MONOGRAPHS IN MODERN LOGIC

Edited by Geoffrey Keene

Modern Logic, like mathematics, is a highly-specialised subject with an extensive technical literature. But unlike mathematics it is a relative newcomer to the academic scene. While it is true that the initial stages of its development took place several hundred years ago, it made its debut only as recently as the beginning of the present century. Since then it has progressed so rapidly that few text-books on the subject can hope to be comprehensive. A compromise between breadth of coverage and depth of treatment seems inevitable. The present series is designed to meet this problem. Each monograph devotes separate attention to a particular branch of modern logic, the level of treatment being intermediate between the elementary and the advanced. In this way a wider coverage of this subject will, it is hoped, be [illegible] accessible to a larger number of readers.

ROUTLEDGE & KEGAN PAUL

MONOGRAPHS IN MODERN LOGIC

ROBERT BLANCHÉ

Axiomatics

7s 6d

Professor Blanché's book places the central ideas of the axiomatic method in their historical perspective —an approach which succeeds better than any other in bringing out their real significance. He writes in a clear informal style and in a manner well-calculated to sustain the reader's interest. What is so frequently regarded as an "impossibly technical" subject is here treated simply, readably, and without resort to symbolism. Yet the level of treatment is sufficiently detailed to interest the professional reader. Mr. Keene's translation fills a vital gap in the literature (in English) of this subject.

AXIOMATICS

BY

R. Blanché

Translated by

G. B. KEENE

LONDON: Routledge & Kegan Paul Ltd
NEW YORK: Dover Publications Inc

Translated from the French
L'AXIOMATIQUE

This translation published 1962
in Great Britain by
Routledge & Kegan Paul Ltd
Broadway House, 68–74 *Carter Lane*
*London, E.C.*4
and in the USA by
Dover Publications Inc.
180 *Varick Street*
New York, 10014

Second impression 1965

Printed in Great Britain

TRANSLATOR'S PREFACE

Prof. Blanché's book *L'Axiomatique*, first published in 1955, has no counterpart in the English language. The subject is one on which much has been written but always either sketchily in an introductory chapter to the relevant section of a logic textbook, or else at a technical level which presupposes full acquaintance with the basic ideas. In *L'Axiomatique*, on the other hand, the subject is discussed at a level intelligible to readers with no previous knowledge of it, while at the same time going far enough into it to interest the professional logician. This monograph presents the first three out of the five chapters of the text of the second edition (1959). The remaining two chapters which deal with the axiomatic method in science and the philosophical import of axiomatics, were omitted as not strictly relevant to the content of this series. The present fragment, therefore, cannot hope to give the English reader a fair impression of the carefully planned continuity of argument of the original work. These three chapters are, however, sufficiently self-contained to stand by themselves as an introductory text on axiomatics.

In many places very free renderings have been given of the original, on the grounds that the clarity of a logical text should not be sacrificed to the ideal of phrase-for-phrase equivalence in translation—especially when, as in logic, a terminological similarity between the two languages often masks the fact that the concepts involved are logically quite distinct. Similar considerations have led to the omission here and there of a word or phrase the English translation of which tends to blur rather than underline the point being made.

I am greatly indebted to Prof. Blanché for his valuable criticisms of the typescript.

G. B. KEENE

CONTENTS

Chapter One

THE DEFECTS OF EUCLID'S FORMALISM

1. Introduction. Classical geometry, as presented by Euclid in his *Elements*, has long been regarded as a paradigm of deductive theory which few have succeeded in emulating. New theoretical terms are never introduced without being defined; nor are any propositions asserted without being demonstrated, with the exception of a small number stated at the outset as basic principles. If demonstration is to be a finite procedure it must be based on some given initial propositions and, in the *Elements*, these have been selected as being of a sort which no sane person could doubt. Moreover, although all the assertions may be empirically true, there is no appeal to experience for their justification: geometry proceeds only by way of demonstration, basing its proofs only on what has been previously established and obeying the laws of logic alone. This ensures that every proposition is linked to certain other propositions from which it is deduced as a consequence, so that a close network is progressively established in which all the propositions are directly or indirectly related. The result is a system of which no part can be transposed or altered without affecting the whole. Thus 'the Greeks reasoned with the greatest possible justice in mathematics, and they have left to the human species models of the art of demonstration'.[1] With them, geometry ceased to be a selection of practical descriptions or at best empirical statements, and became a rational science. Hence the pedagogical role which has ever since been recognized as peculiar to it. In making it compulsory for schoolchildren, the aim is not so much to teach them truths but rather to discipline the mind, its practice being reputed to promote and develop the habit of rigorous reasoning. As L. Brunschvicg says: 'Euclid, for the numerous generations who have been brought up on his book, has been less, perhaps, a teacher of geometry than a teacher of logic.[2]' The expression *more geometrico* has in fact come to signify *more logico*.

However, it has become more and more apparent that although Euclidean geometry has been for so long the accepted model for a deductive theory, the logical apparatus on which it is based is not impeccable. Some of these imperfections soon came to light but it was only in the nineteenth century that the disparity between the traditional exposition and an ideal deductive theory, was recognized. One of the outstanding features of mathematics in this era is, indeed, a remarkable increase in the desire for logical rigour. Examined with this new severity deduction in classical geometry showed many faults. The axiomatic method in general, or axiomatics, is largely a result of the need to rectify these faults. Reflection on the nature of geometrical deduction, especially on its logical and formal character, led to its being separated entirely from all geometrical content, and to the possibility of its being applied quite generally to any deductive theory whatever. It is now common practice for a deductive theory to be pre-

[1] Leibniz, *Nouveaux essais*, IV, II, 13.
[2] *Les étapes de la philosophie mathématique*, Chap. VI, § 49.

sented in the form of an axiomatic system (sometimes known as an axiomatized theory). Such a system is totally different from the fanciful system of which Pascal dreamed, for superhuman minds, in which all the terms would be defined and all the propositions demonstrated. It is a system in which the undefined terms and the undemonstrated propositions are made completely explicit, the latter being put forward simply as hypotheses on the basis of which all the propositions of the system can be constructed according to fixed and completely explicit rules of logic.

Of course any method will seem groundless as long as the reasons for its adoption are ignored. In order to understand the function of axiomatics we must begin by considering the inadequacies which it is supposed to remedy (Chapter 1). But it would be misleading to suppose that it sprang into existence in perfect form. In actual fact, the needs of rigour which had brought it into being became in their turn more pressing through its subsequent use, and this has led to further developments in line with its original aim and purpose. Without attempting, in this book, to examine these transformations in all their historical detail, we can at least distinguish two main stages in its development: the first occurring at the turn of the century (Chapter 1), the second beginning about 1920 (Chapter 2).

2. The Postulates. The first thing which must have irritated the rigorously-minded readers of Euclid, is the inclusion of the postulates. Not that there was really any objection to the three postulates, given along with the definitions and axioms at the beginning of the *Elements*. These have a very general operational character and merely serve to permit constructions with ruler and compass. But, after having begun his chain of deductions Euclid repeatedly invokes, in the very course of a demonstration and as essential to it, a certain proposition which he requires us to admit, without being able to justify it otherwise than by an appeal to intuitive evidence. Thus, in order to demonstrate his twenty-ninth proposition we are required to admit that, from a point outside a straight line, there passes but a single line parallel to that straight line. The resemblance between the assertion that through some given point there passes at least one parallel (which is a theorem) and the assertion that only one such line passes through it (which is a postulate) makes the disparity between their justifications even more striking. It began to look as if the postulate of parallels were something foreign to the system, a mere device to fill a gap in the logical chain. In the eyes of geometers, it seemed to be an empirical theorem, the truth of which was not in question, but the demonstration of which remained to be discovered. Alexandrian, Arabic and modern thinkers made successive attempts to demonstrate it, but the alleged demonstration always turned out on analysis to be founded on some other, usually implicit, supposition: they had merely changed the postulate. Hence the path which led from this failure in direct demonstration to the idea of a demonstration by absurdity, and from a failure in demonstration by absurdity to a reversal in viewpoint and the setting-up of the first geometrics called non-Euclidean.

These new theories are of considerable epistemological importance. In particular they helped to bring about a change in the focus of attention of speculative geometers, from content to structure, from the material truth of isolated propositions towards the internal coherence of the system as a whole. Is the sum of the angles of a triangle equal to, less than, or greater than two right angles? Of the three possible cases the classical geometer would have replied that the first is true, the other two false. For the modern geometer, it is a matter of three distinct theorems which are mutually exclusive only within one and the same system, according as to whether the number of parallels is postulated to be equal to, greater than, or less than[1] one, and which may even occur together within a weaker and more general system, in which the number of possible parallels is left undecided. Whether experience in this world verifies one and only one of these three propositions is a matter solely of applied science, not of the pure (that is, formal) sciences.

The idea which originated with the difficulty over the parallels was naturally extended to cover all the postulates. Here we see two aspects of geometrical truth becoming separated, which were until then intimately interwoven. A geometrical theorem had always been thought of as at one and the same time a piece of information about things and a mental construction, a law of physics, and a part of a logical system, a truth of fact, and a truth of reason. From these paradoxical pairs, theoretical geometry nowadays entirely relinquishes the first, which is assigned to applied geometry. There remains, for the theorems, simply truth, separated and so to speak atomic: their truth is solely their integration into the system, and that is why theorems incompatible with one another can both be true, provided that they are related to different systems. As to the systems themselves, it is no longer a question

[1] The theorem, mentioned on the preceding page which establishes the existence of the parallel, presupposes that one can prolong a straight line indefinitely—a proposition which it is not self-contradictory to deny.

for them of truth or falsity, except in the logical sense of consistency or inconsistency. The principles which govern them are simple hypotheses, in the mathematical acceptation of this term: they are merely presupposed, not asserted; not open to doubt in the way the conjectures of the physicists are, but lying outside the realm of truth and falsity, like a decision or a convention. Mathematical truth thus acquires a new much more general meaning: as something which characterizes a vast implication in which the conjunction of all the principles constitutes the antecedent, and the conjunction of all the theorems the consequent.

From the traditional point of view, mathematical demonstration was categorical and apodeictic. According to this view, since the principles concerned were absolutely true, whatever proposition we may deduce from them must therefore also be true. Nowadays we say simply: if we postulate arbitrarily this or that set of principles, such-and-such consequences are derivable from them. The necessity is found only in the logical link which unites the propositions; it is entirely divorced from the propositions themselves. Mathematics has become an axiomatized system.

3. The Diagrams. In Euclid, the postulate of parallels is based on an explicit, but apparently exceptional, appeal to spatial intuition. In fact, far from being exceptional, this intuition is invoked throughout the demonstrations, and Poincaré could justly claim that this vast construction, in which the ancients could find no logical defect, was based at every point on intuition. In a sense nothing could be more manifest: the diagrams themselves make it sufficiently clear. But in the text it is not quite so apparent. For we are led to believe that the diagrams are there simply to supplement the reasoning without being indispensable to it, and somehow reduplicate the logical demonstrations by visual aids. Nothing could be further from the truth: omit the diagram, drawn or imagined, and the demonstration collapses. We need go no further than the first proposition of Euclid, which is concerned with the problem of constructing an equilateral triangle on a segment of a given line AB. We describe two circles of radius AB, one with A as centre, the other with B: the point of intersection M, whose distance from either A or B is that of the radius AB, will be the required third apex. But for anyone who cannot see or imagine the diagram, the demonstration is defective. What assurance is there that the two circles cut one another? The existence of the point M has been exhibited not demonstrated.

