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PROGRAMMED STATISTICS

with chapters on probability, computer theory, and programmed instruction

RICHARD BELLMAN • JOHN C. HOGAN • ERNEST M. SCHEUER

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With Chapters on Probability, Computer Theory, and Programmed Instruction

RICHARD BELLMAN
University of Southern California

JOHN C. HOGAN
The RAND Corporation

ERNEST M. SCHEUER

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Preface

The minimum amount of information about statistics and measurement as applied to education, including probability theory, electronic computers, and programmed instruction that every high school and elementary school teacher should know is presented in this book. No mathematics beyond the simplest arithmetic or algebra is used.

This is, in part, a programmed textbook, but not all of the chapters have been written in that format. Although the authors believe that statistics is a subject that lends itself especially well to presentation as programmed instructional material; the subjects of probability and the theory of computers and computational aids for educators can perhaps better be studied from conventionally written texts. Accordingly, the book has a variety of formats, each adapted to the particular subject matter being discussed. We feel that this contributes to the book's readability, and makes it appropriate as a supplement to the conventional textbook used in any of the general college level courses in education, learning psychology, or the social sciences where probability, statistics, or the use of computers for instructional purposes are taught.

The authors are aware that some readers of this book will be unfamiliar with the technical and scientific language normally used in discussions of electronic computers and programmed instruction; accordingly, a glossary of common computer and programmed instruction terminology has been included at the end of the book. Additionally, a list of mathematical symbols is provided. There are also selected readings at the end of each chapter for those who wish to investigate further any of the topics that are presented in the book.

Our aim has been to write an elementary textbook for classroom teachers and for persons who are preparing themselves to become teachers, to introduce them to the fundamentals of probability, statistics, computers, and progammed learning as these subjects are related to the field of education.

Many persons, too numerous to mention, have helped with the writing of this book by contributing valuable ideas and suggestions that have been incorporated in the text, and to them we wish to express our deep appreciation. Special acknowledgments, however, should be made to Mrs. Bernice Brown of the Mathematics Department, The RAND Corporation, who helped plan the chapter on statistics and then engineered its early development, and to Dr. Richard E. Beckwith, Professor of Statistics and Dean of the School of Business Administration, Georgia State University, whose perceptive reading of an early version of the manuscript resulted in numerous suggestions for improvements. Our thanks also go to Miss Shirley L. Marks, Mr. Malcolm R. Davis, and Mr. Joe Clayton of the Computer Sciences Department, The RAND Corporation, who read the chapter on new computational aids for educators. Mrs. Rebecca Karush, University of Southern California, ably prepared the final manuscript for publication. Chapter 7 draws upon ideas and issues presented at a 1970 Seminar conducted by Professor Lawrence E. Vredevoe, Graduate School of Education, U.C.L.A.

The faults that will be found in this book, however, are entirely ours; and the authors, following the useful precedent of Allendoerfer and Oakley,[1] cheerfully blame each other for them.

RICHARD BELLMAN
JOHN C. HOGAN
ERNEST M. SCHEUER

Santa Monica, California
January 1970

[1]C. B. Allendoerfer and C. O. Oakley, ***Principles of Mathematics***. New York: McGraw-Hill Book Company, 1955.

Contents

PROGRAMMED STATISTICS

Chapter One

ELEMENTARY PROBABILITY THEORY

INTRODUCTORY REMARKS

The purpose of this chapter is to discuss some elementary concepts of probability theory. No mathematics beyond the simplest arithmetic or algebra is used. However, the reader should have pencil and paper at hand and be prepared to work through the examples presented here, and in addition he should work all the exercises to test his understanding of the material.

The theory of probability is an approach to some kinds of uncertainty. It is a mathematical idealization of certain aspects of reality in much the same way that geometry is an idealization, or model, of other aspects of reality. Such models are valid to the extent that they are useful, that is, to the extent that they help us solve certain types of problems we encounter. We know, for example, that there are no physical entities corresponding to the abstract geometric concept of "line" and that the earth is not truly spherical. Nevertheless, geometrical results are extremely useful in real life. The consequences of Euclidean geometry are, for many purposes, a satisfactory explanation of observed phenomena and enable us to build bridges, construct freeways, and so on. Similarly, probability theory, although not an exact mirror of reality, adequately describes many real situations and is, accordingly, useful.

