual order relation is unextensible [L. E. J. Brouwer 1927].

Proof. If $x \succ y \in \sum$, then $\neg (x \prec y)$ can be deduced from $\sum$ by (1), so if $\neg (x \prec y)$ cannot be deduced from $\sum$ by (1)–(4), then $\neg (x \succ y \in \sum)$, and similarly, $\neg (x = y \in \sum)$.
Thus, by (5c), $x \prec y \in \sum$.

The proof for $=$ is analogous.

Theorem 7. Every unextensible partial order relation is a virtual order relation.
For the proof, see Brouwer [L. E. J. Brouwer 1927].

7.3.2. *The negative theory of convergence*

2.4. Def. 1 can be formulated as follows:
The sequence $\{a_n\}$ of real numbers is positively convergent to the limit a if

(1) $\quad (\forall k)\,(\exists n)\,(\forall p)\,(|a-a_{n+p}|<2^{-k})$.

As the corresponding negative definition I choose:

Definition 1. The sequence $\{a_n\}$ is *negatively convergent* to the limit a ($^{-}\lim a_n=a$) if

(2) $\quad (\forall k)\,\neg\neg\,(\exists n)\,(\forall p)\,(|a-a_{n+p}|<2^{-k})$ [J. G. Dijkman 1952].

Cauchy's general criterion for convergence is

(3) $\quad (\forall k)\,(\exists n)\,(\forall p)\,(|a_{n+p}-a_n|<2^{-k})$.

Definition 2. The sequence $\{a_n\}$ is *non-oscillating*, if

(4) $\quad (\forall k)\,\neg\neg\,(\exists n)\,(\forall p)\,(|a_{n+p}-a_n|<2^{-k})$.

It is well-known that if a sequence $\{a_n\}$ satisfies (3), then a real number a can be found so that (1) is true. The corresponding relation does not hold between (4) and (2), as example (ii) shows.

Examples. (i) $a_n=1$ if the n-th digit after the decimal point in the decimal expansion of π is the 9 of the first sequence 0123456789 in this expansion (let us call this value of n, if it exists, the sequence-number); otherwise $a_n=0$. In order to show that this sequence satisfies (2) with $a=0$, I first remark that (2) is equivalent to

(5) $\quad \neg\,(\exists k)\,(\forall n)\,\neg\,(\forall_p)\,(|a-a_{n+p}|<2^{-k})$.

Suppose we had found k_0 such that

(α) $\quad (\forall n)\neg\,(\forall p)\,(|a_{n+p}|<2^{-k_0})$.

Suppose further that a sequence occurs in π and that s is the sequence-number; then

(β) $\quad (\forall p)\,(a_{s+p}=0)$, so $(\forall p)\,(|a_{s+p}|<2^{-k_0})$.

So (α) cannot be true. We have now proved:
If (α) is true, then no sequence 0123456789 can occur in π; but then $(\forall n)\,(a_n=0)$, and (α) is false. Thus, (α) has led to a contra-

diction, and (5) is proved. Yet it is clear that we cannot assert that the sequence $\{a_n\}$ is positively convergent.

(ii) $a_n = 0$ if no sequence 0123456789 occurs in the first n digits after the decimal point in π; $a_n = 1$ if such a sequence does occur in the first n digits. This sequence $\{a_n\}$ is non-oscillating; the proof is analogous to the preceding one. Yet we cannot assert that it is negatively convergent, for it is unknown whether a limit, if it exists, will be 0 or 1.

Theorem. Every negatively convergent sequence is non-oscillating.

Proof. Let us abbreviate

$(\exists n)\,(\forall p)\,(|a_{n+p} - a_n| < 2^{-k})$ by $\mathfrak{A}(k)$, and

$(\exists n)\,(\forall p)\,(|a_{n+p} - a\,| < 2^{-k})$ by $\mathfrak{B}(k, a)$. Then $\mathfrak{B}(k+1, a) \rightarrow \mathfrak{A}(k)$.

$\neg\neg\,\mathfrak{B}(k+1, a) \rightarrow \neg\neg\,\mathfrak{A}(k)$.