There has been a great deal of discussion as to whether the appeal to diagrams is essential to geometrical theory. If the demonstrations of classical geometry are taken as models, then it is clear that intuition (whether in the form of reflection on or of addition to to the diagrams) must be allowed. Kant, it will be remembered, insists on this point in the foundations of his *Critique*. Let a philosopher be given the concept of a triangle, he says, and however hard he may try to analyse it, by examining the more elementary concepts of Straight Line, Angle, the number 3, he will never be able to discover in them the property of having the sum of its angles equal to two right angles. But suppose the question is submitted to a geometer: he constructs the triangle, extends one of the sides, etc., and arrives at the result by a chain of reasoning which is continuously guided by intuition. Analogous theses have been maintained by Cournot, Goblet and, in a more refined form, by the intuitionist mathematicians of today. But there is an alternative conclusion which can be drawn: if appeal to intuition is regarded as a defect in an allegedly logical construction, then some attempt should be made to correct the classical methods of demonstration by substituting for intuition its intellectual equivalent. This is a matter of vital importance for the new geometries, where an intuitive representation of the spaces concerned can hardly be expected.

The use of diagrams is simply to make clear visually matters which the text, addressing itself solely to the intelligence, only hints at. Intuition is so powerful that it blinds us to their absence. For example, it is barely a century since it was noticed that Euclid nowhere states the following proposition, on which he nevertheless relies: 'If a straight line has two points in a plane it is entirely contained in that plane.' Many other propositions implicit in the classical exposition of geometry are open to similar criticism. Propositions of existence are a case in point. The possibility of an intuitively representable construction can certainly show that a given concept involves no contradiction, but that is a form of factual verification not a deductive proof. Much the same holds true for propositions concerning congruence which have to be assumed in geometry if various imaginary operations are allowed: for example, making a figure coincide with its own boundaries. The *Elements* state only a single proposition of this kind, and that is counted as an axiom. Nor must we overlook propositions stating topological properties, concerning in other words, order and continuity, independently of all consideration of angles and measurements.[1] Euclid and

[1] Consider a figure of some kind, traced on a sheet of rubber which can be buckled and stretched: the properties of the figure which remain invariant are the topological properties.

his successors up to the last century have uniformly passed over these properties in silence, as being obvious from the diagrams, but they nevertheless make use of them at each step. Such continual appeal to intuition is clearly incompatible with strict rigour. If we are to proceed rigorously, all properties which are presupposed should be stated in the explicit form of propositions: those to be demonstrated should be asserted as theorems, the rest as postulates.

4. The Axioms. Alongside the postulates it was customary, for a complete presentation of the principles of geometry, to set out the axioms, (Euclid's 'common notions') and the definitions. But is this arrangement justified from a strictly logical point of view?

The distinction between axioms and postulates was never examined very critically. Frequently, the two words themselves have been, and still are, used interchangeably: hence the very name, axiomatics, could very well be replaced by 'postulatics'. The editors of Euclid who inserted at the beginning of the *Elements* the properties which Euclid had postulated in the course of his demonstrations, listed them sometimes under the heading of 'required properties', and sometimes under the heading of 'common notions'. In so far as it is distinguishable from a postulate, an axiom involves, to begin with, the idea of intellectual self-evidence. While the postulate is a synthetic proposition, the contradictory of which though difficult to imagine, nevertheless remains conceivable, the axiom would be an analytic proposition, the denial of which is

For example, this one: of 4 points on a continuously open curve, if C is between A and D, and if B is between A and C, then B is between A and D.

absurd. Furthermore, it would function as a p formal principle, governing the steps in the reaso but not adding to their content in any way. Thes‹ ideas come together in the view, now widely acc‹ (but never justified by precise analysis) accordi‹ which axioms are simple instances of the laws of as applied to quantities.

Now the idea of self-evidence was becoming les‹ less popular with mathematicians. The feelin obviousness is unreliable and its domain varies ac‹ ing to the intellectual outlook of different people. ‹ decide to rest our case on it, the intuitively mi would certainly require us to reject many a de‹ stration as less evident for them than the the‹ which it is supposed to justify. Others, on the cont more demanding, would refuse to recognize such-such an axiom as unconditionally necessary. Thus tain Euclidean axioms have been subjected in mo mathematics to a kind of degradation: for example one which states that the whole is greater than the holds (in one sense) only for finite sets,[1] and could serve, as has been suggested, to define such set‹ this sense, it is no longer an analytic proposition ‹ convention which delimits a certain field, and to w the intellect is in no way committed. All the same age-old reliance on self-evidence is closely related t‹ ideal of a categorical mathematics in which whatev not demonstrated must nevertheless produce in s way or other its certificate of truth. Something like ideal is realized in the axiomatic method, based o‹ idea of logical coherence rather than on that of abs‹

[1] In the sense in which 'is greater than' is intended to ‹ 'has a greater power than', the axiom ceases in effect to hol infinite sets—where, nevertheless, the whole 'contains tog‹ with a remainder' the part.

truth. In giving priority to the demands of systematization we are committed to reducing the number of independent propositions to a minimum. This may involve attempting to demonstrate axioms, but not in the way Leibniz attempted it when he formulated this requirement. For if, no longer having an eye to their self-evidence, we discard the idea of turning them into propositions of identity, it will be simply a matter of providing an irreducible basis for the system, even if the principles from which we deduce the superfluous axioms seem intuitively less evident than those axioms.

Such considerations, admittedly, apply mainly where it is a question of the truth or falsity of propositions, and lose something of their force when applied to principles which are formal and regulative. But, on this point again, the classical theory lacks precision. The axioms are given an intermediate status, between the logical propositions and the geometrical propositions: regulative like the first, bearing upon quantity like the second. But if they should turn out to be derivable by applying the principles of logic to the basic concepts of mathematics, they ought to be so derived and then reclassified as no longer basic propositions of geometry but propositions of applied logic. If this cannot be done they will have been shown to be genuine postulates. Each of the axioms, therefore, is really either a postulate or else a non-geometrical proposition. They can no longer be regarded as principles of geometry to be stated at the beginning with the postulates.

5. **The Definitions.** Still less can we count the definitions among the basic principles. To do so would be a plain logical blunder. The point of taking certain propositions as basic or primitive is simply that we can-

not demonstrate everything. Now the same is true, *mutatis mutandis*, for definition. We define one term by other terms, the latter in their turn by others, and if we are to avoid an infinite regress we have to stop at some undefined term, just as demonstrations must rest on some undemonstrated proposition. These irreducible terms constitute, to use a comparison of Russell's, a sort of geometrical alphabet: being the ultimate elements out of which definitions are constructed, they enable us to spell out, or unpack, the defined terms, while remaining themselves indefinable. It is these indefinables which should be stated at the outset of a deductive theory, and not the definitions. The latter come in later, where a new, simpler term, is to be substituted for a constructed expression directly or by means of intermediate definitions, with the help of the primitive terms—exactly as demonstrations justify new propositions with the help of primitive propositions.[1]

Thus the initial definitions of Euclid are only apparent definitions. They are in fact simple empirical descriptions, comparable with those given in a dictionary intended simply to focus attention on the concept in question. They are, strictly speaking, *descriptions*. That is why they hardly serve the purpose that they were thought to serve, namely: to state the fundamental properties with which the system is concerned in such a way as to imply all the other properties, by means of propositions containing the defined term. Euclid defines a straight line as that which lies evenly with respect to its points; Heron substitutes for this the following definition, at first sight clearer: the shortest

[1] This functional analogy between definition and demonstration was indeed noticed by Pascal, in his fragment *De l'esprit géometrique.*

distance between two points. Leibniz remarks, reasonably enough, that the majority of the theorems concerning straight lines make no use of either of these two properties. In the first place, then, such definitions are superfluous. In the second place they conceal the fact that propositions stating the essential properties have been omitted, for example, the one which later editors of Euclid make explicit as: two straight lines do not enclose a space. This discrepancy between the properties stated in the pseudo-definition and the properties which are in fact used in the proofs, constitutes a grave logical defect since it leaves the identity of the property in doubt: how can we be sure that the straight line referred to in the theorems is a straight line in the sense which the definition is intended to authorize?

In this matter of definition, the classical expositions of geometry have too often misrepresented both false initial definitions and true subsequent ones as simple formulas rather than as formulas in which two statements of very different kinds are combined, an assertion and an appellation; and undoubtedly it is this confusion which is the origin of the view, for a long time widely accepted, that definitions are fruitful principles from which the theorems draw all their substance. Take the fifteenth definition of Euclid: the circle is a plane figure bounded by a line such that all straight lines joining it to a certain point within the figure are equal. It signifies two things: firstly, that it is possible to delimit a plane figure by a line such that . . . etc.; secondly, that such a figure shall be called a 'circle'. This second statement—the one for which it would clearly be more apposite to reserve the name 'definition', since the first is, strictly speaking, an assertion—is concerned only with language, and, strictly speaking, introduces no new content of any kind into geometrical science. It is a decision or a convention for abbreviational purposes, which can therefore be justified by its convenience, but which has nothing to do with truth. It does not follow that we can arbitrarily assert the correlative proposition: that is true or false and, hence, the source of truths or of subsequent contradictions. If we discard the implicit appeal to intuition as inadequate, it follows that we must either demonstrate it as a theorem or lay it down as a postulate.

The advantage of this insistence on the logical minutiae is even clearer in the case of definitions which bring together under one term a large number of heterogeneous properties: for then it is not enough that each separately be a possible property, they must be compossible. If their compatibility is not established, we are liable to commit what Saccheri denounced as the 'error of complex definition'; as in, for example, trying to define a regular polyhedron having hexagonal sides.

6. Demonstration and Definition. It is clear, then, that if a deductive theory is to satisfy the demands of logic, there must occur, at the outset, not the three sorts of 'principles' traditionally placed there: definitions, axioms and postulates, but undemonstrated propositions (which we may call axioms or postulates as we wish) and undefined terms: all subsequent operations consisting in the construction, on that basis, of new propositions justified by means of demonstrations, and of new terms introduced by means of definitions. Demonstration and definition are, therefore, two fundamental operations by which the deductive theory is developed. But what conditions must be satisfied by a good demonstration, or a good definition? Our answer will depend on the purpose we attach to these operations and, on this point again, the classical expositions of geometry frequently lack precision. They seem to envisage simultaneously two different things which are not necessarily compatible. Admittedly, the confusion here is due not so much to Euclid himself as to the customary pedagogical use to which his work has been put. But it stems also from the attempt by classical geometers to combine the factual truth of propositions with the formal truth of their logical relationships, empirical exactness with logical rigour.

