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Operational research, however, is a long way from being simply a procedure which turns out to be industrially useful: it also constitutes a new field of study of very great interest and significance in pure mathematics. However, it is a feature of the subject that the mathematics involved can be understood without any very deep knowledge of other branches. Undergraduates and those professionally engaged on the scientific or technical side of management are therefore likely to have acquired the necessary minimum of mathematical understanding to be able to appreciate the arguments employed without difficulty.

It is not easy to provide in one short volume a survey of this rapidly growing field, but Dr Houlden has succeeded in including everything which is really fundamental.

D. G. Christopherson, O.B.E., D.Phil., F.R.S.

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Some Techniques of Operational Research

Prepared by
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Coal Board

Houlden

SOME TECHNIQUES OF
OPERATIONAL RESEARCH

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OPERATIONAL RESEARCH

Prepared by
the operational research group of
the National Coal Board

Edited by
B. T. HOULDEN, Ph.D.

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FOREWORD

The aim of this small book is to bring together in convenient and economical form the basic theory of the various techniques of importance in the field of operational research, which have been developed so rapidly and so successfully in the last twenty years.

Operational research was of course born of war-time needs. Its name still betrays its origin. The first contribution by John von Neumann to the formal theory of games dealt with idealisations of particular tactical problems, and much of the work published in this field over the years has been concerned with military applications. For example, attention was first directed to the problems of queueing by the need to study situations which arose when a large number of bombers had to land as quickly as possible on few air-fields, and so on.

But the military applications of the subject have now become much less prominent than those which refer directly to industrial practice and to the organisation and control of production. A glance at the table of contents of this book shows how immediate is the application of the theory to the problems of management.

Operational research, however, is a long way from being simply a routine procedure which turns out to be industrially useful. It also constitutes a new field of study of very great interest and significance in pure mathematics. Many of those who have made contributions to the subject have been in the front rank as mathematicians, and the basic theorems which they have devised are equal in their generality and importance to any contributions to other branches of mathematics made within the same period.

It is not easy to provide in one small volume a complete survey of this rapidly growing field, but Dr. Houlden has, I think, succeeded in including everything which is really fundamental. It is a feature of the mathematics employed in the field that it can be understood and appreciated without any very deep knowledge of other branches of mathematics. Anyone professionally engaged on the scientific or technical side of industrial management is likely during his education to have acquired the necessary minimum of mathematical understanding to be able to appreciate the arguments employed without difficulty.

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CHAPTER TWO: ALLOCATION

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CHAPTER THREE: THEORY OF GAMES

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CHAPTER FOUR: DYNAMIC PROGRAMMING

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CHAPTER FIVE: QUEUES

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CHAPTER SIX: STOCK CONTROL

J. C. Beresford, G. G. Stephenson, Dr. E. Sunderland.

CHAPTER SEVEN: REPLACEMENT

Dr. B. T. Houlden, J. J. Bissell-Thomas, A. D. J. Flowerdew.

CHAPTER EIGHT: THEORY OF SEARCH

A. M. C. McDonald, J. G. Fergusson, R. W. Elliott.

PREFACE

Operational research will only live and prosper if worthwhile advances are made and can be shown to have been made as a result. While it is important to be able to recognise quickly which basic technique is likely to be required for a particular problem, the good operational research scientist must also be able to modify the basic technique to suit the practical problem. Initially the basic technique and the practical problem are usually poles apart. Trying to apply techniques such as those described in this book, without being aware of the limitations to their use imposed by the assumptions made, would be courting disaster—an academic answer would be reached which would not be acceptable to management as a practical solution to the problem.

This book should really be used as an initial part of the training of an operational research scientist. There are many other skills which are required, some of which can only be learnt by practical training on projects. Not least of these skills to be added is the ability to understand and allow for the practical difficulties associated with any problem and the ability to appreciate the points of view of management and others concerned in a problem. To do this the operational research scientist must be able to judge things as if he were in their shoes. Once this difficulty of getting over the other side of the fence is overcome, the difficulties of convincing management sufficiently, not only to accept the recommendations but also to apply them, become relatively small.

The authors wish to acknowledge the kindness of:—

(1) John Wiley & Sons Incorporated, 44th Avenue, New York, 16, N.Y. for permission to reprint on pages 124–126 parts of the book *Introduction to Operations Research (1957)* by Churchman, C.W., Ackoff, R. L., and Arnoff, E. L.

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(6) Princeton University Press, Princeton, New Jersey, for permission to use material on pages 120–121 from the book *Theory of Inventory Management* by T. M. Whitin, 1953.

The authors of the various chapters of this book have been listed separately on page vi. All of them are indebted to Professor B. H. P. Rivett, who was the Head of the National Coal Board's operational research group (The Field Investigation Group) from 1951 until early 1960. Without his perseverance and encouragement this book might never have been produced and certainly would not have been available for publication until much later.

B. T. HOULDEN

CONTENTS

CHAPTER 1

INTRODUCTION

Operational research has been defined by Sir Charles Goodeve as "a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control". It owes its being mainly to two possible situations:

(*a*) The necessary financial or numerical statistics may not be available for management to make the right decision (this may arise, for example, when marginal costing is necessary, it may also arise when the existing statistics are subject to error or misinterpretation).

(*b*) The statistics are available but the best efficiency can only be obtained by compromise between two or more objectives, and the analysis requires some mathematical expertise not otherwise available to management.

In some ways operational research is a method of approach to problems; looking at the whole system first and considering whether altering the factors directly involved in the problem posed will affect other parts of the system (e.g. if the problem was "How can the stocks of machine spares be reduced?" the operational research scientist would first consider what would be the effect on production due to any delays in repairing machines if the stores were unable to supply the parts when they were required). Most of the problems tackled by operational research have this character. Once these interactions have been understood, the operational research scientist can then look into the finer detail of the problem if this is necessary.

Operational research also has the characteristic that it is done by teams of scientists drawn from the various graduate disciplines such as mathematics, statistics, economics, engineering, physics, etc. Without operational research training, the mathematician tends to reach the academic answer which is frequently impossible under practical conditions and the engineer tends to oversimplify and reach too coarse an answer. Between the two a workable good answer can be achieved, and by this process of interaction both the mathematician and engineer finally become operational research scientists with very little remaining of these disadvantages originating from their separate graduate disciplines.

The scientific method is basically founded on measurement and passes through the following stages:

(*a*) Statement of the problem.
(*b*) Collection of the data, experimental and observational, relevant to the problem.
(*c*) The sorting and analysis of these data to produce a hypothesis.
(*d*) The use of the hypothesis to estimate what should happen in various circumstances.
(*e*) Continuous experimental and observational check on the hypothesis in the light of fresh data.

Operational research projects follow the same approach and the hypothesis which appears in quantitative or qualitative terms is called the model.

While there have been isolated instances of operational research being used earlier this century (1), it only began to be recognised as a separate branch of science during the 1939–45 war. Groups of scientists of various disciplines were thrown together to study problems occurring in the armed forces.

Among the first of these was a study of the efficiency of a Royal Observer Corps system consisting of a number of observer posts, a communications network and a human executive control system. Obviously in such a system there are many parts but the objective of a study of this type is to improve the whole system efficiency. Thus the study covered such factors as instrument efficiency, observer efficiency, efficiency of the communication network and decision process up to making an attack on the enemy aircraft, the siting, size and spacing of the posts. Only by looking at each part of the system and considering the effect of any changes on the efficiency of the whole system could the problem be efficiently solved. It was found that this approach of grouping graduates of different disciplines to study this and similar problems was extremely effective and led to considerable improvements in both tactics and strategy.

Another well-known wartime operational research study is that by Professor Blackett (2) and associates on the setting of fuses for depth charges dropped by Coastal Command aircraft. Again the approach was first to look at the particular problem in relation to the whole system and then to analyse operational data which had already been recorded.

During the study of these and many other wartime problems the operational research scientists found that they needed to develop new mathematical techniques and it was realised that both the method of approach and these techniques would be applicable to industrial as well as military problems. Soon after the end of hostilities a few of the larger concerns in this country started operational research groups and the National Coal Board was one of these.

Now most of the major industries in the United States of America and in the United Kingdom use operational research. Among the major concerns in the United Kingdom with operational research teams are, Courtaulds Ltd. (textiles), the British Iron and Steel Research Association, United Steel Companies Ltd., Steel Company of Wales Ltd., Richard, Thomas & Baldwins Ltd. (steel), British European Airways, London Transport Executive (buses and underground trains), Electricity Generating Board, British Railways, Albert E. Reed & Co. Ltd. (paper manufacturers), Shell Petroleum Company Ltd., British Petroleum Company Ltd., Esso Petroleum Company Ltd. and Road Research Laboratories, as well as the various branches of the armed forces.

In the succeeding chapters we will work through some of the most useful basic techniques of operational research. For those wishing to read more widely on the whole subject of operational research, attention is drawn to some of the references given below (3), (4), (5), (6).

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CHAPTER 2

ALLOCATION

Introduction

A business or industrial process may consist of a number of activities, e.g. processing a number of different products in a factory or producing a mixture of a number of different ingredients. If these activities are interdependent (i.e. a change in the output of any product influences the outputs of the other products or a change in the amount of any ingredient influences the amounts which can be used of the other ingredients) a very large number of combinations or allocations of the activities exist. Under these circumstances it is a complex problem to find the allocation of the activities which optimises costs, profit or some other factor. It will be appreciated that minimum costs do not necessarily give maximum profits and that in general only one factor can be optimised.

A particular class of these allocation problems can be solved by linear programming. This method is applicable if the problem has the following characteristics:

(*a*) The objective (i.e. profit, costs, etc.) can be expressed as a linear function in terms of the activities, i.e. a fixed cost or profit, etc., per unit of activity can be given to each of the activities.

(*b*) The objective function described in (*a*) is limited by a set of constraints which can be expressed as linear equations or inequations in terms of the activities, e.g. the sum total weight of the ingredients in a mixture must be 100 tons, the mixture must contain less than 5% of some impurity which may be present in some or all of the ingredients.

Some examples of possible applications of linear programming are given below.

(i) What proportions of available coals, having different production costs, should be used in order to maximise the profit from the manufacture of a patent fuel in which the ash and phosphorous and sulphur contents must be less than specified amounts?

(ii) How should goods be allocated from certain producers to certain consumers so that the total transport cost is minimised?

(iii) How should a number of different operators, who have different performances on various machines, be allocated to a group of machines so that the total output is maximised?

Mathematical formulation

The general linear programming problem can be expressed as follows:

Find the values of $x_1, x_2, \ldots x_n$ which satisfy the constraints

$$\begin{aligned} a_{11}x_1+a_{12}x_2 \;\ldots\; +a_{1n}x_n &\leqq b_1 \\ a_{21}x_1+a_{22}x_2 \;\ldots\; +a_{2n}x_n &\leqq b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1+a_{m2}x_2 \;\ldots\; +a_{mn}x_n &\leqq b_m \end{aligned} \qquad (2.1)$$

$$x_i \geqq 0 \quad \text{for} \quad i = 1 \ldots n \qquad (2.2)$$

and make $f = c_1x_1+c_2x_2 \ldots +c_nx_n$ a maximum (or minimum) (2.3)

This set of inequations (2.1) can be converted into equations by the addition of further variables which take up the slack by which the inequations are allowed to depart from equations. These additional variables are called slack variables.

The problem now becomes,

Find values $x_1, x_2, \ldots x_{n+m}$ which satisfy the constraints

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2 \;\ldots\; +a_{1n}x_n+x_{n+1} &= b_1 \\ a_{21}x_1+a_{22}x_2 \;\ldots\; +a_{2n}x_n+x_{n+2} &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1+a_{m2}x_2 \;\ldots\; +a_{mn}x_n+x_{n+m} &= b_m \end{aligned}\right\} \qquad (2.4)$$

$$x_i \geqq 0 \quad \text{for} \quad i = 1 \ldots n+m \qquad (2.5)$$

and make $f = c_1x_1+c_2x_2 \ldots +c_nx_n+0x_{n+1}+0x_{n+2} \ldots +0x_{n+m}$ a maximum (or minimum) (2.6)

Methods of solution

The equations (2.4) and (2.5) form a convex set in $m+n$-space. It can be shown that the optimum solution for equation (2.6) lies at one or more vertices of this convex set (1), (2).

Three different computation procedures have been developed for the solution of linear programming problems. They are:

(*a*) The method of leading variables—Beale (1).
This involves the introduction of variables one at a time according to the contribution which they make to equation (2.6).

(*b*) The logarithmic potential method—Frisch (2).
Any combination of the variables lying inside the convex set is taken as the starting point. This solution is improved upon by moving in the direction which gives the greatest potential improvement (to equation (2.6)). The movement is limited by the boundary of the convex set.

(*c*) The simplex technique—Dantzig (3), (4).
One of the vertices of the convex set is taken as the starting point. This solution is improved upon by moving to a neighbouring vertex.

The simplex technique seems to be more straightforward and easier to handle than the first two methods and it is unlikely that these methods will have practical application for hand computation. This chapter will therefore concentrate on the simplex technique.

A simplified version of the simplex technique has been developed for the solution of a special case of the general problem. This is called the transportation technique. The simplex and transportation techniques are illustrated in this chapter by means of worked examples. Also illustrated is the assignment technique (developed for the solution of personnel assignment problems) which is a useful alternative to the transportation technique in certain circumstances.

A general introduction and treatment of linear programming is given in references (5) and (6).

The simplex technique

The simplex technique is a general method which can be used to solve all linear programming problems, but it is not used if transportation or a simpler method can be employed.

The problem is solved by a systematic study of the vertices of the convex set formed in $m+n$-space by equations (2.4) and (2.5). The simplex computation procedure requires exactly m non-zero variables in the solution at any time and if a vertex with less than m non-zero variables has to be considered the problem is said to be degenerate. This difficulty to computation is overcome by assuming that the zero variables in the solution at this stage have very small positive values.

A set of x_i which satisfies equations (2.4) and (2.5) is defined as a feasible solution. A feasible solution with m non-zero values is defined as a basic feasible solution. A maximal solution, i.e. with (2.4), (2.5) and (2.6) satisfied, is one of the basic feasible solutions and is described as a basic maximal solution.

The first step is to find a basic feasible solution and carry out a test to see whether this is a basic maximal solution. If the first solution is not a maximal, a second basic feasible solution is derived from the first according to a set of rules and the new solution is tested. These steps are repeated until a basic maximal solution is found or the calculations indicate that f does not possess a maximum (or minimum).

Example

Three grades of coal A, B, C contain as impurities phosphorus and ash. In a particular industrial process up to 100 tons of fuel is required which does not contain more than 3% ash nor more than 0·03% phosphorus. We wish to maximise the profit whilst satisfying these conditions. There is an unlimited supply of each grade.

To meet these requirements it is necessary to blend the 3 grades of coal. It is required to find the proportions of the grades which will give the maximum profit.

The percentage of the impurities and the profit of the grades are shown below:

Coal	% *Phosphorus*	% *Ash*	*Profit in shillings per ton*
A	0·02	3·0	12·0
B	0·04	2·0	15·0
C	0·03	5·0	14·0

We suppose that x tons of A, y tons of B and z tons of C are used in the mixture.

The phosphorous content must not be more than 0·03%,

$$\text{i.e.}\quad (0{\cdot}02x+0{\cdot}04y+0{\cdot}03z) \leqq 0{\cdot}03(x+y+z)$$

$$\therefore\quad -0{\cdot}01x+0{\cdot}01y \leqq 0$$

$$\therefore\quad -x+y \leqq 0. \tag{2.7}$$

Similarly the ash restriction gives

$$-y+2z \leqq 0. \tag{2.8}$$

Since demand is for not more than 100 tons we get

$$x+y+z \leqq 100. \tag{2.9}$$

Also we cannot have negative tonnages

$$x \geqq 0; \quad y \geqq 0; \quad z \geqq 0. \tag{2.10}$$

The profit (in shillings) $= 12x+15y+14z.$ (2.11)

The problem is now a mathematical one of maximising function (2.11) subject to the conditions (2.7), (2.8), (2.9) and (2.10).

In order to solve the problem by the simplex method the inequations (2.7), (2.8) and (2.9) are made into equations by introducing new non-negative variables (called slacks) r, s, t, to each equation. Equations (2.12), (2.13) and (2.14) result. The slacks contribute zero to the profit.

$$-x \quad +y \qquad\qquad +r = 0 \tag{2.12}$$

$$-y \quad +2z \quad +s = 0 \tag{2.13}$$

$$x \quad +y \quad +z \quad +t = 100 \tag{2.14}$$

A table (2.1) is drawn up containing the coefficients of the variables in these equations and in the function (2.11):

TABLE 2.1

W	12	15	14	0	0	0	P
	x	y	z	r	s	t	
r	−1	1	0	1	0	0	0
s	0	−1	2	0	1	0	0
t	1	1	1	0	0	1	100
w	−12	−15	−14	0	0	0	0

In Table 2.1 row W gives the coefficients of function (2.11) (this row will be omitted from the succeeding tables). The second, third and fourth rows give the coefficients of equations (2.12), (2.13), (2.14) respectively. The w row (at this stage) contains the negatives of the elements in the W row.

The variables in the basic feasible solution are written in the W column and the values of these variables are given in the P column. The basic feasible solution is selected by choosing one variable from each equation and making it positive; those variables not chosen are made equal to zero. This is usually done by equating the slacks to the right-hand sides of the equations. The last figure in the P column is the profit given by this solution. In our example the first solution is $r = 0$, $s = 0$, $t = 100$ (profit $= 0$). Although two of these variables are zero (i.e. the solution is degenerate) the problem can still be solved by imagining $r = E$, $s = E$ where E is a very small positive number (Table 2.2).

One of the variables will be interchanged with one of the slacks and the profit increased. The variable to be introduced first is the one with the largest negative value in the w row (i.e. y). To decide which variable to replace, the elements in column P are divided by the corresponding elements in column y. The variable replaced is the one lying in the row which gives the *smallest positive* value for this quotient, i.e. variable r in the first row, which yields E where E is substituted for 0 (see Table 2.2). This row will be called the pivot row and the element lying in the pivot row and the chosen column (y) the pivot element (which is circled in Table 2.2).

TABLE 2.2

	x	y	z	r	s	t	P	
r	−1	(1)	0	1	0	0	E	Pivot row
s	0	−1	2	0	1	0	E	
t	1	1	1	0	0	1	100	
w	−12	−15	−14	0	0	0	0	

When y is introduced into the solution in place of r the coefficient of y in the pivot row must be one, and in the other rows, zero. This is achieved by first dividing the pivot row by the pivot element and then subtracting multiples of the pivot row from the other rows. For our problem the pivot element is 1 so it is only necessary to add the elements of the pivot row to the elements of the s row, subtract them from the elements of the t row and add 15 times them to the w row. Now y is substituted for r in the pivot row. This solution is shown in Table 2.3.

TABLE 2.3

	x	y	z	r	s	t	P
y	−1	1	0	1	0	0	E
s	−1	0	2	1	1	0	$2E$
t	2	0	1	−1	0	1	$100-E$
w	−27	0	−14	15	0	0	$15E$

The first stage of the calculation has now been performed.

The negative numbers in the w row show by how much the profit will increase if one unit of the variable is introduced in place of one of the variables already in the solution. The positive numbers indicate how much the profit will decrease. The object is therefore to make all the terms in the w row non-negative. The second stage in the calculation is performed in exactly the same way as the first. The variable x will be introduced in place of t,

$$\left(\frac{100-E}{2} \text{ is the only positive value of the division of the } P \text{ column}\right).$$

Table 2.4 is the position when the pivot row has been divided by the pivot element and Table 2.5 shows the situation at the end of the second stage.

TABLE 2.4

	x	y	z	r	s	t	P	
y	-1	1	0	1	0	0	E	
s	-1	0	2	1	1	0	$2E$	
t	(1)	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$50-\frac{1}{2}E$	Pivot row
w	-27	0	-14	15	0	0	$15E$	

TABLE 2.5

	x	y	z	r	s	t	P
y	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$50+\frac{1}{2}E$
s	0	0	$2\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$50+1\frac{1}{2}E$
x	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$50-\frac{1}{2}E$
w	0	0	$-\frac{1}{2}$	$1\frac{1}{2}$	0	$13\frac{1}{2}$	$1350+1\frac{1}{2}E$

At the third stage the variable z is introduced in place of s

$$\left(\frac{50+1\frac{1}{2}E}{2\frac{1}{2}} \text{ is the least of the positive values of the division}\right).$$

Table 2.6 shows the position when the pivot row has been divided by the pivot element. In Table 2.7 it will be seen that all the elements in the w row are positive and hence the answer has been obtained.

TABLE 2.6

	x	y	z	r	s	t	P	
y	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$50+\frac{1}{2}E$	
s	0	0	(1)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$20+\frac{3}{5}E$	Pivot row
x	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$50-\frac{1}{2}E$	
w	0	0	$-\frac{1}{2}$	$1\frac{1}{2}$	0	$13\frac{1}{2}$	$1350+1\frac{1}{2}E$	

TABLE 2.7

	x	y	z	r	s	t	P
y	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$40+\frac{1}{5}E$
z	0	0	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$20+\frac{3}{5}E$
x	1	0	0	$-\frac{3}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$40-\frac{4}{5}E$
w	0	0	0	$\frac{8}{5}$	$\frac{1}{5}$	$\frac{68}{5}$	$1360+\frac{9}{5}E$

Hence the mixture is 40 tons of grade A
40 tons of grade B
20 tons of grade C.

The phosphorous content $= \frac{1}{100}[40(\cdot 02)+40(\cdot 04)+20(\cdot 03)] = 0\cdot 03\,\%$

ash content $= \frac{1}{100}[40(3\cdot 0)+40(2\cdot 0)+20(5\cdot 0)] = 3\cdot 0\,\%$

Profit $= 40(12)+40(15)+20(14) = 1360/\text{–}$.

First basic feasible solution

If the inequations are in the "greater than" form the slack variables have negative coefficients, and the derivation of a first basic feasible solution might be difficult. This difficulty can be overcome by adding an artificial variable to these equations. These artificial variables are given a very large cost in the objective function to ensure that they do not appear in the final solution.

A method for discrete linear programming (simplex)

The optimum solution to a simplex programme may not be in integers. A solution in which some or all of the variables are integers may, however, be required and several methods are being developed for computing the best integer solution in these circumstances. The work published so far deals with the case where all the variables are required to be integers.

The procedure is to solve the problem using the simplex method. If the solution is not in integers an additional constraint, which excludes this non-integer solution but will be satisfied by the as yet unknown integer solution, is introduced. A new optimum which satisfies the modified set of constraints is computed. If this optimum is not in integers a further constraint is introduced and the computation is repeated. This rounding off is repeated until either an integer optimum is obtained or an indication is given that the problem has no solution in integers.

One method for generating the additional constraints is by Ralph E. Gomory (7). The method gives a solution in which all the variables (basic and slack variables) are in integers. All the coefficients and values of the original constraints must therefore be in integers.

The problem is first solved by the simplex method giving the following table,

W	x_1	...	x_n	P
x_1^1	a_{11}^1	...	a_{1n}^1	b_1^1
...	...	...	...	...
x_m^1	a_{m1}^1	...	a_{mn}^1	b_m^1
w	c_1^1	...	c_m^1	C^1

The solution is $x_i^1 = b_i^1$, $i = 1$ to m and the value of the objective function is C^1.

If any of the b_i^1 are non-integer a new constraint is introduced based on the row with the b_i^1 with the largest fractional part.

Assume that the row $i = k$ is selected. A new constraint,

$$x_{n+1} = b_{m+1} - \sum_{j=1}^{j=n} a_{m+1,j} x_j$$

is now introduced, where

$$a_{m+1,j} = a_{kj}^{11} - a_{kj}^1 \quad \text{for} \quad j = 1 \text{ to } n,$$

a_{kj}^{11} being the largest integer $\leqq a_{kj}^1$

$$\text{and,} \quad b_{m+1} = b_k^{11} - b_k^1,$$

b_k^{11} being the largest integer $\leqq b_k^1$

The table now becomes

W	x_1	...	x_n	x_{n+1}	P
x_1^1	a_{11}^1	...	a_{1n}^1		b_1^1
...	...	...	...	...	...
x_m^1	a_{m1}^1	...	a_{mn}^1		b_m^1
x_{n+1}	$a_{m+1,1}$	...	$a_{m+1,n}$	1	b_{m+1}
w	c_1^1	...	c_m^1		C^1

The solution given by the table at this stage is not feasible since $x_{n+1} = b_{m+1}$ is negative. The next step is to obtain a feasible solution and optimise by the simplex method. If the new optimum solution is not in integers the procedure for generating a new constraint is repeated.

No solution in integers

The problem has no solution in integers if the computation reaches a stage where either

(i) the solution at this stage is not in integers and all the a_{ij}^1 are integers. No additional constraints can be generated;

or (ii) no feasible solution can be obtained after introducing a new constraint.

Discarding constraints

The only constraints which need to be satisfied are the original ones and the additional constraints can be dropped at any time.

If one of the additional variables (x_{n+1} ...) appears in the solution the row which gives it a value can be dropped. This has no effect on the further computation. The number of rows in the simplex table need not therefore exceed $n+2$.

Example

Maximise $F = 2x_1 + x_2$, subject to

$$\tfrac{3}{4}x_1 - \tfrac{1}{2}x_2 \leqq \tfrac{13}{8}$$
$$x_1 + 2x_2 \leqq 8$$
$$x_2 \leqq 3$$
$$x_i \geqq 0$$
$$x_i = \text{integer}.$$

The first restriction is not in integers and must be modified. The restriction is identical to $3x_1 - 2x_2 \leqq 6\tfrac{1}{2}$ and since x_1 and x_2 must be integers the problem is unchanged if $3x_1 - 2x_2 \leqq 6$ is considered. (Alternatively $6x_1 - 4x_2 \leqq 13$ could be considered but this introduces large numbers unnecessarily.)

The problem is first solved by the simplex method as follows:

W	x_1	x_2	x_3	x_4	x_5	P
x_3	(3)	-2	1			6
x_4	1	2		1		8
x_5		1			1	3
w	-2	-1				0

W	x_1	x_2	x_3	x_4	x_5	P
x_1	1	$-\frac{2}{3}$	$\frac{1}{3}$			2
x_4		$(\frac{8}{3})$	$-\frac{1}{3}$	1		6
x_5		1			1	3
w		$-\frac{7}{3}$	$\frac{2}{3}$			4

W	x_1	x_2	x_3	x_4	x_5	P
x_1	1		$\frac{1}{4}$	$\frac{1}{4}$		$\frac{7}{2}$
x_2		1	$-\frac{1}{8}$	$\frac{3}{8}$		$\frac{9}{4}$
x_5			$\frac{1}{8}$	$-\frac{3}{8}$	1	$[\frac{3}{4}]$
w			$\frac{3}{8}$	$\frac{7}{8}$		$\frac{37}{4}$

At this stage the solution is non-integer $x_1 = \frac{7}{2}$, $x_2 = \frac{9}{4}$, $x_5 = \frac{3}{4}$, $x_3 = x_4 = 0$ and $F_{max} = \frac{37}{4}$. b_3 has the largest fractional part and a new constraint based on row 3 is introduced. The new constraint is,

$$x_6 = [0-\tfrac{3}{4}]-[(0-0)x_1+(0-0)x_2+(0-\tfrac{1}{8})x_3+(-1-(-\tfrac{3}{8}))$$
$$x_4+(1-1)x_5] = -\tfrac{3}{4}-[-\tfrac{1}{8}x_3-\tfrac{5}{8}x_4].$$

This constraint is identical to $x_1+x_2+x_6 = 5$. The simplex table becomes:

W	x_1	x_2	x_3	x_4	x_5	x_6	P
x_1	1		$\frac{1}{4}$	$\frac{1}{4}$			$\frac{7}{2}$
x_2		1	$-\frac{1}{8}$	$\frac{3}{8}$			$\frac{9}{4}$
x_5			$\frac{1}{8}$	$-\frac{3}{8}$	1		$\frac{3}{4}$
x_6			$-\frac{1}{8}$	$(-\frac{5}{8})$		1	$-\frac{3}{4}$
w			$\frac{3}{8}$	$\frac{7}{8}$			$\frac{37}{4}$

The solution can be made feasible and optimal by putting x_4 into solution at row 4.

W	x_1	x_2	x_3	x_4	x_5	x_6	P
x_1	1		$\frac{1}{5}$			$\frac{2}{5}$	$\frac{16}{5}$
x_2		1	$-\frac{1}{5}$			$\frac{3}{5}$	$\frac{9}{5}$
x_5			$\frac{1}{5}$		1	$-\frac{3}{5}$	$\frac{6}{5}$
x_4			$\frac{1}{5}$	1		$-\frac{8}{5}$	$\frac{6}{5}$
w			$\frac{1}{5}$			$\frac{7}{5}$	$\frac{41}{5}$

At this stage the solution is non-integer. $x_1 = \frac{16}{5}$, $x_2 = \frac{9}{5}$, $x_5 = \frac{6}{5}$, $x_4 = \frac{6}{5}$, $x_3 = 0$ and $F_{max} = \frac{41}{5}$. b_2 has the largest fractional part and a new constraint based on row 2 is introduced. The new constraint is,

$$x_7 = [1-\tfrac{9}{5}]-[(0-0)x_1+(1-1)x_2+(-1-(-\tfrac{1}{5}))x_3+(0-0)x_4 \\ +(0-0)x_5+(0-\tfrac{3}{5})x_6] = -\tfrac{4}{5}-[-\tfrac{4}{5}x_3-\tfrac{3}{5}x_6].$$

This constraint is identical to $3x_1-x_2+x_7 = 7$. The simplex table becomes:

W	x_1	x_2	x_3	x_4	x_5	x_6	x_7	P
x_1	1		$\frac{1}{5}$			$\frac{2}{5}$		$\frac{16}{5}$
x_2		1	$-\frac{1}{5}$			$\frac{3}{5}$		$\frac{9}{5}$
x_5			$\frac{1}{5}$		1	$-\frac{3}{5}$		$\frac{6}{5}$
x_4			$\frac{1}{5}$	1		$-\frac{8}{5}$		$\frac{6}{5}$
x_7			$(-\frac{4}{5})$			$-\frac{3}{5}$	1	$-\frac{4}{5}$
w			$\frac{1}{5}$			$\frac{7}{5}$		$\frac{41}{5}$

The solution can be made feasible and optimal by putting x_3 into solution at row 5.

W	x_1	x_2	x_3	x_4	x_5	x_6	x_7	P
x_1	1					$\frac{1}{4}$	$\frac{1}{4}$	3
x_2		1				$\frac{3}{4}$	$-\frac{1}{4}$	2
x_5					1	$-\frac{3}{4}$	$\frac{1}{4}$	1
x_4				1		$-\frac{7}{4}$	$\frac{1}{4}$	1
x_3			1			$\frac{3}{4}$	$-\frac{5}{4}$	1
w						$\frac{5}{4}$	$\frac{1}{4}$	8

The optimal integer solution is $x_1 = 3$, $x_2 = 2$, $x_3 = x_4 = x_5 = 1$ and $F_{max} = 8$.

The transportation technique

The transportation technique can be used to minimise the cost of distributing goods from a number (m) of despatch points to a number (n) of receiving points.

The following conditions must be satisfied:

(a) The amounts of goods to be sent from each point and received at each point are known.

(*b*) The total amount of goods despatched equals the total amount received.

(*c*) The cost of transporting each unit of goods from each despatch point to each receiving point is known.