$(\forall k)\,\neg\neg\,\mathfrak{B}(k, a) \rightarrow (\forall k)\,\neg\neg\,\mathfrak{A}(k)$

$(\exists a)\,(\forall k)\,\neg\neg\,\mathfrak{B}(k, a) \rightarrow (\forall k)\,\neg\neg\,\mathfrak{A}(k)$.

This proves the theorem.

Theorem 2. If $^-\lim a_n = a$ and $^-\lim b_n = b$, then $^-\lim (a_n + b_n) = a + b$.

Proof. Set $a_n + b_n = c_n$, $a + b = c$.

Let us abbreviate

$(\exists n)\,(\forall p)\,(|a_{n+p} - a| < 2^{-k})$ by $\mathfrak{A}(k)$,

$(\exists n)\,(\forall p)\,(|b_{n+p} - b| < 2^{-k})$ by $\mathfrak{B}(k)$,

$(\exists n)\,(\forall p)\,(|c_{n+p} - c| < 2^{-k+1})$ by $\mathfrak{C}(k)$.

Then $\mathfrak{A}(k) \wedge \mathfrak{B}(k) \rightarrow \mathfrak{C}(k)$, thus $\neg\neg\,\mathfrak{A}(k) \wedge \neg\neg\,\mathfrak{B}(k) \rightarrow \neg\neg\,\mathfrak{C}(k)$ (Application of 7.1.2, (11) and (3)).

$(\forall k)\,\neg\neg\,\mathfrak{A}(k) \wedge (\forall k)\,\neg\neg\,\mathfrak{B}(k) \rightarrow (\forall k)\,\neg\neg\,\mathfrak{C}(k)$.

This proves the theorem.

Theorem 3. If $^-\lim a_n = a$ and $^-\lim a_n = b$, then $a = b$.

Proof. From $^-\lim a_n = b$ it follows that $^-\lim -a_n = -b$. Thus by Th. 2, $^-\lim c_n = a - b$, where every $c_n = 0$.

$(\forall k) \neg\neg (\exists n) (\forall p) (|a-b| < 2^{-k})$.
$(\forall k) \neg\neg (|a-b| < 2^{-k})$.
$(\forall k) (|a-b| \not> 2^{-k})$.
$a = b$.

The following theorem has no parallel in the positive theory.

Theorem 4. Every bounded monotone sequence of real numbers is non-oscillating.

For the sequence $\{a_n\}$, we have by hypothesis,

(1) $(\forall n) (a_n < M)$,

(2) $(\forall n) (a_{n+1} \not< a_n)$.

It must be proved that

(X) $(\forall k) \neg\neg (\exists n) (\forall p) (a_{n+p} - a_n < 2^{-k})$,

which is equivalent to

(Y) $\neg (\exists k) (\forall n) \neg (\forall p) (a_{n+p} - a_n < 2^{-k})$.

Let us suppose that we have found a number k_1 such that

(5) $(\forall n) \neg (\forall p) (a_{n+p} - a_n < 2^{-k_1})$.

It suffices to deduce a contradiction from (1), (2) and (5). Now from $\neg (\exists p) (a_{n+p} - a_n > 2^{-k_1-1})$, it follows that $(\forall p) (a_{n+p} - a_n < 2^{-k_1})$, so from (5) it follows that

(6) $(\forall n) \neg\neg (\exists p) (a_{n+p} - a_n > 2^{-k_1-1})$.

The theorem will be proved if we deduce a contradiction from (1), (2) and (6).

As a special case of (6) we have

(7) $\neg\neg (\exists p) (a_p - a_1 > 2^{-k_1-1})$.

Let us suppose that we have found a number p_1 such that $a_{p_1} - a_1 > 2^{-k_1-1}$; then, by (6), $\neg\neg (\exists r) (a_r - a_{p_1} > 2^{-k_1-1})$, and further $\neg\neg (\exists r) (a_r - a_1 > 2.2^{-k_1-1})$.

We have now proved from (1), (2) and (6),

(8) $(\exists p) (a_p - a_1 > 2^{-k_1-1}) \to \neg\neg (\exists r) (a_r - a_1 > 2.2^{-k_1-1})$.