If we give priority to factual truth, then demonstration and definition become simply a means of establishing it. The role of definition will be to clarify the meanings of the terms of which the propositions are composed, that of demonstration, the gaining acceptance for those propositions. From this point of view, definition and demonstration belong, properly speaking to rhetoric; their function, being a pedagogical or didactic one, is essentially psychological. From the other point of view they have, on the contrary, a purely logical function, namely, to interrelate all the terms and all the propositions as a systematic whole. Now it is clear, first of all, that the two demands of psychological efficiency and logical rigour, pull at times in opposite directions, and also that, as soon as the first is accepted the value of a demonstration or of a definition becomes relative, even doubly relative: a demonstration or a definition is no longer good or bad, it is only better or worse than another; and this quality in its turn varies with the reader or the hearer. Pedagogically, the good definition, the good demonstration, is the one which a student understands. It is fairly clear where this will lead us. For the child the true definition of an ellipse is not the one he learns by heart, but something like: *an elongated ring*; the good demonstration is not the one which he writes down in his exercise book, it is the diagram accompanying it. But, if a good demonstration means simply an argument which is effective, where are we to stop? There is a well-known story of a nobleman's tutor who, at the end of his tether, was determined nevertheless to get his theorem accepted, and exclaimed in exasperation: 'Sir, I give you my word of honour.'

It would seem then that even among the mathematicians themselves, the logical and the psychological have not always been clearly distinguished. If this were not so it would be difficult to understand why some of them share the astonishment which so many of Euclid's demonstrations arouse in the layman: why go to the trouble of convincing ourselves by a piece of abstruse reasoning, of things which we never doubted in the first place, or even of demonstrating what is more evident, by means of what is less evident? The Porte-Royal *Logic* counts among the 'defects commonly met with in the method of the geometers' that of 'proving that for which no proof is needed'. Some have even sought explanations and excuses as for instance Clairaut:[1] 'It is not surprising that Euclid goes to the trouble of demonstrating that two circles which cut one another do not have the same centre, that the sum of the sides of a triangle which is enclosed within another is smaller than the sum of the sides of the enclosing triangle. This geometer had to convince obstinate sophists who glory in rejecting the most evident truths; so that geometry must, like logic, rely on formal reasoning in order to rebut the quibblers.' And Clairaut adds: 'But the tables have been turned. All reasoning concerned with what common sense knows in advance, is today disregarded, and serves only to conceal the truth

[1] *Eléments de géometrie*, 1741: quoted by F. Gonseth, *La géometrie et le problème de l'espace*, t, II, p. 141.

and to weary the reader.' The philosopher Schopenhauer's fundamental conception of the role of demonstration is less indulgent and judges Euclid's method and his obsession for substituting reasoning for intuition, as frankly 'absurd': it is, he says, as if a man should cut off both legs so as to be able to walk on crutches.

Perhaps, however, the very fact that it seemed absurd should have led to the suspicion that Euclid's intentions had been misunderstood. Taking geometrical reasoning, as Pascal did, to be a model of the art of convincing, itself a part of the art of persuasion, does not entail that this is its primary and essential function. In fact it is generally accepted that many of Euclid's propositions were known before him, and it can hardly be doubted that they were admitted to be true by all the experts. But they had yet to be related to one another in a systematic way. This is apparently what Euclid wished to achieve, and is in any case what he actually succeeded in doing. And this is becoming more and more the avowed intention of the mathematician. Since the time of Clairaut, the tables have once again been turned. 'In the system of all true judgements,' Bolzano had already written, 'there is an objective connection which is independent of the fact that we are subjectively aware of it; in virtue of it certain judgements become the basis for others.'[1] Sifting out these objective connections, became hereafter the true end of demonstration in a deductive theory. Along with subjective certainty, the factual truth of the propositions is discarded, and mathematics becomes axiomatized. At the beginning of the nineteenth century this separation of the two concepts of knowledge and mathematical demonstration

[1] *Philosophie der mathematik*, 1810; quoted by J. Cavaillès: *Méthode axiomatique et formalisme*, pp. 46–7.

was emphasized, with perfect clarity, by a philosopher nowadays largely neglected, a victim of the discredit into which the Scottish school fell. 'Our reasonings,' remarks Dugald Stewart,[1] 'in mathematics, are directed . . . not to ascertain *truths* with respect to actual existences, but to trace the logical filiation of consequences which followed from an assumed *hypothesis*. If from this *hypothesis* we reason with correctness, nothing, it is manifest, can be wanting to complete the evidence of the result; as this result only asserts a necessary connection between the supposition and the conclusion. . . . The terms *true* and *false* cannot be applied to them; at least in the sense in which they are applicable to propositions relative to facts. . . . If we choose to call our propositions *true* in the one case, and *false* in the other, these epithets must be understood merely to refer to their connection with the *data*, and not to their correspondence with things actually existing, or with events which we expect to be realized in future.'

Just as views on the nature of demonstration vacillate between giving it a psychological role (determining assent) and giving it a logical role (organizing the propositions into a system), so also definition is sometimes claimed to be a matter of thought, sometimes a matter of language, and is most often alleged to be both at once. It covers, as the name suggests, the delimiting of the meaning of a concept, but also the establishing of a logical equivalence between a new term and a set of terms already introduced: it is mistakenly supposed that the same means can be used to achieve both ends. Hence the vacillation which is to be found even in quasi-contemporary mathematics. We may also recall the jibes of Poincaré about the definition of the number

[1] *Elements of the Philosophy of the Human Mind*, 2nd edition, Volume II, Edinburgh, 1816, pp. 157–9. Author's italics.

1 in the symbolized arithmetic of the school of Peano: 'A very fine definition,' he says ironically,[1] 'to give a meaning to the number 1, that no one could ever have heard expressed.'

One immediate advantage of the axiomatic method is that it avoids these confusions by distinguishing pure mathematics, the formal science, from applied mathematics, the science of fact; or more precisely, necessitates a clear choice between the two interpretations of one and the same mathematical theory, according as to whether our prime concern is with logical coherence or empirical truth.

[1] *Science et méthode*, p. 168.

Chapter Two

AXIOMATICS: THE FIRST STAGES

7. The Birth of Axiomatics. As long as geometry continued to be thought of as factually informative its formal presentation could be regarded as an intellectual luxury. The chains of argument being, from this point of view, a means of arriving at true propositions or of gaining acceptance for them by rhetorical argument *ex praecognitis et praeconcessis*, something short of logical perfection could be tolerated at those points which required the backing of intuition: the end was achieved and the certainty of the conclusion was unaffected. The case is rather different when, faced with more than one geometry, we are no longer concerned with the factual truth of the propositions, but with giving the geometry a logical foundation. The slightest defect is then sufficient to make the whole edifice collapse: to rely on intuition would be to violate the rules of the game.

Those who continue to insist on the factual truth of the propositions feel driven to it by another reason: the ever-growing distrust in spatial intuition. The entire history of geometry testifies to a constant tendency to restrict its domain and to intensify the demands of logic. But since the nineteenth century the movement has acquired, with the 'arithmetization of analysis', a fresh impetus to which the proliferation of geometries which exclude intuition, can only have contributed. Some astonishing divergences have thus become apparent between the fallacious suggestions of intuition and the unquestionable conclusions of demonstration. A proposition of which everyone was convinced turus out to be untenable, another which we would have unhesitatingly discarded, is shown nevertheless, to be susceptible of proof. To cite only two notable examples: it is not true that a tangent can always be drawn to a continuous curve (Weierstrauss), nor is it false that a curve (a line without width) can cover the entire surface of a sphere (Peano).

It was Pasch who, in 1882, attempted the first axiomatization of geometry. Even if his solution has all the defects of a classical empiricist approach, it at least presents the problem clearly: 'if geometry is to become a genuine deductive science, it is essential that the way in which inferences are made should be altogether independent of the *meaning* of the geometrical concepts, and also of the diagrams; all that need be considered are the relationships between the geometrical concepts, asserted by those propositions which play the role of definitions. In the course of deduction it is both advisable and useful to bear in mind the meaning of the geometrical concepts used, but this is *in no way essential;* in fact it is precisely when this becomes necessary that a gap occurs in the deduction and (when it is not possible to supply the deficiency by modifying the reasoning) we are forced to admit the inadequacy of the propositions invoked as the means of proof.'[1]

Here then are the fundamental conditions which a deductive presentation must satisfy if it is to be fully rigorous:

[1] H. Pasch. *Vorlesungen über neueer Geometrie*, 1882, p. 98.

1. Explicit enumeration of the primitive terms for subsequent use in definitions.
2. Explicit enumeration of the primitive propositions for subsequent use in demonstrations.
3. The relations between the primitive terms shall be purely logical relations, independent of any concrete meaning which may be given to the terms.
4. These relations alone shall occur in the demonstrations, and independently of the meaning of the terms so related (this precludes in particular, relying in any way on diagrams).

8. The Presuppositions of a System. The rules thus put forward by Pasch involve a sharp distinction between the terms or propositions *peculiar* to an axiomatic system, and those which are logically *prior*. In the case of geometry, for example, the strictly geometrical terms which occur in the primitive propositions, clearly cannot be combined to form propositions unless they are linked together by other words, having a logical function, such as: *the*, *and*, *all*, *not*, *is a*, *if . . . then*, etc. In the same way, no demonstrations can rely solely on the propositions of the system, for in order to make use of these in constructing demonstrations, we require rules of inference such as the rule of the transitivity of implication (if *a* implies *b*, and if *b* implies *c*, then *a* implies *c*). Some knowledge of logic—practical if not theoretical, is, therefore, presupposed here. Hence logic is said to be *prior* in relation to the axiomatized science.

Besides logic, a geometrical system normally presupposes arithmetic. In order to define a triangle we have to use the number 3; in order to demonstrate that the sum of its angles is two right angles, we have to admit the validity of arithmetical theorems concerning addition. Generally speaking, whatever knowledge the system relies on in this way, we call prior to the axiomatic system. It is to be noted that if an axiomatized science is presented as a purely formal system, the presuppositions required for this purpose are concepts understood in their full significance and theorems understood as materially true.

This usually unacknowledged reliance on principles which are logically prior, is opposed to the spirit of axiomatics, where everything must be made explicit and nothing presupposed. We could, of course, resolve the difficulty by enumerating at the outset of our axiomatic construction, those sciences which it presupposes. But this simple formality is really not sufficient to resolve the difficult problems which arise in this connection and which were in fact crucial for the subsequent development of axiomatics. One such problem is this: would it be possible, as the demand for logical rigour seems to suggest, to carry the axiomatization of science to the limit, from geometry to arithmetic, from arithmetic to logic, so as to absorb what is at each stage presupposed by (and hence external to) the axiomatization, and in this way, entirely eliminate any intuitive presuppositions? Or are we, after all, bound for technical reasons to make use of logic and even of arithmetic when the axiomatic method is applied to logic and arithmetic? It is difficult Poincaré points out, 'to formulate a phrase without making use of a name of a number, or at least either of the word *several* or of a word in the plural'.[1] The arithmetician and the logician *enumerates* his propositions and his theorems, he *counts* the number of his primitive concepts. What is

[1] H. Poincaré, *Science et Méthode*, p. 166.

true of arithmetical concepts holds even more clearly for logical concepts.