DEFINITION OF PROBABILITY

We begin with the idea of an experiment in which there are a finite number, N, of distinguishable, equally likely elementary outcomes. For example, if the experiment is to draw one card from a standard pack of cards, then the 52 different cards of the deck constitute $N = 52$ distinguishable, equally likely elementary outcomes; if the experiment is to roll a die, then the six faces of the die constitute the $N = 6$ distinguishable, equally likely elementary outcomes; if the experiment is to toss a coin, then the two sides, heads and tails, constitute the $N = 2$ distinguishable, equally likely elementary outcomes; if a class of 100 students of unknown ability takes an arithmetic aptitude test, then we have $N = 100$ equally likely right or wrong answers to any particular question.

Note that these examples are already *models* of reality in that we *assume* that the various elementary outcomes are equally likely. In any given situation the model may or may not conform to reality. A particular card may have some physical property (being slightly longer than the other cards, or somewhat sticky) that makes it more, or less, likely to be drawn than some other cards, so that the assumption of equally likely elementary outcomes is not precisely correct here. Or, a particular die or coin may be asymmetric and the respective outcomes not be equally likely. Or, a particular student may have superior or inferior arithmetical ability. One aspect of mathematical statistics (*goodness of fit*) deals with examining the validity of assumptions of this sort. This topic is, however, beyond the scope of the chapter on statistics found in this book.

A comment is in order on the use of the term *elementary* with regard to outcomes. One could, for example, be interested in the outcome of drawing, say, a facecard, such as a king, or a heart from a deck of cards. Such outcomes are called *compound events*, being composed of several elementary outcomes. "Drawing a king" is realized if any of four elementary outcomes (drawing the king of clubs, the king of diamonds, the king of hearts, or the king of spades) occur; "drawing a heart" is realized if any of 13 elementary outcomes (drawing the 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, or A of hearts) occur. *An elementary outcome* is one that can be realized in precisely one way, not in more than one way as in the examples just cited.

We are now ready to define the probability of an event. An *event*, first of all, is the set of all elementary outcomes having a specified property. For example, we have already characterized the events of drawing a king or drawing a heart from a pack of cards. The event of rolling an even-numbered face on a die is composed of the equally likely elementary outcomes {2,4,6}.[1]

Let us denote the event of interest by E and the number of elementary outcomes comprising E by $\#(E)$. *We define the probability* P(E) *of an event* E *by*

$$P(E) = \frac{\#(E)}{N}. \tag{1}$$

(Recall that N is the total number of distinguishable, equally likely elementary outcomes.)

The number of outcomes favorable to an event E clearly is an integer that cannot be less than zero and cannot exceed N; Therefore the probability of any event must be at least equal to zero and cannot exceed one. This statement can be expressed in symbols as:

$$0 \leq P(E) \leq 1.$$

For example, a student cannot receive a negative grade on a test, or a mark of more than 100 percent.

If an event E is so specified that every elementary outcome is favorable for E, that is, $\#(E) = N$, then $P(E) = 1$. In this case E is called a *certain event* or a *sure event*. If an event E is so specified that no elementary outcome is favorable for E, then $\#(E) = 0$ and $P(E) = 0$. In this case E is called an *impossible event*.

In some instances it will turn out to be convenient to calculate the probability $P(E)$ of an event E by first obtaining the probability that E does not occur. Let us denote the event of the nonoccurrence

[1]One way of designating a set is, as here, to display its members between braces.

of E by E'.[2] Since every elementary outcome not favorable for E is favorable for E', then $\#(E') = N - \#(E)$, and

$$P(E') = \frac{\#(E')}{N} = \frac{N - \#(E)}{N} = 1 - \frac{\#(E)}{N} = 1 - P(E). \qquad (2)$$

This is an elementary, but important, result—namely that for any event E and its complement E',

$$P(E) + P(E') = 1. \qquad (3)$$

In order to define probability we have spoken of the number of distinguishable elementary outcomes in an experiment and of the number of elementary outcomes favorable to an event. In some situations it is simple to get these numbers. In others it can be a complicated task. As in plane geometry, a particular problem may be hard to solve even though the fundamental ideas are straightforward. We next discuss some procedures for counting that will be an aid in probability calculations.