Using the propositional calculus (7.2.1, (3), (4), (2)), we infer from

(7) and (8) that

(9) $\neg\neg\,(\exists r)\,(a_r - a_1 > 2.2^{-k_1-1})$.

By repeating $h-1$ times this reasoning, we find

(10) $\neg\neg\,(\exists s)\,(a_s - a_1 > h.2^{-k_1-1})$.

Now take $h = \left[\frac{M-a_1}{2^{-k_1-1}}\right]+1$; we find

(11) $\neg\neg\,(\exists s)\,(a_s > M)$.

This contradicts (1).

Remark. It is easily seen that instead of (1), the hypothesis $\neg\neg\,(\exists M)\,(\forall n)\,(a_n < M)$ is sufficient.

7.3.3. *Negative interpretation of classical analysis*

It was first proved by Gödel [K. Gödel 1932] that the classical propositional calculus and classical arithmetic can be developed as parts of the corresponding intuitionistic systems. In order to establish such theorems it is necessary first to formalize the relevant part of intuitionistic mathematics. The remarks which I made concerning the intuitionistic calculus of propositions apply to every formal system which is constructed with the intention of representing an intuitionistic mathematical theory: it can never be proved rigorously that such a system is adequate. However, formal systems have been constructed for the propositional calculus, predicate calculus and elementary arithmetic, such that every provable formula of one of these systems, if interpreted in the right way, expresses a theorem of intuitionistic mathematics. These systems are described in detail by Kleene in his book [S. C. Kleene 1952, p. 492]; he gives them in such a way that, by adjoining to each of them the axiom $\neg\neg\,p \to p$, we obtain the corresponding classical systems.

Kleene proves various extensions of Gödel's theorem, of which I mention the theorems 1, 2 and 3 below.

Definition 1. $p+q$ means $\neg\,(\neg\,p \wedge \neg\,q)$.
$p \supset q$ means $\neg\,(p \wedge \neg\,q)$.
$(\forall x)p(x)$ means $\neg\,(\forall x)\neg\,p(x)$.

By 7.3.1. (3) and (4), the latter is equivalent to $\neg\neg\,(\exists x)p(x)$.

Theorem 1. For the propositional calculus with $\wedge$ as conjunction, $+$ as disjunction, $\supset$ as implication and $\neg$ as negation, every classically provable formula is true intuitionistically.

Theorem 2. For the number-theoretic formal system (elementary arithmetic) with $\wedge$ as conjunction, $+$ as disjunction, $\supset$ or $\rightarrow$ as implication, $\neg$ as negation, and $(\vee\ )$ as existential quantifier, every classically provable formula is true intuitionistically.

Theorem 3. For the predicate calculus, with $\wedge$ as conjunction, $+$ as disjunction, $\supset$ or $\rightarrow$ as implication, $\neg$ as negation, and $(\vee\ )$ as existential quantifier, every classically provable formula becomes true intuitionistically after every prime formula $\mathfrak{p}$ which occurs in it, has been replaced by $\neg\neg\mathfrak{p}$.

Remark. The transformation of $\mathfrak{p}$ into $\neg\neg\mathfrak{p}$ is unnecessary in the propositional calculus, because $\neg\neg\mathfrak{p}\supset\mathfrak{p}$, that is $\neg(\neg\neg\mathfrak{p}\wedge\neg\mathfrak{p})$, is true, and in the number-theoretic system, because the prime formulas have the form $a=b$ or $a>b$, where a and b are natural numbers, so that $\neg\neg(a=b)\rightarrow a=b$ and $\neg\neg(a>b)\rightarrow a>b$ are true.

These theorems contain consistency proofs for the classical systems relatively to the corresponding intuitionistic systems.

I shall now apply these theorems to the arithmetic of real numbers. A number-generator was defined as a Cauchy sequence of rational numbers. For the application of Th. 2, in the definition of a Cauchy sequence, $(\forall x)$ must be substituted for $(\exists x)$, so that we obtain the definition of a non-oscillating sequence.

Definition 2. A *weak real number-generator* is a non-oscillating sequence of rational numbers.