Furthermore, it is not always easy to delineate precisely the boundary between the concepts peculiar to a science and those logically prior to it. In geometry books we are told, for example,[1] that 'The straight line *a* passes through the point A'. The term 'passes through' seems to belong to the vocabulary of geometry; but since we can avoid the use of it by saying: 'The point A belongs to the straight line *a*,' and since membership of an individual in a class (a line being regarded as a class of points) is a logical concept, the term 'passes through' must here be counted as a logical term. We find, further on, the following two phrases: 'Given a point lying outside a plane, etc.,' and, 'Given a point lying outside a spherical surface, etc.' How are we to classify the expression 'lying outside'? In the first case, we simply say that the point does not belong to the plane; the expression is, in that context, a logical one. But in the second, something more is meant: not only that it does not belong to the surface of the sphere but, in addition, that it does not *fall within* the latter: the same term must therefore be regarded, here, as strictly geometrical.

It might seem reasonable, incidentally, to regard the distinct enumeration of the primitive terms of a system as superfluous, since these terms are precisely those to be found in the primitive propositions. In the earlier axiomatic systems, certainly, this precaution was not always taken.[2] But the difficulty which we sometimes

[1] The following examples are taken from Padoa, *La Logique Déductive*, Rev. de Met., et de Morale, Nov. 1911, pp. 830–1.

[2] This difference of treatment between terms and propositions is a result of that curious backwardness in the theory of terms, an example of which we have already met in the layman's

have in recognizing which terms in the propositions are peculiar to the theory, clearly makes a precise listing of them imperative.

9. Undefinables and Undemonstrables. Equivalent Systems. One of the most striking features which characterize the axiomatic presentation of a deductive theory is, as we have seen, that the undefinables and the undemonstrables of the theory are, at the outset, explicitly and exhaustively enumerated. This way of putting it, however, calls for some interpretative comment, if not for correction.

In the first place, it is not logically essential that the entire set of primitive terms and postulates be presented *en bloc* at the beginning before the definitions and demonstrations are introduced. For axiomatized theories above a certain level of complexity such a procedure would be likely to encumber the exposition, without any compensating logical advantage. In that case, it is often regarded as preferable to proceed by stages and to introduce new primitive terms progressively, either in isolation or in groups, together with the postulates corresponding to them, only as the need arises: providing, of course, that this is always done in an explicit manner. At the same time the introduction of terms which are not defined and of propositions

habit of counting definitions as principles. Padoa remarks, in this connection, that although we have for a long time made use of the technical word 'postulate' to signify undemonstrated propositions, no word has been invented for undefined terms: this latter expression being so little used that there has been no need to abbreviate it. [In English the word 'primitive' has been adopted for this purpose—Tr.] The word 'theorem' corresponding to which we have no word to signify defined terms, invites a similar comment.

which are not demonstrated, must always precede the introduction of terms and propositions derived by definition or demonstration, and that it is only in this relative sense that they are called primitive.

Just as the words 'primitive' and 'outset' are to be understood only in a relative sense, so also should the words 'undefinable' and 'undemonstrable' be understood, and for this reason we tend, as far as possible, to avoid them for fear of being misunderstood. A term is undefinable, a proposition undemonstrable, only within a system constructed in a particular way, and can always become the object of a definition or a demonstration if the basis of the system is suitably modified. We should always bear in mind the example of Euclidean geometry. It is by no means impossible to demonstrate the postulate of the parallels: instead of using the postulate to demonstrate that the sum of the angles of a triangle is two right angles, or that corresponding to any figure a similar figure, of any size whatever, can be constructed, or that through any point in the interior of an angle a straight line can be drawn which cuts both sides, we have only to reverse the procedure and we can demonstrate the uniqueness of the parallel by taking as postulate one or other of the latter propositions. In the same way it is a matter of choice which terms are taken as fundamental in the theory. But any alteration in the list of primitive terms entails a corresponding alteration in the postulates, since the latter state the relations which hold between these terms.[1]

[1] The Italian school has clarified the relativeness of the status *primitive* with regard to terms by analogy with what has already been recognized with regard to the propositions. It is to be noticed, however, that although the priority of a proposition or a term in relation to another is, logically speaking, arbitrary, this does not mean that they can be related in any order whatever.

Thus we have to be careful, in speaking of a deductive system, not to confuse two meanings of the word 'system': the totality of concepts and propositions primitive or derived, of which it is composed, and such-and-such a logical organization which it can be given. A system, understood in the first sense, lends itself to a variety of axiomatic presentations; it is, to use a comparison of Nicod's, comparable with a polyhedron, capable of standing on several different bases. These different systems, in the second sense of the word, are then called *equivalent*. Thus all the axiomatic reconstructions of Euclidean geometry are equivalent since they contain, basically, the same set of terms and propositions: the difference is simply in the way in which the latter are divided up into primitive and derived. More generally and also more precisely: two systems of propositions are equivalent if every proposition of the one can be demonstrated solely on the basis of the propositions of the other, and vice versa; two systems of terms are equivalent if every term of the one can be defined solely on the basis of the terms of the other and vice versa.

10. Definition by Postulates. The logical status of the postulates is clear: far from being asserted as truths which are productive of other truths, they are merely adopted as hypotheses for the purposes of deriving some given set of propositions, or in order to find out what consequences are implied by them. And we recognize that it is not in the least imperative that, in order to reason correctly from them, they should be true and be known to be so, the validity of reasoning being independent of the truth of its content. The position with regard to primitive terms seems less straight-forward. For if we can abstract from the truth of the propositions with which we are operating, can we in a similar way, completely abstract from the meaning of the terms? How can we say something even hypothetical, if they have been completely deprived of meaning? And how are we to agree upon a meaning if we are, on the one hand, unable to define them and, on the other, unwilling to allow them their original intuitive meaning? For unless we insist on ignoring their pre-axiomatic empirical meaning, we are only too liable to refer to it unwittingly in subsequent reasoning, and thereby to introduce in a surreptitious manner more or less vague implicit elements, of a subjective kind. There is but one answer: their meaning is determined by the use we make of them in the postulates, which state the logical relations holding between these concepts. This procedure for specifying the meaning of a term is not, properly speaking, a definition, it does not establish a logical equivalence between the new term and the known expression. But since it fulfills the function of a definition, which is to delimit the meaning of a term, we can regard it as an implicit definition.

This idea was introduced by Gergonne. He writes: 'If an expression contains a single word the meaning of which is unknown, the assertion of this expression is sufficient to make its meaning clear to us. If, for example, someone who is acquainted with the words "triangle" and "quadrilateral" but who has never heard of the word "diagonal", is told that each of the two diagonals of a quadrilateral divide it into two triangles, he will see immediately what a diagonal is, and he will see it the more easily since this is the only line by which a quadrilateral can be split up into triangles. These kinds of expressions which bring out, in this way, the sense of one of the words contained in them by virtue of the known meaning of the other words, could be called *implicit definitions*, by contrasting with ordinary definitions, which could be called *explicit definitions*.'[1] There is nothing out of the ordinary about such a procedure. The child learns the meaning of most of the words in his language in this way. It is normal practice in the physical sciences, that a law which has been established with the help of temporarily adopted concepts, helps in its turn to give these concepts a precise meaning. On this fact is founded the view of scientific nominalism, that laws are often only disguised definitions: the law of falling bodies defines *free* fall, the law of definite proportions typifies *combination* as opposed to intermixture, etc. Such indirect definitions are comparable to equations with one unknown, the value of which is fixed by the equation as a whole.

This method of determination is unambiguous when, as in the example given by Gergonne, a single value satisfies the equation. It is not always as simple as this. In particular, if we consider a system of equations with several unknowns, several systems of roots will satisfy the equations, or even an infinity of roots, as for instance when we take:

$$y=2x$$
$$z=y+x$$

In a sense, such a system is nevertheless determinate, for once we assign any arbitrary value to one of the unknowns, the values of the others are immediately fixed. Instead of being individual the determination is here, so to speak, aggregative, and takes on a more abstract character; in our example, y will always be the double of x, and z the triple. It is, clearly, not so much terms themselves as *relations* between terms, that are here

[1] Gergonne, *Essai sur la théorie des definitions*, 1818, pp. 22–3; quoted by F. Enriques, *L'évolution de la logique*, Fr. tr. p. 94.

exactly determined. The characterization of primitive terms by relations which the postulates enunciate as holding between them, presents us with an analogous situation. A system of postulates is like a system of equations with several unknowns, the unknown corresponding to the primitive terms of the axiom system in question: their value is not anything in particular, but is determined *implicitly*, *jointly* and with *systematic ambiguity*. This method of delimiting the meanings of terms is a case of implicit definition known as *definition by postulates*. This explains what Poincaré meant when, in speaking of the postulates of Euclidean geometry, he refers to them as disguised definitions: the totality of the Euclidean postulates constitutes, in effect, an implicit definition of the totality of the Euclidean primitive terms.[1]

We can now see more clearly that the postulates of a theory are not propositions which can be true or false, since they contain relatively indeterminate *variables*.

[1] The ambiguity of definitions by postulates—which, for axiomatic systems is quite the opposite of a defect—explains the *duality* which had previously been recognized in different scientific systems. Thus Gergonne expounded (1826) the first stages of projective geometry (without parallelism) by writing, in two columns (where the words *point* and *plane* are interchangeable in passing from the right-hand column to the left-hand one, without affecting the truth of the propositions) for example: *Two points determine a straight line, Two planes determine a straight line, Three points not in a straight line determine a plane, Three planes having no straight line in common, determine a point*, etc. A similar duality holds where the primitive terms of the theory, those of *point* and *straight line* (a series of points) continue to satisfy the postulates in which they occur, when given, respectively, the meanings *plane* and *collection of planes* (passing through a straight line): that is why every theorem valid for points and straight lines (which join them) are equally valid for planes and straight lines (which are the intersection of them) and *vice versa*.

Only when we give these variables particular values, or in other words, when we substitute constants for them, do the postulates become propositions, true or false, according to the constants chosen. But here we are getting outside axiomatics and into its applications. The postulates like the equations of a given system (and there could be no better comparison) are simply *propositional functions*: an expression for which we do not need an explicit definition since it has, in fact, been implicitly defined by the preceding sentences.

'Mathematics is a science in which we never know what we are talking about nor whether what we are saying is true': this well-known sally of Russell's which occurred to him in reflecting on axiomatized mathematics, holds equally well for axiomatics in general. Similarly, it is really towards axiomatics that this other sally of Poincaré's is directed: 'Mathematics is the art of giving the same name to different things.'