In order to understand the difference between the use of the simplex and transportation technique the following mathematical development of the transportation equations is given.

Let x_{ij} be the tonnage sent from despatch point (i) to destination (j) and c_{ij} be the cost per ton of sending along this route. If despatch point (i) can send a_i tons then

$$x_{i1}+x_{i2}+\ \dots\ +x_{in} = a_i \quad \text{(for each } i\text{)},$$

i.e. total amount sent from (i) = total amount available at (i). (2.15)

Similarly if destination (j) requires b_j tons then

$$x_{1j}+x_{2j}+\dots+x_{mj} = b_j \text{ (for each } j\text{)} \tag{2.16}$$

i.e. total amount received at (j) = total amount required at (j)

also

$$a_1+a_2\ \dots\ +a_m = b_1+b_2\ \dots\ +b_n. \tag{2.17}$$

(Equations (2.15), (2.16) and (2.17) give $m+n-1$ independent equations.)

$$\begin{aligned}\text{Total cost} = {} & x_{11}c_{11}+x_{12}c_{12}\dots\ +x_{1n}c_{1n} \\ & +x_{21}c_{21}+x_{22}c_{22}\ \dots\ +x_{2n}c_{2n} \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & +x_{m1}c_{m1}+x_{m2}c_{m2}\ \dots\ +x_{mn}c_{mn}\end{aligned} \tag{2.18}$$

$$x_{ij} \geqq 0 \quad \text{(all } i \text{ and } j\text{)}. \tag{2.19}$$

It will be seen that equations (2.15), (2.16) and (2.17) are equivalent to equations (2.4) (see page 5), except that the coefficients in (2.15) and (2.16) are all unity and not any value a_{ij} as in (2.4). Similarly the total cost functions (2.6) and (2.18) and non-negative conditions (2.5) and (2.19) are the same.

A basic feasible solution satisfies the capacities and requirements of the despatch and receiving points by the use of $m+n-1$ routes.

A comprehensive description of the transportation technique applied to a large distribution problem is given in reference (8).

Example

There are three collieries which can supply coal in the following quantities:

Colliery *a*	14 tons
Colliery *b*	12 tons
Colliery *c*	5 tons
Total	31 tons

There are three consumers at which coal is required in the following quantities:

Consumer *A*	6 tons
Consumer *B*	10 tons
Consumer *C*	15 tons
Total	31 tons

The costs for transporting 1 ton of coal from any one point, *a*, *b* or *c*, to any other point, *A*, *B* or *C*, are as follows (in £'s):

	a	*b*	*c*
A	6	8	4
B	4	9	3
C	1	2	6

The problem is to minimise the total transport cost.

Step 1. The first step in the solution is to fill in on a blank chart an allocation of the coal which is a basic feasible solution.

One systematic method is to start at the top left-hand corner of the chart (Tables 2.8 and 2.9) and work from there allocating as much as possible by each route. This is called the north-west corner rule. The total cost of this allocation is £136.

Step 2. It is now assumed that the cost of transporting is made up of two parts, namely, a cost of despatch and a cost of reception. It is assumed (in Table 2.9) that one of the costs of reception is zero, say at *A*; then, according to the cost in cell *Aa* (i.e. 6), the cost of sending from *a* must be 6. The cost in cell *Ba* is 4 and thus the cost of reception at *B* is −2.

This procedure is followed for cells in which there is an allocation so that all three costs of sending and the three costs of receiving are found. The unused cells can now be considered and, where the sum of the receiving and sending costs for any unused cell exceeds the true cost which is marked within the cell, there is a saving in cost to be made by using this so far unused cell. The next step is to transfer goods into the unused cell which will give the largest saving, i.e. cell *Ac* in Table 2.9.

The goods to be transferred to cell *Ac* must come from either cell *Aa* or cell *Cc*. If we transfer θ units from *Cc* to *Ac*, then *Aa* must be decreased by θ units to keep the row total correct. Similarly, to maintain the correct row and column totals, cells *Ba* and *Cb* must be

		a	*b*	*c*	*Total*
A		6	8	4	6
B		4	9	3	10
C		1	2	6	15
Total		14	12	5	31

TABLE 2.8

Basic chart with senders *a*, *b* and *c* to send 14, 12 and 5 tons respectively, receivers *A*, *B* and *C* to receive 6, 10 and 15 tons respectively.

Transport cost is shown in the bottom right-hand corner of each route cell.

		a	*b*	*c*	*Total*
		6	11	15	
A	0	$6-\theta$ 6	8	$+\theta$ [11] 4	6
B	-2	$8+\theta$ 4	$2-\theta$ 9	3	10
C	-9	1	$10+\theta$ 2	$5-\theta$ 6	15
Total		14	12	5	31

TABLE 2.9

Step 1—basic feasible solution obtained by N.W. corner rule. $m+n-1=5$ routes in use. Total cost = £136.

Step 2— [11] indicates sum of part costs exceeds actual cost by £11, i.e. total cost can be reduced by £11 × θ (if a quantity θ is sent by route *Ac*). $\theta = 2$ tons. Total cost = £114.

increased by θ and cell *Bb* must be reduced by θ. The maximum value of θ, i.e. 2 tons (determined by cell *Bb*), is entered into the solution. The resulting solution has a total cost of £114.

Step 3, etc. The part costs must be adjusted and Step 2 repeated (Table 2.10).

Successive steps are carried out until all the real costs in the cells either equal or exceed the sum of the part costs for sending and receiving. When this state is reached the optimum solution has been achieved (Table 2.11).

It is essential that only one unused cell which can show a cost saving is dealt with at a time.

		a	*b*	*c*	*Total*
		6	0	4	
A	0	$4-\theta$; 6	8	$2+\theta$; 4	6
B	−2	10; 4	9	3	10
C	2	$+\theta$ [7]; 1	12; 2	$3-\theta$; 6	15
Total		14	12	5	31

TABLE 2.10

Step 3—θ = 3 tons.
Total cost = £93.

		a	*b*	*c*	*Total*
		6	7	4	
A	0	1; 6	8	5; 4	6
B	−2	10; 4	9	3	10
C	−5	3; 1	12; 2	6	15
Total		14	12	5	31

TABLE 2.11

No further improvement possible. Minimum cost £93.

		a	*b*	*c*	*Total*
A		6	6 / 8	4	6
B		4	5 / 9	5 / 3	10
C		14 / 1	1 / 2	6	15
Total		14	12	5	31

TABLE 2.12

Step 1—basic feasible solution by commencing the allocation at the cell with the lowest cost and continuing to the next lowest cost cell, etc. Total cost = £124.

As an alternative to the north-west corner rule the first basic feasible solution can be constructed by commencing the allocation at the cell with the lowest cost and continuing to the cell with the next lowest cost, etc. A better first solution (i.e. lower cost) is generally obtained in comparison with the solution obtained with the north-west corner rule. This is illustrated in Table 2.12.

Degeneracy

If at any stage of the computation the allocation uses less than $m+n-1$ routes the problem is degenerate. With only, for example, $m+n-2$ routes in use it is not possible to compute all the part costs. The difficulty is overcome by increasing the requirements of each customer by a very small quantity, x, and the production of one colliery by nx. There will always be $(m+n-1)$ routes in use if the problem is worked with these slight adjustments. When the final answer has been obtained the x's are ignored.

Demand and capacity unequal

If the total that can be sent exceeds the total that is required, then introduce a dummy customer who requires the difference. The transport costs to this customer are all zero. Similarly if the total requirement is greater introduce a dummy colliery.

Transhipment

Many problems have a choice of methods of transport between supplier and consumer, e.g. road and rail. The methods of transport are often restricted in capacity.

Problems of this type can be set up for solution by the transportation technique by introducing artificial depots for each form of transport for each supplier and consumer. The transport costs between the suppliers and their respective depots and the consumers and their respective depots are zero. The transport costs between depots are the same as those between the suppliers and consumers. The distribution of goods is now considered between depots.

To illustrate the method consider two collieries A and B supplying two washeries (where coal is cleaned and graded) 1 and 2 by either road or rail. Artificial road and rail depots are introduced at the collieries

TABLE 2.13

		Washery 1	*Washery* 2	*Road Depots*				*Rail Depots*				*Total*
				Col. A	*Col. B*	*Wash.* 1	*Wash.* 2	*Col. A*	*Col. B*	*Wash.* 1	*Wash.* 2	
Colliery A				0				0				b_1
Colliery B					0				0			b_2
Road Depot	*Col. A*			0		c_{11}	c_{12}					b_{11}
	Col. B				0	c_{21}	c_{22}					b_{21}
	Wash. 1	0				0						a_{11}
	Wash. 2		0				0					a_{12}
Rail Depot	*Col. A*							0		d_{11}	d_{12}	b_{12}
	Col. B								0	d_{21}	d_{22}	b_{22}
	Wash. 1	0								0		a_{21}
	Wash. 2		0								0	a_{22}
Total		a_1	a_2	b_{11}	b_{21}	a_{11}	a_{12}	b_{12}	b_{22}	a_{21}	a_{22}	

Where b_i = colliery capacity
a_j = washery requirement
c_{ij} = road costs
d_{ij} = rail costs
a_{1j} = washery road capacity
a_{2j} = washery rail capacity
b_{i1} = colliery road capacity
b_{i2} = colliery rail capacity

The cells without a cost are impossible routes.

and washeries and it is assumed that it is necessary to send from each colliery to one of its depots and to receive at each washery from one of its depots. The resulting table (2.13) can now be solved by the transportation technique.

The assignment technique

A special class of allocation problems is the problem of finding the optimal allocation of a number of facilities (e.g. men) to an equal number of tasks (e.g. machines). Each facility may be assigned to only one task. The performance of each facility at each task is known.

The problem can be solved by the transportation technique by considering each facility as a despatch point with one unit of goods available and considering each task as a receiving point requiring one unit of goods. An alternative, and in general a simpler, method of solution is given by the assignment technique (9).

The assignment technique consists of a series of arithmetic operations on the $N \times N$ matrix. The steps of the procedure are repeated until at least N lines, i.e. rows or columns, are required to cover all the zeros in the modified table. When this condition is satisfied, a selection of N zeros such that there is only one of these zeros in each row and column gives the optimal allocation.

Example

A team of 5 horses and 5 riders is entered for a show jumping contest. The number of penalty marks to be expected when each rider rides any horse is shown in Table 2.14. How should the horses be allocated to the riders so that the team loses the smallest number of points?

Step 1. Subtract the smallest element in each row from every element in its row. There will now be at least one zero in each row.

Find the smallest number of lines N_1 that contains *all* the zeros.

If $N_1 = N$ (i.e. 5) an optimal solution has been reached.

If $N_1 < N$ carry out next step.

The position at the end of Step 1 is shown in Table 2.15.

Step 2. Subtract the smallest element in each column from every element in its column.

Find the smallest number of lines N_2 that contains all the zeros.

If $N_2 < N$ carry out next step.

The position at the end of Step 2 is shown in Table 2.16.

Step 3. Let x = the smallest of the elements not covered by the lines N_2.

Subtract x from each of the elements not covered by the lines N_2.
Add x to the elements at the intersections of the lines N_2.
Find the smallest number of lines N_3 that contains all the zeros.
If $N_3 < N$ carry out next step. The position at the end of Step 3 is shown in Table 2.17.

Step 3 is now repeated until a table for which $N_n = N$ is obtained. In this example Step 4 gives the optimal allocation. A set of 5 zeros can now be chosen such that there is only one zero in each row and column. The allocation is shown in Table 2.18.

It should be noted that Steps 1 and 2 can be interchanged, i.e. Step 1 could have been an operation on columns and Step 2 an operation on rows. Whether the first operation is on rows or columns can be chosen for convenience.

If the problem is one of maximisation it can be turned into a minimisation problem by subtracting each element in the matrix from the largest element. The procedure outlined above would then apply. The maximisation/minimisation transformation and the first step of the procedure can be combined by subtracting each element from the largest element in its row (or column).

Rider	*A*	*B*	*C*	*D*	*E*	*Row Minima*
Horse a	10	5	9	18	11	5
b	13	19	6	12	14	6
c	3	2	4	4	5	2
d	18	9	12	17	15	9
e	11	6	14	19	10	6

TABLE 2.14

Number of penalty marks expected from each horse/rider combination. $N = 5$.

	5	0	4	13	6
	7	13	0	6	8
	1	0	2	2	3
	9	0	3	8	6
	5	0	8	13	4
Col. Minima	1	0	0	2	3

TABLE 2.15

Step 1 Row minima subtracted from row elements. $N_1 = 2 \quad (< 5)$.

4	0	4	11	3
6	13	0	4	5
0	0	2	0	0
8	0	3	6	3
4	0	8	11	1

TABLE 2.16

Step 2 Col. minima subtracted from col. elements. $N_2 = 3$.

3	0	3	10	2
6	14	0	4	5
0	1	2	0	0
7	0	2	5	2
3	0	7	10	0

TABLE 2.17

Step 3 Minimum of elements not included in $N_2 = 1$.

1 subtracted from elements not covered by N_2. 1 added to N_2 intersection elements. $N_3 = 4$.

*0	0	3	7	2
3	14	*0	1	5
0	4	5	*0	3
4	*0	2	2	2
0	0	7	7	*0

TABLE 2.18

Step 4 Minimum of elements not in $N_3 = 3$. 3 subtracted from elements not in N_3. 3 added to N_3 intersection elements. $N_4 = 5$ . . . optimal solution reached.

Minimal allocation is shown by asterisks.

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CHAPTER 3

THEORY OF GAMES

Introduction

It will always be necessary to make decisions without being in complete control of all the relevant factors. The situation existing after such a decision has been made depends not only on the decision itself, but also on what happens to the uncontrollable factors. These uncontrollables can take the form of decisions made by outside interested parties, or the workings of disinterested forces. A simple example of this can be seen by considering the question of colliery reorganisation. From the past performance and behaviour of the colliery it may be decided that a substantial reorganisation should be carried out. The behaviour of the colliery after the reorganisation will, however, not only depend on this reorganisation but will also depend on, say, the attitude of the men and the natural conditions prevailing in the colliery: men may object to redeployment or an extensive thinning of a seam may be discovered.

The theory of games is being developed to assist decision making in situations of this kind. At present, the theory can be used to find that decision which will ensure that the ensuing gain is the best possible, on the assumption that the uncontrollable factors have the greatest possible adverse effect. The gain under these conditions can also be found.

There are six properties necessary to the situation in order that any one of the interested parties may apply the theory of games; these are:

(*a*) There are a finite number of participants (i.e. interested and disinterested parties).

(*b*) Each participant has a finite number of possible courses of action.

(*c*) The participant wishing to apply the theory must know all the courses of action available to the others but must not know which of these will be chosen.

(*d*) After all have chosen a course of action their respective gains are finite.

(*e*) The gain of the participant depends upon the actions of the others as well as himself.

(*f*) All possible outcomes are calculable.

Situations which have all of these properties are called games. In the present state of development of the theory only games in which two parties are involved and in which the losses of one are the gains of the other can be solved. These are called two person zero sum games.

In this chapter we are primarily concerned with two person zero sum games and situations involving one interested party in which the other participants are disinterested forces. Simple methods of solving small games are given first and then general methods using linear programming by which all types of two person zero sum games can be solved. The last part of the chapter contains a short discussion of situations involving more than two interested parties.

Description of two person zero sum games

In this section are discussed situations involving two interested parties each of which is trying to gain as much as possible at the expense of the other.

Useful terminology

The terminology often used in the theory of two person zero sum games is given below:

(*a*) Each interested party is called a player.
(*b*) A play of the game results when each player has chosen a course of action.
(*c*) After each play of the game one player pays the other an amount determined by the courses of action chosen.
(*d*) The decision rule by which a player determines his course of action is called a strategy. (To reach the decision regarding which strategy to use, neither player needs to know the other's strategy.)
(*e*) A mixed strategy is one where a player decides in advance to use all or some of his available courses of action in some fixed proportion. If a player decides to use only one particular course of action he is said to use a pure strategy.
(*f*) The value of a game is the average amount that one of the players would win in the long run if both players use their best strategies.
(*g*) A gain matrix is a table showing the amounts received by the player named at the left-hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

Setting up the gain matrix

For convenience we will call the players A and B. If they have N and M courses of action respectively, there are $N.M$ possible outcomes to every play. A table is constructed in which the columns are identified by B's courses of action and the rows are identified by A's courses of action. At the intersection of each row and column is indicated the amount that A would receive were the corresponding courses of action chosen by A and B.

This is the gain matrix of the game. This game is known as an $N \times M$ game, since A has N possible courses of action and B has M.

For example, the table below represents the gain matrix of a 3×4 game.

		B's courses of action			
		T	U	V	W
A's courses	X	a	b	c	d
of action	Y	e	f	g	h
	Z	k	l	m	n

Gain Matrix of a 3×4 game.

This matrix shows that were A to choose his course of action Z, and B choose his course of action T, then A would be paid an amount k by B and similarly for other cells in the table.

The gains given in the table represent payments by B to A. If we redraw the table so that B's courses of action identified rows and A's courses of action identified columns and if we also reversed the signs of the gains, the new table would represent the gain matrix for payments by A to B.

If the rules of the game are such that the payments of one player to another depend on chance, the gains shown would be the expected value of the gain in each case. For example, if when B chose course U and A chose Y, a coin was tossed to decide whether B should pay p or q units to A, the figure representing the gain to A in the table would be put equal to $\frac{1}{2}(p+q)$, i.e. its expected value.

If in the course of a series of plays A chooses courses X, Y and Z at random in the proportions $x:y:z$ respectively and B similarly chooses courses T, U, V and W in the porportions $t:u:v:w$ respectively, then A's strategy is written $A(x, y, z)$ and B's strategy is written $B(t, u, v, w)$.

A game is solved when the following has been determined:

(*a*) The average amount per play that A will win in the long run if A and B each uses his best strategy. This is the value of the game.
(*b*) The strategy to be used by A to ensure that his average gain per play is at least equal to the value of the game.
(*c*) The strategy to be used by B to ensure that his average loss per game is no more than the value of the game.

Saddle points

Suppose a game has the following gain matrix:

		B 1	2	3	
	1	3	−1	2	−1
A	2	−4	−1	13	−4
	3	2	−2	−1	−2
		3	−1	13	

Now player B is trying to minimise the value of the game, and player A to maximise it.

On studying the matrix, player A will note that by always playing A_1 he cannot lose more than 1, A_2 restricts his losses to 4, and A_3 to 2. Thus, although A may be tempted to play A_2 and stand a chance of winning 13, if player B is using B_1 A may lose 4.

Player B notices that by sticking to B_2 throughout he cannot lose no matter what A may do.

Thus in this game each player has a best strategy which entails using the same course of action throughout. The solution to this game is written $A(1, 0, 0)$; $B(0, 1, 0)$. The value of the game to A is -1. Note that if A deviates from his best strategy he will lose 1 or 2, and if B does he will lose 3 or 2; this assumes in each case that the opponent continues to play his best strategy.

When the solution involves each player using only one course of action throughout (i.e. a grand strategy) the game is said to have a saddle point. The saddle point is the point of intersection of the two courses of action and the gain at this point is the value of the game.

Rule to detect a saddle point

		B 1	2	3	4	min
	1	1	7	3	4	1
A	2	5	6	4	5	(4)
	3	7	2	0	3	0
	max	7	7	(4)	5	

(i) At the end of each row write the row minimum and ring the largest of them.
(ii) At the bottom of each column write the column maximum and ring the smallest.
(iii) If these two elements are the same, the cell where the row and column meet is a saddle point, and the element in that cell is the value of the game.
(iv) If the two ringed elements are unequal, there is no saddle point, and the value of the game lies between these two values.

If there is no saddle point neither player can optimise his chances by using a pure strategy, and they must mix some or all of their courses of action. Methods of finding their strategies in this case are a little more complicated, and are shown below.

It is important to look for a saddle point before attempting to solve a game.

Dominance

Consider the following gain matrix for *A*.

		B 1	2	3	4
A	1	2	2	3	4
	2	4	3	2	2

There is no saddle point.

B's strategy 1 loses him 2 and 4 against $A1$ and $A2$ respectively, whereas by playing B_2 he loses 2 and 3.

Thus he should ignore strategy 1 since he can always do better by using B_2. B_1 is said to be dominated by B_2.

Similarly B_4 is dominated by B_3.

Thus the game reduces to

	B	
A	2	3
	3	2

Rule for dominance

(*a*) If all the elements in a column, are greater than or equal to the corresponding elements in another column, then that column is dominated.

(*b*) Similarly, if all the elements in a row are less than or equal to the corresponding elements in another row, that row is dominated.

Dominated rows or columns may be deleted which reduces the size of the game.

Always look for dominance when solving a game.

Special methods for two person zero sum games

Two by two games.

If there is no saddle point, it means that both players must mix their courses of action to optimise their returns.

For example:

		B 1	B 2	
A	1	a_1	a_2	$\lvert b_1 - b_2\rvert$
	2	b_1	b_2	$\lvert a_1 - a_2\rvert$
		$\lvert a_2 - b_2\rvert$	$\lvert a_1 - b_1\rvert$	

This game is solved as follows:

(i) Subtract the two digits in column 1 and write them under column 2, ignoring sign.
(ii) Subtract the two digits in column 2 and write them under column 1, ignoring sign.

Similarly with the rows.

These are called oddments. They are the frequencies with which the players must use their courses of action in their best strategy.

To obtain the value of the game any one of the following expressions may be used:

B plays B_1; $$V = \frac{a_1|b_1-b_2|+b_1|a_1-a_2|}{|b_1-b_2|+|a_1-a_2|}$$

B plays B_2; $$V = \frac{a_2|b_1-b_2|+b_2|a_1-a_2|}{|b_1-b_2|+|a_1-a_2|}$$

or using B's oddments.

A plays A_1; $$V = \frac{a_1|a_2-b_2|+a_2|a_1-b_1|}{|a_2-b_2|+|a_1-b_1|}$$

A plays A_2; $$V = \frac{b_1|a_2-b_2|+b_2|a_1-b_1|}{|a_2-b_2|+|a_1-b_1|}.$$

The above values of V are only equal if the sums of the oddments vertically and horizontally are equal. Cases in which this is not so must be treated as described later.

Example

A has two ammunition stores, one of which is twice as valuable as the other. B is an attacker who can destroy an undefended store but he can only attack one of them. A can only successfully defend one of them.

What should A do so as to maximise his return from the situation, no matter what B may do?

If the value of the small store is 1, the value of the larger store is 2.

If both stores survive, A loses nothing.
If only the larger survives, A loses 1.
If only the smaller survives, A loses 2.

Since both parties have only two possible courses of action, the gain matrix for A is as follows:

		B 1 Attack the smaller store	B 2 Attack the larger store
A	1 Defend the smaller store	0 Both survive	−2 The larger one destroyed
A	2 Defend the larger store	−1 The smaller one destroyed	0 Both survive

There is no saddle point.

Using the rules described to solve 2×2 games we get:

	B 1	B 2	
A	0	−2	1
A	−1	0	2
	2	1	

Thus A should mix his courses of action in the ratio of 1:2.

The value of the game to A is:

Against B_1 $\quad \dfrac{1\times 0-1\times 2}{3} = -\frac{2}{3}$

Against B_2 $\quad \dfrac{-2\times 1+0\times 2}{3} = -\frac{2}{3}$.

The full solution is $\quad A(1, 2)$

$B(2, 1)$

$V = -\frac{2}{3}$.

This means that each time A has to make a decision in this situation he should choose at random from a distribution which has 1 one and 2 twos.

Two × N games

These are games in which one player has only two courses of action open to him, whilst his opponent may have any number.

Thus:

		B 1	B 2
	1	1	5
	2	2	4
A	3	3	3
	4	4	2
	5	5	1

or

		B 1	2	3	4	5	6
A	1	0	9	8	7	4	2
	2	10	1	2	3	6	8

Their solution involves picking out a 2 × 2 sub-game which fits the 2 × N game; it is therefore desirable to reduce the size of the game as much as possible before the solution is tackled. If it has no saddle point and cannot be reduced by dominance consideration to a two by two, there will be a 2 × 2 sub-game within the 2 × N which is a solution of the 2 × N game.

Consider the following game:

		B 1	B 2	B 3
A	1	−6	4	−1
A	2	7	−5	−2

Since there is no dominance or saddle point, we pick out each 2×2 game in turn, and find its solution.

Trial 1. Ignoring B_3

		B 1	B 2
A	1	−6	4
A	2	7	−5

The oddments are $A(12, 10)$; $B(9, 13)$.
The best strategies are $A(6, 5)$; $B(9, 13)$.

$$\text{The value to } A = \frac{6\times(-6)+5\times 7}{11} = \frac{-36+35}{11} = \frac{-1}{11}.$$

We now test this solution against B's ignored strategy.
So long as the value is the same, or better, for A the solution will satisfy the 2×3 game.

$$\text{Value to } A \text{ against } B_3 = \frac{6\times(-1)+5(-2)}{11} = \frac{-6-10}{11} = \frac{-16}{11}.$$

Thus B can improve his chances by playing 3.

Trial 2. Ignoring B_2

		B 1	B 3
A	1	−6	−1
A	2	7	−2

The oddments are $A(9, 5)$; $B(1, 13)$.

$$\text{The value to } A = \frac{-9-10}{14} = \frac{-19}{14}.$$

$$\text{The value to } A \text{ against } B_2 = \frac{36-25}{14} = \frac{11}{14}.$$

This is a solution to the 2×3 game, since on average B can cause A to lose $\frac{19}{14}$ by playing his best strategy; if, however, he decides to play B_2 he will lose $\frac{11}{14}$ so long as A sticks to his best strategy.

The full solution is $A(9, 5)$; $B(1, 0, 13)$.

The value to $A = -1\frac{5}{14}$.

Example of a $2\times N$ *game*

This type of game situation where player A has N courses of action is very similar to the previous example.

Consider the following game:

A \ B	1	2
1	-6	7
2	0	-3
3	6	-8
4	$-\frac{3}{2}$	2

This game has no saddle point, and it cannot be reduced by dominance.

Trial 1. A plays A_1 and A_2.

A \ B	1	2	
1	-6	7	3
2	0	-3	13
	10	6	
	5	3	

$V_A = -\frac{9}{8}$

Now test B's strategy against A's two unused courses of action.

$$A_3;\quad V_A = \frac{30-24}{8} = +\frac{6}{8}$$

$$A_4;\quad V_A = \frac{-\frac{15}{2}+6}{8} = -\frac{3}{16}.$$

$\therefore$ A can do better by playing either of these.

Trial 2. A plays A_1 and A_3.

		B 1	B 2	
A	1	-6	7	14
	3	6	-8	13
		15 / 5	12 / 4	

$V_A = -\frac{2}{9}$

Test against A_2 and A_4:

$$A_2;\quad V_A = \frac{0-12}{9} = -\frac{12}{9}$$

$$A_4;\quad V_A = \frac{-\frac{15}{2}+8}{9} = +\frac{1}{18}.$$

$\therefore$ A can do better by playing A_4.

Trial 3. A plays A_1 and A_4.

		B 1	B 2
A	1	-6	7
	4	$-\frac{3}{2}$	2

This has a saddle point at A_4, B_1.
Since the 2×4 game has no saddle point, this cannot be a solution.

Clearly if B plays B_1 continually, A can improve on $-\frac{3}{2}$ by playing A_2 or A_3.

Trial 4. A plays A_3 and A_4.

		B 1	B 2			
A	3	6	-8	$\frac{7}{2}$	1	$V_A = 0$
	4	$-\frac{3}{2}$	2	14	4	
		10	$\frac{15}{2}$			
		4	3			

Test against A_1 and A_2:

$$A_1;\quad V_A = \frac{-24+21}{7} = -\frac{3}{7} \qquad\qquad A_2;\quad V_A = \frac{0-9}{7} = -\frac{9}{7}.$$

$\therefore$ A cannot improve by playing his two ignored strategies.
Thus the full solution is:

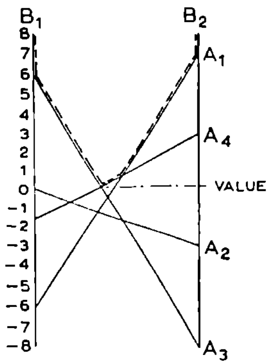
$A(0, 0, 1, 4)$ Value to $A = 0$.
$B(4, 3)$

Clearly as the $2 \times N$ or $N \times 2$ game gets larger, this method gets very tedious but a pointer to the solution can be obtained as follows:

Graphical treatment of a $2 \times N$ *game*

Using the same game as before;

		B 1	B 2
A	1	-6	7
	2	0	-3
	3	6	-8
	4	$-\frac{3}{2}$	3



Plot the points of each of A's strategies on two vertical axes corresponding to B's strategies, connect the points as shown and bound the figure from above.

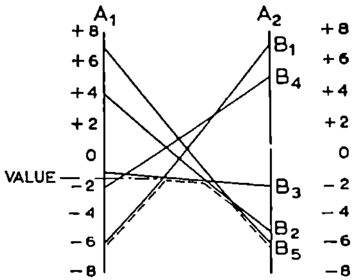
The two lines which intersect at the lowest point of the bound show the two courses of action A should use in his best strategy, i.e. A_4 and A_3.

The figure also shows that the value of the game is approximately zero. We can thus reduce the $2 \times N$ to a 2×2 game immediately.

This method can also be used on games such as:

		B 1	2	3	4	5
A	1	−6	4	−1	−2	7
	2	+7	−5	−2	5	−6

Using vertical axes for *A*'s strategies we now bound the figure from *below* and the *highest* point of the bound shows the two courses of action which B should use in his best strategy.



Solution is: B_1 and B_3 and the value to $A \doteqdot -1{\cdot}5$.

The reduced game is:

		B 1	3	
A	1	−6	−1	9
	2	7	−2	5
		1	13	

The solution is: $A(9, 5)$; $B(1, 0, 13, 0, 0)$

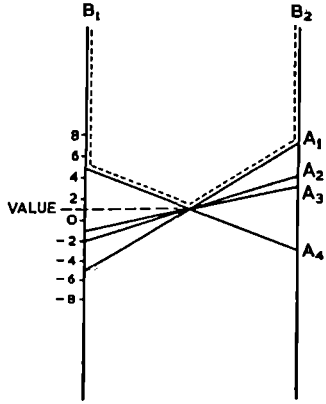
$$V_A = \frac{7-26}{14} = \frac{-19}{14} = -1\tfrac{5}{14}.$$

Against B_2; $V_A = \dfrac{36-25}{14} = \dfrac{11}{14}$ etc.

This method falls down if there is more than one solution.

Example

		B 1	2
	1	−5	7
	2	−2	4
A	3	−1	3
	4	5	−3



As B has only two possible strategies, we look at the diagram and bound it from above. All A's strategies pass through the lowest point. The solution is found to be:

$A(2, 0, 0, 3)$, $A(0, 4, 0, 3)$, $A(0, 0, 2, 1)$;

$B(1, 1)$, $B(1, 1)$, $B(1, 1)$; $V = 1$.

The graphical treatment does not completely solve the game, but indicates the sub-game which has the same solution as the $2 \times N$ game.

The last example has an infinity of solutions, since any linear combination of A's strategies will be a solution to the game.

We see then that although every game has a solution, it is not necessarily unique, even though all solutions must give the same value of the game.

Three by three games

Consider the following game:

		B 1	2	3
	1	6	0	6
A	2	8	−2	0
	3	4	6	5

First check for saddle point and dominance.

If there is dominance remove the dominated Row(s) and/or Column(s) and solve the resulting game by methods already given.