The definition of coincidence for real number-generators was as follows: If $a=\{a_n\}$ and $b=\{b_n\}$, then $a=b$ means that

$$(\forall k)\,(\exists n)\,(\forall p)\,(|a_{n+p}-b_{n+p}|<2^{-k}).$$

Definition 3. The weak number-generators $a=\{a_n\}$ and $b=\{b_n\}$ *coincide in the weak sense* $(a\doteq b)$ if

$$(\forall k)\,(\forall n)\,(\forall p)\,(|a_{n+p}-b_{n+p}|<2^{-k}).$$

Definition 4. A *weak real number* is the species of the weak

number-generators which coincide in the weak sense with a given weak number-generator.

The definition of the order relation between real number-generators was as follows: $a > b$ means that

$$(\exists k)\,(\exists n)\,(\forall p)\,(a_{n+p} - b_{n+p} > 2^{-k}).$$

Definition 5. For weak number-generators $a = \{a_n\}$ and $b = \{b_n\}$, *a is weakly greater than b* ($a \succ b$) means that

$$(\forall k)\,(\forall n)\,(\forall p)\,(a_{n+p} - b_{n+p} > 2^{-k}).$$

This is equivalent to the double negation of $a > b$, for from 7.2.2, (11) and (12) we infer easily that $\neg\neg\,(\exists x)p(x)$ is equivalent to $\neg\neg\,(\exists x)\,\neg\neg\,p(x)$.

As a corollary of Th. 2 we have

Theorem 4. Let S be the system which is obtained from the number-theoretic formal system by adjoining to it free variables for real numbers and the relations $=$ and $\succ$ for real numbers. Let T be a theorem of classical arithmetic which can be formalized in S. Let T' be the theorem which is obtained from T by replacing $\vee$ by $+$, $(\exists x)$ by $(\forall x)$, the variables for real numbers by variables for weak real numbers, $=$ by $\doteq$ and $>$ by $\succ$, then T' is true intuitionistically.

In order to extend theorem 4 to analysis, it will be necessary to extend Th. 2 or Th. 3 to a predicate calculus of higher order or to some other calculus in which the main part of classical analysis can be formalized. Probably a consistency proof for analysis relatively to intuitionistic mathematics can be derived by this method.

VIII

CONTROVERSIAL SUBJECTS

8.1. Infinitely proceeding sequences, depending upon the solving of problems

8.1.1. *The method*

Since 1948 Brouwer has published a number of papers, many of them in Dutch [L. E. J. Brouwer 1948, p. 1246; 1948A; 1948B; 1949; 1949A; 1950; 1950A; 1951; 1952B; 1954B; 1954C], in which he gives counterexamples to classical theorems. All these examples are based on a principle which he indicated in his conference for the 10-th Int. Congress of Philosophy, Amsterdam 1948. I begin by illustrating it by means of the following definitions.

We shall say that a mathematical proposition $\mathfrak{p}$ has been *tested* if either $\neg\,\mathfrak{p}$ or $\neg\neg\,\mathfrak{p}$ has been proved. Let $\mathfrak{p}$ be a proposition which has not been tested (e.g. "a sequence 0123456789 occurs in the decimal expansion of π"). Now I define a real number-generator $a=\{a_n\}$ by an ips as follows. As long as $\mathfrak{p}$ has not been tested, I choose $a_n=2^{-n}$, but if $\mathfrak{p}$ is tested between the choice of a_m and that of a_{m+1}, then I choose $a_{m+q}=2^{-m}$ for every q.

SIGN. This does not sound like a mathematical definition. Can a sequence of rational numbers be considered as mathematically well-defined if its components depend upon material facts, such as the existence at a given moment of a proof for a certain proposition?

INT. I agree to this objection; and, indeed, I doubt whether it is advisable to adopt such definitions as mathematical. As I stressed before, no verbal definition can be perfectly unambiguous; we see now that the definition of an ips left us some free play. In such a case we may decide freely which interpretation we adopt.

FORM. As van Dantzig has remarked [D. van Dantzig 1949], Brouwer's definition and the reasoning which he based on it can be fully justified from a formal point of view.

INT. Van Dantzig's methods, as he indicates, also throw some light on the subject as it is seen by the intuitionists. I shall try to characterize briefly his point of view.