11. Two Examples of Axiomatics. Although it is not concerned with geometry and its author was primarily occupied with the problem of symbolization, we shall take as our first example of an axiomatized system that which Peano constructed for the theory of natural numbers: first because its brevity enables us to view it in its entirety, secondly because it serves as a simple and striking illustration of the nature of systematic ambiguity. It involves only three primitive terms: zero, number, successor of, and five primitive propositions which are translated from their symbolic notation into ordinary language below:

1. Zero is a number.
2. The successor of a number is a number.
3. Different numbers do not have the same successor.

4. Zero is not the successor of any number.
5. If a property belongs to zero and if, when it belongs to a given number it belongs also to the successor of that number, then it belongs to all numbers (Principle of Induction).

We can see how, with the help of the first two propositions the number *one* can be defined, then the number *two* and so on. On this basis, the elementary concepts and propositions of arithmetic can all be defined or demonstrated. However the normal interpretation of the primitive terms is not the only one which satisfies this set of axioms, since it does not determine unambiguously some one set of concrete propositions. As Russell points out if, for example, we give 'successor' its usual meaning, but understand by 'zero' any given number, say 100, and by 'number' each of the numbers starting with 100, the five axioms remain true together with, of course, all the theorems deducible from them. In the same way we could, by giving 'zero' its ordinary meaning, understand by 'number' pairs of numbers only, and by 'successor' the next but one after; or again, with 'zero' standing for the number 1 and with 'successor' meaning a half, 'number' would denote each of the terms in the series: 1, $\frac{1}{2}$, $\frac{1}{4}$, etc. All these interpretations and others similar to them which are easily conceived, assume a common formal structure which is made explicit in the above axiom set. What it characterizes is not, strictly speaking, arithmetic in any limited sense, but, quite generally, a certain structure —that of *progressions*. The series of natural numbers is only one illustration of the structure among many. Furthermore, these others are not, as the preceding examples might suggest, confined to sub-domains within the general domain of arithmetic: a progression can occur quite as well among entities other than numbers, such as points or instants.

As a second example, we shall sketch the axiomatization which Hilbert gave to Euclidean geometry.[1] Hilbert's interest is primarily in the propositions. He is not very much concerned with reducing the number of primitive terms to a minimum, and has in any case allowed them to be incorporated in the axioms without listing them separately in a systematic fashion.[2] But there are two features of his system which deserve some notice.

In the first place, he is not content with simply sifting out the axioms (several of which had until then occurred implicitly), and enumerating them: he divided them up in accordance with the fundamental concepts used in them, into five groups, and insisted on precisely delimiting the range of theorems governed by each of these groups or by combinations of them. Those of the first group establish a *relationship* between the concepts of point, straight line and plane: they are the axioms

[1] *Grundlagen der Geometrie*, 1899. In subsequent editions the author has made a number of minor alterations; the edition consulted was the 3rd edition, 1909. [An English translation is now available, published by Open Court, 1959—Tr.] It should be recalled, once and for all, that the term 'axiom' no longer carries with it the idea of self-evidence and of law, but simply that of a principle hypothetically adopted, or a postulate. The substitution of the former term for the latter has, in view of the word 'axiomatics', become almost inevitable.

[2] As early as 1882, Pasch succeeded in defining all the terms by means of four primitive terms (*point*, *segment*, *plane*, *is superimposible on*). Subsequently (1889, 1894) Peano, starting with these, succeeded in reducing the terms to three (*point*, *segment*, *movement*). Soon after, Pieri (1899) and Padoa (1900) reduced them to two (*point* and *movement*, *point* and *distance*, respectively). This kind of reduction is not carried through to any great extent by Hilbert. A little later (1904) Veblen published a reduced axiomatization of the same geometry.

characteristic of projective geometry (8 axioms, for example: Two points determine a straight line; On one straight line there are always at least two points, and: On one plane there are at least three points not on a straight line). Those of the second group, the axioms of *order*, fix the meaning of the word 'between': they are the topological axioms (4 axioms, for example: If A, B, C are points on a straight line and if B is between A and C, it is between C and A). The third group contains the 6 axioms of congruence or geometrical equality (for example: if A and B are points on a straight line *a*, and A′ is a point on a straight line *a*′, there exists on *a*′ and on a given side of A′, one and only one point B′ such that the segment A′B′ is congruent to the segment AB). The fourth group comprises only one axiom, that of the *parallels*. Finally, one last group deals with *continuity* and consists of two axioms, one of which, known as the axiom of Archimedes, says in effect that by repeated laying off of a segment along a line starting from a point A, it is always possible to pass beyond any given point B on the line.

Secondly, Hilbert inaugurated a type of research which was to become of the first importance for any work in axiomatics, namely, the systematic investigation of the non-contradiction of his axiom system, and the mutual independence of its components. To prove non-contradiction he constructed an arithmetical interpretation of the system in such a way that any contradiction which might occur among the consequences of the axioms, would be bound to show up in the interpretation: this, given the consistency of arithmetic, guaranteed the consistency of the axioms. The independence of an axiom, on the other hand, is proved by the possibility of constructing a consistent system without including it: the first non-Euclidean geometries had already testified to the independence of the parallel axiom; in a similar way, Hilbert proved the independence of the continuity axioms by constructing a non-Archimedean geometry.

12. Models. Isomorphism. A theory which remains at the pre-axiomatic stage is called *concrete*, *empirical*, or *intuitive*,[1] that is to say, it retains its connection with the knowledge which it organizes, and is not divorced either from meaning or from empirical truth. This is the case with ordinary geometry as traditionally taught in schools. Given a concrete deductive theory it is always possible, as we have seen, to reconstruct it on different bases: thus each of the various authors of elementary textbooks of geometry, while presenting the same body of knowledge, have, down the centuries, modified to a greater or lesser extent this Euclidean formulation. These differences of form, which are quite unimportant as long as the content of the theory is given priority over all else, take on a new significance as soon as this content is ignored. It can truly be said that their full importance was realized only as a result of the abstraction achieved through axiomatics. In this sense, we may contrast a given concrete theory with the plurality of axiom systems which correspond to it. The axiomatic system of Hilbert for example, is only one among all those to which Euclidean geometry lends itself.

Let us now consider one among the plurality of axiomatizations of some concrete theory. Since the meanings of the terms and consequently of all the pro-

[1] These terms are, of course, being used only in a relative sense to bring out the contrast with the more abstract, formal and logical character of the corresponding axiomatization.

positions is only partially limited by the postulates we can, if there are several systems of values which equally well satisfy the relations stated by the postulates, give them different concrete interpretations or, in other words, choose between different realizations. These concrete realizations of an axiomatic system are called *models*.[1] It goes without saying that the original concrete theory, the one which furnished the data of the logical structure outlined by the axiomatization, will be one of the models, but not the only one. An axiomatization thus lends itself, as we saw in connection with Peano's axiom system, to different realizations which can be taken from fields of study very far removed from the initially given domain. Thus, what we are now concerned with is a plurality of interpretations or concrete models of one and the same axiomatization.

When models are distinguishable from each other only by the differences in the concrete interpretations given to their terms, and exactly coincide when we abstract from the latter for the purposes of formal axiomatization, we call them *isomorphic*: they have in effect the same logical structure. The axiomatic method aims specifically at establishing isomorphisms between apparently heterogeneous concrete theories, thereby exhibiting the unity of the abstract system underlying them all. In fact, any one of these theories can, if we extend the usage of the word, serve as a model for the

[1] This word is not intended to suggest any kind of archetypal priority. It refers to the assimilation of these various interpretations to the concrete primitive theory which can itself properly be called a *model* of the axiomatic system which has been constructed from it. The idea of the 'mechanical models' of English physicists has also, perhaps, played an important part in the concrete meaning of this term.

others as much as for the corresponding abstract theory itself.[1]

There are, then, three different levels at which deductive theories can be differentiated. We should always bear in mind the example of Euclidean geometry. In the first place, if we modify one or more of its postulates in different ways, we obtain besides Euclidean geometry, other theories (Lobatchovskian geometry, non-Archimedean geometry, etc.) which are, as it were, *neighbours* or *relatives* of it: it is in this sense that we speak of the plurality of geometries. If we now take any one of these geometries, then, since there are several ways of presenting a logical reconstruction of it, we have a further proliferation in the form of several axiomatizations which are *equivalent* to each other. Finally, if we select any one of these axiomatizations we can generally find different interpretations for it: whence yet another diversification, according to whether or not the models are *isomorphic*. Thus over and above the diversity of geometries, we have alternative axiomatizations of a given geometry, and in addition, alternative models of a given axiomatization. The word 'theory' being just as appropriate for the axiomatization as for one of its concrete interpretations, we have, clearly, to guard against confusing a case of related theories, a case of equivalent theories and a case of isomorphic theories.

[1] An axiomatic system all the models of which are isomorphic with each other, is called *monomorphic* [or *categorical*—Tr.]. There are also *polymorphic* systems. It should be noticed that the different models of a system which is not *complete* (§ 15) are not all isomorphic, since non-completeness means precisely the possibility of at least two formally distinct models. But systems which are complete can also, paradoxically, possess non-isomorphic models. In other words, completeness is a necessary but not a sufficient condition of monomorphism [but monomorphism is a sufficient condition of completeness—Tr.].

13. Consistency and Completeness. Decidability. Although the choice of postulates for the basis of an axiomatic system is, in a sense, arbitrary they are not simply chosen at random: the choice is dependent on certain internal considerations of varying importance.

The most important of them is clearly coherence. If the various postulates of a system are not compatible with each other, the system is contradictory. Admittedly it is sometimes desirable, for theoretical purposes, to waive this requirement or even to construct a contradictory system deliberately—just as we sometimes argue by assuming what is absurd. But this is the exceptional case, and normally non-contradictoriness or, as it is called, *consistency*,[1] is an absolute requirement for an axiomatization. One property of a contradictory system is, in effect, that it permits the derivation in it of any proposition whatever: not only can any proposition in the system be derived, but also its negation. Such indeterminacy renders the system completely uninteresting.

Now how are we to determine that a system of postulates really is consistent? Intuition is not enough. On the other hand, if we have derived a long series of consequences without ever encountering a contradiction, it seems reasonable to allow a presumption of consistency; and where the axiomatization is that of a concrete theory well entrenched through centuries of development, even complete conviction: no one, for example, doubts the consistency of elementary arithmetic or of Euclidean geometry. Such a presumption, however (particularly in cases which have not even been put to the test), does not amount to absolute certainty

[1] A more detailed analysis enables us to distinguish between non-contradictoriness and consistency, between different concepts of consistency, etc.

there is nothing to safeguard us from the unexpected, nothing to assure us that beyond a certain stage of development we shall not come upon an absurdity. This is in fact just what happened, for example, in the case of the paradoxes of set theory. The practice of axiomatics, aimed originally at satisfying a long-felt need for logical rigour, itself makes that very need the more urgent, and leads to the replacement of the empirical kind of test, by a genuine demonstration. Such a demonstration can be effective in practice, it is true, only at the final stages of symbolic and formalized axiomatizations (Chapter 3); also, as we shall see, it can be successful only within very narrow limits.