COUNTING

In this section we study some aspects of how the number of outcomes favorable to some complicated events can be obtained from the number of outcomes favorable to simpler, related events. This subject is very important and an entire branch of mathematics, called *combinatorial theory*, is devoted to it. Combinatorial theory rests on two basic principles. We state the first now and discuss some of its ramifications. The second principle and some of its consequences are discussed later.

First Basic Combinatorial Principle

Consider an event E that is specified by the joint occurrence of two conditions. If there are n_1 elementary outcomes favorable for the

[2]The event E'c is alled the *complement of E* or the *complementary event to E*. Clearly $(E')' = E$.

first condition and, having selected one of those outcomes, there are n_2 elementary outcomes favorable for the second condition, then

$$\#(E) = n_1 n_2. \tag{4}$$

This first basic principle of combinatorial theory can be extended to counting the number of outcomes specified by the joint occurrence of more than two conditions in the obvious way. Let us consider some examples to illustrate the principle.

(1) Two coins are tossed. How many distinguishable outcomes are possible? We apply the first basic combinatorial principle. Here $n_1 = 2$ since there are two outcomes (heads, tails) possible for the first coin. Similarly, $n_2 = 2$. Therefore there are $n_1 n_2 = 2 \times 2 = 4$ distinguishable outcomes possible.

This was rather a trivial example because it is easy to enumerate all the possible outcomes[3]: {HH, HT, TH, TT} and observe that they are four in number. Let us look at a more complicated example.

(2) Ten coins are tossed. How many distinguishable outcomes are possible? Using the obvious extension of the first basic principle with $n_1 = n_2 = \cdots = n_{10} = 2$, we find the number of distinguishable possible outcomes[4] to be $2^{10} = 1024$. While it is possible, in principle, to enumerate all of these possibilities, it is tedious and we should begin to see the power inherent in this basic principle.

(3) A poker hand of five cards is to be drawn (without replacement) from a standard pack of cards. How many different such hands are there? [We distinguish (artificially) here between hands containing the same cards but appearing in a different order.] We apply the extended first basic principle. The first card can be selected in any of 52 ways. Having

[3]Note that we consider the outcomes HT and TH different here. They are different if one is concerned with what happened first and what second. They are *not* different if one merely counts the number of heads (for example). One's point of view governs here.

[4]Compare with footnote 3.

selected the first card, the second card can be selected in 51 ways (the card selected in the first draw is not available for selection at the second draw). Having selected the first two cards, the third card can be any one of the remaining 50 cards. Continuing in the same way, we find that the number of ordered poker hands is $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 =$ 311,875,200. This number is sufficiently large to eliminate enumeration of cases as a practical alternative to the use of the (extended) first basic combinatorial principle.

(4) The idea of the preceding example can be generalized to determine the number of distinct arrangements when r objects are selected without replacement from a set of n objects and order is important. Such ordered arrangements are called *permutations.* The number of distinct ordered arrangements of r objects selected without replacement from n objects, called *the number of permutations of n things taken r at a time*, and denoted $P(n,r)$, can be determined by reasoning identical to that of Example 3 to be:

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1). \tag{5}$$

[There are r factors in the product defining $P(n,r)$.]

In particular, the number of arrangements of all n items is

$$P(n,n) = n(n-1)(n-2)\cdots 3 \times 2 \times 1. \tag{6}$$

This product of all the integers beginning with n and going down to 1 is denoted by a special symbol, $n!$, which is read n *factorial.* Observe that this notation allows us to write

$$P(n,r) = \frac{n!}{(n-r)!}. \tag{7}$$

In order that the preceding formula make sense for the case $r = n$, that is, agree with the result $P(n,n) = n!$ just derived, it is conventional to *define* the symbol $0!$ to equal one.

The numbers $n!$ grow very rapidly with n. You should verify the values in the following table.