Let ω_i $(i=1, 2, \ldots)$ denote for every i a finite set of mathematical deductions. The sequence $\{\omega_i\}$ will be denoted by Ω.

Set $\sigma_n = \bigcup_{i=1}^{n} \omega_i$. Let $\mathfrak{p}$ be a mathematical proposition. The real number-generator a depends upon Ω; $a(\Omega) = \{a_n(\Omega)\}$. If σ_n contains no deduction of either $\neg\, \mathfrak{p}$ or $\neg\neg \mathfrak{p}$, then $a_n(\Omega) = 2^{-n}$. If σ_n contains a deduction of $\neg\, \mathfrak{p}$ or of $\neg\neg \mathfrak{p}$, and if m is the least number such that σ_m contains a deduction of $\neg\, \mathfrak{p}$ or of $\neg\neg \mathfrak{p}$, then $a_n(\Omega) = 2^{-m}$. In this form the definition of a sounds more mathematical; the question remains whether it is adequate for Brouwer's purpose.

Brouwer wished to show that $a \neq 0$ while $a \not\# 0$ has not been proved (This is the example which I promised in 2.2.3). For suppose $a = 0$; then neither $\neg\, \mathfrak{p}$ nor $\neg\neg \mathfrak{p}$ could ever be proved, so $\neg\neg \mathfrak{p}$ and $\neg\neg\neg \mathfrak{p}$ would both be true, a contradictory result. Hence $a \neq 0$. On the other hand, $a \not\# 0$ would mean that we could find a number q so that $a > 2^{-q}$; it follows that $\mathfrak{p}$ would be tested before the choice of a_q. We can only know this if $\mathfrak{p}$ has already been tested. Note that $a \not\# 0$ is not contradictory, for this would imply $a = 0$.

Evidently Brouwer supposes that he does not know beforehand which deductions will be made; if a law is passed throughout the world prohibiting the making of any mathematical deduction whatever, then the proof of $a \neq 0$ fails.

In van Dantzig's version the result is as follows. For a given n, the assertion, that $a = 0$ whatever $\omega_{n+1}, \omega_{n+2}, \ldots$ may be, is false. If $a \not\# 0$, then a number q is known such that a proof of either $\neg\, \mathfrak{p}$ or $\neg\neg \mathfrak{p}$ occurs in σ_q.

SIGN. That is to say, in van Dantzig's terminology we have no example of a real number a such that $a \neq 0$ while $a \not\# 0$ is unproved.

INT. That is so. Personally, I prefer van Dantzig's terminology, which avoids several equivocalities in Brouwer's words. Van Dantzig acutely analyses Brouwer's definition and introduces many refinements, for which I refer to his paper. In my opinion it is not very important whether we express the result in Brouwer's words or in those of van Dantzig, provided we understand what is meant by

it. Nor is it important whether we call it a mathematical result or not. In any case it shows that it would be foolish to seek a proof for the equivalence of the relations $\neq$ and $\#$ between real numbers.

CLASS. I have been convinced of that since 2.2.3.

INT. Brouwer's example analyses one of the subconscious reasons which made you feel this conviction.

Brouwer's example can also be used to demonstrate that the virtual order of the continuum is not a pseudo-order (see 7.3.1). Let me repeat the example. Let $\mathfrak{p}$ be a proposition which has not been tested. I define a real number-generator $a=\{a_n\}$ by an ips as follows. As long as $\mathfrak{p}$ has not been tested, I choose $a_n=2^{-n}$, but if $\mathfrak{p}$ is tested between the choice of a_m and that of a_{m+1}, then I choose $a_{m+q}=2^{-m}$ for every q. Now I define $c=\{c_n\}$ as follows. As long as $\mathfrak{p}$ has not been tested, I choose $c_n=2^{-n}$, but if $\mathfrak{p}$ is tested between the choice of a_m and that of a_{m+1}, *and m is even*, then I choose $c_{m+q}=2^{-m}$ for every q, but if m is odd, then I choose $c_r=2^{-r}$ even for $r>m$.