Here there is none.

		B		
	6	0	6	0
A	8	−2	0	−2
	4	6	5	(4)
	8	(6)	(6)	

The value lies between +6 and +4

Subtract each row from the row above, and each column from the column to the left.

Thus

		B 1	2	3		
	1	6	0	6	6	−6
A	2	8	−2	0	10	−2
	3	4	6	5	−2	1
		−2	2	6		
		4	−8	−5		

To obtain the oddments:

for B_1 evaluate the determinant, $\left\|\begin{matrix} 2 & 6 \\ -8 & -5 \end{matrix}\right\| = 38,$

for B_2 evaluate the determinant, $\left\|\begin{matrix} -2 & 6 \\ 4 & -5 \end{matrix}\right\| = 14.$

Proceeding in this fashion for A and B we obtain:

		B 1	2	3			
	1	6	0	6	6	−6	6
A	2	8	−2	0	10	−2	6
	3	4	6	5	−2	1	48
		−2	2	6			
		4	−8	−5			
		38	14	8			60

Having worked out the oddments, they are written down, regardless of sign. Since both the sums of oddments are here the same, this is a solution to the game. If the sums are different, both players do not use all their courses of action in their best strategy, and an alternative method will be needed.

The solution is:

$$A(1, 1, 8)$$
$$B(19, 7, 4).$$

Finally check that these oddments give the same value to the game against each of the opponent's plays.

Thus

$$A \text{ v. } B_1; \quad V = \frac{1\times6+1\times8+8\times4}{10} = \frac{46}{10} = \frac{23}{5},$$

$$A \text{ v. } B_2; \quad V = \frac{1\times0+1\times(-2)+8\times6}{10} = \frac{46}{10} = \frac{23}{5},$$

etc., and similarly for *B*.

This method only works if *both* players use all their plays in their best strategy.

If this method breaks down it means that the players do not use all of their courses of action, and this has not been detected by dominance.

In order to solve such a game we could proceed similarly to the method of $2\times N$ games. This, however, will be long and tedious, since in a 3×3 there are three 3×2 games each having three 2×2 games. As the gain matrices get larger than 3×3 the amount of work involved increases.

Summary of special methods of solution

Games of any size can be solved by the methods described above but for larger than 3×3 a quicker method of solution is described in the next section.

The procedure necessary to solve two person zero sum games using the methods already described is:

(*a*) Look for a saddle point or points. If one is found, the game is solved and the value of the game is the gain shown at the saddle point.

(*b*) Look for dominance. If dominance is found, delete the dominated row(s) and/or column(s). Each matrix so formed must be checked for dominance.

(*c*) The square matrix or matrices of the highest order in the game remaining after steps (*a*) and (*b*) should be treated as follows:

Subtract each column from the one to its left, and write the result at the side of the matrix. These new columns are used to calculate the oddments for *A*.

Subtract each row from the one above it and write the result below the matrix. These new rows are used to calculate the oddments for *B*.

If the sum of *A*'s oddments equals the sum of *B*'s, the reduced game is solved and the value of the game can immediately be found using any of *A*'s or *B*'s courses of action against the other's mixed strategy.

If these sums are unequal, all the matrices of order one less than the one must be considered; first looking for a saddle point, etc. This process must be repeated until a matrix is formed whose solution is a solution to the original game. The applicability of the solution of a reduced game is tested by applying each player's strategy to the other player's unused courses of action. The solution applies if the values of the game so found are always less for *A*'s unused course(s) of action and always more for *B*'s unused course(s) of action.

Solution of two person zero sum games by linear programming

Linear programming provides a more general method of solution which applies to any two person zero sum game. However, the procedure given in the last section should always be used except in the following cases:

(i) If the gain matrix is 4×4 or higher order and the game must be reduced in order to solve by the methods above.

(ii) When player *B* is Nature and it is possible from past experience to learn what constraints affect Nature's ability to do its worst. This is illustrated in two examples given later. By the linear programming method constraints on nature can be introduced into the solution, and the players' optimum strategy evaluated.

In any case, for games in which neither player has constraints on his actions, saddle points and dominance should be looked for first.

The method of solution by linear programming will be explained by considering the following game:

		B 1	2	3
	1	6	0	3
A	2	8	−2	3
	3	4	6	5

There is no dominance, no saddle point, and the sums of the oddments for *A* and *B* are not equal.

Thus

	B					
	6	0	3	6	−3	0
A	8	−2	3	10	−5	0
	4	6	5	−2	1	0
	−2	2	0			
	4	−8	−2			
	4	4	8			0
						16

A's strategy is indeterminate.

We could now proceed by reducing the game, which in this case would not take long, but with 4×4 and larger games it would be prohibitive.

Consider the game from B's point of view. Let the value of the game to A be V. B is trying to minimise V.

Let $B(y_1, y_2, y_3)$ be his optimum strategy. Then,

$$\left.\begin{array}{lll} \text{against } A_1 & 6y_1+0y_2+3y_3 \leqq V \\ \quad\text{,,} \quad A_2 & 8y_1-2y_2+3y_3 \leqq V \\ \quad\text{,,} \quad A_3 & 4y_1+6y_2+5y_3 \leqq V \end{array}\right\} y_i \geqq 0$$
$$\text{and} \quad y_1+y_2+y_3 = 1.$$

To make the inequations into equations insert slack variables p, q, r which are non-negative.

$$\begin{aligned} 6y_1 \qquad +3y_3+p \qquad\qquad &= V \\ 8y_1-2y_2+3y_3 \quad +q \qquad &= V \\ 4y_1+6y_2+5y_3 \qquad +r &= V \\ y_1+y_2+y_3 \qquad\qquad &= 1. \end{aligned}$$

Re-write these equations as:

$$\begin{aligned} 6y_1 \qquad +3y_3+p \qquad\qquad -V &= 0 \\ 8y_1-2y_2+3y_3 \quad +q \qquad -V &= 0 \\ 4y_1+6y_2+5y_3 \qquad +r-V &= 0 \\ y_1+y_2+y_3 \qquad\qquad &= 1. \end{aligned}$$

V (the value of the game) can be either positive or negative and to ensure that all the variables in the problem are non-negative (a requirement of any L.P. problem), V is expressed as the difference of two positive variables, i.e. $V = V_1 - V_2$.

In the Allocation chapter the difficulty in obtaining a first feasible solution is mentioned and is resolved by introducing an artificial variable s with very high value, M. This is to ensure that s will be zero in the final solution. We can now write the equations as:

$$\begin{aligned} 6y_1 \qquad +3y_3+p \qquad\qquad\qquad -V_1+V_2 &= 0 \\ 8y_1-2y_2+3y_3 \quad +q \qquad\qquad -V_1+V_2 &= 0 \\ 4y_1+6y_2+5y_3 \qquad +r \qquad -V_1+V_2 &= 0 \\ y_1+y_2+y_3 \qquad\qquad +s \qquad\qquad &= 1, \end{aligned}$$

where $y_1, y_2, y_3 \geqq 0$; $p, q, r, s, \geqq 0$; $V_1, V_2 \geqq 0$ and we wish to minimise $V_1 - V_2 + Ms$.

Setting up these equations in a simplex tableau we have:

	y_1	y_2	y_3	p	q	r	s	V_1	V_2	P
p	6	0	3	1				-1	1	0
q	8	-2	3		1			-1	1	0
r	4	6	5			1		-1	1	0
s	1	1	1				1			1
w	0	0	0	0	0	0	$-M$	-1	1	

In order to start solving this problem the value in the w row and s column must be made zero. This is done by adding M times the s row to the w row.

	y_1	y_2	y_3	p	q	r	s	V_1	V_2	P
p	6	0	3	1				-1	1	0
q	8	-2	3		1			-1	1	0
r	4	6	5			1		-1	1	0
s	1	1	1				1			1
w	M	M	M	0	0	0	0	-1	1	M

As will be seen any of y_1, y_2 or y_3 may be introduced and one is chosen at random. Since the problem is one of minimising, the simplex iterations are continued until all the w row is negative or zero. The iterations are performed in the manner of Chapter 2 and either of the following two solutions arises:

	y_1	y_2	y_3	p	q	r	s	V_1	V_2	P
p	0	0	0	1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$
y_1	1	-1	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$
y_3	0	2	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$
V_1	0	0	0	0	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{14}{3}$	$+1$	-1	$\frac{14}{3}$
w	0	0	0	0	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{14}{3} - M$	0	0	$\frac{14}{3}$

i.e. $$y_1 = \tfrac{1}{3},\quad y_3 = \tfrac{2}{3},\quad V = V_1 - 0 = +\tfrac{14}{3}.$$

Player A's best strategy appears in the w row, under p, q, r with negative signs. Thus $A(0, 1, 5)$; $B(1, 0, 2)$ is the solution.

	y_1	y_2	y_3	p	q	r	s	V_1	V_2	P
p	0	0	0	1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$
y_1	1	0	$\frac{1}{2}$	0	$\frac{1}{12}$	$-\frac{1}{12}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$
y_2	0	1	$\frac{1}{2}$	0	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$
V_1	0	0	0	0	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{14}{3}$	$+1$	-1	$\frac{14}{3}$
w	0	0	0	0	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{14}{3}-M$	0	0	$\frac{14}{3}$

i.e. $y_1 = \frac{2}{3}, \quad y_2 = \frac{1}{3}, \quad V = V_1 - 0 = +\frac{14}{3}.$

Thus $A(0, 1, 5)$; $B(2, 1, 0)$.

An alternative method of solution by linear programming

The method of solution by linear programming is a general method which can be used to solve any two person zero sum game. The special significance of it will be seen when games against Nature are discussed in the next section. The following method can be used so long as there are no restrictions on Nature's courses of action.

Consider the following game:

		B: 1	2	3
A	1	3	-4	2
	2	1	-3	-7
	3	-2	4	7

This game has no saddle point, no dominance and it cannot be solved by the method shown in the section on 3×3 games.

When the matrix is tested for a saddle point we see that the value must lie between -2 and $+3$.

If 2 is added to every element in the matrix, the value of the new game will be non-zero and positive.

The method of solution as shown below cannot be used if the value of the game is negative or zero.

The new game is thus:

		B 1	B 2	B 3
	1	5	−2	4
A	2	3	−1	−5
	3	0	6	9

Let B's best strategy be (y_1, y_2, y_3).

$$\left.\begin{aligned} \therefore\quad 5y_1-2y_2+4y_3 &\leqq V \\ 3y_1-y_2-5y_3 &\leqq V \\ 6y_2+9y_3 &\leqq V \\ y_1+y_2+y_3 &= 1 \end{aligned}\right\} \quad (1)$$

Dividing through equations 1 by $V (V > 0)$ and putting $\frac{y_i}{V} = Y_i$ we get:

$$\left.\begin{aligned} 5Y_1-2Y_2+4Y_3 &\leqq 1 \\ 3Y_1-Y_2-5Y_3 &\leqq 1 \\ 6Y_2+9Y_3 &\leqq 1 \end{aligned}\right\} \quad (2)$$

$$Y_1+Y_2+Y_3 = \frac{1}{V} \quad (3)$$

Since B is trying to minimise V, he must maximise $\frac{1}{V}$;

∴ The solution will be given by maximising function (3) subject to the restrictions (2).

In this case, inserting non-negative slack variables p, q and r, we get:

1st simplex tableau

	Y_1	Y_2	Y_3	p	q	r	P
p	5	−2	4	1			1
q	3	−1	−5		1		1
r		6	9			1	1
w	−1	−1	−1				0

The first iteration replaces p by Y_1.

∴ *2nd simplex tableau*

	Y_1	Y_2	Y_3	p	q	r	P
Y_1	1	$-\frac{2}{5}$	$\frac{4}{5}$	$\frac{1}{5}$			$\frac{1}{5}$
q		$\frac{1}{5}$	$-\frac{37}{5}$	$-\frac{3}{5}$	1		$\frac{2}{5}$
r		6	9			1	1
w		$-\frac{7}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$			$\frac{1}{5}$

The second iteration replaces r by Y_2.

∴ *3rd simplex tableau*

	Y_1	Y_2	Y_3	p	q	r	P
Y_1	1		$\frac{7}{5}$	$\frac{1}{5}$		$\frac{1}{15}$	$\frac{4}{15}$
q			$-\frac{77}{10}$	$-\frac{3}{5}$	1	$-\frac{1}{30}$	$\frac{11}{30}$
Y_2		1	$\frac{3}{2}$			$\frac{1}{6}$	$\frac{1}{6}$
w			$\frac{19}{10}$	$\frac{1}{5}$		$\frac{7}{30}$	$\frac{13}{30}$

$$\therefore \quad V = \tfrac{30}{13}.$$

$$\therefore \quad y_1 = \tfrac{4}{15} \times \tfrac{30}{13} = \tfrac{8}{13}$$

$$y_2 = \tfrac{1}{6} \times \tfrac{30}{13} = \tfrac{5}{13}.$$

Also from the values in the w row under p, q and r:

$$x_1 = \tfrac{1}{5} \times \tfrac{30}{13} = \tfrac{6}{13}$$

$$x_3 = \tfrac{7}{30} \times \tfrac{30}{13} = \tfrac{7}{13}.$$

∴ The solution to the original game is

$A(6, 0, 7)$
$B(8, 5, 0)$
Value to $A = \frac{30}{13} - 2 = \frac{4}{13}$.

This method is much simpler than the previous one but care must be taken to ensure that the value of the game is neither zero nor negative before using it.

Games against Nature

Games against Nature are rather different from a normal two person zero sum game because Nature cannot be considered as an interested party. Nature does not know anything about the theory of games; thus we can often arrive at a better result for the player by imposing restrictions on nature's courses of action based upon previous experience.

When these restrictions are being used, saddle points, and dominant columns must be ignored, since Nature will not necessarily ignore dominated courses of action; Nature is not *trying* to do her best.

Bearing this in mind, and using all the knowledge at our disposal regarding Nature's activities, we can calculate what player *A* should do in order to maximise his winnings against the worst that Nature is able to do. These principles are illustrated by an example solved by linear programming.

Example

A firm making wireless sets discovers that a valve in one of their sets wears out too quickly. The firm which supplies the valves recommends that a new type be installed. The new valve costs less than the old one but has not been tested in these sets. The question is, would the change-over be economically worthwhile to the wireless firm?

The wireless firm has three alternative courses of action:

(1) To change over immediately to the new valves.
(2) Test the new valves in some sets and make the change if the trial is successful.
(3) Ignore the suggestion and carry on installing the old valves.

Nature has the following alternative courses of action:

(1) The new valves will appear to be a success from the trial but they will not overcome the problem.
(2) The new valves will appear to be a success from the trials and will overcome the problem.
(3) The trial will indicate that the new valves will be unsuccessful and they will in fact be successful.
(4) The trial will indicate that the new valves will be unsuccessful and they will in fact be unsuccessful.

If the present valves are used, an additional cost of 5 units will be incurred on the manufacture of these sets.

A trial would cost 1 unit to carry out and the cost of a complete change-over to the new valves would be 3 units.

If the new valves were used and found to be successful, a reduction of 10 units in the cost of manufacturing the sets would result.

If the trial is unsuccessful the wireless firm will be credited with 2 units by the valve manufacturers.

The gain matrix for the wireless firm is given below.

		Nature 1	Nature 2	Nature 3	Nature 4
Wireless firm	1	$(-3-5)$	$(10-3)$	$(10-3)$	$(-3-5)$
	2	$(-1-3-5)$	$(10-1-3)$	$(-1-5+2)$	$(-1-5+2)$
	3	(-5)	(-5)	(-5)	(-5)

		Nature 1	Nature 2	Nature 3	Nature 4
Wireless firm	1	-8	7	7	-8
	2	-9	6	-4	-4
	3	-5	-5	-5	-5

Let Nature's strategy be written $N(x_1, x_2, x_3, x_4)$.

The valve firm informs the wireless firm of the following:

(*a*) There is at least a 10% chance that the new valves will be successful,

i.e. $$x_2+x_3 \geqq 0{\cdot}1$$

or $$x_1+x_4 \leqq 0{\cdot}9.$$

(*b*) If the valves would be unsuccessful, the trial will indicate this with a probability of at least 0·7,

i.e. $$\frac{x_4}{x_1+x_4} \geqq 0{\cdot}7$$

or $$0{\cdot}7x_1-0{\cdot}3x_4 \leqq 0.$$

(*c*) There is at least an 80% chance that if the new valves would be successful the trial will indicate this,

i.e. $$\frac{x_2}{x_2+x_3} \geqq 0{\cdot}8$$

or $$-0{\cdot}2x_2+0{\cdot}8x_3 \leqq 0.$$

Hence the restrictions on Nature are:

$$\begin{aligned} x_1+x_4 &\leqq 0{\cdot}9\\ 0{\cdot}7x_1-0{\cdot}3x_4 &\leqq 0\\ -0{\cdot}2x_2+0{\cdot}8x_3 &\leqq 0\\ x_1+x_2+x_3+x_4 &= 1. \end{aligned}$$

The game is now solved from Nature's point of view using the restrictions and the inequations

$$\begin{aligned} -8x_1+7x_2+7x_3-8x_4 &\leqq V\\ -9x_1+6x_2-4x_3-4x_4 &\leqq V\\ -5x_1-5x_2-5x_3-5x_4 &\leqq V \end{aligned}$$

where V is the value of the game to the wireless firm.

We now proceed as shown before. Let $V = V_1-V_2$, and insert non-negative slack variables p, q, r, etc., to transform the inequations into equations

$$\begin{array}{llllllllll} -8x_1+7x_2+7x_3-8x_4+p & & & & & & & -V_1+V_2 & =0\\ -9x_1+6x_2-4x_3-4x_4 & +q & & & & & & -V_1+V_2 & =0\\ -5x_1-5x_2-5x_3-5x_4 & & +r & & & & & -V_1+V_2 & =0\\ x_1+x_2+x_3+x_4 & & & +s & & & & & =1\\ x_1+x_4 & & & & +t & & & & =0{\cdot}9\\ 0{\cdot}7x_1-0{\cdot}3x_4 & & & & & +u & & & =0\\ -0{\cdot}2x_2+0{\cdot}8x_3 & & & & & & +y & & =0 \end{array}$$

To minimise V_1-V_2+Ms
where M is a large positive number.

Simplex Tableau 1

	x_1	x_2	x_3	x_4	p	q	r	s	t	u	y	V_1	V_2	P
p	−8	7	7	−8	1							−1	1	0
q	−9	6	−4	−4		1						−1	1	0
r	−5	−5	−5	−5			1					−1	1	0
s	1	1	1	1				1						1
t	1			1					1					0·9
u	0·7			−0·3						1				0
y		−0·2	0·8								1			0
w								$-M$				−1	1	0

adding M times row s to row w we get

Tableau 2

	x_1	x_2	x_3	x_4	p	q	r	s	t	u	y	V_1	V_2	P
p	−8	7	7	−8	1							−1	1	0
q	−9	6	−4	−4		1						−1	1	0
r	−5	−5	−5	−5			1					−1	1	0
s	1	1	1	1				1						1
t	1			1					1					0·9
u	0·7			−0·3						1				0
y		−0·2	0·8								1			0
w	M	M	M	M								−1	1	M

Any of the x_1, x_2, x_3 or x_4 columns can now be chosen as the pivot column, and the usual simplex iterations performed.

The final tableau is:

	x_1	x_2	x_3	x_4	p	q	r	s	t	u	y	V_1	V_2	P
x_1	1								0·03	0·1				0·27
x_2		1						0·8	−0·08		−0·2			0·08
x_3			1					0·2	−0·02		0·2			0·02
x_4				1					0·07	−0·1				0·63
p					1	−1		−3	0·55	−0·5	−0·2			1·95
r						−1	1	9	−0·95	−0·5	−2			0·45
V_2						1		−4	0·95	0·5	2	−1	1	4·55
w						−1		+4 −M	−0·95	−0·5	−2			−4·55

The values of x_1, x_2, x_3, x_4 are given in column P and can be seen to be

$$x_1 = 0{\cdot}27$$
$$x_2 = 0{\cdot}08$$
$$x_3 = 0{\cdot}02$$
$$x_4 = 0{\cdot}63.$$

The numbers, ignoring sign in row w under columns p, q, r, show the wireless firm's strategy. In this case it is (0, 1, 0), i.e. the firm should adopt decision (*b*) and test the new valves before deciding to install them. The value of the game to the firm is $-4{\cdot}55$, which is a little better than the cost of the present difficulties. The value of the game is shown twice in the tableau. Once at the bottom right-hand corner and once more in the P column opposite V_2.

The value of the game against the firm's decision (*b*) is $-4{\cdot}55$, whereas against decision (*a*) the value is $-6{\cdot}5$, and against decision (*c*) it is -5.

Whatever Nature's strategy may be, the value will always be $-4{\cdot}55$ or more.

The next section deals briefly with games involving more than two interested parties. These games are usually called N-person games.

N-person games

These games are usually treated as if two coalitions are formed by the N persons involved. The characteristics of such a game are the value of the various games between every possible pair of coalitions.

For 4 players there are 7 possible pairs of coalitions that can form, e.g. if the players are A, B, C, D the possible coalitions are:

A against BCD
B against ACD
C against ABD
D against ABC
AB against CD
AC against BD
AD against BC.

The value of the game to a coalition of one is considered to be the minimum value that player is prepared to accept.

A solution of such a game is said to be any set of values which is at least equal to the set of values of the game to the coalitions of one, e.g. if for the games A v. BCD, B v. ACD, C v. ABD and D v. ABC the values to A, B, C and D respectively are, say, 4, 6, 8 and 5, a solution to the main game would be any set of numbers V_1, V_2, V_3, V_4 where

$$V_1 \geqq 4$$
$$V_2 \geqq 6$$
$$V_3 \geqq 8$$
$$V_4 \geqq 5.$$

The coalitions that are formed determine which strategy should be adopted by each player.

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CHAPTER 4

DYNAMIC PROGRAMMING

Introduction

This account is largely based on the work of Richard Bellman. In many papers and in his book *Dynamic Programming* (Princeton Univ. Press, 1957), Bellman has presented this approach to multi-dimensional problems, i.e. problems involving optimisations of functions involving many variables.

Dynamic programming is basically concerned with the theory of multi-stage decision processes. By this rather formidable phrase we mean processes in which a sequence of choices has to be made, each choice being between two or more possibilities. These processes may be of a deterministic nature, i.e. once a decision is made the outcome is uniquely determined, or stochastic in the sense that the outcome is drawn from a set of possible outcomes according to some given frequency distribution.

An example of the deterministic type might be the periodic decision of a haulage contractor as to how many of his fleet of lorries he should replace. An example of the second is the amount of a commodity to be ordered on successive occasions by a retailer, when the demand for the commodity is known in terms of a probability distribution.

The processes we consider may be concerned with either a finite number of stages or a continual process involving an infinite number of successive decisions. Similarly the processes may be either discrete or continuous in that there may at each stage be either a finite number or an infinite number of choices possible.

Though natural multi-stage decision processes themselves arise sufficiently frequently for a theory dealing with their solution to be valuable, it is also often possible to regard other problems in this light, and to obtain solutions more readily as a result of applying the dynamic programming approach. Thus the theory is not confined to multi-stage decision processes and has a far wider area of application. We shall attempt to indicate its scope in this chapter.

We shall initially solve a simple problem making use of the dynamic programming approach and then present the basic principle of the theory. Subsequently, we shall consider more complex problems which we hope will throw more light on the nature of the dynamic programming approach and shall indicate its application to various practical problems. We shall also see how the theory is related to the linear programming approach. The section on formulation of the processes, and the subsequent theoretical sections, are mainly mathematical and may be omitted on a first reading.

A multi-stage deterministic problem

To illustrate the principle of the theory and perhaps to clarify the type of problem to which it may be applied, we shall consider the following simple example:

Suppose a farmer harvests z tons of wheat in a certain year. He obtains his next year's crop by sowing x tons of the z tons harvested. These x tons will yield ax tons at the next harvest, $a > 1$. The remaining wheat is sold, y tons selling for £$f(y)$. ($f(y)$ is monotonic increasing—i.e. the more wheat sold the greater the revenue.) If the farmer wishes to sell out after N further harvests, what policy of selling and sowing should he adopt in order to maximise his total income over the period? This is analogous to the economic problem—what proportion of income should be reinvested?

Let us first consider the problem in the following way. Suppose at the end of the first year the farmer sells y_1 tons and sows x_1 tons (naturally $y_1+x_1 = z$), then at the end of the second year he will have ax_1 tons to either sell or sow. Again, dividing these into x_2 and y_2, say, we have $x_2+y_2 = ax_1$. Continuing this process until the final harvest when ax_{N-1} tons are available, the farmer will obviously sell the whole crop so that $x_N = 0$ and $y_N = ax_{N-1}$. We may represent this situation in the following table:

Year	*Sells*	*Sows*	*Available the following year*
1	y_1	x_1	ax_1
2	y_2	x_2	ax_2
.	.	.	.
.	.	.	.
$N-1$	y_{N-1}	x_{N-1}	ax_{N-1}
N	y_N	0	—

We thus have the series of equations:

$$\left.\begin{aligned} z &= x_1+y_1 \\ ax_1 &= x_2+y_2 \\ ax_2 &= x_3+y_3 \\ &\cdot \\ &\cdot \\ ax_{N-2} &= x_{N-1}+y_{N-1} \\ ax_{N-1} &= y_N \\ x_N &= 0 \end{aligned}\right\} \quad (4.1)$$

Eliminating the x_i from equations (4.1) above we obtain:

$$\left.\begin{aligned} x_1 &= z-y_1 \\ x_2 &= za-(y_2+ay_1) \\ x_3 &= za^2-(y_3+ay_2+a^2y_1) \\ &\cdot \\ &\cdot \\ x_j &= za^{j-1}-(y_j+ay_{j-1}+ \ldots +a^{j-1}y_1) \\ &\cdot \\ &\cdot \\ x_{N-1} &= za^{N-2}-(y_{N-1}+ay_{N-2}+ \ldots +a^{N-2}y_1) \end{aligned}\right\} \quad (4.2)$$

From (4.2) we have:

$$ax_{N-1} = za^{N-1}-(ay_{N-1}+a^2y_{N-2}+ \ldots +a^{N-1}y_1) = y_N.$$

$$\therefore \quad za^{N-1} = y_N+ay_{N-1}+ \ldots +a^{N-1}y_1.$$

But the total income of the farmer over the period is

$$P = f(y_1) \ldots +f(y_N).$$

Thus, we have to maximise

$$P = \sum_{i=1}^{N} f(y_i)$$

subject to the constraint

$$za^{N-1} = y_N+ay_{N-1}+ \ldots +a^{N-1}y_1, \quad \text{and} \quad y_i \geqq 0 \text{ for all } i.$$

If $f(y)$ is linear or concave upwards, the problem is trivial, as it is obviously desirable to sell nothing until the final harvest. If, however, $f(y)$ is of any other form, it is impossible to solve by the techniques of linear programming and is extremely difficult to solve using other classical methods.

Let us now see how readily the problem breaks down by adopting an alternative approach. Let us define $g_k(q)$ as the total income obtained when starting with q tons and adopting an optimal policy of sowing and selling over a k year period; then we see immediately that

$$g_1(q) = f(q)$$

since, if the farmer is in business for one year only he obviously sells all his stock at the end of that year. Now consider $g_2(q)$. If the optimal policy is such that the farmer should sell y tons at the end of the first year then we have:

$g_2(q) = f(y) +$ the return from $a(q-y)$ tons over the one remaining year.

But as $g_2(q)$ represents the optimal policy, the remaining $a(q-y)$ tons must clearly be utilised themselves in an optimal manner, i.e.,

$$g_2(q) = f(y) + g_1[a(q-y)], \quad \text{if the } y \text{ is optimal.}$$

If we now consider the y as that which we wish to determine we have:

$$g_2(q) = \max_{0 \leq y \leq q} [f(y) + g_1\overline{a(q-y)}]$$

a simple maximisation problem involving only one variable. This may be immediately generalised to give:

$$g_m(q) = \max_{0 \leq y \leq q} [f(y) + g_{m-1}\overline{a(q-y)}]. \tag{4.3}$$

But we already have that $g_1(q) = f(q)$ so that we may proceed step by step considering successive single variable optimisations to determine $g_2(q)$, $g_3(q)$, etc.

As an example, take $f(y) = K\sqrt{y}$. Then,

$$g_1(q) = f(q) = K\sqrt{q},$$

and this may be calculated over a range of q. Then,

$$g_2(q) = \max_{0 \leq y \leq q} [f(y) + g_1\overline{a(q-y)}]$$

$$= \max_{0 \leq y \leq q} [K\sqrt{y} + K\sqrt{a(q-y)}].$$

For any q, the value of y which maximises this expression may be calculated (in this case by differentiation), and hence $g_2(q)$ determined over a range of q.

Similarly, $g_3(q)$, and in general $g_m(y)$, for all m. In particular we may find $g_N(z)$.

With more complicated forms of $f(y)$, the successive maxima might need to be determined numerically rather than by differentiation, but apart from this the method of solution would be identical. Hence, the computation is very simple, using the recurrence relation (4.3) and the problem may be solved whatever the form of $f(y)$. Though this method does, of course, involve computing in this example N separate functions, each of these, as has been pointed out, is very easily obtained, and the total amount of computation is very much less than that required when adopting the classical approach, of a single maximisation over N dimensions.

This provision of a simple algorithm for computing the total return in a relatively simple manner is typical of the application of dynamic programming. The approach does, in fact, reduce the dimensions of a problem, thereby making a solution more feasible. The progressive build up to the final function in easy stages is characteristic of the method.

With the final function $g_N(z)$ we now have the maximum return over the N year period. To obtain the policy of sowing and selling in each year which yields this maximum return we have to use the intermediate functions $g_1(q)$ to $g_N(q)$.

To obtain the first policy decision, consider

$$g_N(z) = \max_{0 \leq y \leq z} [f(y) + g_{N-1}\overline{a(z-y)}].$$

The particular value of y (say y_1) which in this equation yields a maximum for $g_N(z)$, represents the wheat sold in the first year of the N year period.

The amount of wheat available the following year is, therefore, $a(z-y) = z_1$, say. We now consider the function $g_{N-1}(z_1)$ and find the particular value of the dummy variable (i.e. y in equation (4.3)) which yields the maximum in the equation relating $g_{N-1}(q)$ to $g_{N-2}(q)$, for an argument value of $q = z_1$. This represents the amount of wheat sold in the second year. In this way we progress through the problem, using the successive "return functions" to determine the corresponding "policy functions".

It is also worth noting that in computing the solution in this way we necessarily obtain the solution to the problem for varying numbers of stages and for different amounts of wheat, and thereby gain insight into the form of the solution. Thus the problem is not solved in isolation, but its solution is extracted from the solution to a set of similar problems of the same form but varying parameters. This again is typical of dynamic programming.

The key to the solution of the above problem was contained in the statement: "clearly the remaining $a(q-y)$ tons must be utilised in an optimal manner". This observation is an application of Bellman's "Principle of Optimality".

The principle of optimality

Intrinsic in the above problem was the notion that if the farmer wanted to maximise his receipts with k years remaining then he had to make the best decision (i.e. select the best values of x and y) and following this decision he had the next year to make the (then) best decision with respect to the remaining $k-1$ year period and the wheat available at that time. Bellman states his principle of optimality in the following terms: "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." A proof by contradiction is immediate.

This principle forms the whole basis of the theory of dynamic programming and enables problems of many dimensions (which may be otherwise intractable) to be reduced to problems of far fewer dimensions and often, as in the wheat problem, to one dimension.