$1! = 1$	$6! = 720$
$2! = 2$	$7! = 5040$
$3! = 6$	$8! = 40{,}320$
$4! = 24$	$9! = 362{,}880$
$5! = 120$	$10! = 3{,}628{,}800$

Calculate 15! and 20! to further appreciate the rapid growth of $n!$ with n. It will save you some effort to realize that $n! = n \times (n-1)!$, so that you need not do each calculation afresh, but can proceed recursively.

(5) If r objects are selected without replacement from n objects and order is *not* important, the result of the selection is called a *combination.* A combination differs from a permutation in that only the composition, but not the order of selection, is important.

How many combinations of n things taken r at a time are there? Having selected such a combination, we could make from it $r!$ distinct ordered arrangements, that is, permutations. Thus, denoting the number of combinations of n things taken r at a time by $\binom{n}{r}$,[5] we have $r!\binom{n}{r} = P(n,r)$

or

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}. \tag{8}$$

A convenient way of calculating these quantities, called the *binomial coefficients,* is the "tree of Pascal":

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

[5]This symbol is read "n binomial r." (It is not a fraction. Do not put in a fraction line when you write it.)

Observe that each number, apart from the 1's, is obtained by adding the two immediately above. The ath entry in the bth row is $\binom{b-1}{a-1}$. The top of the tree is $\binom{0}{0}$; the first entry in the bth row is $\binom{b-1}{0}$.

EXERCISES

1. Calculate the next three rows.
2. Read off (a) $\binom{7}{3}$, (b) $\binom{6}{4}$, (c) $\binom{8}{3}$ from the tree and verify by using Equation (8).

Applying Equation (8) to determining the number of poker hands (eliminating the previous artificial restriction that the order in which the cards appeared mattered in terms of distinguishing hands) we find the number to be:

$$\binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960.$$

In what has preceded we dealt with selections made without replacement from a set of objects. Both ordered and unordered configurations have been examined. In the following two examples we consider ordered and unordered selections made with replacement.

(6) California automobile license plates contain three letters followed by three digits. Ignoring the fact that certain letter combinations will never be used, how many distinct California license plates are there? Each of the three letters can be chosen in 26 ways and each of the three digits can be chosen in 10 ways. By the (extended) first basic combinatorial principle the number is $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$. Note that each license plate is designated by an ordered drawing with replacement (from the 26 letters and the 10 digits).

(7) The idea used in the preceding example yields directly the number of ordered selections with replacement of r items selected from n items. Each of the r items can be selected in n ways. Therefore the number sought is:

$$\underbrace{n \times n \times \cdots \times n}_{r \text{ factors}} = n^r. \tag{9}$$

EXERCISES

1. For each unused combination of letters, how many license plate designations are lost?
2. How many four-letter "words" (pronounceability not considered) are there?
3. How many three-letter "words" are there whose first and third letters are consonants and whose middle letter is a vowel (a, e, i, o, u, y)?
4. How many seven-digit telephone numbers are possible?
5. How many numbers could there be in the GRanite exchange?

(8) The reasoning leading to the number of unordered selections with replacement is too complicated to appear here, so we content ourselves with stating the result. If r objects are selected with replacement from n objects, the number of unordered selections is $\binom{n+r-1}{r}$.

To illustrate this, consider an urn with four balls numbered 1, 2, 3, 4 from which two balls are to be drawn. The first ball is to be returned before the second draw. Only the numbers on the balls, without regard for order, are to be considered. How many such distinct selections are there? Using the formula above with $n = 4$ and $r = 2$ we find the number to be $\binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$. This number might be verified by actually displaying all the unordered selections with replacement:

(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4), and noting that they are 10 in number.[6]

We are now ready to state and discuss the second basic combinatorial principle. To do this we first must define the concept of mutually exclusive events: *Two events are mutually exclusive if the realization of either event precludes realization of the other.* For example, if, in drawing one card from a standard pack, D is the event of drawing a diamond and C is the event of drawing a club, events D and C are mutually exclusive. However, if T is the event of drawing a 10, events T and D (as well as events T and C) are not mutually exclusive.

Second Basic Combinatorial Principle

If an event E occurs if either of two mutually exclusive events, F and G, occurs, then

$$\#(E) = \#(F) + \#(G). \tag{10}$$

Upon dividing through by N in Equation (10), the total number of outcomes possible for the experiment, it follows that for mutually exclusive events F and G

$$P(F \text{ or } G) = P(F) + P(G). \tag{11}$$

The following examples illustrate this second basic combinatorial principle.