In default of a demonstration properly so-called, there remain two ways of establishing the non-contradictoriness of a theory. Firstly, *reduction* to a prior theory. Here we postulate the non-contradiction of some system which is well established in practice, such as classical arithmetic or Euclidean geometry, and then construct an interpretation of the system under investigation so as to give it an application to (or to a part of) the prior theory: the non-contradiction postulated for the one is thereby transmitted to the other. Clearly, this kind of proof is only conditional, but if the testifying theory be suitably chosen, it is adequate for practical purposes. When Poincaré gave a Euclidean interpretation to Lobatchevskian geometry, doubts about the consistency of the latter were laid to rest. Euclidean geometry itself has been given an arithmetical interpretation by Hilbert, which adds to the already high probability of its own consistency. It is usual to take classical arithmetic as the testifying theory.

The second procedure consists in finding, for the theory in question, a *realization* in the world of things. Instead of relating the theory to some prior theory the consistency of which is better founded, we here, by contrast, descend towards the concrete, and construct a physical model of the theory. Since everything which exists is *a fortiori* possible, the existence of this model guarantees the consistency of the axiomatization to which it corresponds. Is it not ultimately the success of the empirical interpretation of classical geometry which forces us to admit, without further proof, the coherence of this geometry, and hence of the axiomatization which reflects the logical skeleton of it?

Of two contradictory propositions p and not-p the principle of contradiction tells us that they cannot both be true together: one at least is false. With this principle we are accustomed to associating that of the excluded middle, which states that the two propositions cannot both be false: one at least is true. The conjunction of these two principles gives us what we might call the principle of alternativity: of two such propositions, one is true, the other false. To the consistency of a system, founded on the principle of contradiction there corresponds its completeness, founded on the principle of excluded middle. A system of postulates is called *complete* when, of two contradictory propositions correctly formulated in the terms of the system, one at least, can always be demonstrated. If, in addition, a system is consistent we can see that, of every pair of propositions formed within the system by taking a proposition together with its negation, we can always demonstrate one and one alone. In other words, given any proposition of the system we can always demonstrate or refute it and, consequently, assert its truth or its falsity in relation to the set of postulates. We say of such a system that it is *strongly complete*.

There exists, in addition to this strong form of completeness which is a feature of only a very few systems, a weaker form of completeness: in the sense that, given any one of the expressions of the system, we can always, if not demonstrate or refute it, at least decide whether or not it is demonstrable or refutable. Such a system is then characterized as *decidable*.[1] Even this characteristic belongs only to a limited number of relatively simple systems.

Certainly non-completeness, and even more so non-decidability are imperfections, but not logical defects in the way in which inconsistency, for example, is a logical defect; and for this reason, the requirement of completeness is, normally speaking, regarded as much less urgent than that of consistency.

14. Independence. Economy. It is often desirable also, that the various postulates of a given system be *independent* of each other, that is to say such that we can modify any one of them without rendering the system contradictory. To assure ourselves of the independence of an axiom, we put it to the test by modifying it without altering the others and deriving the consequences of the new system: if the latter remains consistent, the independence of the postulate is thereby established. If, on the other hand, a contradiction should arise and if in addition, as is very often the case, the modification of the postulate consisted in replacing it by its negation, then the result is not a purely negative one; for the sequence of propositions thus obtained constitutes a demonstration by *reductio ad absurdum* of the original postulate. This shows the connection between an independence proof and a

[1] With the help of further distinctions it becomes clear that certain systems can be at once both complete but nevertheless undecidable.

demonstration by *reductio ad absurdum*: the failure of the one ensures the success of the other. So that the outcome of the vain attempt to demonstrate the postulate of the parallels by *reductio ad absurdum* was, quite unintentionally, the construction of the first of the non-Euclidean geometries, and hence the proof, in virtue of their consistency, of the postulate's independence. As we have seen, Hilbert proceeded in the same way, but this time deliberately, when he established the independence of the postulate known as Archimedes' postulate.

The independence of the postulates of a given system is not absolutely essential. It is simply that if this condition is not satisfied, there is an over-abundance of primitive propositions, and it is normally regarded as desirable in the interests of economy to reduce their number to a minimum. To say that two postulates are not independent is to say that one of them can be demonstrated either directly or by *reductio* from the other: in that case it would accord better with the spirit of the deductive method, to produce this demonstration and to list the proposition among the theorems.

Considerations of economy, although they are of an aesthetic rather than a logical kind, nevertheless play an important part in the construction of axiomatic systems. The ideal for such systems and for deductive theories in general, would surely be to reduce to a minimum the number of primitive terms and of primitive propositions. A great deal of energy has certainly been directed towards this end. But the simplicity gained at one stage is often gained only at the expense of increasing complexity at other stages; the dilemma can be resolved only on aesthetic or pedagogical grounds.

It is difficult to reduce the number both of primitive terms and of axioms, and at the same time reduce the length of the axioms: impoverishing the basis of the language generally results in protracting discourse in it. Besides, by simplifying a system we only succeed in making its concrete application more complicated if, as a result, no entity in the intended domain now corresponds directly to any primitive term of the system—except in so far as the use of the axiomatization gradually accustoms us to the meaning. Even apart from questions of interpretation, we can be led, for purposes of convenience of exposition, to sacrifice to some extent the ideal of maximum simplicity.

15. Systems Weak and Strong. Given a system of compatible and independent postulates we may, instead of modifying one of them, also try merely detaching it, without altering any of the others. The system would then be *weakened* since we should have eliminated certain derivations; at the same time we enlarge it by allowing certain possibilities which are precisely those which the omitted postulate serves to exclude. In other words, the system becomes impoverished in logical content but its range of consequences is enlarged. If, for example, we deny the uniqueness of the parallel while retaining intact the other Euclidean postulates, we obtain Lobatchevskian geometry which, though differing from that of Euclid, has the same logical characteristics. But if, on the other hand, we allow the number of possible parallels to be completely undetermined, that is to say if, instead of replacing the postulate concerning the parallels by another, we content ourselves with simply omitting it, leaving as it were a gap in the system, then we obtain the principles of a more general geometry of which the Euclidean and the Lobatchevskian appear as particular specializations.

We can proceed in the opposite direction: strengthening and limiting a given system by adding to it one or more postulates which are independent of the original ones. However, we usually find ourselves very soon faced with an obstacle: we reach the point at which the addition of any other independent postulate renders the system contradictory. The system is then *strongly complete*. This is the case, for example, with Euclidean geometry—provided, of course, we do not count among the additional postulates those which, without being explicitly formulated at the outset, have nevertheless been admitted in the demonstrations.

Chapter Three

FORMALIZED AXIOMATICS

16. Symbolization. By presenting a deductive theory in axiomatic form we aim to eliminate the concrete and intuitive meanings on which it was originally built so as to exhibit clearly its abstract logical structure. Now in this respect the earlier axiomatizations suffer from many defects, as we saw in the case of Hilbert. For example, we are asked to forget the meanings of the technical terms of the theory and regard points, lines and planes simply as 'things' satisfying the axioms. But the very fact that these terms are retained fosters rather than counteracts our natural inclination to put a certain specific interpretation on them. The temptation becomes almost irresistible when geometrical diagrams are freely used to illustrate the text. This renders us liable to commit precisely that mistake from which we ought to safeguard ourselves: of preserving a kind of flexibility of meaning for the terms explicitly governed by the postulates, as if they carried with them from the start a more or less indeterminate meaning, and then referring unwittingly to it, in the course of the demonstrations. Retaining the familiar terms, then, would be a fatal obstacle to our hopes of eliminating all intuitive content from the logical core of the theory.

It immediately becomes apparent that what is needed is the replacement of words which denote the fundamental theoretical concepts and which are still endowed with their intuitive meaning, by symbols entirely divested of meaning and therefore admirably suited to convey exactly, and no more than, the meaning conferred on them by the axioms. Instead of asserting that a point lies on a straight line, we use for example the letter J to denote the relation of incidence, the capital letters of the alphabet for points, the lower-case letters for straight lines, and write simply:

$$J(\mathrm{A}, a)$$

We can already see from this example that symbolization is not confined merely to the concepts peculiar to the theory—in our example geometrical concepts—but includes, in addition, the symbolism of relational logic. Admittedly, from a theoretical point of view, this is not absolutely necessary, since those theories which are presupposed by the given theory—in this case arithmetic and logic—are brought in for operational purposes and carry with them their usual significance. Nevertheless it would be very paradoxical if, in the very act of creating a symbolism for a theory which had not so far possessed one, we failed to take advantage of existing symbolisms, such as those which arithmetic has had for a long time and logic for a short time. It is in fact well known that since the middle of the nineteenth century logic has been completely revised and expanded at the hands of mathematicians who, imitating the method of their own science, directed it along the path of symbolization. While Boole and his disciples set themselves the task of constructing a logical calculus on the model of algebra, the Italian school, following Peano, aimed at the establishment of an algorithm for logic, specially designed to meet the needs of mathematical expression. Naturally enough when this second line of investigation came to the notice of the pioneers of axiomatized mathematics the result was an axiomatization presented entirely in symbolic form, and it was indeed in this way that Peano expounded his arithmetic, towards the end of the nineteenth century.

A rather different but even more important factor which influenced this move towards total symbolization, was the demand for formalization. Although symbolization and formalization are two distinct and theoretically separable operations, they are in fact closely connected; the second makes the first a very much easier procedure, and indeed requires it.

17. Formalization. Once we have convinced ourselves that the ultimate logical requirements have been satisfied, a new and more subtle requirement appears and calls for further attention. From empirical geometry to deductive geometry, from the Euclidean to the axiomatic form, from ordinary axiomatics to symbolized axiomatics, at each step we seem to have eliminated intuition, with very profitable results from a logical point of view. Have we now reached the limit and is the last stage really the last? Have we succeeded in dismissing every intuitive and subjective element from our approach to the validity of deductive theories?

The theory presents us with primitive propositions stating in symbolic form certain logical relations which hold between the primitive terms: since they are put forward only as hypotheses, we admit them as such, demanding only that they be compatible. But from this point on we shall admit a new term only if it is defined by means of the primitive terms, and we shall admit a new proposition only if it is demonstrable on the basis of the primitive propositions. There will then be no uncertainty, no possibility of questioning the adoption of a new term or of a new proposition—always assuming one condition is satisfied, namely, that the rules of definition and of demonstration be themselves agreed upon and admit of no ambiguity, that the technique of deductive procedures, in other words the logic, be both absolutely precise and completely universal, governing every detail and obligatory for all. Otherwise, if it were possible for disagreement to arise on this score, if a dispute were possible as to the logical acceptability of some step in a demonstration or definition, then the axiomatization itself could be, for one person a logically irreproachable whole, while judged by another to be logically defective.