A multi-stage stochastic problem*

Let us consider a problem in which the outcome of any decision is not uniquely determined but depends upon a probability distribution, and see how once again the dynamic programming approach is able to shed much light on the problem. The problem we shall consider is simpler than the wheat problem in that there are only two decisions possible at each stage.

Suppose we are concerned with using a bomber so as to inflict maximum damage on some enemy. Our bomber has two possible enemy targets, A and B, to attack. A raid on target A will result either in a fraction r_1 of the enemy's resources in A being destroyed or in the bomber being shot down (before inflicting damage), the probability of the bomber surviving a mission to A being p_1. Target B is similarly

* This is a shortened and rephrased version of the gold-mining problem, given in Bellman, Chapter II.

associated with a fraction r_2 and a probability p_2. The enemy's resources initially are x at A and y at B. Then our problem might be, for instance, to determine what policy of attack we should follow in order to maximise the total expected damage to the enemy resources. (Other objectives are also possible: we might, perhaps, wish to maximise the probability of destroying at least half of each target.)

Let $$f_N(x, y)$$

be the expected damage if the optimal policy is followed when a maximum number of N raids is possible and the system starts with resources x at A and y at B.

Then $$f_1(x, y) = \max\,[p_1 r_1 x, p_2 r_2 y].$$

Now suppose the optimal N-raid policy starts with an attack on A. Then by applying the principle of optimality, the total expected damage is,

$$a = p_1(r_1 x + f_{N-1}[x(1-r_1), y]).$$

Similarly, if we attack B first the total expected damage is

$$b = p_2(r_2 y + f_{N-1}[x, y(1-r_2)])$$

and $$f_N(x, y) = \max\,(a, b).$$

If we are going to have an unlimited number of raids (except in so far as the aircraft may be shot down on any raid), the function $f_N(x, y)$ is replaced by the function $f(x, y)$ and the above equation becomes

$$f(x, y) = \max \begin{bmatrix} p_1(r_1 x + f[x(1-r_1), y]) \\ p_2(r_2 y + f[x, y(1-r_2)]) \end{bmatrix}$$

Let us now consider the case of an unlimited number of raids. For given values of p_i and r_i, if the resources at each target are increased in the same ratio, the damage resulting from any policy is also increased in this ratio. The choice of policy is, therefore, not affected by a change of resources from x, y to kx, ky, and depends only on the ratio $x:y$. It is clear that if, at any stage, when the resources are x, y, the best policy is to attack A, then it will also be best to attack A when the ratio of resources at A to those at B exceeds $x:y$. There must therefore be a value λ such that for $x:y > \lambda$, the best policy is to attack A, while for $x:y < \lambda$, B should be attacked.

To determine this value of λ, consider the condition when $x:y = \lambda$. When this applies either target may be attacked initially. However, if

A is attacked initially we come to the point $(x(1-r_1), y)$ which is necessarily in the B region so that our next attack will be on target B. Similarly, if we attack B first we land in the A region and must then choose A for our next attack.

In each case we finish at the point $(x(1-r_1), y(1-r_2))$ provided that the aircraft has not been destroyed. As the results of these two procedures must be equal we have the equation (considering the unbounded process):

$$p_1r_1x+p_1p_2r_2y+p_1p_2f[x(1-r_1), y(1-r_2)]$$
$$= p_2r_2y+p_1p_2r_1x+p_1p_2f[x(1-r_1), y(1-r_2)],$$

i.e.
$$x/y = p_2r_2(1-p_1)/p_1r_1(1-p_2) = \lambda.$$

We thus have a rule which enables us to determine which target to attack first and though we have not evaluated the final pay-off function we have all the information required to make the correct policy decision.

In this problem, therefore, we have determined a policy which we can use for making our decisions but have no knowledge of the pay-off to be achieved from using this policy. We shall consider the implications of this in the next section.

Returning to the limited case, it may also be shown that for a finite series of attacks the plane is again divided into two regions by a line $x = \lambda_N y$ where the value of λ_N is dependent on the number of stages (N). λ_N tends to λ as N tends to infinity and, in fact, for any values of p_1, p_2, r_1, r_2 there exists a number N_0 such that

$$\lambda_N = \lambda \quad \text{for} \quad N > N_0 \quad (N_0 \text{ dependent on } p_i, r_i).$$

So we may use the policy in the infinite process as an approximation to that in the finite one.

The mathematical formulation of dynamic programming processes

Deterministic processes

Suppose we have a system whose ith state is defined by a vector $P_i = (p_{i1} \dots p_{in})$. For example, in the wheat problem, the state variables (p_{ij}) are the quantity of wheat available, and the total income to date.

Let T_j be a transformation as a result of a decision j (on, say, how much wheat to sow) such that $T_j(P_i)$, giving some new state, say P_j, represents the effect of the decision, j, on the system in an initial state P_i.

Then in an N stage process we have a series of N decisions such that if P_0 is the initial state of the system:

$$
\begin{aligned}
P_1 &= T_1(P_0) \\
P_2 &= T_2(P_1) \\
&\dots \\
&\dots \\
P_N &= T_N(P_{N-1})
\end{aligned}
$$

and we wish to choose our transformations $T_1 \dots T_N$ so as to optimise some function of the final state vector $R(P_N)$.

Defining $f_N(P_0)$ as the optimum obtained from an N stage process commencing in state P_0, we can put $f_N(P_0) = \max R(P_N)$, where the maximum is taken over all states resulting from all possible sequences $T_1 \dots T_N$.

Then, in terms of these variables, Bellman's principle of optimality becomes

$$
\left.
\begin{aligned}
&f_N(P) = \max F_{N-1}(T(P)) \quad \text{for} \quad N \geqq 2 \\
\text{and} \quad & \\
&f_1(P) = \max R(T(P))
\end{aligned}
\right\}
\text{ where the maximum is taken over all possible } T.
$$

The above follows from our previous statements of the principle of optimality since clearly in order to optimise over N stages, the first stage must be chosen so as to give a new state which, when optimised over an $N-1$ stage process, itself gives an optimal final state. This equation is the basic recurrence relation for processes of this type.

If the process is unbounded (i.e. if N tends to infinity), then the above equation may be replaced by the functional equation

$$f(P) = \max f(T(P)).$$

Stochastic processes

Suppose the process under consideration is not deterministic but the result of a decision is dependent upon a probability distribution function of the state vector. Then the result of a decision q on an initial state p gives a result in $(z, z+dz)$ with probability $dG_q(p, z)$. The distribution $G_q(p, z)$ of the transformed state is dependent on both p and q.

Then if we wish to optimise the expected return we have, as a result of the decision q on the initial state p (this being considered as the first stage of an N stage process),

$$f_N(p) = \max_q \int_z f_{N-1}(z)\, dG_q(p, z)$$

with

$$f_1(p) = \max_q \int_z R(z)\, dG_q(p, z),$$

where we are maximising the expected value of $R(p_N)$.

Again, if the process is unbounded, the equation becomes

$$f(p) = \max_q \int_z f(z)\, dG_q(p, z).$$

The duality in dynamic programming processes

The mathematical formulations above may make it more apparent that there are always, in these problems, two alternative ways of approaching the solution.

For while $f(p)$ (or $f_N(p)$) is a pay-off function, since it represents the optimal return, we also have the decision at any time $q = q(p)$ as a policy function.

Thus we have a continuous duality throughout the process.

(*a*) Finding the pay-off obtained from using the optimal policy.
(*b*) Determining the policy which gives the optimal pay-off.

This ability to use either space to obtain the best solution proves useful when solving the functional equations given above.

The relationship of dynamic programming to linear programming

Consider the following example which is very similar to the classic nut-mix problem solved by the simplex method of linear programming.

Each year the walnut crop consists of different grades (1, 2, ... k) in quantities Q_1, Q_2, ... Q_k. Using various quantities of each grade assortments of walnuts are put together for commercial sale at different prices. Assuming there are fixed demands d_i for the ith assortment ($i = 1, \ldots n$) and that each assortment mixes walnuts of different grades in its own fixed ratios, how much of each assortment should be made in order to maximise the total profit?

Suppose the proportion of grade j in the ith assortment is p_{ij}. Then the total amount of grade j used is $\sum_{i=1}^{n} p_{ij} w_i$ where w_i is the weight of the ith assortment produced.

Then $w_i \leqq d_i$ for all i, and $\sum_{i=1}^{n} p_{ij}w_i \leqq Q_j$ for all j, and we wish to maximise $R = \sum_{i=1}^{n} r_i w_i$ where r_i is the selling price per unit weight of the ith assortment.

Let us now approach the problem using dynamic programming. If we produce only the first assortment, and have quantities $q_1, q_2, \ldots q_k$ of each grade available, the maximum revenue under this condition is given by

$$R_1(q_1 \ldots q_k) = r_1 w_1, \tag{4.4}$$

where $$w_1 = \min(d_1, q_1/p_{11}, q_2/p_{12}, \ldots q_k/p_{1k}).$$

Introducing the second assortment, we have, if a weight w_2 of this second assortment is produced,

Revenue $= r_2 w_2 +$ revenue from producing assortment 1 with the remaining $(q_1 - p_{21}w_2, \ldots q_k - p_{2k}w_2)$ walnuts,

and if $R_2(q_1 \ldots q_k)$ is the *maximum* revenue from producing only the first two assortments, we have, using the principle of optimality,

$$R_2(q_1 \ldots q_k) = \max_{w_2} \{(r_2 w_2 + R_1(q_1 - p_{21}w_2, \ldots q_k - p_{2k}w_2)\}$$

where $$0 \leqq w_2 \leqq \min(d_2, q_1/p_{21}, q_2/p_{22} \ldots q_k/p_{2k}).$$

More generally we can put

$$R_s(q_1 \ldots q_k) = \max_{w_s} \{r_s w_s + R_{s-1}(q_1 - p_{s1}w_s, \ldots, q_k - p_{sk}w_s)\} \tag{4.5}$$

where $$0 \leqq w_s \leqq \min(d_s, q_1/p_{s1}, \ldots q_k/p_{sk}).$$

Equations (4.4) and (4.5) suffice to compute $R_1, R_2 \ldots R_n$ for any quantities $(q_1 \ldots q_k)$, and in particular $R_n(Q_1, Q_2, \ldots Q_k)$.

Thus the example can be solved by dynamic programming although the simplex technique will in this instance be much more direct. It is clear that any linear programming problem can be formulated in dynamic programming terms, though the linear programming technique will generally be much simpler. However, the dynamic programming formulation of such problems is not bound by the linearity of the functions involved.

Considering the above example, we might have the revenue as a non-linear function of the amount produced. In this case, we merely replace $r_i w_i$ by $r_i(w_i)$ in equations (4.4) and (4.5), and the method of solution remains unchanged. Furthermore, the demands d_i might not be known exactly, but known to conform to some probability distribution. In this case, the problem could still be tackled by dynamic programming, using, for example, the expected revenue as that to be maximised. These various complications do not essentially affect the basic property of this type of approach, in that the problem is still reduced to a series of single variable maximisation problems.

In this ability to handle functions of a general nature lies the advantage of using dynamic programming in this type of problem, though it should be repeated that, where applicable, linear programming techniques will generally yield the solution more readily.

Inventory problems

Another class of problems to which dynamic programming may be applied is that of inventory. For example, suppose the demand for an item is given by

$\phi(s)\,ds =$ the probability of a demand within the range $(s, s+ds)$.

Consider the situation in which we make up the stock from the existing level x to a level y at fixed intervals. Suppose that the cost of ordering z items is $k(z)$ and that the cost of a stock-out of z items is $p(z)$.

Then, if we are concerned only with one period, the minimum expected cost over that period is given by

$$f_1(x) = \min_{y \geq x} \left\{ k(y-x) + \int_y^\infty p(s-y)\phi(s)\,ds \right\}$$

while for n periods,

$$f_n(x) = \min_{y \geq x} \left\{ k(y-z) + \int_y^\infty p(s-y)\phi(s)\,ds + f_{n-1}(0)\int_y^\infty \phi(s)\,ds + \int_0^y f_{n-1}(y-s)\phi(s)\,ds \right\}.$$

This simple equation yields recurrence relations sufficient to solve the problem as formulated for a finite stage process. Similar equations may be derived for cases when an unbounded process is considered and when time lags occur in the delivery of items. For an account of the formulation of these equations the reader is referred to Bellman, Chapter V.

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More generally we can put

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This simple equation yields recurrence relations sufficient to solve the problem as formulated for a finite stage process. Similar equations may be derived for cases when an unbounded process is considered and when time lags occur in the delivery of items. For an account of the formulation of these equations the reader is referred to Bellman, Chapter V.

Some practical applications

We conclude by giving in this section three examples of the application of dynamic programming to industrial and commercial problems.

A bidding problem

Suppose we are bidding for items of known value $v_1, v_2, \dots v_n$, against a group of bidders each submitting a single bid for each, or some of the items, simultaneously.

From the records of previous auctions let us suppose we know the distribution of winning bids w_i about the value v_i of the items.

Let
$$\int_0^{w_i} \phi(w;\, v_i)\, dw$$
be the probability of winning v_i with a bid w_i.

If there is a limit C to the amount that we can bid on all the items then
$$\sum_{i=1}^{n} w_i \leqq C,$$
where w_i is the amount we bid for v_i. We wish to obtain the maximum expected return $R_n(C)$ so that:
$$R_n(C) = \max \sum_{i=1}^{n} \left\{(v_i - w_i) \times \int_0^{w_i} \phi(w;\, v_i)\, dw\right\}.$$

Though for certain forms of ϕ it may be possible to solve this problem analytically, in general this will not be possible. However, a solution can always be obtained using the dynamic programming approach.

Considering one item only we have:
$$R_1(y) = \max_{w_1} \left\{(v_1 - w_1) \int_0^{w_1} \phi(w;\, v_1)\, dw\right\}$$
and we can readily maximise this one dimensional function. Hence we obtain $R_1(y)$.

Introducing the second item and employing the principle of optimality we have:
$$R_2(y) = \max \left\{(v_2 - w_2) \int_0^{w_2} \phi(w;\, v_2)\, dw + R_1(y - w_2)\right\}$$
and on the introduction of the ith item we have the equation:

$$R_i(y) = \max\left\{(v_i - w_i)\int_0^{w_i} \phi(w\,;\,v_i),\, dw + R_{i-1}(y-w_i)\right\}.$$

Thus, we may proceed step by step, computing

$$R_1(y),\ R_2(y) \quad \text{and finally} \quad R_n(y) \quad \text{and particularly} \quad R_n(C).$$

*A replacement problem**

In the manufacture of rubber tyres, a machine is used which simultaneously produces a tyre on each of two bladders. Sometimes a bladder fails while a tyre is being produced on it and then the tyre must be scrapped and the bladder replaced immediately.

In order to replace a bladder it is necessary to strip down the machine and it may then pay to replace the other bladder also while the machine is stripped down. In some circumstances it may even be worth while replacing both bladders before either of them bursts.

Thus there are four categories of cost:

c_1 the cost of replacing a bladder.
c_2 the cost of scrapping a faulty tyre.
c_3 the labour cost of stripping down the machine—this is the same whether one or both bladders are replaced.
c_4 the cost of lost production time when a bladder fails during use.

The older a bladder, that is the greater the number of tyres which have been produced on it, the greater the chance of it failing. Thus the problem is to determine the best policy for replacing bladders.

We define:

$f_N(i,j)$ = total expected cost, using an optimal policy, of producing N additional tyres when bladders 1 and 2 have already produced i and j tyres respectively.

p_i = probability of successfully producing a tyre on a bladder which has already made i tyres. We assume that these p_i are known from records.

At each stage we have four choices:

(1) Replace both bladders.
(2) Replace bladder 1.
(3) Replace bladder 2.
(4) Continue to produce without replacing either bladder.

* This problem was posed by M. Sasieni in *Opns. Res. Quart.* 7, 4, December 1956. The following solution by S. Dreyfus appeared in *Opns. Res. Quart.* 8, 4, December 1957.

Considering these four choices in turn, we have:

(1) If we replace both bladders, our total expected cost will be the cost of replacing two bladders, $2c_1$, plus the cost of stripping down the machine, c_3, plus the total expected cost of producing N tyres starting with two new bladders and using an optimal policy $f_N(0, 0)$.

(2) If we replace bladder 1, our total expected cost will be the cost of replacing one bladder, c_1, plus the cost of stripping the machine, c_3, plus the total expected cost of producing N tyres starting with one new bladder and one bladder which has already produced j tyres and using an optimal policy, $f_N(0, j)$.

(3) If we replace bladder 2, the same costs as in 2 above are incurred. but $f_N(i, 0)$ replaces $f_N(0, j)$.

(4) If we continue to produce, four occurrences are possible:

(*a*) A tyre may be produced on each bladder,
(*b*) bladder 1 may produce a tyre and bladder 2 fail,
(*c*) bladder 2 may produce a tyre and bladder 1 fail,
(*d*) both bladders may fail.

So to find the total expected cost of continuing to produce we calculate the expected cost for each possible occurrence and multiply by its probability, obtaining:

$$\begin{aligned} &p_i p_j f_{N-2}(i+1, j+1) \\ &+p_i(1-p_j) \min [c_1+c_2+c_3+c_4 f_{N-1}(i+1, 0), \\ &\qquad\qquad 2c_1+c_2+c_3+c_4+f_{N-1}(0, 0)] \\ &+p_j(1-p_i) \min [c_1+c_2+c_3+c_4+f_{N-1}(0, j+1), \\ &\qquad\qquad 2c_1+c_2+c_3+c_4+f_{N-1}(0, 0)] \\ &+(1-p_i)(1-p_j)\times(2c_1+2c_2+c_3+c_4+f_N(0, 0)). \end{aligned}$$

The alternatives given in the second and third terms arise since either 1 or 2 bladders may be replaced in these cases, the course of action being dependent on the values of i and j.

This completes the analysis of the four possible decisions at each stage, and the optimal decision is that for which the total expected cost is least. It applies for all $N \geqq 1$, if we ignore the possibility of producing two tyres successfully when only one is required. This transient case is not important.

To compute this for any particular N we first note that $f_0(i, j) = 0$. So that $f_1(i, j) =$

$$
\min \begin{cases} \text{replace both } (2c_1+c_3+f_1(0,0)) \\ \text{replace } B.1(c_1+c_3+f_1(0,j)) \\ \text{replace } B.2(c_1+c_3+f_1(i,0)) \\ \text{produce } (1-p_i)(1-p_j)(2c_1+2c_2+c_3+c_4+f_1(0,0)) \\ +[p_i(1-p_j)+p_j(1-p_i)]\times(c_1+c_3). \end{cases}
$$

Hence we calculate $f_1(i,j)$, and then $f_2(i,j)$ using $f_1(i,j)$ then $f_3(i,j)$ using $f_1(i,j)$ and $f_2(i,j)$ and so on.

For small n, the bladder replacement policy is dependent on the value of n, but for all $n \geqq$ some N_0, the policy remains the same, irrespective of the exact number of tyres to be produced. This "steady state" policy is the one which we require.

With the values

cost of bladder, $c_1 = 10$
cost of scrapping tyre, $c_2 = 1$
labour cost of stripping machine, $c_3 = 2$
cost of lost production time, $c_4 = 3$

number produced i	0	1	2	3	4	5	6
probability of success p_i:	1	0·9	0·8	0·6	0·4	0·2	0

and using the above recurrence relations we obtain the "steady state" policy shown in the following table for the possible values of i and j:

i	0	1	2	3	4	5	6
0	$P-1$	$P-1$	$P-2$	$P-2$	$R-1$	$R-1$	$R-1$
1	$P-1$	$P-1$	$P-2$	$P-2$	$R-1$	$R-1$	$R-1$
2	$P-2$	$P-2$	$P-2$	$P-3$	$P-3$	$R-1$	$R-1$
3	$P-2$	$P-2$	$P-3$	$P-3$	$P-3$	$P-3$	$R-2$
4	$R-1$	$R-1$	$P-3$	$P-3$	$P-3$	$R-2$	$R-2$
5	$R-1$	$R-1$	$R-1$	$P-3$	$R-2$	$R-2$	$R-2$
6	$R-1$	$R-1$	$R-1$	$R-2$	$R-2$	$R-2$	$R-2$

where $P-1$ means produce, and if a bladder fails replace that bladder only.

$P-2$ means produce, and if the newer bladder fails replace both bladders, if the older bladder fails replace it alone.

$P-3$ means produce, and if either bladder fails replace both.

$R-1$ means replace the older bladder immediately.

$R-2$ means replace both bladders immediately.

The standardisation problem

This problem may arise in various guises, each amenable to a similar approach. The application of dynamic programming to this class is demonstrated by the following example:

Suppose a buyer for an organisation has to provision for a certain type of equipment over a range of sizes. A demand for a given size may be satisfied by supplying a size greater than or equal to that demanded. The buyer is able, if he wishes, to take advantage of bulk buying discounts, by buying large quantities of a few sizes. Given the demand for each size, and the discount rates, how many and what sizes should be purchased in order to minimise the cost of satisfying the demand?

Suppose $f(w)$ is the demand for all items of size less than or equal to w and that $g(m, w)$ is the total cost of purchasing m items of size w.

Assume:

(i) That $g(m, w)$ increases with w (i.e. that the cost per item increases with size) and with m.

(ii) That $\frac{1}{m}g(m, w)$ decreases with increasing m (i.e. that the cost per item decreases as the number purchased increases).

These two assumptions will generally hold in practice and simplify the formulation of the problem. They are not necessary, however, for this type of approach.

The problem, then, is to balance the discount for bulk buying against the expense of providing a larger item than demanded.

If only one size of equipment was purchased, to satisfy demand up to size w, this would have to be the maximum size demanded and the cost would be given by:

$$c_1(w) = g[f(w), w], \quad \text{for all } w \text{ in the range of demand, } (0, w_{max}), \text{ say.}$$

Suppose we now allow two sizes to be bought to satisfy demand for sizes up to w. If these sizes are w and w'

$$\text{cost} = g[f(w) - f(w'), w] + c_1(w').$$

Let $c_2(w)$ = minimum cost of provisioning for all items in the size range 0 to w, when only two sizes are allowed.

Then $$c_2(w) = \min_{w' \leq w} \{g[f(w) - f(w'), w] + c_1(w')\}.$$

If $c_2(w_{max}) < c_1(w_{max})$, it is worth while buying two sizes rather than one.

In general, if $c_n(w)$ = minimum cost of buying n sizes to satisfy demand in the size range 0 to w, we have, for all $n > 1$,

$$c_n(w) = \min_{w' \leqq w} \{g[f(w)-f(w'), w]+c_{n-1}(w')\}. \tag{4.6}$$

Thus we can compute $c_1(w)$, $c_2(w)$ ... etc. and in particular,

$$c_1(w_{max}),\ c_2(w_{max}),\ \ldots \text{ etc.}$$

Now if $g(0, w) = 0$, we obtain

$$c_1 > c_2 > c_3 \ldots c_{N-1} > c_N = c_{N+1} = c_{N+2} \ldots \text{ for some } N,$$

while if $g(0, w) > 0$, we obtain

$$c_1 > c_2 > c_3 \ldots c_{N-1} > c_N \leqq c_{N+1} < C_{N+2} < \ldots \text{ for some } N.$$

In either case, N sizes are required, the actual sizes being obtained in sequence from c_N, c_{N-1} ... etc.

This problem is rather different from those considered previously, in that it is treated as a multi-stage decision problem, although the number of stages is not known initially.

As an example, take

$$f(w) = w \quad \text{for} \quad w \leqq w_0$$

or

$$= w_0 \quad \text{for} \quad w > w_0$$

and

$$g(m, w) = w(m+k).$$

Then we have, if buying only one size, $m = f(w) = w$, so that

$$c_1(w) = w(w+k), \quad \text{and from (4.6), putting } n = 2$$

we obtain

$$c_2(w) = w(w+k)-\tfrac{1}{4}(w-k)^2 \quad \text{for} \quad w > k,$$

and

$$w' = \tfrac{1}{2}(w-k)$$

is the intermediate size,

or

$$= c_1(w) \quad \text{for} \quad w \leqq k,$$

and no intermediate size is required.

Similarly, putting $n = 3$ into (4.6), and minimising, we obtain

$$c_3(w) = w(w+k)-\tfrac{1}{3}(w^2-3wk+3k^2) \quad \text{for} \quad w > 3k$$

and $\qquad w' = 2w/3-k$

or $\qquad = c_2(w) \quad \text{for} \quad 3k \leqq w.$

In general, we obtain, for all $n \geqq 1$

$$c_n(w) = \frac{(n+1)}{2n}w(w+nk)-n\,.\frac{(n^2-1)}{24}\,.k^2 \quad \text{for} \quad w > \tfrac{1}{2}n(n-1)k$$

and $\quad w' = (n-1)(w/n-k/2)$

or $\qquad = c_{n-1}(w) \quad \text{for} \quad \tfrac{1}{2}n(n-1)k \geqq w. \qquad (4.7)$

Thus we see that the number of sizes required depends on the values of k, and of w_0.

If we take $k = w_0/20$, we see from (4.7) that

$$\tfrac{1}{2}n(n-1)k \geqq w_0 \quad \text{for} \quad n \geqq 7,$$

so that
$$c_n(w_0) = c_{n-1}(w_0) \quad \text{for} \quad n \geqq 7.$$

Thus $c_6(w_0)$ gives a minimum cost and 6 sizes are required. The largest is clearly w_0. The next largest is, from (4.7), and putting $n = 6$,

$$w_1 = 17w_0/24.$$

Putting $n = 5$ in (4.7) and $w = 17w_0/24$, we obtain for the next size down,

$$w_2 \text{ (say)} = 7w_0/15.$$

Similarly, $n = 4$, 3 and 2 give sizes

$$11w_0/40, \quad 2w_0/15 \quad \text{and} \quad w_0/24.$$

Thus we standardise on sizes

$$w_0, \quad 17w_0/24, \quad 7w_0/15, \quad 11w_0/40, \quad 2w_0/15 \quad \text{and} \quad w_0/24.$$

CHAPTER 5

QUEUES

Introduction

After he has selected the goods he wants, a shopper in a self-service store must take them to a desk to be checked and paid for. He may be served immediately, or may have to wait for other customers to be served first. The customer would like always to have immediate service; the store manager would like to see the cashier busy most of the time. These requirements cannot both be met, as customers do not arrive regularly and some take longer to serve than others. If the cashier is usually busy, a queue will develop when customers arrive more frequently than usual or take longer than usual. If customers find that they often have to wait they may decide to shop elsewhere in future; if the queue gets too long it will begin to block the shop. For these reasons the manager might consider employing another cashier. He knows that it will double the cost of the checking service but what effect will it have on average waiting time or queue length? Will the extra cost be justified by avoiding loss of customers or by the extra shop space made available?

This is the kind of problem to be studied in this chapter. The basic features are that an input of some kind of unit is provided with a service by a service channel. The problem arises because of the irregularity in the rate of input or service or both, so that they cannot be matched exactly and the best compromise is sought. This queue situation can occur in a wide range of operations and its existence is not always obvious. One of the earliest examples to be studied was the automatic telephone exchange where the units are callers and the service channels are switches at the exchange; in this case if the channels are all busy, unsuccessful callers do not form a queue. In a factory a product may be processed by a number of machines in turn; if the processing time is not constant a queue problem arises, and the input to one service channel is the output from the previous one. Finished products may accumulate in the despatch bay and be sent away in batches by lorry; in this example, the service channel (lorry) serves several units simultaneously, a process referred to as bulk service.

Similar problems arise in which the input is continuous and not a series of discrete units—for example, in the storage of water in reservoirs (1)—but continuous inputs are not considered in this chapter.

Before a queueing problem can be analysed, information is needed about the input, the service and the behaviour of the units. The rate of arrival of units (or the intervals between arrivals) and the service time may be specified by frequency distributions. The arrival and service rates may be affected by the length of queue, or may change with time, and the service rate may depend on the type of unit being served. In many problems mathematical expressions can be found which give a good enough approximation to the actual distributions but sometimes the actual distributions, found by observation, must be used. The behaviour of the units is described by the queue discipline. When a queue forms, the units may be served in the order in which they arrive (as in the shop example) or in arbitrary order (as may happen with goods in a factory) or some types of unit may have priority. A unit is said to have pre-emptive priority if it not merely goes to the head of the queue but displaces any unit already being served when it arrives. Provided that the order of service is not related to service time, it does not affect the queue length or average waiting time but it does affect the time an individual unit has to wait. Other types of queue discipline are for units to leave a queue after waiting a certain time—they may be, for example, impatient customers or perishable materials—or for them not to join a queue which is greater than a certain length.

If the average rate of arrival is less than the average rate of service, and both are constant, the system will eventually settle down into a steady state; the probability of finding a particular length of queue will then be the same at any time. If the rates are not constant, the system will not reach a steady state but it could remain stable. If the rate of arrival is not less than the rate of service the system is unstable and the probability of a long queue steadily increases; this happens even if the two rates are equal on average, because the time lost when the service channel is idle can never be made up. Imposing a limit on the maximum length of queue (so that further arrivals are not accepted) automatically ensures stability. Queueing situations which are unstable for a limited time are common in practice—rush-hour traffic is an example.

The solution of a queueing problem consists in selecting the best compromise for the factors which can be controlled. In different problems different features of the system are important in measuring the effectiveness of the compromise. In the self-service store example, average and maximum waiting time of a customer, or queue length, may be important. The effectiveness of an automatic exchange is measured by the probability of obtaining a connection.

This chapter will be mainly concerned with estimating these and other measures of queueing situations from the known, or assumed, information on inputs and service. To complete the solution of a problem it is necessary to examine the economics of the whole system and choose the optimum operating conditions. In practice this optimisation (or at least approximate optimisation) is often straightforward as the reasonable variation of the controlled factors is likely to be very limited; it is then only necessary to examine a few possible sets of conditions and to choose the best. Some simple problems can be solved by mathematical analysis and other problems by simulation experiments or a combination of both methods.

In general, mathematical solutions are most readily obtained for queues which have settled down to a steady state; that is, where the probability of a given state of the system is independent of time. Much of the analysis in this chapter will deal with this state. Steady state solutions are often useful in practical problems, but they should be used with caution; in some types of problem the operating conditions do not remain constant long enough for the situation to become steady.

Stock control is essentially a queueing problem and in this case the optimising process may be the more difficult part of the solution; this subject is not covered here but is dealt with separately in the next chapter. One other problem should be mentioned as it looks like a queueing problem but is not included in the description above. The characteristic example is the arrival of taxis and potential customers at a taxi rank. Depending on the relative rates of arrival, either may form a queue. In this problem there are two inputs but no service channels in the sense used above; the situation is unstable unless the inputs adjust themselves to be exactly equal on average (see Kendall (2)).

To summarise, a queue problem has the following basic components and there may be one or more of each:

(*a*) Input, specified by the type of unit and the time distribution of arrivals of units into the system.
(*b*) Queue (which may be of zero length). The queue discipline describes how the units leave the queue.
(*c*) Service channel, which serves the units and is specified by the distribution of service times.

To solve the problem, appropriate measures of the state of the system are calculated, such as:

Average
- (i) number in the system
- (ii) queue length
- (iii) waiting time.

Probability of
- (i) n units in the system
- (ii) more than n units in the system
- (iii) waiting time of t to $t+dt$
- (iv) waiting time greater than t
- (v) service channels being idle.

These measures are then used to decide the optimum operating conditions.