If $c\#0$, then $c=a$; hence, if $a>c$, then $c=0$. If $c>0$, then $c\neq 0$. Hence, $a>c$ or $c>0$ would imply that $c=0$ or $c\neq 0$ and therefore either that $\mathfrak{p}$ will not be tested after an even number of choices for a, or that it is impossible that $\mathfrak{p}$ will not be tested after an even number of choices for a. But as long as $\mathfrak{p}$ has not been tested, we can know nothing about the number of choices for a after which it will perhaps be tested. Although $a>0$, we cannot assert that $a>c$ or $c>0$, so that rule (6) of 7.3.1 is not fulfilled.

8.1.2. *Contradictoriness of classical real number arithmetic*

From here on I shall use Brouwer's terminology, which has now been sufficiently clarified to enable you to substitute another one if you prefer to do so.

The theorems of this section are stronger than those of the preceding one, in this respect, that they express the contradictority of some classical results.

Theorem 1. It is contradictory that for every real number a, $a\neq 0$ would imply $a\#0$.

Proof. Let J be a finitary spread which coincides with the interval [0, 1] (3.4.1, Th. 1). We define simultaneously an element

f of J and a real number-generator $a(f)=\{a_n(f)\}$ as follows. If $f \in J$, let $\mathfrak{p}(f)$ be the proposition "f is rational". For every n, f_n is chosen between the choice of $a_n(f)$ and that of $a_{n+1}(f)$. As long as $\mathfrak{p}(f)$ has not been tested, we choose $a_n(f)=2^{-n}$. If $\mathfrak{p}(f)$ is tested between the choice of a_m and that of a_{m+1}, we choose $a_{m+q}=2^{-m}$ for every q. As in the preceding example, $a(f) \neq 0$ for every f. Let us suppose that $a(f) \# 0$ for every f; then for every f a natural number $r(f)$ is known, so that $\mathfrak{p}(f)$ is tested before the choice of $a_{r+1}(f)$, that is before the choice of f_{r+1}. By the fan-theorem, a maximum s for $r(f)$ can be found, consequently $\mathfrak{p}(f)$ is tested for every f before the choice of f_{s+1}. Now let f be an ips which after the choice of f_s is subject to no other restrictions than those contained in the definition of J; we are free to continue f in such a way that either a rational or an irrational number results. Hence, it is contradictory that $\mathfrak{p}(f)$ is tested before the choice of f_{s+1}.

Thus $(\forall f)\,(a(f) \neq 0)$ and $\neg\,(\forall f)\,(a(f) \# 0)$ are both true, where f ranges over J.

FORM. Evidently, in this formal notation the quantifiers must be understood in an enlarged sense, corresponding with Brouwer's enlarged notion of an ips.

INT. The proof shows also that it is contradictory that, for every real number a, $a \neq 0$ and $a \not< 0$ would imply $a > 0$, or, in the terminology of 7.3.1, that $a \gtrdot 0$ would imply $a > 0$.

In order to prove the contradictoriness of classical elementary arithmetic, Brouwer proves the following extension of Th. 1.

Theorem 2. It is contradictory, that for every real number a, $a \neq 0$ would imply $a \gtrdot 0 \vee a \not< 0$.

Proof. As before, let J be a finitary spread coinciding with $[0, 1]$, f any element of J, and $\mathfrak{p}(f)$ the proposition "f is rational". We define f and the real number-generator $b(f)=\{b_n(f)\}$ as follows. For every n, f_n is chosen between the choice of $b_n(f)$ and that of $b_{n+1}(f)$. As long as $\mathfrak{p}(f)$ has not been tested, we choose $b_n(f)=(-1)^n 2^{-n}$. If between the choice of $b_m(f)$ and that of $b_{m+1}(f)$ it is proved that f is irrational, we choose $b_{m+q}(f)=2^{-m}$ for every q. If between the choice of $b_m(f)$ and that of $b_{m+1}(f)$ it is proved that f cannot be irrational, we choose $b_{m+q}(f)=-2^{-m}$ for every q. As before, $b(f) \neq 0$. Suppose that for every f in J, either $b(f) \gtrdot 0$ or $b(f) \not< 0$.

If f is irrational, then $b(f) > 0$, so if $b(f) \not> 0$, then f cannot be irrational.