Now this is indeed exactly what happened and in a particularly acute form just at the time when axiomatics systems were first being constructed. The so-called Crisis in the Foundations of Mathematics which occurred in connection with the Cantor's Theory of Sets, led to profound disagreements between mathematicians. These disputes differed from the sort which commonly occur in the sciences; they were not of the sort which is confined to a particular problem and which is soon resolved by a unanimous agreement to which an expert can, in all honesty hardly fail to subscribe. They arose from an apparently fundamental disagreement over questions of principle, a disagreement stemming from basically opposed attitudes of mind. A definition which seems perfectly clear to one theorist, is regarded by another as totally devoid of meaning; a demonstration which is impeccable in the eyes of one is deemed by another to be quite unacceptable; a logical principle which, according to some, is a *sine qua non* of all thought, is for others valid only in a restricted domain.

What steps can be taken in such cases to delimit the disagreement while at the same time ensuring that there is some ground in common to both sides of the dispute? The only possible way is to undertake a detailed investigation of the rules of logic *in accordance with which* we reason: to formulate them explicitly and exhaustively. In doing so we must adopt the same detached attitude to them as we adopted in dealing with axioms; in other words we should not lay them down in a categorical or assertive way but treat them rather as assumptions. For we can accept at the outset of an abstract axiomatization, different systems of logical rules and hence different ways of developing the same axiomatic system, just as we can allow for different incompatible systems of postulates (Euclidean, Lobatchevskian, etc.) without having to decide in advance which of them is true. As Carnap says, logic is a-moral; there is no question of laying down what shall or shall not be done but only of setting up conventions. Everyone is free to construct his logic to his own requirements as long as he expounds it clearly and thereafter adheres rigorously to it (Principle of Syntactical Tolerance). Thus the correction of logical mistakes in the development of an axiomatized theory no longer has any absolute meaning; it acquires, nevertheless, some degree of objectivity in so far as it is a 'mistake' relative to such-and-such a set of regulative principles. The situation we are in, when faced with an axiomatic system, is like that of two players between whom there is disagreement about the rules of the game: if they fail to take the precaution of seeing that these are explicitly formulated and agreed upon at the start, they will not be playing one and the same game, or indeed any game. If, on the other hand, their disagreement is made explicit at the start and they decide, for example, to use alternatively two sets of rules, they can play successive games without being impelled to accuse each other of cheating. Questions of validity are in this way answerable at a new level. Just as, in proceeding from the concrete theory to the axiomatized theory, a proposition of the system may become hypothetical (occupying a neutral position as a member of some postulate set) so we may now regard the formal validity of an axiomatization as being shifted back one stage and as becoming, in its turn, hypothetical through its dependence on the choice of logical norms.

The idea that deductive systems should be formalized in this way became more and more widely accepted after about 1920. Since that date it has been the accepted practice to eliminate the possibility of subjective criticisms of validity and to forestall disagreement, by stating in precise detail the rules of definition and demonstration governing the construction of any given system. Even those who do not accept the dictates of logic as absolute and who support the case for intuition have found themselves compelled to adopt this method. For without it they would have been unable to justify themselves in the eyes of their opponents. As a result we find them setting out, somewhat paradoxically, the 'formal rules of intuitionistic logic' and establishing an 'intuitionistic formalism'.

18. From Reasoning to Calculation. It would clearly be impossible in practice to satisfy such strict demands if we continued to express ourselves in ordinary language, with its imprecision and its innumerable irregularities. Hence formalization presupposes symbolization. For a formalized axiomatic system consists in a set of signs, some of them peculiar to the theory in question, others being logically prior to it, together with a statement of the rules to be applied in the manipulation of these signs. These rules are commonly divided into two groups: rules of construction governing the formation of expressions (among these are included the rules of definition) and rules of deduction, governing their transformation (on which the proofs rest). The purpose of these rules is to ensure, in the case of the first group, that no possible doubt can arise as to whether or not an expression (of any kind) is well formed and hence admissible, and in the case of the second group, that there is no ambiguity as to whether or not a deduction is properly constructed and hence renders its conclusion a theorem of the system. At the same time these rules impose no restrictions whatever on the interpretation which is eventually to be given to the terms and formulas, including the purely logical ones. They are concerned only with the formal structure of the expressions of the system, the succession of the printed marks which we read from left to right, line after line, on the page. Strictly speaking they are merely prescriptions for a calculus. They are comparable, say, with the rules of chess which tell us how we are initially to place the pieces, and which are the various moves allowed for each piece. In accepting a given logical sequence we can no longer rely on any feeling of intuitive self-evidence. It will now be a matter of proceeding by successive stages from one or more formulas initially accepted either as axioms or theorems, and performing a series of basic transformations until the required formula is reached; every step being shown and its authorization given by reference to the number of the relevant rule. This approach differs from that of earlier logicians in the complete reorientation involved. For the mind, instead of treating the symbols as deputizing for objects symbolized, now focuses on the symbols themselves, setting aside their intended interpretation and concentrating for the moment on their operational role, as if they themselves were the final subject matter.

The demands of logical rigour so successfully undermined all faith in sensible intuition, particularly where diagrams were concerned, that reliance came to be placed on strict deduction alone. The unreliability of sensible intuition makes it imperative that reasoning itself, whether silent or spoken, be replaced by calculation with fixed and visible signs on paper. Yet it might seem that in taking this stand on the visible evidence of written signs we are no longer working at the level of thought at which we originally started. However, we have from the very beginning allowed for progress towards the abstract and general in our insistence on the possibility of an ultimate interpretation for our symbols or rather, of many alternative interpretations. At the same time we are assured of a high standard of certainty and objectivity. As long as the number of signs used is fairly restricted, as long as they are so chosen that their shapes are unlikely to breed confusion, and finally, provided there are explicit rules to prevent any incoherence or ambiguity in their use, then no serious disagreement can arise. As in a well-regulated game, any given position of the pieces is either admissible or it is not, and the same holds for any given move. To quote Cavaillès: 'When the reasoning is written down the visible structure of the argument will betray any improper steps.'[1] Mistakes will be as immediately apparent as errors in an arithmetical calculation, or an improper move in chess or a solecism in a language which has a definitive grammar. Formal calculation has, as Leibniz hoped, become accepted as a rightful heir to ordinary reasoning.

[1] J. Cavaillès': *Méthode axiomatique et formalisme*, p. 94.

19. Metamathematics. Now that symbols were a subject of study in their own right and were no longer regarded as intermediaries, new horizons were opened up. Interest now centred on an entirely new system of entities which could be interrelated or dissociated from one another according to precise laws, and subjected to transformations which were, for the mathematician, reminiscent of the operations of geometry or, even more so, of combinatory problems. The signs themselves together with the laws governing their use 'define a type of abstract space having as many dimensions as there are degrees of freedom in the construction of unforeseen combinations'.[1] Thus there arose the idea of an entirely new science having as its subject-matter, not mathematical entities to which formulas are supposed to refer, but the formulas themselves in abstraction from their content. These formulas, constructed wherever mathematical entities would normally be presupposed, are entirely dissociated from such entities and accepted as an ultimate subject-matter for separate investigation. *Metamathematics* stands in the same relation to mathematical expressions, as that in which ordinary mathematics stands to numbers themselves. It was Hilbert again, who from 1917 onwards was the driving force behind this new level of research which originated at Göttingen under his direction. His name, in fact, has become associated with the second stage in the historical development of axiomatics, as with the first.

This development was not purely fortuitous. Metamathematics originated at the meeting point of several different lines of research. We have only to consider, in the first place, the merging of the two lines of thought which we have already discussed: the one originating from reflection on the logical basis of geometry and

[1] J. Cavaillès, op. cit., p. 93.

which, by attempting to perfect it, arrived at axiomatics—the other, aiming at the reform of logic itself, with the help of algebraic methods, and which succeeded in reformulating it as a calculus. Under these reciprocating influences axiomatics then transformed itself into a calculus while logic, in its turn, became axiomatized. Secondly, the direction which discussions on the vital problem of the foundations of mathematics took, led towards the adoption of formalism, while at the same time enabling the controversial issues to be formulated in terms acceptable to adversaries of the purely formal approach. Zermelo's early attempt to resolve the controversies by means of what might be called retrospectively, naïve axiomatics, resulted merely in a hardening of the attitude of empiricist and intuitionist mathematicians so that the doctrines of, for instance, Brouwer and his school gained an even stronger foothold. Of course, this attention solely to the written signs is, in a sense, a return to intuitive evidence. So that, if it is possible to examine any contested demonstration by strictly scientific means and abstracting from the mathematical meaning of the terms (which according to the intuitionist is totally absent in some cases, such as where the idea of an actual infinity is involved) to focus attention solely upon concrete relationships among symbols, the problem is completely solved in a way that should give satisfaction to both parties. This change of approach which involves relinquishing the domain of mathematical entities in favour of the domain of signs used to represent them, and operating with symbols which are open to immediate inspection, instead of with ideas which many people find obscure or void of meaning, puts us on a footing which the intuitionist will recognize as familiar, with no sacrifice of formal rigour. Problems about the denumerable infinite now become problems concerning a finite number of signs immediately given. At the same time, even the most demanding of logicians must welcome the formation of a 'theory of demonstration' which is itself demonstrative.

Furthermore it is a complete misunderstanding to suppose that metamathematics has arbitrarily invented new problems. On the contrary, metamathematics was itself called into being by certain problems which Hilbert, and indeed all theorists of axiomatics, have had to face at the outset of their researches; in particular the proofs of consistency and independence of axiom-sets. These problems and their accompanying ones (completeness, decidability and so on) are not, properly speaking, mathematical problems, since they concern, not mathematical entities themselves but propositions which refer to these entities. Being thus central to all axiomatic research it is hardly surprising that their elevation to the status of scientific issues requiring rigorously methodical treatment, should have been regarded as imperative. This was precisely the task which metamathematics set itself. Consider for example, the problem of non-contradiction, which together with decidability is a metamathematical problem of the first importance. We have already seen how it was solved by the earlier writers on axiomatics. This was either by appeal to a concrete model or realization (which, besides being empirical only, it is not always possible to find), or else by reduction to a prior abstract theory whose non-contradiction is presupposed (which merely shifts the problem back one stage). The alternative is to transform the question altogether and instead of seeking a coherent interpretation to examine the possibility or impossibility, given a set of formulas expressing the axioms of the system, which is strictly governed by well-defined rules, of constructing within that system, expressions of such-and-such a form; for instance, a pair of propositional expressions differing only in this —that the one becomes transformed into the other by prefixing to it the sign of negation. If this possibility or impossibility is provable one will thereby have proved, respectively, the inconsistency or consistency of the theory.

20. The Limitations of Consistency Proofs. In regard to all such proofs there is one reservation to be made: whatever the complexity and uncertainty of the mathematical theory in question and of the formulas in which it is expressed the metamathematical proof which is brought to bear on it, must, if it is to avoid a vicious circle or *petitio principii*, use only the simplest and most incontrovertible chains of deductive reasoning and in a manner calculated to ensure their acceptance by the attentive reader. While this concern with signs alone leads us back from abstract entities to visible data, nevertheless in reasoning about these signs we are relying on the intellect not the senses (if only in understanding the rules, judging whether they have been correctly applied, etc.). But any such reliance on intuition either of the senses or of the intellect will be legitimate only if it is confined to immediate intuitions which no one calls in question.