Solution by differential difference equations

The earliest method of analysis applied to queueing problems was the use of differential difference equations. In this section the method is described and is applied to the simplest queueing situation. More recently integral equations and Markov chain analysis have been developed to solve queueing problems and these are mentioned in later sections.

The method depends on deriving the rate of change of the probability of a given state of the system. This is done by considering the probabilities of transitions (e.g. by the arrival of a unit or the completion of service) in an infinitesimal interval. In order that a problem can be solved explicitly by this method it is necessary that these probabilities should be fairly simple functions. This means in practice that the method is limited to a few types of input and service time distributions.

The simplest queueing situation has an input of identical units, an unlimited queue served in order of arrival and a single service channel, the probabilities of an arrival and of a completion of service (of a unit already in the service channel) being constant, independent of time or of the state of the system. The constant probability of occurrence implies a Poisson distribution of the events and an exponential distribution of intervals between events. Thus, if the average rate of arrival of units is λ in unit time,

Probability of one arrival between time t and $t+dt$ is $\lambda\, dt$.

Probability of n arrivals in time t is $\frac{1}{n!}(\lambda t)^n e^{-\lambda t}$.

Probability of an interval between t and $t+dt$ before the next arrival is $\lambda e^{-\lambda t}\,dt$.

Probability of an interval greater than t is $e^{-\lambda t}$.

Average interval between arrivals is $1/\lambda$.

If the average service time is $1/\mu$, similar expressions, with μ in place of λ, apply to completions of service, provided that the service channel is busy; if the channel is unoccupied the probability of a completion of service is zero.

In this simple situation, the state of the system is completely described by the number of units in the system (including any being served). Let $P(n, t)$ be the probability of the system being in the state n at time t (i.e. that there are n units in the system). Consider the ways in which the state $n-1$ may be reached at time $t+dt$; they can be tabulated as follows:

Way	*State at time t*		*Arrivals in dt*		*Completions of service in dt*	
	No. in System	*Probability*	*Number*	*Probability*	*Number*	*Probability*
1	$n-1$	$P(n-1, t)$	0	$1-\lambda\,dt$	0	$1-\mu\,dt$
2	$n-1$	$P(n-1, t)$	1	$\lambda\,dt$	1	$\mu\,dt$
3	n	$P(n, t)$	0	$1-\lambda\,dt$	1	$\mu\,dt$
4	$n-2$	$P(n-2, t)$	1	$\lambda\,dt$	0	$1-\mu\,dt$

All other ways (and the second one in the table) involve two or more events in the short time dt so that their probability is of the order of dt^2. The probability of the state $n-1$ at $t+dt$ is the sum of the probabilities of the possible ways of reaching the state. If terms in dt^2 are neglected it can be written:

$$P(n-1, t+dt) = P(n-1, t)(1-\lambda\,dt-\mu\,dt)+P(n, t)\mu\,dt \\ +P(n-2, t)\lambda\,dt \quad \text{for} \quad n > 1.$$

If $n = 1$ the state $n-2$ cannot occur and there is no possibility of a service finishing in the state $n-1$ so the equation is slightly altered and becomes

$$P(0, t+dt) = P(0, t)(1-\lambda\,dt)+P(1, t)\mu\,dt.$$

When dt tends to zero, the fraction $[P(n-1, t+dt)-P(n-1, t)]/dt$ becomes the rate of change of $P(n-1, t)$ and by dividing the above equations by dt we obtain:

$$\frac{dP(n-1,t)}{dt} = -(\lambda+\mu)P(n-1,t)+\mu P(n,t)+\lambda P(n-2,t)$$

$$\frac{dP(0,t)}{dt} = -\lambda P(0,t)+\mu P(1,t).$$

These equations can be solved to give explicitly the probability of n in the system at any time. It can be shown that, provided μ is greater than λ, the system tends to a steady state in which the probabilities are independent of time. In this section, only this steady state will be considered; the rates of change of P are zero so that:

$$\mu p(n) = (\lambda+\mu)p(n-1)-\lambda p(n-2)$$
$$\mu p(1) = \lambda p(0),$$

where $p(n)$ is the steady state probability of n units in the system.

Hence $\quad p(1) = \lambda p(0)/\mu = \rho p(0) \quad$ putting $\lambda/\mu = \rho$

and $\quad p(n) = (1+\rho)p(n-1)-\rho p(n-2) \quad$ for $n > 1$,

so that

$$\begin{aligned} p(2) &= (1+\rho)p(1)-\rho p(0) \\ &= \rho(1+\rho)p(0)-\rho p(0) \\ &= \rho^2 p(0). \end{aligned}$$

Similarly n can be made 3, 4, 5 in turn, giving

$$p(n) = \rho^n p(0) \quad \text{for } n \geqq 0.$$

The sum of the probabilities from $n = 0$ to ∞ must be 1

$$1 = \sum_0^\infty p(n) = p(0)\sum_0^\infty \rho^n$$
$$= p(0)/(1-\rho).$$

Hence $\quad p(0) = 1-\rho,$

and $\quad p(n) = \rho^n(1-\rho).$

The average number of units in the system is:

$$N = \sum_0^\infty n p(n)$$

$$= (1-\rho)\sum_{0}^{\infty} n\rho^n$$

$$= \sum_{0}^{\infty} n\rho^n - \sum_{0}^{\infty} n\rho^{n+1}$$

$$= \sum_{0}^{\infty} n\rho^n - \sum_{1}^{\infty} (n-1)\rho^n$$

$$= 0\rho^0 + \sum_{1}^{\infty} (n-n+1)\rho^n$$

$$= \rho/(1-\rho).$$

The probability that the service channel is idle is $p(0)$. The average utilisation of the channel is $1-p(0) = \rho$. This may be regarded as the proportion of time the channel is busy or the average number of units in the channel. The ratio of input rate to service rate, ρ, is often known as the traffic intensity, which in this case is equal to the utilisation.

The average number of units in the queue (excluding any in the channel) is then $N-\rho = \rho^2/(1-\rho)$.

The probability that a queue exists is the probability that n exceeds 1, which is $1-p(0)-p(1) = \rho^2$.

More generally, the probability that the number of units in the system exceeds n is

$$p(>n) = 1-\sum_{0}^{n} p(r)$$

$$= 1-\sum_{0}^{n} \rho^r(1-\rho)$$

$$= \rho^{n+1}$$

(by a calculation like that above for N).

The waiting time of a unit, from its arrival to the time it starts to be served, can be considered in two parts. There is a finite probability that the waiting time will be zero; this is the probability that there are no units already in the system, or $p(0) = 1-\rho$. There is also the probability that the waiting time will be from t to $t+dt$; let this be $w(t)\,dt$. This waiting time occurs only if the last of the units already in the system finishes its service in the interval t to $t+dt$, all the others having been served during the period 0 to t. During this time the service channel is continuously busy so that the random completions of service give a Poisson distribution of completions in a given time. That is, the probability of $n-1$ completions in the time t is $(\mu t)^{n-1}\,e^{-\mu t}/(n-1)!$. The probability of one completion from t to $t+dt$ is $\mu\,dt$. The probability of n units being in the system when one unit arrives is $p(n) = \rho^n(1-\rho)$. Thus the probability that this unit will have to wait a time t to $t+dt$ as a result of waiting for n previous units to be served is the product of these three probabilities (i.e. the joint probability of all three events occurring). Finally, the total probability of this waiting time arising in any way is the sum of all these products for all values of n from 1 to ∞.

$$w(t)\,dt = \sum_{n=1}^{\infty} \rho^n(1-\rho)(\mu t)^{n-1}\,e^{-\mu t}\,\mu\,dt/(n-1)!$$

$$= \mu\rho(1-\rho)\,e^{-\mu t}\,dt \sum_{r=0}^{\infty}(\rho\mu t)^r/r! \quad \text{(putting } r \text{ for } n-1\text{)}$$

$$= \mu\rho(1-\rho)\,e^{-\mu t(1-\rho)}\,dt$$

or $$w(t) = \rho(\mu-\lambda)\,e^{-t(\mu-\lambda)} \quad \text{as} \quad \rho = \lambda/\mu.$$

The average waiting time of a unit is

$$W = 0p(0)+\int_0^{\infty} t w(t)\,dt$$

$$= \rho\int_0^{\infty} t(\mu-\lambda)\,e^{-t(\mu-\lambda)}\,dt$$

$$= -\rho\Big[t\,e^{-t(\mu-\lambda)}\Big]_0^{\infty}+\rho\int_0^{\infty} e^{-t(\mu-\lambda)}\,dt$$

$$= 0+\rho\left[\frac{-e^{-t(\mu-\lambda)}}{(\mu-\lambda)}\right]_0^{\infty}$$

$$= \frac{\rho}{\mu-\lambda} = \frac{\rho}{\mu(1-\rho)}.$$

The probability of a waiting time greater than t is

$$w(>t) = \int_t^{\infty} w(x)\,dx = \int_t^{\infty} \rho(\mu-\lambda)\,e^{-x(\mu-\lambda)}\,dx$$

$$= \Big[-\rho\,e^{-x(\mu-\lambda)}\Big]_t^{\infty}$$

$$= \rho\,e^{-t(\mu-\lambda)}.$$

It should be noted that this last result includes the case where $t = 0$.

The total time spent by a unit in the system, including being served, can be obtained in a similar manner. In this case the probability of zero time is zero, and the probability $s(t)\,dt$ of a time from t to $t+dt$ is given in the same way as $w(t)\,dt$ except that there are now n completions in the interval 0 to t instead of $n-1$. Following through the arithmetic it will be seen that

$$s(t) = (\mu-\lambda)\,e^{-t(\mu-\lambda)}.$$

Average time spent is

$$S = 1/(\mu-\lambda) = W + 1/\mu$$

and

$$s(>t) = e^{-t(\mu-\lambda)}.$$

All the probabilities and averages which have been derived so far refer to the whole of the time considered. Similar results referring only to the time when queue length or waiting time are greater than zero can be obtained by dividing by the probabilities of these conditions (ρ^2 or ρ respectively).

Example

The results which have been derived can be used to examine the example mentioned in the introduction—that of a self-service store with one cashier. In this case it is not unreasonable to assume random arrival and service times. (The average rate of arrival may depend on time of day, in which case the methods mentioned in the section on transient solutions would be needed.) Suppose that 9 customers arrive on the average every 5 minutes and the cashier can serve 10 in 5 minutes. That is,

$$\lambda = 1{\cdot}8, \quad \mu = 2{\cdot}0, \quad \rho = 0{\cdot}9 \quad \text{(measuring time in minutes).}$$

The features which may be important to the manager are:

(*a*) The average number of customers waiting for service (which may decide whether a shopper comes into the store), which is

$$Q = \rho^2/(1-\rho) = 0{\cdot}81/0{\cdot}1 = 8{\cdot}1.$$

(*b*) The chance of having more people waiting than there is convenient room for, say 10 altogether,

$$p(>10) = \rho^{n+1} = 0{\cdot}9^{11} = 0{\cdot}31.$$

(c) The chance of a customer having to wait long for service, say more than two minutes,

$$w(>2) = \rho\, e^{-t(\mu-\lambda)} = 0{\cdot}9e^{-2\times 0\cdot 2} = 0{\cdot}60.$$

If service can be speeded up to 12 in 5 minutes by using a different cash register, the quantities then become

$$\lambda = 1{\cdot}8, \quad \mu = 2{\cdot}4, \quad \rho = 0{\cdot}75$$

$$Q = 0{\cdot}5625/0{\cdot}25 = 2{\cdot}25$$

$$p(>10) = 0{\cdot}75^{11} = 0{\cdot}042$$

$$w(>2) = 0{\cdot}75\, e^{-1\cdot 2} = 0{\cdot}23.$$

It can be seen that this slight change in speed of service makes a great difference to the operation of the store.

Other applications of differential difference equations

The queue situations to which this method is applicable can be grouped in four classes according to the type of distributions of arrival intervals and service times.

(a) Exponential. The simplest case has been dealt with; other cases may have more complex queue discipline or more than one service channel.

(b) Erlang. Erlang distributions are derived by combining a number of identical exponential distributions. A constant rate can be considered as a special case of an Erlang distribution.

(c) Exponential with varying parameter. The parameter λ or μ may be a function of time or of the number of units in the system.

(d) Hyper exponential. The variance is greater than for an exponential distribution. These distributions are not further considered here.

In this section, special features of the method of solution for types (a) to (c) are discussed. The main results are included in the list at the end of the chapter.

Exponential distribution, single channel and limited queue length

Suppose that the maximum number of units in the system is limited to L. Any units arriving when there are L already there do not join the queue. The situation is otherwise taken to be identical with that considered in the previous section.

The basic transition equations given before still apply when n is less than $L+1$, but in addition

$$P(L, t+dt) = (1-\mu\, dt)P(L, t)+\lambda\, dt\, P(L-1, t)$$

as there is no possibility of a transition to or from the state $L+1$. As before, the solution for the steady state can be obtained by putting the rates of change equal to zero. In this case, as the number in the system is limited, a steady state is possible for any value of ρ and not only when the input rate is less than the service rate. However, if L is large and ρ is greater than one the steady state may be reached very slowly and in practice may not be achieved before the operating conditions change. In using the results it is necessary to check that the steady state is reached. In this state, as before,

$$p(1) = \rho p(0)$$

$$p(n) = (1+\rho)p(n-1)-\rho(n-2) \text{ for } 1 < n < L$$

and from the equation above, $p(L) = \rho p(L-1, t)$.

Hence $$p(n) = \rho^n p(0)\; n \leqq L.$$

$$\text{The total probability of any state} = 1 = \sum_0^L \rho^n p(0)$$

$$= p(0)(1-\rho^{L+1})/(1-\rho)$$

$$p(0) = (1-\rho)/(1-\rho^{L+1})$$

$$\text{If } \rho = 1, \quad \sum_0^L \rho^n = L+1 \quad \text{so that} \quad p(0) = 1/(L+1)$$

The various measures of the state of the system can be obtained by calculations similar to those above. An important quantity in this case is the proportion of units lost to the system because they arrive when the system is full; this proportion is $p(L)$. Because some units are lost, the utilisation of the service channel is no longer equal to ρ. It is

$$1-p(0) = \rho(1-\rho^L)/(1-\rho^{L+1}).$$

Exponential distribution and multiple channels

It is again supposed that there is an input of identical units with mean rate λ but that there are c service channels in parallel, each with service rate μ. The units form a single queue and go to any channel which becomes free. The basic transition equations are

$$P(0, t+dt) = P(0, t)(1-\lambda\, dt)+P(1, t)\mu\, dt$$

$$P(n-1, t+dt) = P(n-1, t)[1-\lambda\, dt-(n-1)\mu\, dt]+P(n-2, t)\lambda\, dt + P(n, t)n\mu\, dt \quad \text{for } 1 < n \leqq c$$

$$P(n-1, t+dt) = P(n-1, t)(1-\lambda\, dt-c\mu\, dt)+P(n-2, t)\lambda\, dt + P(n, t)c\mu\, dt \quad \text{for } n > c$$

as the probability of a service finishing in the interval dt is $r\mu\, dt$ where r is the number of channels occupied. From these equations the steady state probability $p(n)$ can be derived. For a steady state to be possible λ must be less than $c\mu$.

In this state

$$p(n) = \rho^n p(0)/n! \quad \text{for } 0 < n \leqq c \quad \text{where} \quad \rho = \lambda/\mu$$

$$= c^{c-n}\rho^n p(0)/c! \quad \text{for } n \geqq c.$$

With a multiple channel system there may again be an upper limit L to the number in the system, leading to an additional transition equation,

$$P(L, t+dt) = P(L, t)(1-c\mu\, dt)+P(L-1, t)\lambda\, dt$$

the last equation above applying only for $c < n \leqq L$.

Hence $$cp(L) = \rho p(L-1)$$

so that the expressions given above for $p(n)$ apply up to $n = L$ but with a different value of $p(0)$.

An important particular case is that where no queue is allowed; that is, $L = c$. The steady state results are then considerably simplified and

$$p(n) = (\rho)^n \Big/ n! \sum_{r=0}^{c} \frac{(\rho)^r}{r!} \quad \text{for } n \leqq c \text{ and } c = L.$$

As for a single channel, an important quantity is the proportion of units lost to the system, which is $p(L)$, or $p(c)$ when no queue is allowed.

Example

The example of the self-service store can now be completed by considering the effect of an additional cashier. Suppose, as before, that $\mu = 2{\cdot}0$ for each cashier and $\lambda = 1{\cdot}8$. Then $\rho = 0{\cdot}9$ and $c = 2{\cdot}0$. The results needed are listed at the end of the chapter.

$$p(0) = 1\Big/\left\{\sum_{i=0}^{c-1}\rho^i/i!+\rho^c/(c-\rho)(c-1)!\right\} = 1/(1+0{\cdot}9+0{\cdot}9^2/1{\cdot}1\times 1)$$

$$= 0{\cdot}38$$

The probability of a customer having to wait at all is

$$p(>1) = c^{c-n}\rho^{n+1}p(0)/(c-\rho)c! = 2^1\times 0{\cdot}9^2\times 0{\cdot}38/1{\cdot}1\times 2 = 0{\cdot}28.$$

The measures used previously can now be calculated.
The average number in the queue is

$$Q = \rho^{c+1}p(0)/(c-\rho)^2(c-1)! = 0{\cdot}9^3\times 0{\cdot}38/1{\cdot}1^2\times 1 = 0{\cdot}23.$$

The probability of having more than 10 customers in the system is

$$p(>10) = 2^{-8}\times 0{\cdot}9^{11}\times 0{\cdot}38/1{\cdot}1\times 2 = 0{\cdot}00021.$$

The probability of having to wait more than 2 minutes for service is

$$w(>2) = p(>1)\,e^{-\mu t(c-\rho)} = 0{\cdot}28\times e^{2\times 2\times 1{\cdot}1} = 0{\cdot}0034.$$

The introduction of an extra cashier thus reduces the queueing to a very low level.

It is of interest to notice what would happen if the number of customers now increased so as to bring the utilisation of the cashiers back to the original level. In this case $\lambda = 3{\cdot}6$, $\mu = 2{\cdot}0$, $c = 2{\cdot}0$ and the utilisation is again 0·9. It will then be found that

$$Q = 7{\cdot}7,\quad p(>10) = 0{\cdot}33,\quad w(>2) = 0{\cdot}38$$

which may be compared with the values (8·1, 0·31, 0·60) derived before for half the number of customers and one cashier. Although the number of customers is doubled, the numbers waiting are about the same and the probability of a long wait is less. The system is thus not equivalent to the sum of two separate stores each like the original example.

These examples illustrate three features which are common to most queueing situations:

(*a*) Unless the utilisation of the service is fairly high (over ½, say) the congestion is very slight.

(*b*) When the utilisation approaches unity a small change in it makes a large difference to the congestion.

(*c*) When service channels can be shared by one queue the congestion is much less than it would be for the same rates of input and service but with an independent queue for each service channel.

Exponential distribution and service in series

It is supposed that each unit entering the system requires a number of services, which are carried out in succession (3). After receiving the first service at the first stage the unit joins a queue for the second service at the second stage and so on. At each stage there may be one or more service channels. The input for one stage is thus the output from the previous stage. With exponential distributions of input and service intervals, it can be shown that the output is also exponentially distributed so that each stage has an exponential input. In this case the state of the system is not completely described by the number of units present; it is necessary to specify the number at each stage (i.e. the number waiting for service or being served at that stage).

All the units pass through each stage so that for a steady state to be possible the average rate of input must not exceed the total rate of service at any stage. In the steady state the average rate of input to each stage is the same and the stages can be considered separately. Suppose the rate of input is λ and at the jth stage there are c_j channels each with average service time $1/\mu_j$. Then, as in the previous section, the steady state probability of n_j units in the stage is

$$p(n_j) = v(n_j)p(0_j)$$

where

$$v(n_j) = \rho_j^{n_j}/n_j! \quad \text{for} \quad n_j \leqq c_j$$

$$= c_j^{c_j - n_j}\rho_j^{n_j}/c_j! \quad \text{for} \quad n_j \geqq c_j$$

and

$$\rho_j = \lambda/\mu_j.$$

The probability of the state of the whole system of k stages with n_j at the jth stage is the product of the probabilities for each stage and can be written

$$p(n_1, n_2, \ldots n_j \ldots n_k) = p(0) \prod_{j=1}^{k} v(n_j).$$

As before, $p(0)$ can be found by equating to unity the sum of the probabilities of all states and it is

$$p(0) = \prod_{j=1}^{k} \left\{ 1 \bigg/ \sum_{n_j=0}^{\infty} v(n_j) \right\}.$$

Erlang distributions

The examples so far have been limited to those where the arrival interval and service time both have exponential distributions. With these distributions the probability of an arrival at any instant is quite independent of previous arrivals, and the probability of service being completed at any instant is quite independent of the time when service began. It is this property which allows simple transition equations to be written.

In practice the unrelated arrivals of customers might well be exponential, but the distribution of service times may be quite unlike the exponential. Transition equations cannot be written using such distributions because the probability of service being completed depends on when service began.

There are a number of ways of dealing with this. One way is to extend the use of the simple method by building up service time distributions from combinations of exponential distributions. This method is considered here.

We consider service to be in a number of exponential phases immediately following each other; all phases have to be completed before another unit can enter the service channel. The distribution of the whole service time is quite different from that of its components. When all k phases are identical, the service time has a chi-square distribution with $2k$ degrees of freedom. Because of this physical interpretation of chi-square, formulated by A. K. Erlang for just the reasons given above, the distributions are widely known as Erlang distributions. The family of Erlang distributions for $k = 1, 2, 3, \ldots$ is scaled so that the mean of the whole service time is the same for each. When $k \to \infty$, the service time is constant. Erlang also considered the cases where the phases are different from each other, and obtained much more general distributions.

The transition equations for systems with such distributions of service times can now be written, because the probability of passing from the jth to the $(j+1)$th phase of service is $\mu_j\,dt$, and it depends only on the presence of a unit in the jth stage.

For random arrivals and Erlang service time, a single-channel system is characterised by $p(n, j)$ which is the probability that there are n units waiting and the unit being served is in the jth phase of service. The equations for the steady state, with an arrival rate λ and a service rate $k\mu$ for each phase, and therefore μ for the whole system, are

$$\begin{aligned}
\lambda p(0, 0) &= k\mu p(0, k) \\
(\lambda+k\mu)p,(0, 1) &= \lambda p(0, 0)+k\mu p(1, k) \\
(\lambda+k\mu)p(0, j) &= k\mu p(0, j-1) && \text{for } 1 < j \leqq k \\
(\lambda+k\mu)p(n, 1) &= \lambda p(n-1, 1)+k\mu p(n+1, k) && \text{for } n \geqq 1 \\
(\lambda+k\mu)p(n, j) &= \lambda p(n-1, j)+k\mu p(n, j-1) && \text{for } n \geqq 1,\ 1 < j \leqq k.
\end{aligned}$$

In general, the solution of these equations requires the introduction of probability generating functions and numerical computation. For $k = 2$ it can be shown that

$$p(n) = \sum_{i=0}^{n} \frac{(n+i+1)!(1-\rho)}{(2i+1)!(n-1)!} \left(\frac{\rho}{2}\right)^{n+i} \quad \text{where } \rho = \frac{\lambda}{\mu}$$

but for larger values of k the probabilities have to be found numerically. Average results can be expressed generally and the main ones are given in the list at the end of the chapter.

It is seen that the average queue length and average waiting time for exponential service ($k = 1$) are twice the queue length and waiting time for constant service ($k = \infty$).

A similar approach can be used to analyse systems with an Erlang arrival time distribution. Units are imagined to pass through k phases of an arrival channel before joining the queue. As soon as one unit is through the arrival channel, the next one is introduced.

The use of Erlang distributions does not necessarily imply that the phases of service and arrival have any physical reality, though that may be so.

Methods of solution of queueing problems with Erlang distributions are described in some detail by Morse (4). Tabulations of some of the more complicated examples have been published (5).

Exponential distributions with varying parameters

The queueing situations to be discussed in this section have input and service time distributions of the same form as those given earlier but the parameters λ and μ are not constant. Differential difference equations can be derived in exactly the same way as for exponential distributions, but λ and μ are replaced by the appropriate functions.

The parameters may depend on the number of units in the system. The values of the parameters when there are n in the system can be written $\lambda(n)$ and $\mu(n)$. The steady state probabilities can be derived in the same way as before. It will be found that for a single channel

$$p(n) = p(0) \prod_{i=1}^{n} \frac{\lambda(i-1)}{\mu(i)}$$

and for multiple channels corresponding results apply. By a suitable choice of the functions $\lambda(n)$ and $\mu(n)$ several of the situations already described can be derived as special cases of this model.

Some other special cases have been examined. For example, if $\mu(n) = n\mu$ and $\lambda(n) = \lambda$ where μ and λ are constant, a fairly simple expression can be derived for the time dependent probability,

$$P(n, t) = e^{-R} R^n/n! \quad \text{where} \quad R = (1-e^{-\mu t})\lambda/\mu \quad \text{and} \quad P(0, 0) = 1.$$

The average number in the system at time t can be derived directly from the differential difference equations and is equal to R. It may be noted that, except for the waiting time, this case is equivalent to a system with an infinite number of exponential service channels.

The other type of variation which has been studied is dependence on time; there can then be no steady state in the sense used previously. It is necessary to solve the equations to give the time dependent probabilities of the state n (which depend also on the initial state of the system). This has been done for a single service channel and distribution parameters which are general functions of time; the results are rather complex (6).

Other mathematical methods of solution

The previous sections have used the classical approach to the theory of queues, by the formation of differential-difference equations for the distribution of queue size. In his book on queueing theory, Morse uses only this approach (4).

Other analytical techniques have been used to give both general and particular results. Some of the more important of these techniques are briefly described in this section.

Integral equations

One approach leads to integral equations for the distributions of the waiting time. It is valid for general distributions of arrival and service time and could be used to study transient states of the system. It was first used by Lindley (7) for a single service channel. The analysis for such a queue is given here.

If W_n is the time the nth unit has to wait for service if it waits at all, and is minus the time the server has been idle if the nth unit does not have to wait, a_n is the interval between the arrival of the nth and $(n+1)$th units, with distribution $a_n(t)\,dt$, and b_n is the service time of the nth unit, with distribution $b_n(t)\,dt$,

then
$$\begin{aligned} W_{n+1} + a_n &= W_n + b_n \quad \text{for} \quad W_n \geqq 0 \\ &= b_n \quad\quad\quad \text{for} \quad W_n \leqq 0 \end{aligned}$$

and on writing $U_n = b_n - a_n$,

$$\begin{aligned} W_{n+1} &= W_n + U_n \quad \text{for} \quad W_n \geqq 0 \\ &= U_n \qquad\quad \text{for} \quad W_n \leqq 0. \end{aligned}$$

The distribution of W_{n+1} can be simply written down in terms of the distributions of W_n and U_n. If these are $W_n(t)\,dt$ and $U_n(t)\,dt$ respectively,

then $$W_{n+1}(t) = \int_0^\infty W_n(x)U_n(t-x)\,dx + U_n(t)\int_{-\infty}^0 W_n(x)\,dx.$$

In writing this equation it is assumed that the time the nth unit waits is independent of the subsequent service time and of the time when the next customer arrives.

The service time and arrival interval distributions are known, so the distribution of U_n is known,

$$U_n(t) = \int_0^\infty a_n(x)b_n(t+x)\,dx.$$

In the steady state the distributions of arrival, waiting and service times are the same for each unit, and we can write

$$W(t) = \int_0^\infty W(x)u(t-x)\,dx + u(t)\int_{-\infty}^0 W(x)\,dx\cdot$$

where $$u(t) = \int_0^\infty a(x)b(t+x)\,dx.$$

The integral equation for the waiting time distributions is soluble in terms of the distribution of u, which in turn is completely given by the known arrival time and service time distributions. Although the equation is soluble analytically for simple forms of $u(t)$, numerical computation would usually be required. The equation is of the Wiener-Hopf type, and also arises in the study of random walks and sequential tests of hypotheses.

With this technique Lindley studied queues with the following distributions:

(*a*) Regular input and Erlang service.
(*b*) Poisson input and general service.
(*c*) Erlang input and general service.

Smith (8) showed how to solve the integral equation for general distributions of service time and arrival interval. As one example of the method of solution he considered a service time that was a constant delay plus an Erlang delay, and an arrival interval that was a constant delay plus an exponential delay.

Markov chains

A sequence of events forms a Markov chain if the probability of each event can be expressed in terms of the previous event, irrespective of any information about earlier events. Hence a knowledge of any event makes all information about previous events irrelevant to the prediction of subsequent events.

In a similar way, a continuous process is Markovian if a knowledge of any instantaneous state of the process makes all previous history irrelevant in predicting its future behaviour.

The instantaneous state of a queueing system may be described by the number of units in it. If the arrival and service distributions are exponential, this number of units is all that is needed to determine the probabilities of future instantaneous states, and the process is Markovian. For other distributions these probabilities depend also on the previous history, and the process is non-Markovian. However, the process can still be regarded as Markovian if the times since the previous arrival and the last completion of service are included in the description of the instantaneous state of the system.

If the queueing system is described in a Markovian way, it can be studied as a discrete-time system, using the well-developed theory of Markov chains (9). There is, however, a further approach using Markov chains.

A queueing system with random arrivals is considered. If we know the queue size when a customer leaves, then previous information on queue size is of no predictive value. This is true about all instants when a customer leaves. This set of particular instants can then be thought of as a Markov chain embedded in the queueing process (10).

The analysis of embedded Markov chains has proved a fruitful approach to the analysis of queueing systems. It was used, for instance, by Bailey in the analysis of queueing for bulk service (see later).

An interesting use of the Markovian properties of a queueing process was made by Winsten and more recently by Mercer (11). They consider the equilibrium condition of a queue with regular input and a single exponential service channel. The numbers in the system immediately before each arrival form a Markov chain. The probability of a transition from N_a before one arrival to N_{a+1} before the next (where both are greater than zero) depends only on $N_{a+1}-N_a$. A sequence of transitions, such that the number does not drop below its initial value, has a probability which is the product of the individual transition probabilities. This probability is the same whatever the number in the system initially, provided the sequence of changes at each transition is the same. From this property it can be shown that $p(n+1) = kp(n)$ where k is a constant, that is that the number in the system has a geometric distribution.

The method can be extended to consider arrivals with a probability distribution about the regular time, but if the distributions of successive arrivals overlap the calculation is complicated. The usual method of specifying the input to a queueing situation is by the probability distribution of the arrival intervals; unless the distribution is exponential this implies that the time of an arrival depends on the previous arrivals. In many problems this is not so and a distribution about a regular arrival time is a more appropriate description. Winsten's method of analysis may therefore prove to be a useful general approach.

Ad hoc *methods*

Some quite general results can be obtained without carrying out the analysis necessary to give a complete description of the system. A good example of this type of *ad hoc* approach is D. G. Kendall's derivation of the average queue length and waiting time for a single queue with random arrivals and general service time distribution (2).

Let λ be the arrival rate
μ be the service rate
q be the size of queue a departing unit leaves behind it
q' be the size of queue the next unit leaves behind it
v be the service time of this second unit
r be the number of units arriving during the service time v
(r is conditionally a Poisson variable of mean value λv).

Then $q' = q-1+\delta+r$ where $\delta = 1$ when $q = 0$
and $\delta = 0$ when $q \neq 0$.