(9) In drawing a pair of cards without replacement from a standard deck, how many pairs contain exactly one red card? The answer involves both basic combinatorial principles. First note that the event E = "exactly one red card" occurs if the draw resulted in one of the following two mutually exclusive, ordered events: F = "red card first, black card second" or G = "black card first, red card second." Then

[6]We should make it clear, however, that we have displayed the selections here solely to verify the result given by the formula. Only if other methods of counting fail do we suggest this exhaustive (and sometimes exhausting) technique. Indeed, the whole purpose of this section is to show how to *avoid* enumeration.

note that $\#(F) = 26^2 = 676$ (verify this) and similarly $\#(G) = 676$, so that $\#(E) = 676 + 676 = 1352$.

(10) In rolling two dice, what is the probability that a "seven" will be rolled? We must determine the total number of these outcomes favorable for "seven." By the first basic combinatorial principle we see that there are a total of $6 \times 6 = 36$ outcomes possible when rolling two dice. Of these, the following six combinations yield "seven": (1,6),[7] (2,5), (3,4), (4,3), (5,2), (6,1). (We have used here an extended version of the second basic combinatorial principle. Of course, we would have been able to count the outcomes favorable for seven even if we had never heard of the second basic combinatorial principle, but the present example gives some instances of mutually exclusive events.) Thus

$$P(\text{seven}) = \frac{6}{36} = \frac{1}{6}.$$

EXERCISES

1. Verify for yourself that there are 36 possible outcomes, that there are no outcomes favorable for seven beyond those listed, and that these outcomes are mutually exclusive.
2. Two students each take an examination containing five questions. Assume that three is a passing grade. In how many ways can both students pass?

NON-MUTUALLY EXCLUSIVE EVENTS

It is possible to extend the second basic combinatorial principle to count the number of outcomes favorable for an event, E, where E occurs if either of two perhaps non-mutually exclusive events, F or G, occurs. Mathematicians use the word "or" in its nonexclusive sense. That is, the compound event "F or G" will occur if F occurs, or if G

[7] The first number in the parentheses refers to the face showing on the first die; the second number refers to the second die.

occurs, or if both F and G occur. The last event is possible if F and G are not mutually exclusive. Let us consider that situation, and let us denote "F or G" by E. If, in counting the number of outcomes favorable for E, we add $\#(F)$ and $\#(G)$, we will have taken account twice of those outcomes favorable for both F and G. Thus, in general,

$$\#(E) = \#(F \text{ or } G) = \#(F) + \#(G) - \#(F \text{ and } G). \qquad (12)$$

Upon dividing through in Equation (12) by N, the total number of outcomes possible for the experiment, it follows that in general

$$P(F \text{ or } G) = P(F) + P(G) - P(F \text{ and } G). \qquad (13)$$

Note that if F and G are mutually exclusive, then $\#(F \text{ and } G) = 0$ and we have agreement with the second basic combinatorial principle.

Two examples of the use of Equations (12) and (13) follow.

(11) In drawing one card from a standard deck, what is the probability of selecting a red card (R) or a face card (F)? We observe: $\#(R) = 26, \#(F) = 12, \#(R \text{ and } F) = 6, N = 52$, so that

$$\begin{aligned} P(R \text{ or } F) &= P(R) + P(F) - P(R \text{ and } F) \\ &= \frac{\#(R)}{N} + \frac{\#(F)}{N} - \frac{\#(R \text{ and } F)}{N} \\ &= \frac{26}{52} + \frac{12}{52} - \frac{6}{52} = \frac{32}{52} = \frac{8}{13}. \end{aligned}$$