But as long as there is some margin left within which subjective judgements of validity can operate, the strict formalist will still be dissatisfied. The question then arises as to whether it is not possible so to arrange things that the steps of the metamathematical demonstration be integrated with the very theory whose consistency is being proved, so that the results established for the theory are simultaneously established also for the metamathematical reasoning itself. Progress along this line of thought has become possible thanks to the ingenious procedure known as 'arithmetization of syntax' invented by Godel, which enables us to formulate the logical syntax of arithmetic within arithmetic itself. This is achieved by stipulating that a certain correspondence shall hold between the symbols in terms of which the syntax of arithmetic is expressed and the ordinary arithmetical symbols, it being at the same time so arranged that every expression of the syntax language has for its translation one and only one arithmetical expression. Furthermore, this correspondence has only to be set up in such a way that every proposition which expresses arithmetically a proposition of the syntax language, should itself be arithmetically demonstrable, and we shall have expressed the syntax of arithmetic within arithmetic.

The crucial question is: can the consistency of arithmatic be proved in this syntax language? One of the first results obtained by Godel through his device of arithmetization was to establish the impossibility of any such proof. What he succeeded in establishing in two of the best known theorems of metamathematics (1931), was firstly, that a consistent arithmetic was bound to be an incomplete system which necessarily includes some undecidable statements and secondly, that the very assertion that the system is consistent, is itself one of those undecidable statements.

This apparently negative result which was obtained by the application of strictly formal methods and which was soon after corroborated by analogous results reached in closely related fields, was in fact a matter of unparalleled importance. It was far from being a mere episode in the history of metamathematics. The study of metamathematics had revived in a new form the old ideal of an absolutely valid demonstration, directed towards the construction of a formalism whose self-sufficiency would rest in its somehow containing itself within its own boundaries. It is the unrealizability of this ideal which has now been finally proved. Even in the paradigm case of deductive science, namely axiomatized mathematics, theorists must resign themselves to the distinction which they had hoped to obliterate, between truth and provability. The first of these concepts is very much wider than the second. For since, on the one hand, even the most elementary of all mathematical theories contains not only propositions which are as yet undecided, but propositions which are essentially undecidable (that is propositions of which it can be established that both they and their negations are alike unprovable); and on the other hand the law of excluded middle, whose validity the formalists maintain against their intuitionist opponents, assures us that of any two such propositions one must be true, even if we cannot decide which: we are forced to the conclusion that there are unprovable truths within any axiomatized mathematics. Thus even for a formal language as restricted as arithmetic is, consistency can be proved only if we allow ourselves to go beyond its boundaries.

21. The Axiomatization of Logic. Problems and difficulties analogous to those with which metamathematics is concerned arise in logic, the two levels of study being indeed very closely interrelated. When axiomatic theory was in its infancy logic was regarded as occupying a privileged position as the most basic discipline. An axiomatized theory denuded the terms and postulates on which it was built of their ordinary meaning and truth, but in doing so appealed to prior theories whose meaning and truth were presupposed. Logic seemed to be the one theory which was, in this sense, prior to all others. Certainly, it could be claimed for this theory that it had axiomatized itself, since from the time of Frege and particularly in the monumental synthesis of Whitehead and Russell, it had taken the form of a deductive system in which the primitive terms and primitive propositions were explicitly enumerated at the start. But it was still, unfortunately, only a concrete axiomatization. The terms in it retained to a greater or lesser degree their ordinary meaning which was merely made more precise through the relationships expressed in the postulates. These were indeed axioms proper, in the sense that they were, at one and the same time, primitive propositions and self-evident truths. The system as a whole had a genuine significance and a necessary truth which were disseminated through definitions and demonstrations to the defined terms and theorems. In attempting to base arithmetic and through arithmetic the whole of mathematics on logic alone, the 'logicism' of Frege and Russell was in fact directed at something very different from a mere compliance with a fashionable demand for explicitly formulated axioms. It aimed to lay bare the ultimate source and foundation of this demand. The primitive terms of Peano's axiomatic system had been left relatively indeterminate, allowing for any number of different interpretations; the primitive propositions suffered from the same indeterminacy and, being propositional functions rather than propositions, they were not categorical assertions, nor could such assertions be derived from them. By defining those terms which had hitherto been regarded as essentially variable, with the help of logical constants conceived as ultimate timeless essences, and by demonstrating postulates until then regarded as independent of truth and falsity, with the help of logical principles conceived as so many absolutely inviolable laws of thought, Russell imagined he had endowed the principles of mathematics and all deductions made from them, with a final meaning and an absolute truth. So that mathematics no longer had to be considered as the science in which 'we never know what we are talking about nor whether what we are saying is true', but was thus to become as categorical and deductive as logic, the latter being the very source of all its material.

But scepticism with regard to the self-evidence of axioms was soon to infect logic itself. Once the paradoxes of set theory had made their appearance and it was realized that their origin lay in its very foundations, the ensuing storm of controversy over the validity of one principle after another, led for the first time to the questioning of the absolute authority of logic. The new orientation which certain logicians began to give their work, in about 1920, initiated a gradual disintegration of logic from within. Logic had in fact to go through the same transformation as geometry had been through a few decades earlier. Just as the uniqueness of the latter was undermined by the discovery of non-euclidean geometries, and its reliance on intuition eliminated by the adoption of the axiomatic approach, so logic began to multiply and to axiomatize itself. Having become strictly deductive, its transformation into an abstract axiomatic system was inevitable. The grounds for eliminating the intuitive meanings of the terms, in the setting up of a deductive system, to prevent their occurring tacitly in the subsequent reasonings, held good for logic as for any other deductive system. The terms of the theory should not be regarded as serving any purpose other than to indicate the field of the relations occurring in the postulates. This being so, the propositions of logic now entirely deprived of their usual logical significance (just as those of geometry were deprived of their normal geometrical significance) become purely formal expressions. They are, as Wittgenstein explained, simple tautologies; that is to say expressions which tell us nothing about the world but which, for that very reason, hold true no matter what interpretation is given to them. Moreover, this formal approach to logic encourages the construction of non-classical logics which in their turn, by a kind of reciprocal action, reinforce the effectiveness of precisely that approach. For as long as principles are given only hypothetical authority, there is nothing to prevent our proposing alternatives, modifying this one, suppressing that one, and in this way passing from one logic to many quite arbitrarily constructed logics. Faced with this plurality of logics, classical logic can no longer claim any privileged status, since it is seen to be only one system among others and like them to be no more than a formal structure whose validity depends entirely on its internal consistency.

There is however one important point at which the analogy with geometry breaks down, namely that logic is not related to any prior body of knowledge whose formal structure is to be axiomatized. Yet one hardly needs to advance far up the hierarchy of the sciences before finding oneself faced with the ever increasing difficulty of giving them axiomatic form without presupposing some knowledge of the science in question. To take a simple example: numerical plurality has to be introduced at the very outset of arithmetic. With logic itself the difficulty becomes an absolute impossibility, for how is the reasoning of the axiomatic theorist to be judged if not by logical laws? With sufficient care, of course, one can arrange matters so that the logic governing the theorist's own reasoning is reflected in the axiomatized system of logic which he is constructing, or in other words so that the logic used is an application of the axiomatized logic in the sense of being one of its possible models. Nevertheless awkward objections can still be raised. In the first place, how can we be sure that there is a complete correspondence between the two? Even the earliest of the symbolic logicians had not failed to notice the fact that certain rules of formal deduction could not themselves be included in the formalism: for instance, the rule of substitution permitting the replacement of variables by individual constants in a formula. Without such a rule the formula would be useless, yet the permissive force of the rule itself would clearly be presupposed by any attempt at a symbolized formulation of it. It is essential, therefore, in the case of any calculus, to make clear a distinction between axioms and rules, between the assertions of which the calculus itself is made up and assertions about that calculus, the latter being regulative of the calculus and external to it. Precisely the same distinction will have to be made in attempting to axiomatize logic. This seems to suggest that we can never hope finally to eliminate all intuitive presuppositions, that the procedure of axiomatization involves an infinite regress. For if the propositions *of* the calculus can, and must, be regarded as purely formal, the propositions *about* the calculus cannot possibly be treated in this way; they have to retain their ordinary meaning. Given a logic which is assumed to be unique and absolute, then the correspondence between its axiomatized form and its informal use, even though perhaps only partial, seemed a matter of course. This is no longer possible once logics start to be constructed *ad lib.* Their multiplicity and diversity rules out any attempt to equate them with the logic we actually use in constructing them, unless we quite absurdly assume it to be indefinitely flexible.

22. Metalogic. Thus the axiomatization of logic leads inevitably to a kind of duplication. It leads not only to the kind of duplication which is an integral part of any axiomatic construction where we have in any case to allow for either an abstract or a concrete interpretation, but also to that duplication stemming from the fact that a purely formal construction presupposes a corresponding constructive activity of mind. Every formal axiomatic system is in effect bounded on all sides by the domain of intuition. Above it lie the various concrete interpretations that can be given to it (known as models) one of which is usually selected as the intended interpretation; below it we have those sciences which are logically prior and which, being endowed in the normal way with both meaning and truth, contribute effectively to the work of construction. Now logic, being at the bottom of the hierarchy, cannot rest on any other more fundamental discipline. If despite this we insist on an explicit formulation of the principles assumed in the procedure of axiomatizing logic, we can achieve this result only by stepping outside logic altogether and relying on an entirely new discipline whose subject-matter is the formulas of axiomatized logic and the rules governing them. Metalogic, in this sense, is related to logic in the same way as metamathematics to mathematics. It would of course be an exaggeration to say that it was born out of the axiomatization of logic: in one sense logicians have always made use of metalogic to some extent, but without realizing it. The effect of axiomatizing logic was to force this on their attention and to reveal more clearly the distinction between the metalogic and the logic to which it was being applied. In short, a metalanguage is superimposed on the formal calculus or object-language and its sole concern is with the syntactical and semantic (interpretation) rules of that calculus.

There is of course nothing to prevent us from taking the metalanguage itself as an object of study, formulating its syntax and arranging this in the form of a deductive system which can be axiomatized, symbolized, and formalized. It must, however, be remembered that if we proceed in this way a new metalanguage will be required, or to put it another way, that we shall have created a new object-language. We can, indeed, at least in theory proceed indefinitely up this hierarchy, the word 'indefinitely' signifying the impossibility of ever reaching a limit to the regress and of ever finally eliminating intuition from the foundations of axiomatics.

INDEX

24.9.21

7'5.07

MONOGRAPHS IN MODERN LOGIC

Edited by Geoffrey Keene

The Development of Mathematical Logic P. H. NIDDITCH

Axiomatics ROBERT BLANCHÉ

Propositional Calculus P. H. NIDDITCH

Truth-Functional Logic J. A. FARIS

Quantification Theory J. A. FARIS

First-Order Functional Calculus GEOFFREY KEENE

The Logic of Commands NICHOLAS RESCHER