For the steady state, we can equate expected values of both sides of the equation and the expected values of the squares of both sides. As $\delta^2 = \delta$ and in the steady state, $E[q'] = E[q]$ and $E[q'^2] = E[q^2]$,

$$E[\delta] = 1-E[r] = 1-\frac{\lambda}{\mu} = 1-\rho$$

and $$E[2q(1-r)] = E[(r-1)^2]+E[\delta(2r-1)].$$

Noting that r is independent of q and δ it follows that

$$E[q] = E[r]+\frac{E[r(r-1)]}{2(1-E[r])}$$

and hence $$E[q] = \rho+\frac{\rho^2(1+\mu^2 var(v))}{2(1-\rho)}.$$

From this expression for mean queue length at a departure, the mean waiting time can be obtained. If a departing unit has waiting and service times w and v and leaves q units behind, then q is the number of arrivals in time $w+v$, and so

$$E[q] = \lambda E[w]+\lambda E[v] \text{ whence } \frac{E[w]}{E[v]} = \frac{\rho}{2(1-\rho)}(1+\mu^2 var(v)).$$

From this formula it is clear that, for a given arrival rate and service rate, waiting time will be a minimum when service time is constant. A similar result for the Erlang family of service times might have led us to expect this.

Special queue situations

The theory of queues has been extended to take in more and more features which appear in real-life queueing situations, whether the queues be people, manufactured parts, repair jobs or perishable goods. Some of these situations are dealt with in this section.

Limited input

It may be that units arrive from a limited pool of potential customers. Once a unit joins the queue, there is one less unit which could arrive, and therefore the probability of an arrival is lowered. When a unit is served it rejoins the pool of potential customers, and the probability of an arrival is thereby increased.

The theory of queues with limited input was first formulated to give a model of machine interference. In some industries, banks of machines are supervised by one operator. When a machine stops it has to wait for service if the operator is already busy. If an operator looks after too few machines the labour cost is high, and if the operator looks after too many machines the cost of lost output becomes appreciable. These costs have to be balanced so that the total cost per unit of production is minimised.

Systems with limited input have been analysed for random arrivals and negative exponential service. In the general case there are m units in the system, and c servers each with mean service rate μ. Each unit in the pool of potential customers has a probability $\lambda\, dt$ of joining the queue in the short interval dt. This means that the time spent in the pool between completing service and joining the queue again is exponentially distributed. Writing $p(r)$ for the probability that there are r customers waiting or being served, and noting that the probability of an arrival is then $(m-r)\lambda\, dt$, we can write the steady state equations

$$\lambda m p(0) = \mu p(1)$$

$$[(m-r)\lambda+r\mu]p(r) = (m-r+1)\lambda p(r-1)+(r+1)\mu p(r+1) \qquad r = 1, 2, \dots, c-1$$

$$[(m-r)\lambda+c\mu]p(r) = (m-r+1)\lambda p(r-1)+c\mu p(r+1) \qquad r = c, c+1, \dots m-1$$

$$c\mu p(m) = \lambda p(m-1)$$

from which it follows that

$$(r+1)\mu p(r+1) = (m-r)\lambda p(r) \qquad r < c$$

$$c\mu p(r+1) = (m-r)\lambda p(r) \qquad c \leqq r < m.$$

The solution is

$$p(r) = \frac{m!}{(m-r)!r!}\left(\frac{\lambda}{\mu}\right)^r p(0) \quad \text{for} \quad r \leqq c$$

$$\frac{m!}{(m-r)!c!c^{r-c}}\left(\frac{\lambda}{\mu}\right)^r p(0) \quad \text{for} \quad c \leqq r \leqq m$$

$p(0)$ being found by the condition that $\sum_{0}^{m} p(r) = 1$. As the maximum queue length is finite there is no limit to the ratio λ/μ.

As in other systems, it is better for two or more servers to work on the pool together as a team, rather than to split the pool into parts and to work independently.

The various measures of effectiveness of the system can be derived from the formula as before, though they are not of simple form. Naor (12) has given the solution in terms of tabulated Poisson functions and their cumulatives, and has discussed the solution in these terms. Cox (13) has given a table of the utilisation of the service for a single service channel when the average time in the pool is at least eight times as long as the service time. In an early paper, Ashcroft (14) gave tables of results for machine interference with constant service time. It can be shown that queue length is independent of the form of the distribution of the input.

In his original paper, Palm (15) gave a diagram showing the optimum number of machines to use in terms of the hourly cost of idle time, the hourly cost of labour, the arrival rate and the service rate for groups of machines tended by one operator.

The assumptions of our theory are an idealisation. With more machines to care for, travelling time between them increases, and so service takes longer. This becomes even more marked if the operators work as a team. It might well be that when much ground is to be covered, the increase in service time could outweigh the theoretical advantage of working as a team. This ought to be checked before limited input—machine interference—formulae are applied, quite apart from the need to check that it is reasonable to use exponential service and running times.

The theory of queues with limited input is applicable to many situations: for example,

breakdown of machinery,
maintenance of a pool of equipment,
allotment of maintenance workers,
shorthand writers attending a limited number of dictators.

Bulk service

Service of batches of units may take a number of different forms:

- the server waits for s units, and then serves them together, e.g. passengers on a coach tour,
- when ready, the server takes the units waiting, though not more than s, and serves them together, e.g. passengers on public transport.

In some instances the time spent waiting may be all that matters, the service time being irrelevant. The interval between successive services could then be regarded as being completely taken up by serving the previous batch, or to be the time when the service channel is ready for a new batch, the actual service time being shorter.

One case has been analysed by Bailey (16). Units arrive at random and are served in batches of not more than s, the server taking zero units if none is waiting. The distribution of intervals between service is chi-square with $2p$ degrees of freedom and mean p/a. The analysis uses embedded Markov chains and probability generating functions. There is no simple analytic solution, but a useful inequality for the average waiting time W is

$$0 \leqq W - \frac{p+s}{2a(s-m)} \leqq \frac{p(s-1)}{2am}$$

where m is the average number of arrivals per service interval. For example, if the arrival rate is nearly equal to the service rate, say $s = m+1$, then the average waiting time, measured in service intervals, is given by

$$\frac{m+1}{2} + \tfrac{1}{2} \leqq W \leqq \frac{m+1}{2} + 1 \quad \text{for exponential service } (p = 1)$$

and $$\tfrac{1}{2} \leqq W \leqq 1 \quad \text{for regular service } (p = \infty).$$

In general,

the greater the traffic intensity, m/s, the greater is the advantage of regular service over irregular service,

the larger the batch size for a given traffic intensity, the greater is the advantage of regular service over irregular service.

The theory was developed to give a model of customers at a clinic, where requests for attention are effectively random and clinics are held each week or at irregular intervals. The theory of bulk service might also be applied to

rubbish removal,
bus and train services,
bulk shipment of goods,
train despatch from marshalling yards,
lifts.

Transient solutions

Most of the previous analysis has been concerned with the steady state of queueing systems. For systems where this condition can occur, the steady state probability of a given state can be defined as the probability of finding the system in that state at a time sufficiently far in the future; that is, the steady state probability of the state n, $p(n)$ is the limit of the time dependent probability, $P(n, t)$, as $t \to \infty$. If such a system is known to be in the state k at a given instant, then immediately afterwards states near k are more likely than would otherwise be expected. The state probabilities tend gradually to the steady state probabilities. These changes in the probabilities are of interest in some problems, particularly after the occurrence of no units or a large number of units in the system, that is after idle or busy periods.

Transient probabilities are also of interest for systems which operate only for limited periods. In this case there is no need for a steady state to be possible. Indeed Bailey (17) gives an instance where it is quite suitable for an appointment system to have $\rho = 1$ because the period for which it operates is short enough for the waiting time of customers to remain small.

The transient solutions can be obtained by solving the time-dependent differential equations of the system. Morse (4) gives one way of doing this and Saaty (6) quotes transient solutions for a few types of system.

For systems having input or service rates which vary with time, all solutions can be regarded as transient. Random distributions with varying parameters have been considered earlier. Another more common problem arises from the rush-hour type of situation; input, and perhaps also service, rate increases for a limited period. This problem can be tackled by splitting the total period into intervals during which the rates can be taken as constant, and solving for each interval in turn. Cox (18) has used this method for a simplified analysis of a rush hour. Usually, however, simulation methods are employed for such problems.

Queue discipline

So far, in our queueing models, units have been served in the order in which they arrive. The queue discipline is thus first-come first-served. This is the discipline which normally comes to mind when we think of queues, but it is not the only one.

Units may be selected for service at random from those waiting. The average waiting time of units is the same as for first-come first-served queues. But because a unit may now be served out of turn and have to wait an excessively long time, there is a greater proportion of very long and very short waiting times, and the variance of the waiting time is greater than for first-come first-served queues. An example of random selection for service is the selection of trunk callers by the operator at a telephone exchange.

A further queue discipline is last-come first-served. The average waiting time is the same as for queues which are first-come first-served. But the variance of the waiting time is even greater than for random selection, and there is a correspondingly greater proportion of very long and very short waiting times. An example of last-come first-served discipline is the handling of goods which have to be stacked: foodstuffs in a warehouse or letters in an in-tray. It is not necessarily a bad policy for perishable goods if their survival time is short compared with the average waiting time.

In short, for these three types of queue discipline the average waiting time is the same, but the variance of the waiting time, and the proportion of units waiting a long time, increase in the order first-come first-served, random, last-come first-served.

Queue discipline might be extremely important in the handling of goods which deteriorate, the discipline which may be convenient to the stockroom being the one causing the greatest loss.

Priority

The units waiting for service may belong to two or more classes having an order of priority. The priority will be either:

(*a*) priority of entry when the service is free, or
(*b*) pre-emptive priority: priority to displace units being served if they are of lower priority.

The average waiting time of all the units remains unaltered if some of them are given priority (unless the priority is related to the service time), but the average waiting time of the various classes is decreased or increased according to their order of priority. In the case of pre-emptive priorities, the behaviour of the class with first priority is as though there is no other type of unit in the system.

It may be that the high priority classes have a shorter average service time than those of lower priority. In this case the average number waiting for service and the average waiting times are reduced. Similarly, if priority is given to classes having longer average service time, the average number waiting for service and the average waiting time are increased. Having a shorter or longer service time may be an incidental characteristic of the classes selected for priority, or it may be the reason why they are given priority.

Saaty (6) gives references to three systems in which priorities have been studied:

Poisson arrival, r classes of priority, non pre-emptive service, general service times (19),
Poisson arrival, 2 classes of priority, pre-emptive service, exponential service times (20),
Poisson arrivals, a continuous number of priorities according to shortness of service, general service time (21).

Impatient customers

Another feature of real life queues is the impatient customer who does not want to queue for too long. Two types of impatient customers have been studied:

the one whose decision to join is influenced by queue length,
the one who decides to leave the queue if the wait is too long.

When the decision to join the queue depends on the length of the queue, the arrival probability in the queue is effectively queue dependent—a situation described earlier. In the case of exponential arrival and service times when the probability of an arriving unit joining the queue is proportional to e^n, the average queue length remains finite, however large the arrival rate is compared with the service rate (4).

Another case studied is that of customers leaving after waiting for a time T in an exponential arrival and service system (22).

But here, as in the previous section, although some particular cases have been studied, there does not as yet appear to have been a systematic study of the loss of customers who are impatient.

The applications of such a study are not only to human queueing systems where impatience is understood, but also to such queueing systems as

storing perishable goods,
processing heated metal,
communication of information,

in each of which the unit loses its value if it waits too long.

Cyclic queues

In closed systems, units may pass from one service facility to the next in continual circulation, either being served or waiting for service. At each stage there may be a single server or a number of servers in parallel.

Koenigsberg (23) has analysed a cyclic system with m units in the system and k single-channel exponential service facilities of mean service rate $\mu_1, \mu_2 \ldots \mu_k$. There are no arrival rates to consider, as the output of one stage is the input of the next stage. The equations governing the transition probabilities can be formulated as in previous examples. The probability of there being n_1 units waiting or being served at the first stage, n_2 at the second stage, and so on, is given by

$$p(n_1, n_2, \ldots n_k) = \frac{x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}}{Z_k^m}$$

where
$$x_i = \frac{\mu_1}{\mu_i}$$

and
$$Z_k^m = \sum_{\substack{\text{All}\\ \text{partitions}\\ \text{of } m}} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} = \sum_{i=1}^{k} \frac{x_i^{m+k-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{k} (x_i - x_j)}.$$

If further
$$_iZ_{k-1}^m = \sum_{\substack{\text{All partitions}\\ \text{of } m \text{ excluding}\\ \text{the } i\text{th term}}} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$$

then the idle time of the ith stage, D_i, can be written as

$$D_i = \frac{_iZ_{k-1}^m}{Z_k^m}.$$

The throughput of the ith stage is $(1 - D_i)\mu_i$. As the throughput of all the stages must be equal, it follows that this expression has the same value for all values of i. It is of interest, too, that the throughput is independent of the order of the servers in this case.

The simplest particular case is that when all k servers have identical service times, μ. In this case

$$Z_k^m = \frac{(m+k-1)!}{m!(k-1)!}$$

$$_iZ_{k-1}^m = \frac{(m+k-2)!}{m!(k-2)!}$$

$$D_i = \frac{k-1}{m+k-1}.$$

The rate of throughput is therefore $\frac{m}{m+k-1}$. More difficult cases have been tackled.

The theory of cyclic queues was developed as a model of the cycle of operations in a group of nine coal faces in room and pillar working. The faces were considered to queue for service by blasting, filling, timbering and drilling teams. The effect of doubling one service stage was also considered. Cyclic queues with a transit time between one output and the input to the next queue have been studied.

An obvious application of the theory would be the movement of trucks on a closed loop. For most circulating systems it will probably be necessary to consider some more general distributions of service times.

Simulated sampling experiment

Many queueing problems cannot be solved explicitly by analytical methods. An approximate solution may then be obtained by experiment. To carry out sufficient experiments in each of several sets of conditions would be expensive and often impracticable; simulated experiments avoid these difficulties. For this technique—an application of the Monte Carlo technique—it is necessary to have random samples from the distributions of arrival and service intervals. These samples should have distributions which tend to those of the parent populations but the order should be random.

Two methods are available for obtaining the samples.

(*a*) If the distribution of intervals is known or assumed to be a smooth curve or mathematical function, let the cumulative distribution be $F(x)$. Then it is required that the probability of any interval in the sample being in the range a to b should be $F(b)-F(a)$. $F(x)$ varies from 0 to 1 so that if a number, y, is taken at random between 0 and 1 the probability that it lies between $F(b)$ and $F(a)$ is $F(b)-F(a)$. Values of x obtained by putting $y = F(x)$ for different values of y thus provide a sample with the required properties.

(*b*) If the distribution has been obtained by observation, the observed intervals can be numbered consecutively from 1 to n. The required sample is then found by taking random numbers in the range 1 to n and using the corresponding intervals.

The simulated experiment consists in using these samples to work out stage by stage the changes in the queue situation. From the results, measures such as average queue length can be calculated. The process is continued until the variance of the answers is small enough for the particular application. Techniques have been derived for reducing the amount of repetition but they do not yet seem applicable to queueing problems. The experiment is carried out for each set of conditions which is to be considered; to reduce the effect of sampling errors on the comparison of the different conditions, the same samples of intervals can be used in each experiment. An example of the method is given below.

Example

Men using a canteen first have to queue to buy meal tickets and then go on to join the queue at the service counter for the meal. The arrivals of men in the canteen are Poisson and occur every minute on average and the service times at the ticket box and service counter are as follows:

Stage I—Ticket Counter. Normal mean 15 sec. S.D. 5 sec.
Stage II—Meal Counter. Exponential with $\mu = \frac{5}{6}$ per min.

What is the average length of time each man spends in the system and how much of this is waiting time?

We consider the history of the first 30 men arriving at the ticket counter.

The cumulative distribution of arrival intervals, in seconds, is $F(x) = e^{-x/60}$.

The sample is obtained by taking random numbers, y, from 0 to 1 and putting $y = F(x)$; the values of x for the sample are then given by $x = 60 \log 1/y$ seconds.

A sample of service times at the second stage can be obtained in a similar way.

The sample service times for the first stage of service can be taken from tables of random normal deviates. These tables are derived from a population of mean zero and standard deviation unity so that samples of first-stage service times are obtained by multiplying each of the values of random normal deviates taken from the tables by 5 and adding 15.

The first step is to prepare a table of arrivals and service times. All times in this and the following table are in seconds.

Arrival	*Arrival time measured from previous arrival*	*First stage service time*	*Second stage service time*
1	0	19	222
2	36	13	119
3	112	16	6
4	6	7	19
5	57	22	82
6	127	18	204
7	47	11	134
8	96	14	77
9	176	22	25
10	100	8	143
11	2	18	150
12	59	16	32
13	33	9	4
14	7	11	61
15	45	17	118
16	99	12	41
17	162	17	11
18	29	16	58
19	40	20	4
20	99	16	40
21	39	19	194
22	27	9	140
23	1	13	4
24	64	17	103
25	29	13	83
26	137	5	27
27	60	12	4
28	17	20	1
29	20	14	215
30	34	6	60
Mean	58·7	14·3	79·4

These numbers are then used to piece together a history of the first 30 arrivals. A man moves into a service stage if it is free, otherwise he waits until the service station is free.

1	2	3	4	5	6	7
Arrival	*Arrival time*	*Time into ticket service*	*Time out of ticket service*	*Time into meal service*	*Time out of meal service*	*Time spent in system*
1	0	0	19	19	241	241
2	36	36	49	241	360	324
3	148	148	164	360	366	218
4	154	164	171	366	385	231
5	211	211	233	385	467	256
6	338	338	356	467	671	333
7	385	385	396	671	805	420
8	481	481	495	805	882	401
9	657	657	679	882	907	250
10	757	757	765	907	1050	293
11	759	765	783	1050	1200	441
12	818	818	834	1200	1232	414
13	851	851	860	1232	1236	385
14	858	860	871	1236	1297	439
15	903	903	920	1297	1415	512
16	1002	1002	1014	1415	1456	454
17	1164	1164	1181	1456	1467	303
18	1193	1193	1209	1467	1525	332
19	1233	1233	1253	1525	1529	296
20	1332	1332	1348	1529	1569	237
21	1371	1371	1390	1569	1763	392
22	1398	1398	1407	1763	1903	505
23	1399	1407	1420	1903	1907	508
24	1463	1463	1480	1907	2010	547
25	1492	1492	1505	2010	2093	601
26	1629	1629	1634	2093	2120	491
27	1689	1689	1701	2120	2124	435
28	1706	1706	1726	2124	2125	419
29	1726	1726	1740	2125	2340	614
30	1760	1760	1766	2340	2400	640

Average time spent in system = 6 min. 38 sec. (from column 7).
Average time queueing for service = 5 min. 4 sec. (from [col. 3 − col. 2] + [col. 5 − col. 4]).

In this example the rate of arrival exceeds the rate of service at the second stage so that the situation is unstable. To obtain representative results the experiment should be repeated starting each time with none in the system. If the steady state of a stable situation is being simulated, the necessary repetitions can be made by continuing the process for a large number of arrivals, without starting each time from the beginning.

Steady state formulae for queues

Most of the quantities likely to be of interest in queueing problems in the steady state are defined, and formulae for them are given, for the simple case; a selection of the more important results for other cases is given. For brevity, the word probability has been abbreviated to P. In all the formulae, $\rho = \lambda/\mu$.

Contents list

Unless otherwise specified, the following conditions are satisfied:

(*a*) unlimited input with Poisson distribution,
(*b*) unlimited number in the system,
(*c*) single queue served one unit at a time,
(*d*) each unit requires service by only one channel (i.e. single stage service).

(i) With service in order of arrival.

Type

(1)	Single channel	Exponential service
(2)	Single channel, limited number in system	Exponential service
(3)	Multiple channels	Exponential service
(4)	Multiple channels, limited number in system	Exponential service
(5)	Multiple channels, multiple stages	Exponential service
(6)	Single channel, service rate proportional to number in system	Exponential service
(7)	Single channel, Erlang service	
(8)	Single channel, general service	

(ii) With priorities.

(9)	Single channel, general service	Priorities in order of service time
(10)	Single channel, exponential service	Priorities in order of service time

(11) Single channel, exponential service, two classes, one with pre-emptive priority.

Formulae

(1) Poisson input, unlimited queue length, single service channel with exponential service time distribution. Average rate of input λ and average rate of service μ; $\lambda < \mu$.

Traffic intensity	$\rho = \lambda/\mu$
P of n units in the system	$p(n) = \rho^n(1-\rho)$
P of more than n units in the system	$p(>n) = \rho^{n+1}$
Average number of units in the system	$N = \rho/(1-\rho)$
P of n units in the queue	$q(n) = \rho^{n+1}(1-\rho)$ (for $n > 0$) $q(0) = 1-\rho^2$
P of more than n units in the queue	$q(>n) = \rho^{n+2}$
Average number of units in the queue	$Q = \rho^2/(1-\rho)$
P of no waiting (up to start of service)	$w(0) = 1-\rho$
P of waiting from t to $t+dt$	$w(t)\,dt = \rho(\mu-\lambda)\,e^{-t(\mu-\lambda)}\,dt$
P of waiting longer than t	$w(>t) = \rho\,e^{-t(\mu-\lambda)}$
Average waiting time	$W = \rho/(\mu-\lambda)$
P of time from t to $t+dt$ spent in the system	$s(t)\,dt = (\mu-\lambda)\,e^{-t(\mu-\lambda)}\,dt$
P of more than time t spent in the system	$s(>t) = e^{-t(\mu-\lambda)}$
Average time spent in the system	$S = 1/(\mu-\lambda)$
P that a queue exists	$p(>1) = \rho^2$
P that a unit needs to wait for service	$p(>0) = \rho$
Utilisation of the service channel	$u = \rho$

The formulae for $q(n)$, $q(>n)$ and Q if divided by ρ^2 apply to the time during which a queue exists; those for $w(t)$, $w(>t)$ and W if divided by ρ refer to the units which need to wait.

(2) Number in the system limited to L but otherwise as type (1) with no restriction on λ and μ.

$$p(0) = (1-\rho)/(1-\rho^{L+1}) \qquad N = \frac{\rho-(L+1)\rho^{L+1}+L\rho^{L+2}}{(1-\rho)(1-\rho^{L+1})}$$
$$p(n) = \rho^n p(0) \quad (\text{for } n \leqq L)$$
$$u = 1-p(0) \qquad Q = N+p(0)-1.$$

(3) Poisson input, unlimited single queue, c service channels in parallel, each with exponential service time distribution. Average rate of input λ and average rate of service by each channel μ; $\lambda < c\mu$.

$$u = \rho/c$$
$$p(0) = 1\Big/\left\{\sum_{i=0}^{c-1}\rho^i/i!+\rho^c/(c-\rho)(c-1)!\right\}$$
$$p(n) = \rho^n p(0)/n! \quad \text{for } n \leqq c$$
$$= c^{c-n}\rho^n p(0)/c! \quad \text{for } n \geqq c$$
$$p(>n) = c^{c-n}\rho^{n+1}p(0)/(c-\rho)c! \quad \text{for } n \geqq c-1$$
$$N = \rho^{c+1}p(0)/(c-\rho)^2(c-1)!+\rho$$
$$Q = N-\rho$$
$$w(>t) = p(>c-1)\,e^{-\mu t(c-\rho)}$$
$$W = Q/\lambda$$
$$S = W+1/\mu = N/\lambda.$$

(4) Number in the system limited to L, no restriction on λ or μ, and otherwise as type (3).

$$p(0) = 1\Big/\left\{\sum_{i=0}^{c-1}\rho^i/i!+\rho^c[1-(\rho/c)^{L-c+1}]/(c-\rho)(c-1)!\right\}$$
$$p(n) = \rho^n p(0)/n! \quad \text{for } n \leqq c$$
$$= c^{c-n}\rho^n p(0)/c! \quad \text{for } L \geqq n \geqq c.$$

In the particular case where $c = L$ (no queue allowed),

$$p(n) = \rho^n/n!\sum_{i=0}^{c}\rho^i/i! \quad \text{for } n \leqq c \text{ and } c = L.$$

(5) Poisson input of average rate λ, k stages of service in series with c_j channels at the jth stage each with exponential service time distribution of mean rate μ_j. $\lambda < c_j\mu_j$ for all j. $\rho_j = \lambda/\mu_j$.

P of $n_1, n_2, \ldots n_j \ldots$ units at stages $1, 2 \ldots j \ldots$ is

$$p(n_1 n_2 \ldots n_j \ldots n_k) = p(0) \prod_{j=1}^{k} v(n_j)$$

where
$$v(n_j) = \rho_j^{n_j}/n_j! \quad (\text{for } n_j \leqq c_j)$$
$$= c_j^{c_j - n_j} \rho_j^{n_j}/c_j! \quad (\text{for } n_j \geqq c_j)$$

and
$$p(0) = \prod_{j=1}^{k} \left\{ 1 \Big/ \sum_{n_j=0}^{\infty} v(n_j) \right\}.$$

When $c_j = 1$ for all j (one channel at each stage) the results for type (1) apply to each stage (putting μ_j for μ).

(6) Poisson input of average rate λ, unlimited queue, single service channel, with probability $n\mu\, dt$ of service finishing in the interval t to $t+dt$.

$$p(n) = \rho^n e^{-\rho}/n!$$
$$N = \rho$$
$$Q = \rho - 1 + e^{-\rho}$$
$$W = Q/\lambda$$
$$S = 1/\mu.$$

(7) Poisson input of average rate λ, unlimited single queue, single service channel with Erlang service time distribution of mean rate μ; $\lambda < \mu$. P of service time from t to $t+dt$ is

$$b(t)\, dt = (\mu k)^k e^{-\mu k t} t^{k-1}\, dt/(k-1)!$$

$$u = \rho$$
$$p(0) = 1 - \rho$$
$$N = Q + \rho \qquad W = Q/\lambda$$
$$Q = (k+1)\rho^2/2k(1-\rho) \qquad S = N/\lambda.$$

When $k \to \infty$ the service time is constant at $1/\mu$ so that

$$Q = \rho^2/2(1-\rho).$$

(8) Poisson input of average rate λ, unlimited single queue, one service channel with a general service time distribution of mean $1/\mu$ and coefficient of variation v; $\lambda < \mu$.

$$p(0) = 1-\rho$$
$$N = Q+\rho \qquad W = Q/\lambda$$
$$Q = \rho^2(1+v^2)/2(1-\rho) \qquad S = N/\lambda.$$

(9) Poisson input of average rate λ, unlimited single queue, single service channel. Units in the queue are served in order of increasing service time (which is known accurately in advance). General distribution of service time, with probability $F(t)$ of a time less than t. Average waiting time of a unit with service time t is

$$W_t = W_0 \Big/ \left[1-\lambda\int_0^t t'\,dF(t')\right]^2 \quad \text{if} \quad t \leqq t_0$$

$$N = W_0\int_0^{t_0}\left\{dF(t)\Big/\left[1-\lambda\int_0^t t'\,dF(t')\right]^2\right\}$$

where units with service time greater than t_0 never reach the service channel, t_0 being given by

$$\lambda\int_0^{t_0} t\,dF(t) = 1$$

and where
$$2W_0 = \lambda\int_0^{t_0} t^2\,dF(t)$$

$$(W_t = \infty \quad \text{if} \quad t > t_0 \quad \text{and} \quad N = \infty \quad \text{if} \quad t_0 < \infty).$$

(10) As for (9) but exponential service of mean rate μ.

$$W_t = W_0/\{1-\rho[1-e^{-\mu t}(1+\mu t)]\}^2 \quad \text{if} \quad t \leqq t_0$$

$$N = \lambda\rho\int_0^\infty e^{-\mu t}\,dt/\{1-\rho[1-e^{-\mu t}(1+\mu t)]\}^2 \quad \text{if} \quad t_0 < \infty,$$

where
$$\rho[1-e^{-\mu t_0}(1+\mu t_0)] = 1,$$

and
$$W_0 = \lambda\rho \qquad \text{if} \quad t_0 = \infty$$
$$= (1-\lambda t_0\, e^{-\mu t_0})/\mu \quad \text{if} \quad t_0 < \infty.$$

(11) Two Poisson inputs, 1 and 2, of rates λ_1 and λ_2. Unlimited single queue, single service channel. Units 1 have pre-emptive priority over units 2. Exponential service of rates μ_1 and μ_2.

The results for units 1 are exactly as for queues type (1) (with λ_1 for λ and μ_1 for μ).

For units 2, the average number in the system is

$$N_2 = \rho_2[1+\mu_1\rho_1/(1-\rho_1)\mu_2]/(1-\rho_1-\rho_2).$$

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CHAPTER 6

STOCK CONTROL

Introduction

When supply and demand for goods are not equal over short periods of time, stocks of goods may be held. Thus raw materials, goods being processed, finished articles and spare parts for machines may be stocked. Whatever the circumstances in which these stocks occur they contribute to the costs of production in tied up capital, storage costs, insurance charges and some wastage costs. These costs may amount to between 10% and 30% of the value of stock held. It is therefore economic to keep stocks as low as possible consistent with satisfying most demands for the goods involved.

According to the circumstances the interests of some Departments may favour high or low stock levels. For example, Production likes high stocks of raw material and spare parts so as to avoid stoppages. At the same time Production Departments cannot favour too many goods in progress standing on a factory floor because this may also lead to interference with production. At any one time Sales Departments like to draw from stocks of finished material to meet any demands from any customer, but on the other hand they may wish to reduce stocks when faced with the problem of redundant goods. There is a whole range of stock control problems where it is necessary to bring together, in some optimal way, two or more conflicting interests, and a scientific approach has been successfully made to these problems. This attempts to take into account the factors which enter into any stock control problem.

The pattern of demand and availability for a product is basic to the problem and affects the method of solution. For demand, it is necessary to know what quantity is consumed in a given period and how that quantity varies. For supply, it is necessary to know how long it takes to replenish stock, that is, what the period (called the lead-time) between placing an order and receiving a delivery is likely to be. The statistician would summarise this basic information briefly by saying he requires the distributions of demand and lead-time. Regular demand and constant lead-time make the problem easier.

To apply mathematics to stock control problems the costs associated with stocks are split into three parts:

(*a*) The cost which is a known function of the number of orders placed.
(*b*) The cost which is a proportion of the stock value.
(*c*) The cost which is a known function of the number of occasions and periods of time when the stock is not available.

These three costs correspond roughly with ones which are termed ordering, holding and stock-out costs.

The cost of ordering is interpreted differently according to the type of stock; to determine the quantity to be produced in one batch, ordering cost is synonymous with the cost of setting up machinery for making that batch, whereas in determining the input of raw material it includes the cost of sending out an enquiry, checking tenders, placing an order, receiving delivery of raw material in the storehouse and clearing payment.

The cost of holding stock depends on the materials stored, their rate of obsolescence, methods of handling in the storehouse and the interest charges on the capital tied up in the material. This cost is often proportional to the value of the stock, in which case the price per unit of stock enters into the problem.

Any stock item has associated with it a cost which is incurred whenever that item is not available. The make-up of this cost depends on the particular stock control problem; stock-out cost of raw material or spare parts is made up of plant down time and special delivery of the necessary material. Stock-out cost of a finished article is made up of a cost of dissatisfied or lost customers. The amount of the stock-out cost depends on the importance of the stock concerned.

Stock control problems may occur in production processes, in stores where thousands of different spare parts are stocked, and in large retail or wholesale distribution depots where ranges of consumer goods are handled.

The problem of provisioning for one article out of several hundreds in a large storehouse is the same in theory as that of provisioning for one raw material occupying a whole warehouse. Whilst it would not be economical to give detailed consideration to the small article and to repeat the work a hundred times for similar small ones in the storehouse, it is possible to group together a large number of items all requiring the same method of provisioning.