If f cannot be irrational, then $b(f) < 0$, so if $b \not< 0$, then f is irrational. (Application of 7.1.2, (2) and (5)). Thus, [0, 1] would be split up into the subspecies of numbers which cannot be irrational and the subspecies of irrational numbers. This contradicts 3.4.3, Th. 2.

Corollary. It is contradictory that for every real number x either $x \not> 0$ or $x \not< 0$.

Theorem 3. It is contradictory that the equation $ax + b = 0$, where $a \neq 0$ and $b \neq 0$, has always a solution.

Proof. Let c be a real number, $c \neq 0$. Take $a = c + 2|c|$, $b = c - 2|c|$, then $a \neq 0$ and $b \neq 0$. Let x_1 be a solution of $ax + b = 0$. If $c > 0$, then $a = -3b$, so $b(3x_1 - 1) = 0$, so $x_1 = {}^1/_3$. If $c < 0$, then $b = -3a$, so $a(x_1 - 3) = 0$, so $x_1 = 3$. Thus, if $x_1 \neq {}^1/_3$, then $c \not> 0$, and if $x_1 \neq 3$, then $c \not< 0$. As either $x \neq {}^1/_3$ or $x \neq 3$, we have either $c \not> 0$ or $c \not< 0$. Hence, if $ax + b = 0$ had a solution for every $c \neq 0$, then $c \neq 0$ would imply $c \not> 0$ or $c \not< 0$, which is contradictory by Th. 2.

Corollary. In euclidean plane geometry it is contradictory that every two lines which can neither coincide nor be parallel, intersect. For if $a \neq 0$ and $b \neq 0$, and if d is any real number $\neq 0$, then $ax + dy = b$ is the equation of a line which does not coincide with the axis of x and is not parallel to this axis. If all these lines intersected the axis of x, $ax = b$ would always have a solution if $a \neq 0$ and $b \neq 0$.

8.1.3. *Example concerning the Bolzano–Weierstrass theorem*

One of the forms of this theorem, discussed in 3.4.4, ran as follows:

(C) Every bounded species of real numbers without a point of accumulation is bounded in number.

Let $\mathfrak{p}$ be a proposition which has not been tested. I form an ips $\{b_n\}$ of rational numbers as follows. As long as $\mathfrak{p}$ has not been tested, I choose $b_n = 2^{-n}$. If $\mathfrak{p}$ is tested between the choice of b_m and that of b_{m+1}, I choose $b_{m+p} = 2^{-m}$ for every p. Let S be the species of the components b_n of this sequence. Suppose c were a point of

accumulation of S. It is clear that $c \not< 0$. But also $c \not> 0$, for from $c > 0$ it would follow that S were finite. Thus $c = 0$, but this implies that $\mathfrak{p}$ can never be tested, which is contradictory. Consequently, S can have no point of accumulation. On the other hand, if S were bounded in number, a natural number m would be known such that p will be tested before the choice of b_{m+1}, which is not the case. So there is no hope of proving (C) intuitionistically [L. E. J. Brouwer 1952B].

8.2. Negationless mathematics

Serious objections against the use of negation in mathematics have been raised by Griss [H. Freudenthal 1936A; A. Heyting 1936; G. F. C. Griss 1946, p. 24 and p. 64; 1948; 1948A]. Though agreeing completely with Brouwer's basic ideas on the nature of mathematics, he contends that every mathematical notion has its origin in a mathematical construction, which can actually be carried out; if the construction is impossible, then the notion cannot be clear.

Brouwer admits such theorems as "a square circle cannot exist"; we can prove this theorem by deriving a contradiction from the supposition that we had constructed a square which were at the same time a circle. According to Griss, this supposition has no clear sense, because it can never be realized. In other words, if a square circle does not exist, how can we have a clear notion of what it would be if it existed? Therefore Griss rejects negation as a mathematical concept.

In many cases the proof of a negative theorem suggests a positive form of it. For instance, in the proof of "no square circle exists" we meet some assertion like the following:

"If S is a square and P any point, then we can find points Q and R on the boundary of S, such that $PQ \# PR$." For Griss, this positive assertion expresses the real content of the negative theorem. Of course in most cases one negative theorem admits several positive translations. Griss tried to rebuild intuitionnistic mathematics without negation and reached some remarkable results in this direction [G. F. C. Griss 1946A; 1950; 1951].