There are two main methods of provisioning. The first is the so-called Two-Bin system, where an order for a fixed quantity is made when stocks fall to a pre-set re-order level. The second is cyclical review, where a variable quantity is ordered at fixed intervals. If orders are placed at regular periodic intervals it is possible to group orders for related articles or to send one order for a list of articles supplied by the same company. Stock control is more thorough if stocks are reviewed by quantity rather than periodically, but it may be less convenient from an organisational point of view. Clearly a decision on choice of routine is needed, and under some conditions some articles will be provisioned by cyclical reviewing by time and others by quantity.

There are further fields of enquiry where stock control becomes more strategic in character. In the usual approach attempts are made to minimise total stores cost by assuming that the cost per order is constant and optimising the number of orders placed. It may be more profitable to study the cost of ordering in more detail with a view to reducing the cost per article ordered.

In many large organisations, there are stores holding the same material, but at different administrative levels. For example, an army will have central storehouses and depots in the home country and subsidiary storehouses, fed from the central stores, wherever forces may be posted. A large company may have a large central stores organisation with separate stores at factory or plant level. In such cases there are problems of finding which items should be centrally stocked and how often items held centrally should be distributed to the subsidiary stores. Very often centralised holding leads to economy in stock holding and operation of the stores and to the better care of material. In such instances factors against central stores would be poorer availability of goods and added transport costs. In any given situation it is possible to enumerate the factors for and against, and to deduce conditions that determine the stores location. Later in this chapter an equation is derived for central holding which ensures that savings by stock reduction exceed added transport costs.

The mathematical models used in solving stock-control problems must make use of existing information and be adaptable to stocks held before, during or after a production process. They can in fact be adapted to problems which involve:

(*a*) Different types of stocks.
(*b*) More than one location of a stock-holding point.

(*c*) Stock of one homogeneous item or stocks of hundreds of items.
(*d*) Small-scale or large-scale enterprises.
(*e*) Stocks other than ones of material, for example, trained personnel.

Problems of stock control and production scheduling are often conveniently grouped together, the production scheduling ones being the more complex. This chapter is largely confined to stock problems. The questions raised are:

(i) When to order; that is, to determine the re-order level or the interval of time between orders.
(ii) How much to order; that is to determine the re-order quantity or a fixed maximum stock level.

In the next section, models of some stock-control problems are given. The last four are described as special models; these examples illustrate how the models can be adapted to consider problems where particular characteristics have to be taken into account. Finally the application of the models is discussed.

Models for stock control problems

Model 1

The first model has demand known and constant, delivery immediate and shortage cannot occur. In this case the buffer stock and the re-order level are both zero.

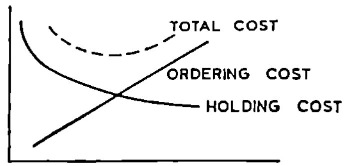
In deriving the optimum ordering quantity or optimum lot size, the buffer stock is left in because it makes the discussion more general without complication. The optimum provisioning policy is one which minimises the total cost of holding and ordering stock.

If the buffer stock is B, and order quantity is Q,

then average stock holding $= B+\frac{Q}{2}$. If demand per annum is D, then the number of orders per annum is $\frac{D}{Q}$. Holding a unit of stores costs C_1 per annum, and the cost of one order is C_2.

$$\text{Then total expected cost} = C_1\left(B+\frac{Q}{2}\right)+\frac{C_2 D}{Q} \text{ per annum.} \qquad (6.1)$$

The next step implies the position shown in the graph:



The total annual cost is minimised by differentiating with respect to Q, assuming that ordering cost varies directly with number of orders and that the buffer stock B is independent of Q.

Thus from equation (6.1),

$$\frac{d\,\text{Total Cost}}{dQ} = \frac{C_1}{2} - \frac{C_2 D}{Q^2}. \tag{6.2}$$

Equating (6.2) to zero gives $Q = \sqrt{\dfrac{2C_2 D}{C_1}}$.

If H is the cost of holding a unit of stock expressed as a fraction of the price of a unit of stock, then

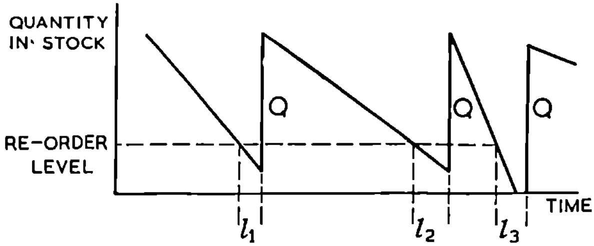
$$H = C_1/P, \quad \text{and} \quad Q = \sqrt{\frac{2C_2 D}{H.P}}, \tag{6.3}$$

where P is the price per unit.

For a given stores, C_2 and H may sometimes be regarded as constants, so that Q becomes $K\sqrt{\dfrac{D}{P}}$ where K is a constant.

Model 2

The second example considers the problem of optimising order quantity and buffer stock level for an article when its lead-time and demand are both variable. The price of the article is 10/–, its average consumption per week is 7 and it follows the Poisson distribution. The lead-time is normally distributed with average 14 days and standard deviation 1 day. The cost of placing an order is 6/–, stock-holding costs 25% per annum of the value of the stock, and £1 for special delivery charges is incurred each time a stock-out occurs. The problem is to find how the total costs can be minimised.



l_1, l_2, l_3 ARE SUCCESSIVE LEAD TIMES

The economic ordering quantity can be obtained from the formula given in equation (6.3) above.

Putting in the values for C_2, D, H and P which are the ordering cost, average consumption, percentage holding cost and price respectively,

$$Q = \sqrt{\left(\frac{2\times 0{\cdot}3\times 7\times 52}{0{\cdot}25\times 0{\cdot}5}\right)} = 42.$$

Therefore, whenever an order is placed 42 items should be ordered. The remainder of the problem is to find the re-order level. This level, say L, has to cater for variable demand in a variable lead-time and will equal D_L the average demand in the lead-time, plus some buffer stock B for the variations in demand and lead-time, i.e.

$$L = D_L + B = 14 + B \quad \text{in this example.} \tag{6.4}$$

The average stock at any time $= B + Q/2 = L - D_L + Q/2$.

The larger B is, the smaller is the probability of running out of stock. The remainder of the problem is, therefore, one of minimising the total cost of holding the buffer stock B and the run-out cost.

The number of orders placed per year is $7\times\frac{52}{42} = 8{\cdot}7$. If the probability of a stock-out is p the average number of stock-outs per year is $8{\cdot}7p$. The stock-out cost per year is thus £$1\times 8{\cdot}7p$ = £$8{\cdot}7p$.

The costs to be minimised are:

$$\text{Stock-out cost} + \text{buffer stock-holding cost} = 8{\cdot}7p + 0{\cdot}125B. \tag{6.5}$$

The value of B making this cost a minimum has to be found, and this can be done by drawing up a table as follows, the numbers 15, 20 and 30 in blocks (4) and (5) being trial values of the re-order level (i.e. $B+14$).

Lead-time (days)	*Probability of given lead-time*	*Average demand in lead-time*	*Probability of demand greater than:*			(5) = (2)×(4) *Probability of given lead-time and demand*		
			15	20	30	15	20	30
(1)	(2)	(3)	(4)			(5)		
12	0·05	12	0·156	0·012	0	0·0078	0·0006	0
13	0·24	13	0·236	0·025	0	0·0567	0·0060	0
14	0·40	14	0·331	0·048	0	0·1323	0·0192	0
15	0·24	15	0·432	0·083	0·0002	0·1037	0·0199	0·00005
16	0·05	16	0·533	0·132	0·0006	0·0267	0·0066	0·00003
						0·3271	0·0522	0·0001

The probabilities 0·3271, 0·0522 and 0·0001 shown at the foot of block (5) in the table are the probabilities of demands exceeding 15, 20 and 30 respectively, whichever one of its possible values the lead-time takes.

The values 15, 20, 30 of the re-order level give buffer stocks of 1, 6 and 16 respectively, and total shortage and stock-holding costs of £2·97, £1·20 and £2·00 respectively, by equation (6.5).

A re-order level of 20 leads to lower costs than levels of 15 or 30.

If the exercise is repeated to find the exact re-order level giving the minimum cost, the following results are obtained:

Re-order level	19	20	21	22	23
Cost £	1·33	1·20	1·16	1·17	1·22

Hence the lowest cost is incurred with a re-order level of 21.

Model 3

It is often useful to set the buffer stock level to give a specified probability of stock run-out, rather than the minimum total cost as in Model 2.

(*a*) With Poisson distribution of demand

In this case the buffer stock will be $K\sqrt{lD}$ where l is the lead-time, D is the average demand per unit of time, and K is a constant depending on the probability level chosen for stock run-out. This problem is solved by using limiting values of the Poisson distribution as lD tends to infinity when it approximates to the Normal. Reference to Normal distribution tables will show that there is:

1 % probability of demand exceeding $lD+2{\cdot}33\sigma$
2·5 % „ „ „ „ $lD+1{\cdot}96\sigma$ and so on

where σ is the standard deviation of demand in the lead-time. The result is a good approximation for all demand values. For the Poisson distribution, $\sigma = \sqrt{lD}$.

(*b*) With Normal distribution of demand

The Normal distribution may be a better fit than the Poisson for large demands. In this case, the re-order level L, that is the average demand in the lead-time plus the buffer stock, is given by

$$L = lD + K\sigma\sqrt{l}$$

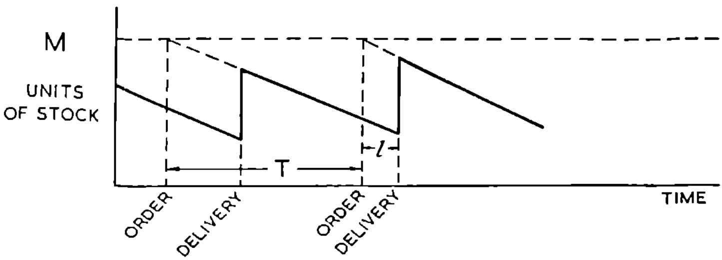
where l is the lead-time in months, D is the average demand per month, σ is the standard deviation of the monthly demand and K the same constant as used previously, that is the Normal deviate for a given probability of stock run-out.

Model 4

The problem is, given variable demand at an average rate D and constant lead-time l,

(i) at what cyclical review intervals should stocks be replenished;
(ii) how much should be ordered on each occasion;

in order to minimise costs, for a given probability of a stock-out.



A maximum stock level M is set and the re-order quantity is such that it would (if delivered immediately) bring stocks up to the level M. Let the interval between re-orders be T years, the yearly cost of holding per item be C_1 and the cost of one re-order be C_2. Then the costs per year will be:

C_2/T for re-ordering.
$(M-lD-\frac{1}{2}DT)C_1$ on average for holding.

With cyclical review there is a longer period during which the reserve stock must guard against fluctuations in demand than there is with a re-order level routine. This period is $l+T$ and the buffer stock will vary according to this period. If the standard deviation of the demand is assumed to be proportional to the average level, D, the buffer stock can be taken as some function $F(\overline{l+T}D)$. This function $F(\overline{l+T}D)$ is the expected value of $(M-lD-DT)$.

The expected total cost is $C_2/T+(M-lD-\frac{1}{2}DT)C_1$

$$= C_2/T+F(\overline{l+T}D)C_1+\tfrac{1}{2}DTC_1.$$

Differentiating this total expected cost with respect to T gives a minimum total cost where

$$T = \sqrt{\frac{2C_2}{C_1D}}.\sqrt{\frac{D}{D+2F'}},$$

where F' is the derivative of $F(\overline{l+T}D)$ with respect to T.

The value of T giving a minimum cost has been found but, unfortunately its value includes a function containing T itself. We can obtain an approximate solution for T by determining how F varies with T, that is F', from knowledge of the distribution of demand. It will normally be necessary to fix some arbitrary probability of stock-out which can be allowed in order to determine the function F.

Example

If the distribution is Poisson, the buffer stock is given by a function

$$F = K\sqrt{(l+T)D}.$$

For 1% probability of stock-out, K is 2·33.

Then by differentiating F with respect to T,

$$F' = \frac{K\sqrt{D}}{2\sqrt{l+T}} = \frac{2{\cdot}33\sqrt{D}}{2\sqrt{l+T}} \text{ for 1\% stock-out probability.}$$

Using the equation above for the optimal value of T, a solution can be obtained by iteration, for any given values of the quantities l, D, C_1, C_2.

Special models

In any practical problem special features may occur which are not covered by the basic models. This section gives four examples. It is probably easier to devise models to suit a particular problem than to attempt to modify a set of basic models to meet all possible circumstances.

Model 5. *Discounts*

Price reductions may be obtained by buying large quantities of goods, and it may prove that, by ordering the fixed quantity Q, and minimising costs with respect to holding and ordering costs, the opportunity of further saving from discounts is being missed. Whether this is the case can be roughly checked easily and quickly.

For example, an item has an annual consumption by value of £A and a stock-holding cost, H, of 12% of value per annum. If ordering quantity is increased by 1 month's stock the average holding is increased by ½ month's stock. If N months' stock are added, the extra stock-holding cost will be £$\frac{N}{2} \times \frac{A}{12} \times H$. Suppose a discount of d% on price is gained, the annual saving will be $\frac{£dA}{100}$. This saving will be worthwhile if $\frac{dA}{100} > \frac{N.A.H}{24}$, i.e. for $H = 12\%$, $2d > N$. Thus for a 12% holding cost, a simple check states that the % discount must exceed half the number of extra months' stock that has to be bought to gain that discount.

Model 6. *Central stores*

More centralised holding or control of expensive stores items may produce savings in total costs by stock-holding reduction. Suppose an item with a cost P and lead-time l has a Poisson demand with average demand D in each of n depots. What condition would allow centralised stores holding for the n depots? For a given level of protection, each depot would hold a stock reserve of $K\sqrt{lD}$ where K is a constant from the Poisson distribution depending on probability of stock-out. Since there are n depots, the total reserve is $nK\sqrt{lD}$. For the same level of protection, stock reserve at one centralised point is $K\sqrt{nlD}$. (It may be desirable to provide a higher level of protection when the stock is all held centrally.) Thus the stock-holding reduction is at least

$$K\sqrt{nlD}(\sqrt{n}-1).$$

If stock-holding cost at the depots is C_1 per item and is the same at the central point the money saving is

$$C_1 \times K\sqrt{nlD}(\sqrt{n}-1).$$

This saving is gained at the expense of some increase in transport cost. The transport cost is likely to vary with the amount of goods involved, so let it cost C_4 per item on average to move goods from the central store to each of the depots.

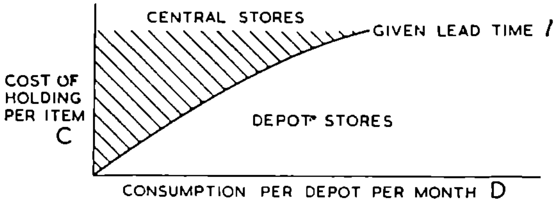
$\therefore$ Total extra transport cost $= nC_4D$.

Centralised holding follows the condition

$$nC_4D \leqq C_1K\sqrt{nlD}(\sqrt{n}-1),$$

i.e.
$$\frac{C_1^2 l}{D} \geqq \frac{nC_4^2}{K^2(\sqrt{n}-1)^2}.$$

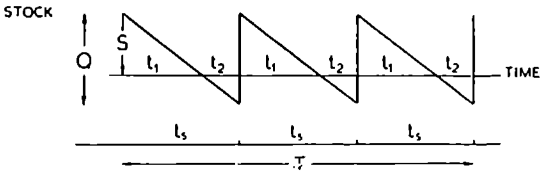
The expression has on the right-hand side a quantity which can be calculated for a given group of stores and regarded as a constant value for that set-up. The condition for centralised stock-holding can, therefore, be shown on a diagram as below; the position of an item on the graph is set by its price and consumption and if that item falls in the shaded portion, that is the portion above its own particular lead-time, then it is worthwhile stocking that item in a central store.



Model 7. The economic order quantity for constant demand and a stock-out cost

This example is the same as Model 1 given earlier except for the inclusion of a stock run-out cost. Model 2 also introduced lead-time, variation in this time and a variable demand. There are, therefore, a number of models possible between Models 1 and 2; these can be regarded as special cases of Model 2.

The present example is of a manufacturer who has to supply units at a constant rate to his customers. The units are manufactured in batches and the problem is to find the size of batch, Q, which minimises the costs. Stocks can be allowed to drop to zero; orders not met as a result are taken from the next batch. Consider a period T during which R units are supplied.



The variable costs are:

C_1 = the cost of holding one unit in stock for a unit of time
C_2 = the set-up cost per run
C_3 = the cost of stock-out per unit time per unit short and the delivery time is zero.

Since the consumption is regular the stock will vary in a regular way as shown in the diagram. The stock is shown varying below zero. Using the letters shown on the diagram, by similar triangles:

$$t_1 = St_s/Q = ST/R, \quad t_2 = (Q-S)t_s/Q = (Q-S)T/R$$

as $t_s = TQ/R$.

The average number of units in stock during t_1 is $S/2$ so that the total holding cost during t_1 is $C_1t_1S/2$.

Similarly the average number of units short during t_2 is $(Q-S)/2$ and the total stock-out cost during t_2 is $C_3t_2(Q-S)/2$.

As there are R/Q set-ups during T, the total holding, stock-out and set-up cost during T is:

$$C = (C_1t_1S/2 + C_3t_2(Q-S)/2 + C_2)R/Q$$

and substituting for t_1 and t_2,

$$C = C_1S^2T/2Q + C_3(Q-S)^2T/2Q + C_2R/Q.$$

By partially differentiating this equation with respect to Q and S and equating the derivatives to zero we find that

$$Q\min = \sqrt{\frac{2C_2R}{TC_1}}\sqrt{\frac{C_1+C_3}{C_3}} \qquad S = QC_3/(C_1+C_3),$$

and $$C\min = \sqrt{2RTC_1C_2}\sqrt{\frac{C_3}{C_1+C_3}}.$$

Model 8. Stock requirements of an expensive spare part

This example considers the decisions on spares holding at the time of placing orders for the erection of the plant itself. The spare in question is one whose failure cannot be foreseen and leads to a machine break-down. If the spare is not available when required there is a very high cost both in standing time of the machine and in getting a new spare part made specially. On the other hand the spare is very expensive and only used with the one machine so that if it is never needed its value has to be written off as scrap. The expected cost of overstocking these spares must balance the expected cost of stock run-outs.

Here again slight differences in the problem give another model. For example, the spare might have been obtainable normally at times other than at installation of the plant, or again it could be used on the subsequent plant or on other plants.

The cost of the part is £500. If a spare part is needed (because of a failure of the part in use) and is not available, the cost of the down time of the plant plus moving the part made specially is £10,000. The planned life of the installation is twenty years, and records of similar parts in other similar plants yield the information shown in the table.

No. of spare parts required in 20 years	*No. of plants requiring indicated no. of spares*	*Estimated probability of indicated no. of failures*
0	90	0·90
1	5	0·05
2	2	0·02
3	1	0·01
4	1	0·01
5	1	0·01
6 or more	0	0·00

We must here balance the costs incurred by buying too many parts against the cost of run-outs.

The cost associated with holding R units if the actual number of parts consumed is r is:

either $\quad (R-r)C_1 \text{ prob } (r) \quad \text{if} \quad r < R$

or $\quad (r-R)C_2 \text{ prob } (r) \quad \text{if} \quad r > R$

where R is the number of spares stocked
r is the actual consumption of spares
C_1 is the cost for overstocking one spare part
C_2 is the cost for being short of one spare part.

Then the total expected cost, for all r, of holding R units is

$$C(R) = C_1 \sum_{r=0}^{R} \text{prob}(r)(R-r) + C_2 \sum_{r=R+1}^{r} \text{prob}(r)(r-R)$$

$$C(R=0) = 10{,}000[(5-0)0{\cdot}01+(4-0)0{\cdot}01+(3-0)0{\cdot}01 +(2-0)0{\cdot}02+(1-0)0{\cdot}05] = £2{,}100$$

$$C(R=1) = 500[(1-0)0{\cdot}90+(1-1)0{\cdot}05] + 10{,}000[(5-1)0{\cdot}01+(4-1)0{\cdot}01+(3-1)0{\cdot}01 +(2-1)0{\cdot}02] = £1{,}550$$

$$C(R=2) = 500[(2-0)0{\cdot}90+(2-1)0{\cdot}05+(2-2)0{\cdot}02] + 10{,}000[(5-2)0{\cdot}01+(4-2)0{\cdot}01+(4-3)0{\cdot}01] = £1{,}525$$

$$C(R=3) = 500[(3-0)0{\cdot}90+(3-1)0{\cdot}05+(3-2)0{\cdot}02 +(2-2)0{\cdot}01] + 10{,}000[(5-3)0{\cdot}01+(4-3)0{\cdot}01] = £1{,}710$$

$$C(R=4) = 500[(4-0)0{\cdot}90+(4-1)0{\cdot}05+(4-2)0{\cdot}02 +(4-3)0{\cdot}01+(4-4)0{\cdot}01] + 10{,}000[(5-4)0{\cdot}01] = £2{,}000$$

$$C(R=5) = 500[(5-0)0{\cdot}90+(5-1)0{\cdot}05+(5-2)0{\cdot}02 +(5-3)0{\cdot}01+(5-4)0{\cdot}01+(5-5)0{\cdot}01] = £2{,}395$$

This shows that the optimum stock level is two parts.

Application of models

Models of stock control form a theoretical basis for starting an investigation, and on application in practice certain limitations appear and have to be overcome. This section discusses some of these difficulties.

Three difficulties that may be encountered in stock control work are:

(a) distributions of demand are not always Poisson;
(b) cost of stock-out is not known;
(c) costs are difficult to obtain and to split into parts dependent on stock value, numbers of orders placed and frequency of stock run-outs.

These are considered below.

Distribution of demand

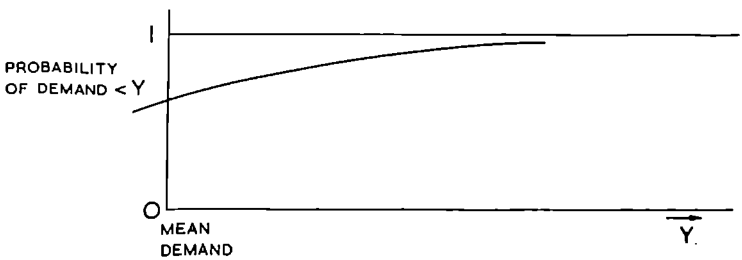
Consumption is often more variable than would be expected with a Poisson distribution. This is especially the case where sub-stores are held below the stores level under consideration. In this instance the total consumption can be regarded as the sum of the separate consumptions, each one of these having a Poisson distribution.

A paper by Dr. Galliher (9) introduces a hyperexponential distribution which is a good fit for consumption data showing too much variation to fit a Poisson distribution. In his paper Galliher uses the hyperexponential distribution to represent the time between demands on a store.

The hyperexponential can also be used to fit the distribution $f(x)$ of demands in a fixed period, and would in this sense be the inverse of Galliher's distribution. This distribution is simpler and is again a better fit than the Poisson when, as is often the case, the standard deviations of demand are considerably greater than the means. It is of the form

$$f(x) = a\lambda_1 e^{-\lambda_1 x} + (1-a)\lambda_2 e^{-\lambda_2 x},$$

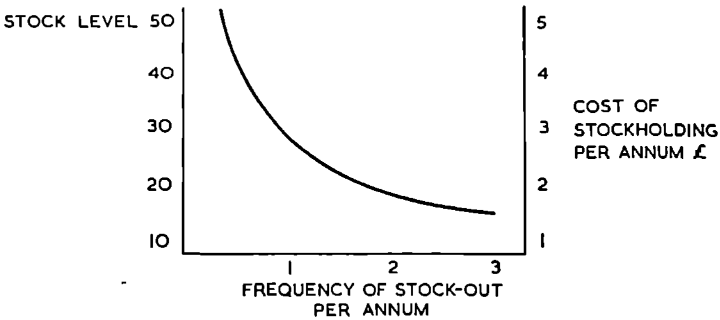
where x is demand in a fixed interval and a, λ_1 and λ_2 are parameters. Usually, only the tail integrals of this distribution are required for varying stock levels; the problem of finding these can be made simpler by assuming that the distributions of demand are all similar in shape. The distribution can be found by fitting average observed values. This enables a limited range of the cumulative distribution as shown below to be obtained:



Unknown cost of stock-out

Stock-out costs are known to exist but are difficult to assess. Models which include values for stock-out cost cannot be universally applied, whereas those including only figures like price and average consumption of an article are always applicable because these figures are usually readily available.

The effect of stock-out can be partially introduced into any problem by illustrating clearly the effect of different stock levels on the frequency of stock-outs. This type of illustration is conveniently done in graph form; a graph of the type shown below can be drawn for any given situation:



Management in using the graph has to base its decision on intuition, or has to put a rational estimate on stock-out cost. The fact that all the available information has been put forward in a simple form makes this job easier. Recent developments show that, where only the order of magnitude of a stock run-out cost is known, good decision rules, based on minimum total costs, can still be given for deciding re-order levels.

Costs difficult to obtain

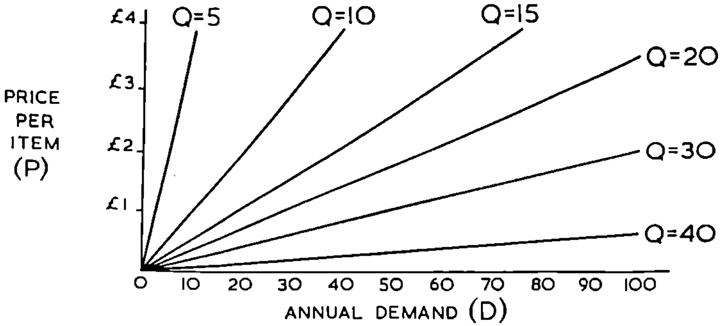
In addition to a difficulty in assessing stock run-out costs, ordering and stock-holding costs may not be clearly related to the number of orders placed and stock value respectively. Detailed study of these costs would enable them to be written as functions of some variables and these functions could usually be introduced into the appropriate models. Alternatively, the ordering and holding costs can be calculated approximately, and observations will show whether the application of the economic ordering quantity formula changes stock levels and present ordering practices. If large savings are made when this rational technique is used, then the apparent difficulties in ascertaining the exact ordering and holding costs will not appear to be so serious.

Practical aids to presentation of results

Economic order quantity

In one store, holding hundreds of different articles, the ordering and holding costs may be approximately the same for all the articles. In this case the economic order quantity becomes $K\sqrt{\frac{\text{Demand}}{\text{Price}}}$, where K is some constant.

This quantity can conveniently be shown on a table or graph for suitable ranges of demand and price. For example, the graph for ordering cost of 15/- and holding cost of 10% per annum would be:



Re-order level nomograms

Nomograms can be drawn giving the re-order level from the lead-time and the average demand. Any one nomogram is for a specified stock run-out probability, and for a given distribution of demand. Where we apply a Poisson distribution for low average demands and a normal distribution for the large demands, a nomogram for re-order level can still be used.

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CHAPTER 7

REPLACEMENT

Introduction

The term replacement covers a field of applications rather than a method of analysis.

Replacement problems occur in their simplest form in one or other of two fairly frequent situations: replacement of items whose maintenance costs increase with time, and replacement of items which have little or no maintenance costs, but are expensive to replace on failure. This chapter treats these two problems in some detail and mentions other possibilities for replacement models. A section on life curves is included.

There are few techniques which apply especially to replacement problems; highly sophisticated tools may be used as well as the more humble (and usual) arithmetic which will suffice for this chapter. The replacement problems we shall discuss can all be solved by tabulating various functions of the costs involved. We shall, however, need some general results, both in order to prove the validity of these tabular methods, and to provide a framework for tackling more complicated replacement problems. These results, which will for the most part be stated without proof, are derived from ordinary calculus, the calculus of finite differences, and from renewal theory, a mathematical theory dealing with replacement problems by a random variable approach, which is illustrated in the final section. Some references on replacement are given at the end of the chapter (1), (2), (3), (4), (5), (6), (7).

Replacement of items whose maintenance costs increase with time

Ignoring changes in the value of money; constant scrap value

For simplicity, we consider first a situation in which we wish to minimise the average annual cost of equipment whose maintenance costs are given as a function increasing with time, and whose scrap value is constant.

If the maintenance costs decrease or remain constant with time, the best policy is never to replace the equipment; unfortunately this desirable state of affairs is rarely met with in practice. If the maintenance costs fluctuate with time, equipment should only be replaced when the maintenance costs are increasing, but a slightly more complicated analysis is required.

Let: C = Capital cost of equipment.
S = Scrap value of equipment.
n = Number of years equipment is to be in use.
T = Average annual total cost.
$f(t)$ = Rate of expenditure on maintenance.

We want to find the value of n that minimises T.
Now the cost incurred in every period of n years is

$$C+\int_0^n f(t)\,dt-S.$$

Hence the average annual cost is given by

$$T=\frac{C-S}{n}+\frac{1}{n}\int_0^n f(t)\,dt.$$

Differentiating with respect to n, we obtain

$$\frac{dT}{dn}=-\frac{C-S}{n^2}-\frac{1}{n^2}\int_0^n f(t)\,dt+\frac{1}{n}f(n).$$

Hence for $\dfrac{dT}{dn}=0,\quad f(n)=\dfrac{C-S}{n}+\dfrac{1}{n}\displaystyle\int_0^n f(t)\,dt=T,\quad n\neq 0.$

It can be shown that this solution, $T = f(n)$, is a minimum for T, provided that $f(t)$ is non-decreasing, and $f(0) = 0$. Hence, in this type of example equipment should be replaced when the average cost to date becomes equal to the current maintenance cost.

Using this result we can decide when to replace equipment provided we have an explicit expression for the maintenance costs. In the following example it will be seen that, if we have the yearly maintenance costs given, it will not be necessary to use this result as we may more simply replace when the average annual cost reaches a minimum. The table confirms that this occurs when the average annual cost becomes equal to the current maintenance cost.

Example 1

Suppose a firm is considering when to replace their snoggle-casting machine, whose cost price is £12,200 but, since the firm hold a monopoly in this type of work, the scrap value is only £200 The maintenance costs are found from experience to be as follows:

Year	*Maintenance cost*
1	£200
2	£500
3	£800
4	£1,200
5	£1,800
6	£2,500
7	£3,200
8	£4,000

The costs required may now be tabulated:

Year	*Maintenance cost*	*Total cost to date (less scrap value)*	*Average annual cost to date*
1	£200	£12,200	£12,200
2	£500	£12,700	£6,350
3	£800	£13,500	£4,500
4	£1,200	£14,700	£3,675
5	£1,800	£16,500	£3,300
6	£2,500	£19,000	£3,167
7	£3,200	£22,200	£3,171
8	£4,000	£26,200	£3,275

The last line is of course superfluous, since the machine should be replaced during the seventh year, but it is of interest to note the comparative flatness of the curve of average cost against time near the minimum. The absolute minimum average annual cost will be less than £3,167, probably about £3,160.

The firm now receives an offer from another firm to sell them a somewhat shoddy snoggle-casting machine for only £7,500. The engineer inspects the machine and reports that it is likely to cost them about £400 in spares costs in the first year and that these costs are likely to rise by about £500 per annum. The scrap value of the second machine is reckoned to be negligible, and their present machine has done five years' service. Should the firm replace with the second machine or not?

We have:

$$C = £7{,}500$$
$$S = 0.$$

We first work out the minimum average annual cost of the new machine just as before:

Year	*Maintenance cost*	*Total cost to date*	*Average annual cost to date*
1	£400	£7,900	£7,900
2	£900	£8,800	£4,400
3	£1,400	£10,200	£3,400
4	£1,900	£12,100	£3,025
5	£2,400	£14,500	£2,900
6	£2,900	£17,400	£2,900

The minimum average cost comes somewhere in the sixth year and is probably about £2,870. Hence this machine is more economical than the previous machine and it should be installed. However, it should not be introduced straight away. The maintenance cost of the old machine does not reach £2,870 a year until half-way through the seventh year, and it is then that the new machine should be installed.

Alternative models

It would not be hard to refine the previous model so as to allow for a variable scrap value, but it is simpler to include the change in the scrap value as an extra maintenance cost, called depreciation. Thus a car-owner, in calculating what it costs him to run his car, would allow so much for maintenance and so much for depreciation.

A more difficult adjustment to make before we can claim to represent a real situation is one which allows us to take account of changes in the value of money. There are several possibilities which we might have to take into account. A manufacturer may have a certain amount of capital to spend or he may have to borrow. He might be allowed to borrow only a limited amount of money, and have to decide how to lay it out on various items. He may also have to take into account the prevailing tax laws. However, for illustration purposes we shall consider a simple case.

We assume that the present value of one pound to be spent in a year's time is £v, where $v = 1/(1+i)$, and i is the interest rate. (i can also be considered as the rate of inflation, or the sum of the rates of interest and inflation.)

If R_n is the running cost of a machine in the nth year of its life, then the total value, $P(r)$, of all expenditure on the machine, if we replace it after r years, is given by:

$$P(r) = C + R_1 + vR_2 + v^2R_3 + \ldots + v^{r-1}R_r.$$

Suppose this is to be paid off in fixed annual payments X, so that after r years we will have paid off the total cost of the machine, then the present worth of r of these payments is:

$$X + vX + v^2X + \ldots + v^{r-1}X = X(1-v^r)/(1-v).$$

This must equal $P(r)$, hence:

$$X = \left(\frac{1-v}{1-v^r}\right)P(r).$$

We wish to find the replacement period r that minimises X, and it can be shown that the required period r is that which gives:

$$\frac{1-v^{r-1}}{1-v}R_r - P(r-1) \leqq 0 \leqq \frac{1-v^r}{1-v}R_{r+1} - P(r).$$

Example 2

For a numerical example let $v = 0{\cdot}9$
$C = £3{,}000$

and the R's be as given in the table.

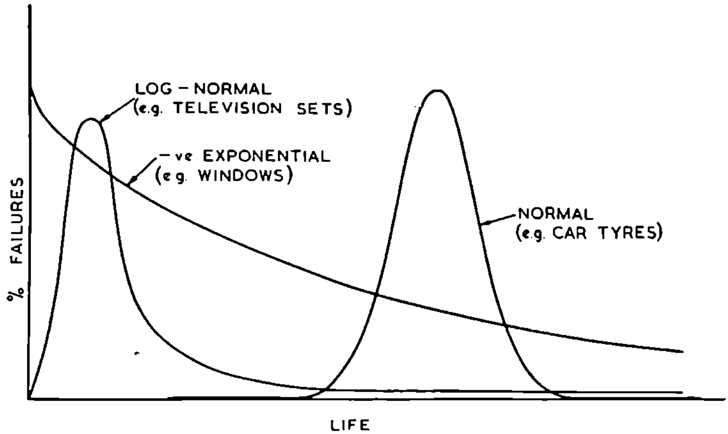
In the table we work out successively v^{r-1}, $v^{r-1}R_r$, $P(r)$ and $(1-v^r)R_{r+1}/(1-v)$. When the entries in the last two columns become the same it is time to replace the machine.

Year r	R_r	v^{r-1}	$V^{r-1}R_r$	$P(r)$	$\left(\frac{1-v^r}{1-v}\right)R_{r+1}$
1	500	1	500	3,500	600
2	600	0·9	540	4,040	1,520
3	800	0·81	648	4,688	2,710
4	1,000	0·729	729	5,417	4,471
5	1,300	0·6561	853	6,270	6,552.
6	1,600				
7	2,000				

By year 5 the entry in the last column has passed that in the fifth. Hence the optimum replacement interval is just under five years, and the discounted cost about £6,000.

Life curves

In dealing with the second kind of replacement problem we need to know the distribution of the life of an item. There are three main types of life curves, normal, log-normal and negative exponential.



Normal

This curve is typical of items which have one main kind of wear and are predominantly composed of one material, such as car tyres, which may have a mean life of 25,000 miles, with 95% between 20,000 and 30,000 miles.

Log-normal

This curve is the result of a series of normal curves added together. It is typical of those items of simple construction which can wear out or fail in various ways, such as rotting, abrasion, etc., and of items which contain many parts any of which can fail, such as radio and television sets.

Negative exponential

This curve results from failures occurring randomly in time and is typical of failures caused by damage, misuse, natural phenomena, etc., such as glass in fire-alarms, milk bottles, chinaware.

In practice combinations of these may occur but forecasting the type of life curve can save much effort.

Replacement of items which fail expensively

We now consider what is the best policy for replacing items of a large collection which are indistinguishable but liable to fail, such as light bulbs. A related problem is the staffing problem.

Light bulb problems

For instance, suppose we have a large number of light bulbs, all of which we must keep in working order. If a bulb fails in service, it costs £1 to replace; but if we replace all the bulbs in the same operation we can do it for only 7 shillings a bulb. If the proportion of bulbs failing in successive time intervals is known, it is then possible to decide on the best replacement policy, under certain simplifying assumptions.

Example 3

The given distribution of bulb lives is as follows:

Proportion	failing	during	first	week	= 0·09
,,	,,	,,	second	,,	= 0·16
,,	,,	,,	third	,,	= 0·24
,,	,,	,,	fourth	,,	= 0·36
,,	,,	,,	fifth	,,	= 0·12
,,	,,	,,	sixth	,,	= 0·03.

If we assume:

(*a*) that all bulbs failing during a week do so just before the end of the week and that group replacements can only be made at the end of a week;

(*b*) that the actual percentage of failures during a week for a sub-population of bulbs with the same age is the same as the expected percentage of failures during the week for that sub-population;

we can then calculate the cost per bulb per week of replacing all bulbs after one week, two weeks, etc.

We need also to know what will be the cost per bulb per week if we do not use group replacement at all. This is equal to the cost of replacing a bulb divided by the average life of a bulb, a result which may be proved rigorously by the calculus of finite differences, or, for the continuous case, by renewal theory.

In this example, the average life equals:

$$1 \times 0{\cdot}09 + 2 \times 0{\cdot}16 + 3 \times 0{\cdot}24 + 4 \times 0{\cdot}36 + 5 \times 0{\cdot}12 + 6 \times 0{\cdot}03 = 3{\cdot}35 \text{ weeks};$$

hence cost per bulb per week if no group replacement is used

$$= 20/3{\cdot}35 = 5{\cdot}97 \text{ shillings.}$$

We may now tabulate the cost per bulb per week for various group replacement policies:

Replace after week	*Proportion failed in week*	*Total proportion of failures to date*	*Total cost of replacing this week*	*Average cost of replacing this week (in shillings per week)*
1	0·09	0·09	$7+0{\cdot}09\times 20$	8·8
2	$0{\cdot}16+(0{\cdot}09)^2 = 0{\cdot}17$	0·26	$7+0{\cdot}26\times 20$	6·1
3	$0{\cdot}24+2\times(0{\cdot}09\times 0{\cdot}16)+(0{\cdot}09)^3 = 0{\cdot}27$	0·53	$7+0{\cdot}53\times 20$	5·87
4	$0{\cdot}36+2\times(0{\cdot}09\times 0{\cdot}24)+(0{\cdot}16)^2+3\times(0{\cdot}09)^2\times 0{\cdot}16+(0{\cdot}09)^4 = 0{\cdot}43$	0·96	$7+0{\cdot}96\times 20$	6·55

The minimum cost per bulb per week is obtained by replacing all bulbs after three weeks, with an average cost of 5·87 shillings per bulb per week.

Staffing problems

Example 4

A research team is planned to rise to a strength of 50 chemists and then to remain at that level. The wastage of recruits depends on their length of service and is as follows:

Year:	1	2	3	4	5	6	7	8	9	10
Total per cent who have left up to the end of the period:	5	36	56	63	68	73	79	87	97	100

What is the recruitment per year necessary to maintain this strength? There are eight senior posts for which length of service is the main criterion. What is the average length of service after which a new entrant can expect to be promoted to one of these posts?

Suppose the intake is 100 per year, then when equilibrium is reached the distribution of length of service of the team will be as follows:

Years	*No. of chemists*
0	100
1	95
2	64
3	44
4	37
5	32
6	27
7	21
8	13
9	3
10	0
Total	436

i.e. an intake of 100 per year gives a total strength of 436. To maintain a strength of 50 requires recruitment of $50 \times 100/436 = 11 \cdot 5$ per year.

We have assumed that all those who completed x years' service but left before $x+1$ years' service actually left immediately before completing $x+1$ years. If we assume that they left immediately after completing x years' service the total becomes 336 and the required intake is $50 \times 100/336 = 14 \cdot 9$. In practice, chemists may leave at any time in the year and 13 is a reasonable answer.

This problem can also be analysed in two other ways. It can be done by the light bulb approach, or by calculating the average length of service of all chemists. This is approximately 4 years, which corresponds to 25% wastage and therefore 25% recruitment per year. The above method does, however, have advantages when the second stage of the problem is tackled.

With 50 chemists in the team the distribution of the completed length of service of the chemists will be:

Years	*No. of chemists*
0	12
1	11
2	7
3	5
4	4
5	4
6	3
7	2
8	2
9	0
10	0

i.e. chemists can expect to be promoted to the senior posts after completing 5 and before completing 6 years of service.

Renewal theory

From the simplicity of these methods it may be concluded that replacement problems are always easy. But we have avoided the complications, which are considerable, both from the economic and the mathematical point of view. Preinreich (4) gives an economist's approach to replacement problems, and Smith (3) gives the mathematician's approach. Renewal theory, which is dealt with there, is a stochastic theory in which, for example, the life of a machine is treated as a random variable. We sketch a proof of the result cited in the light bulb example, for the interest of mathematicians. Many other important results are also obtainable.

Renewal density theorem

To prove that the renewal rate of items tends to the reciprocal of the mean life of the item, suppose we represent each item life by a random variable X_i ($i = 1, 2, \ldots, n, \ldots$), and assume that each item is replaced immediately on failure.

Then if we start with the first item at time $t = 0$, the nth item will start being used at time $t = X_1 + X_2 + \ldots + X_{n-1}$.

We assume that the X_i are identically and independently distributed with probability distribution $F(x)$.

We write $pr[x]$ for the probability of an event x.

Let $h(t)$ be the renewal density function, that is:

$$h(t)\delta t = pr[\text{renewal occurs in time interval } (t, t+\delta t)].$$

This probability equals

$$pr[\text{first machine fails in } (t, t+\delta t)]$$
$$+pr[\text{second machine fails in } (t, t+\delta t)]$$
$$+ \dots .$$

Hence we may write

$$h(t) = f_1(t)+f_2(t)+ \dots = \sum_{r=1}^{\infty} f_r(t), \tag{7.1}$$

where $f_1(t)$ = density function of a random variable with probability distribution $F(x)$

$f_2(t)$ = density function of the sum of 2 random variables with probability distribution $F(x)$

and so on,

i.e. $f_r(t)$ = density function of $X_1+X_2+ \dots +X_r$.

We require to prove that $h(t)$ tends to a constant value as $t \to \infty$, i.e. that the probability of a replacement in any small time interval tends to a constant.

To do this we need to use the Laplace transforms of h and f:

$$h^*(s) = \int_0^\infty e^{-st} h(t)\, dt$$

$$f^*(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

From equation (7.1) we have, by a well-known property of Laplace transforms:

$$h^*(s) = f^*(s)+[f^*(s)]^2+ \dots = f^*(s)/[1-f^*(s)].$$

Now $f^*(s) \to 1-\mu+0(s)$ as $s \to 0$,

when $\mu = EX_i$ = mean life of an item.

Hence as $s \to 0$, $h^*(s) \to \dfrac{1-\mu s+ \dots}{\mu s+ \dots} \to \dfrac{1}{\mu s}.$

Now $$\int_0^\infty \frac{1}{\mu} e^{-st} dt = \frac{1}{\mu s}.$$

Hence* $$h(t) \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty,$$

that is, the rate of renewal of items tends to the reciprocal of the mean life of an item.

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* The rigorous proof of this last step is too involved for inclusion here, but this sketch may serve to illustrate the typical methods of renewal theory.

CHAPTER 8

THEORY OF SEARCH

Introduction

In the theory of search it is difficult to get a complete picture of all the types of problem which arise and the techniques used for their solution. Most of the work in this field has been done by military O.R. groups, which unfortunately are seldom able to publish their results. This chapter cannot therefore be fully comprehensive.

Theory of search is defined more by the field in which the problems it tackles lie, than by a technique or series of techniques. As can be imagined, a large number of search problems involve the geometrical combinations of relative velocities and tracks of the targets and search objects. There is no general technique (or series of techniques) for solving this part of the problem, which is usually done by the normal geometrical or trigonometrical methods. Consequently, this part of theory of search has been excluded from this chapter.

There remain, however, some theorems, and techniques (very largely based on the Poisson distribution), which have a wide field of application, and this chapter deals mainly with them. They are:

(i) the mean free path theorem,
(ii) the optimum distribution of searching effort,
(iii) the use of a clustering technique in surveys.

Mean free path theorem

Consider an object, called the target, which is situated at random in an area A. The target may be an enemy submarine which we wish to locate, or perhaps an enemy position which we wish to destroy by bombardment. To achieve these aims we have search objects, which in the first case may be patrol planes and in the second case bombs or shells. Each of these search objects has an effective range of action against the target. That is, for the patrol plane an effective range of sighting, and for the bomb or shell a lethal radius. After a certain length of time, some portion of the area A will have been covered by one or more of the search objects, so that if the target is within this covered area, it will have been discovered or damaged at least once. We assume that this covered area is distributed at random inside the area A. There may be some overlap, in that part of the area A may be covered more than once, but we assume that this is done in a random manner. For example, if the search object were a patrol plane, the covered area would be equal to twice the effective range of vision, times the speed of the plane, times the length of time it has spent searching in the area A. In the case of a shell the covered area would be the number of shells fired inside A, times the lethal area of the shell for the target considered. The mean free path theorem gives the probability of success as a function of the ratio between the covered area and the total area A.

To find the value of this probability, we consider the situation at some given instant when the covered area divided by the whole area A is equal to ϕ (known as the coverage factor). Let $P(m, \phi)$ be the probability that the target has been discovered or damaged m times. We then increase the covered area by a small amount, so that ϕ will be increased by a correspondingly small amount $d\phi$. If the additional covered area is placed at random inside the area A, then the probability that the target will be discovered or damaged again in this new covered area is $d\phi$.
And so we see that

$$P(m, \phi + d\phi) = (1 - d\phi)P(m, \phi) + d\phi P(m-1, \phi).$$

Taking $P(m, \phi)$ from each side of this equation and dividing by $d\phi$ we get

$$\frac{dP(m, \phi)}{d\phi} + P(m, \phi) = P(m-1, \phi).$$

The solution of this equation which satisfies the initial condition that the probability of no success when the covered area is zero is equal to unity, is

$$P(m, \phi) = \phi^m e^{-\phi}/m!.$$

Hence, the probability that the target has not been discovered or damaged is

$$P(0, \phi) = e^{-\phi}.$$

The probability that the target will have been discovered or damaged at least once is

$$\sum_{m=1}^{\infty} P(m, \phi) = 1 - P(0, \phi) = 1 - e^{-\phi}.$$

$P(m, \phi)$ is, of course, merely the Poisson distribution probability of obtaining m successes when the expected number of successes is ϕ. Only the two-dimensional case is considered here, but the same argument can be applied to any number of dimensions.

Example. Area bombardment

A mortar emplacement is somewhere in an area A. The mortar is considered to have been hit by a shell if it is in a circle of area a centred on the point of impact of the shell. Then if n shells are fired at random into the area A, the chance that the mortar will not be hit is $e^{-\phi}$ where $\phi = na/A$. The probability of the mortar being hit m times is

$$P(m, \phi) = \phi^m e^{-\phi}/m!.$$

Now let us suppose that the probability that a hit on the mortar will damage it beyond repair is P, so that the probability of being able to repair the mortar after m hits is $(1-P)^m$.

The probability of the mortar having m hits and still being repairable is thus

$$(1-P)^m \phi^m e^{-\phi}/m!.$$

But the probability of damaging the mortar beyond repair $= 1 -$ probability of the mortar being repairable after all possible numbers of hits which equals

$$1 - \sum_{m=0}^{\infty} (1-P)^m \phi^m e^{-\phi}/m!$$

$$= 1 - e^{-\phi} e^{(1-P)\phi} = 1 - e^{-P\phi},$$

where $\phi = na/A$.

We thus see that the probability of complete destruction can be expressed in terms of a new coverage factor ϕ' (say) $= Pna/A$. This shows that in this and similar cases the coverage factor for complete destruction can be obtained from the coverage factor for a hit, by multiplying by the probability of complete destruction when hit. This simple property of ϕ is typical of the Poisson distribution.

The optimum distribution of searching effort

Suppose that an object is in an unknown position, but that its probabilities of being in the various possible positions are known. Thus in the one-dimensional case, $p(x)\,dx$ might be the probability that the target lies between x and $x+dx$, where $p(x)$ is known.

Suppose, further, that a limited total amount of searching effort (or time) is available. Thus if the amount of searching to be done between x and $x+dx$ is represented by $\phi(x)\,dx$, then $\phi(x)$ would have the properties

$$\phi(x) \geqq 0, \quad \int_{-\infty}^{+\infty} \phi(x)\,dx = \Phi,$$

where Φ is a positive constant which is a measure of the total amount of search available. Assume* finally that the probability of detecting the target (given to be at x) is $1-e^{-\phi(x)}$. The practical problem is to find the best manner of distributing the available searching effort; that is, the one which will give the greatest chance of finding the object. Thus, since $p(x)\,dx$ is the probability that the target lies in the interval $(x, x+dx)$, the probability that it will be detected in this interval is

$$p(x)[1-e^{-\phi(x)}]\,dx$$

so that the overall probability of detection is

$$P = \int_{-\infty}^{+\infty} p(x)[1-e^{-\phi(x)}]\,dx = P[\phi],$$

which is the function we wish to maximise, or rather we wish to find the function $\phi(x)$ which makes this a maximum. There is a very simple graphical solution to this problem, which is set out below.

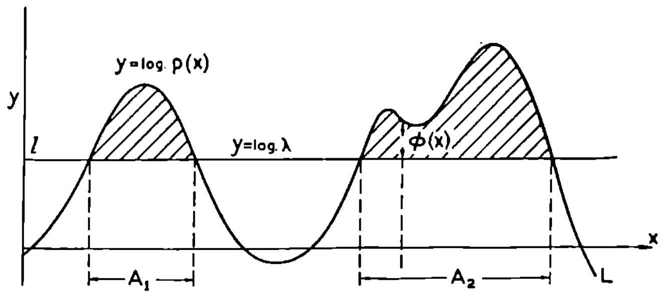

FIG. 8.1

* This one-dimensional case can be explained by the following example: Let n glimpses be made from random positions $x_1, x_2, \ldots x_n$ in an interval R, and suppose that in the x_ith glimpse the target is detected if and only if it is between x_i-r and x_i+r (r = definite detection range, thought of as much less than R), so that its probability is $2r/R$. If, as we assume, the n detection events are independent the probability of finding the target is $1-(1-2r/R)^n$, or $1-e^{-2rn/R}$ for large n. Now suppose that the glimpse positions are so numerous and so smoothly distributed that n/R can be represented approximately by a continuous function of x (the mid-point of R). Then we can also write $2rn/R = \phi(x)$, so that the expression for the probability becomes $1-e^{-\phi(x)}$.

Step 1

Plot the natural logarithm of the given probability $p(x)$ against x as shown in Fig. 8.1 to give the curve L.

Step 2

Draw a line l, parallel to the x-axis in such a way that the area above l but under the curve L (i.e. the shaded area) has a value equal to the total available quantity of searching effort Φ. Mark in A_1, A_2, etc., the perpendicular projections on to the x-axis of the segments cut off from l by the curve L.

Step 3. *The answer*

No search should be made outside the intervals A_1, A_2, etc. Inside the intervals, the density of search $\phi(x)$ should be equal to the length cut off by l and L from the vertical line drawn for the fixed x in question (i.e. $\log p(x) - \log \lambda$. See Fig. 8.1).

These three steps solve the problem, but if the full searching effort has been used up according to the optimum schedule given above we may wish to know

(i) what is the probability P of detection, and
(ii) what does the probability of the position of the target become if the search has failed.

To do this we proceed as follows:

Step 4

Plot $p(x)$ against x, and draw in the line $y = \lambda$ where λ equals the exponential of the constant y-value for points on l (thus in Fig. 8.1 l is the line $y = \log \lambda$). The area (shaded in Fig. 8.2) cut from this graph by the horizontal line is the probability that the search has been successful.

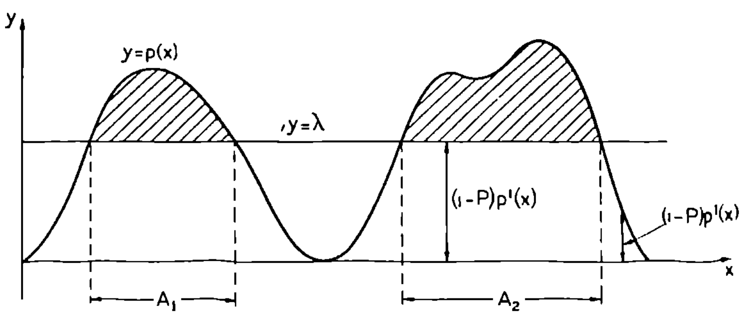

FIG. 8.2

Step 5

Decapitate the region below the curve $y = p(x)$ and above the x-axis by removing the shaded area above the line $y = \lambda$; the residual region will be bounded above by some horizontal segments of this line (projecting on the x-axis into A_1, A_2, etc.), by the remaining parts of the original curve $y = p(x)$; and below by the x-axis. Its area is $(1-P)$, the probability of failure of the search. The distance from any given point on the x-axis to the upper boundary of the residual (decapitated) region is proportional to the new probability density $p'(x)$ of the target's position (given failure). When divided by $(1-P)$, the distance becomes equal to $p'(x)$.

If, after the unsuccessful expenditure of effort Φ, a new quantity of searching effort Φ' becomes available, its optimum distribution is found as in the first case, but starting with $p'(x)$ instead of $p(x)$. It is worth noting that this successive scheduling is no worse than if we had known at the outset that the searching effort $\Phi+\Phi'$ was available; in fact it leads to the same overall optimum search distribution.

Generalisations

(*a*) Suppose that the visibility varied from position to position, then $\phi(x)$ would have to be replaced by a product such as $g(x)\phi(x)$, where $g(x)$ is a measure of the ease with which a unit of search conducted near x detects a target there. The function $g(x)$ would be known, and we would now have to find $\phi(x)$ so as to maximise

$$P[\phi] = \int_{-\infty}^{+\infty} p(x)[1-e^{-g(x)\phi(x)}]\,dx.$$

(*b*) Suppose that the value or usefulness of finding the target depends on where it is found, and that that is the value we wish to maximise. This would usually lead to the replacement of $p(x)$ by $v(x)p(x)$, where $v(x)$ is a weighting factor. However, this is only a difference in interpretation and not in the mathematics.

(*c*) Suppose that a given amount of searching costs more at some positions than at others. If, for instance, $\phi(x)\,dx$ units of search at x cost an effort $h(x)\phi(x)\,dx$ then the requirement that the total amount of effort be Φ (given) takes the form

$$\phi(x) \geqq 0, \quad \int_{-\infty}^{+\infty} h(x)\phi(x)\,dx = \Phi,$$

where $h(x)$ is supposed known. For example, suppose that the searcher has an unlimited amount of time for search, but is constantly exposed to a danger, depending on his position x. If he searches an amount $\phi(x)\,dx$ near x, his chance of being destroyed may be $h(x)\phi(x)\,dx$. We would then let the given quantity Φ represent, not the total available effort, but the total permissible risk.

(*d*) The above case is only the simple one-dimensional case and the same methods can be easily extended to apply to the n-dimensional case.

(*e*) It must be remembered that the basic quantitative ideas and methods of this technique need not be confined to the case of search. Any other form of effort, provided that its pay-off is additive and that each unit is expressible by the exponential formula, as above, is an appropriate field of application of the present material. Some applications in World War II were made to the optimum distribution of destructive effort, e.g. in bombardment, as mentioned previously.

The use of a clustering technique in surveys

Another technique, which can be applied to non-military problems, is that of two-stage search. In this, the objects being sought are small, motionless and distributed at random over a large area. The first stage or preliminary search involves a number of complete coverings of the entire area. It can produce spurious indications of possible prizes (the objects being sought) where there are none and can fail to detect some of the prizes that are present. The second stage or detailed search is perfect but it is difficult, expensive or time-consuming, and is concentrated only in those sub-regions of the total search area that are indicated by the preliminary search.

A practical application of this type of two-stage search may be found in mineralogical explorations, since preliminary surveys for minerals (or other well-hidden unmoving objects) frequently are conducted from moving vehicles equipped with suitable detection devices. Examples of preliminary surveys might be:

(*a*) aircraft, lorries or ships equipped with magnetometers seeking anomalies in the earth's magnetic field that may be associated with oil or mineral deposits,

(*b*) ships equipped with fathometers or other sonic devices searching for marine hazards.

In order to obtain a suitable interpretation of the results of the preliminary survey, a clustering technique is used to take advantage of the tendency for signals caused by a prize to be recorded at or near the actual location of the prize, while false contacts show no greater tendency to cluster than would be expected by chance. The following assumptions are made:

(i) The number of prizes, m, in the entire region being searched is known.

(ii) The probability that any given prize will be detected by the first stage search equipment after a single covering of the region is a known constant, p. Success or failure to detect a prize on one covering is independent of results of other coverings. The probability of occurrence of any specified number of detections of any particular prize after a specified number of coverings of the region is, therefore, determined by a binomial distribution.

(iii) In the first stage, detections of prizes are recorded within a known range, R, of their actual location, so that if a first stage detection is caused by a prize, the prize will be somewhere inside a circle of radius R centred on the recorded location of the contact. Such circles are small compared with the entire region being searched. The ratio of the area of a circle radius R to the entire area being searched is known and is designated as a.

(iv) The false contacts made in the first stage search are indistinguishable from the contacts produced by prizes. The false contacts are scattered randomly at a constant number per unit area throughout the area being surveyed. Consequently the probability of the occurrence of a given number of contacts resulting from a given number of coverings of any given area may be determined by a Poisson distribution. The expected number of false contacts occurring from a single covering of a circle of radius R is known and is designated as λ.

From these assumptions we see that

$$P_f(n, c = s) = (n\lambda)^s e^{-n\lambda}/s! \quad (0 \leqq s), \tag{8.1}$$

where $P_f(n, c = s)$ is the probability that n first stage coverings will produce a number of false contacts, c, equal to s (s is any positive integer and can exceed n) within range R of any given point of the preliminary survey region.

Looking at this in another way, we see that if a circle of radius R is drawn about each false contact (obtained from a particular set of n coverings) some of the circles may overlap. Dividing the area of all regions covered by exactly s circles by the total area of the preliminary survey gives the ratio of the s-cluster area (for a particular set of n coverings) to the total preliminary survey area. $P_f(n, c = s)$ is the average or expected value of the random sample ratios. This result is obtained by neglecting boundary effects which are small.

But
$$P_f(n, c \geqq s) = 1 - \sum_{i=0}^{s-1} P_f(n, c = i). \tag{8.2}$$

The probability that n coverings will produce a number, c, of detections of a specific prize, where c is less than or equal to s, is given by

$$P_t(n, c \leqq s) = \sum_{i=0}^{s} \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \quad \text{if} \quad 0 \leqq s \leqq n, \tag{8.3}$$

or
$$= 1 \qquad \text{if} \quad s > n.$$

From equations (8.1) and (8.3) we see that

$$P_{tf}(n, c \geqq s) = 1 - \sum_{i=0}^{s-1} P_f(n, c = i) P_t(n, c \leqq s-1-i), \tag{8.4}$$

where $P_{tf}(n, c \geqq s)$ is the probability that, as a result of n coverings, s or more true or false contacts are recorded within a range R of any given prize.

$P_{f'}(n, c = s)$ and $P_{f'}(n, c \geqq s)$ are now defined in the same way as the quantities shown in equations (8.1) and (8.2) above with λ replaced by $\lambda' = \lambda + amp$, so that

$$P_{f'}(n, c = s) = (n\lambda')^s e^{-n\lambda'}/s! \tag{8.5}$$

$$P_{f'}(n, c \geqq s) = 1 - \sum_{i=0}^{s-1} P_{f'}(n, c = i). \tag{8.6}$$

$P_{f'}(n, c \geqq s)$ approximates to the probability that as a result of n coverings, s or more contacts (true or false) will occur within range R of any arbitrary point of the preliminary survey area.

Thus it also approximates to the expected proportion of the original survey area which will be subjected to detailed examination during the second stage search if the following particular (n, s) clustering technique is adopted:

(*a*) The first stage search consists of n coverings of the original survey area. The locations of all contacts obtained in or within a distance R from the original survey area are charted. (This is to ensure that the prizes at the periphery of the area being searched have the same likelihood of being in the sub-areas subjected to the second stage search as those in the middle of the area.) A circle of radius R is centred about each such contact point.

(*b*) Those portions of the original survey area that lie within the intersections of at least s of these circles obtained from the first stage search are subjected to detailed examination during the second stage search, and all prizes in the second stage search area are found. It is not until the second stage detailed examination of a cluster area has been made that it can be determined whether or not a prize is present.

Three tasks have to be accomplished to cover the whole operation. The first stage search must be concluded. Then the second stage detailed search of the sub-areas indicated by the preliminary survey must be made, and finally the prizes must be utilised as appropriate. Before showing the best way in which this can be done we make the following further assumptions:

(v) The expenditure of effort in the preliminary survey is proportional to n, the number of coverings of the original survey area.

(vi) The expenditure of effort in the second stage detailed survey is proportional to the size of the area subjected to detailed exploration, as approximated by equation (8.6).

(vii) The expense of exploitation of prizes found and value of prizes found are proportional to the number of prizes found.

We now see that the total expected income is given by

$$I(n, c \geqq s) = mP_{tf}(n, c \geqq s)[V - E_3] - [nE_1 + P_{f'}(n, c \geqq s)E_2],$$

where m is the number of prizes in the whole survey area,
E_1 is the expenditure per covering of the original survey area,
E_2 is the expenditure required for second stage survey of the entire original survey area,
E_3 is the average expenditure per prize for the exploitation of located prizes,
V is the average value of a prize.

Values of $P_{tf}(n\ c \geqq s)$ and $P_f'(n, c \geqq s)$ have been tabulated for various values of n and s, so that it is now possible to find appropriate values of n and s to maximise the expected income (4), (5).

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