In the arithmetic of integers and rational numbers negation is not essentially used. Here $a \neq b$ is the same as $a > b \vee a < b$, which

contains no negation. The first negative notion which occurred in these lectures was that of inequality between real numbers, $a \neq b$. Griss cannot admit this notion as well defined; he uses instead the relation $a \not\# b$, which is defined positively. However, among the basic properties of the relation $\not\#$ is the following: $\neg\, a \not\# b \to a = b$. Instead of this property Griss uses the following.

Theorem. If a and b are real numbers such that every real number c which lies apart from a lies also apart from b, then $a = b$.

It is not easy to give a negationless proof for this theorem; Griss succeeds only by an application of the fan-theorem.

Let a and b satisfy the hypothesis of the theorem. We may suppose that a and b are defined by canonical number-generators, $a = \{a_n 2^{-n}\}$, $b = \{b_n 2^{-n}\}$, such that $|a - a_n 2^{-n}| < 2^{-n}$, $|b - b_n 2^{-n}| < 2^{-n}$. For a given value of n, either $b_n < a_n$, $b_n = a_n$ or $b_n > a_n$. First suppose $b_n > a_n$. Consider the interval $i = [(a_n - 1)2^{-n},\ (b_n - 1)2^{-n}]$, and construct the finitary spread S_i which coincides with i, as in 3.4.1, Th. 1. Let $c = \{c_n 2^{-n}\}$ be an element of S_i, then $c < b$, so, by hypothesis, $c \not\# a$. This means that

$$(\exists k)\ (\exists m)\ (\forall p)\ (|c_{m+p} - a_{m+p}|2^{-m-p} > 2^{-k}).$$

Then, a fortiori,

$$(1) \qquad (\exists k)\ (|c_k - a_k|2^{-k} > 2^{-k}).$$

Thus to every element c of S there is associated a natural number k so that (1) is satisfied. By the fan-theorem we can find r so that k is known after that the first r components of c have been chosen; thus k takes but a finite number of values and we can find $k_0 = \max_{c \in S_i} k$, so that

$$(2) \qquad |c_{k_0} - a_{k_0}|2^{-k_0} > 2^{-k_0}$$

is valid for every c in S_i, that is, for every latticepoint $c_{k_0} 2^{-k_0}$ of order k_0 in i.

$a_{k_0} 2^{-k_0} > (a_n - 1)2^{-n}$; moreover, by (2), $a_{k_0} 2^{-k_0}$ is different from every latticepoint of order k_0 in i ("different" for rational numbers is a positive notion!), so $a_{k_0} 2^{-k_0} > (b_n - 1)2^{-n}$. But also $a_{k_0} 2^{-k_0} < (a_n + 1)2^{-n}$, so $b_n < a_n + 2$. We have supposed that $b_n > a_n$, so $b_n = a_n + 1$. Similarly, if $b_n < a_n$, then $b_n = a_n - 1$. We have proved that $|b_n - a_n| \leqq 1$ for every n, so $b = a$.

In the theory of spreads and species the null species cannot be defined. A species is defined by a property of mathematical objects, as explained in 3.2.1, but such a property can only have a clear sense after we have constructed an object which satisfies it. As a consequence of this exigency, the intersection of two species is only definable if it contains at least one element. For instance, we can speak of the species of algebraic fields, and of the species of species of six elements, but not of their intersection. The relation of difference between species is defined by induction as follows. Two species are different if one of them contains an element which is different from every element of the other one.

The logic of negationless mathematics is difficult to formalize, for several reasons. First of all, there is no calculus of propositions, because only true propositions make sense. Furthermore, the restriction on the use of conjunction between propositional functions (species) must be taken into account. Attempts at a formalization have been made by Griss [G. F. C. Griss 1949; 1950A], Gilmore [P. C. G. Gilmore 1953], and Vredenduin [P. G. J. Vredenduin 1953] and Valpola [V. Valpola 1955].

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