(12) In drawing two cards with replacement from a standard deck, what is the probability of drawing at least one club? Let C_1 denote the event of selecting a club on the first draw and C_2 the event of selecting a club on the second draw. Clearly the event "C_1 or C_2" is the symbolic way of designating the event "at least one club." We observe:

$$\#(C_1) = \#(C_2) = 13 \cdot 52, \#(C_1 \text{ and } C_2) = 13^2,$$

$N = 52^2$, so that

$$\begin{aligned} P(C_1 \text{ or } C_2) &= \frac{\#(C_1)}{N} + \frac{\#(C_2)}{N} - \frac{\#(C_1 \text{ and } C_2)}{N} \\ &= \frac{13 \cdot 52}{52^2} + \frac{13 \cdot 52}{52^2} - \frac{13^2}{52^2} \\ &= \frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \frac{7}{16}. \end{aligned}$$

CONDITIONAL PROBABILITY AND INDEPENDENCE

There are situations in which one wants to calculate the probability of an event, A, given the knowledge that another event, B, has occurred. The term used for this concept is *the conditional probability of A given B*, and it is denoted by $P(A|B)$. Applying the definition of probability, Equation (1), we find

$$P(A \mid B) = \frac{\#(A \text{ and } B)}{\#(B)}. \tag{14}$$

The reasoning supporting Equation (14) goes like this. Since we know that B occurred, the number of possible outcomes is $\#(B)$, not the total number of outcomes possible for the experiment, N. The number of outcomes favorable for A when we know B has occurred is $\#(A \text{ and } B)$.

Equation (14) usually appears in a different form, namely

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}. \tag{15}$$

Equation (15) is obtained from Equation (14) by dividing numerator and denominator by N, the total number of outcomes possible for the experiment, and then appealing to the definition of probability, Equation (1) — thus

$$P(A \mid B) = \frac{\#(A \text{ and } B)}{\#(B)} = \frac{\#(A \text{ and } B)/N}{\#(B)/N} = \frac{P(A \text{ and } B)}{P(B)}.$$

This development of conditional probability involved one tacit assumption which must be made explicit: Since division by zero is not defined, it is necessary to require that $\#(B) \neq 0$ or, equivalently, that $P(B) \neq 0$. (The symbol $\neq$ is read "not equal to.")

It is often convenient to write Equation (15) in the following equivalent form:

$$P(A \text{ and } B) = P(A \mid B)P(B). \tag{16}$$

Some examples follow.

(13) What is the probability of drawing a king in a single draw from a standard deck if it is known that the card drawn will be a face card? Let K denote the event of drawing a king and F the event of drawing a face card. $\#(F) = 12$, $\#(K \text{ and } F) = \#(K) = 4$, so that $P(K|F) = \frac{4}{12} = \frac{1}{3}$. Note that the ordinary probability of drawing a king is $\frac{4}{52} = \frac{1}{13}$. Verify that for this example the event (K and F) is the same as the event K. Generally speaking, the event (A and B) is not the same as the event A.

(14) What is the probability of drawing a king in a single draw from a standard deck if it is known that the card drawn will be red? Let K denote the event of drawing a king and R the event of drawing a red card. $\#(R) = 26$, $\#(K \text{ and } R) = 2$, so that

$$P(K \mid R) = \frac{2}{26} = \frac{1}{13}.$$

Example 14 is but one instance of a general concept involving two events; namely that of the *independence of two events*. A formal definition follows directly from the example.

Definition

Two events A and B (neither of them an impossible event) are *independent* if

$$P(A \mid B) = P(A). \tag{17}$$

If we substitute Equation (15) on the left-hand side of Equation (17) we obtain

$$\frac{P(A \text{ and } B)}{P(B)} = P(A)$$

or, equivalently, we find two events A and B are independent if

$$P(A \text{ and } B) = P(A)P(B). \tag{18}$$

If two events are not independent they are called *dependent*.

EXERCISES

Verify the independence of the following pairs of events.

1. A coin is tossed twice. H_1 is the event of a head on the first toss, H_2 the event of a head on the second toss.

2. A card is drawn from a standard deck and a die is rolled. C is the event of drawing a three from the deck, D is the event of rolling a three with the die.

3. Two dice are rolled. A is the event of rolling a one-spot on the first die, B is the event of rolling a one-spot on the second die.

4. In Exercises 1–3, formulate some other events that can occur and investigate their independence.

5. Can you prove that if two events, A and B — neither of which is impossible — are mutually exclusive, then they cannot be independent?

6. An urn contains ten red and five white balls. Two balls are drawn without replacement. Let R_1 denote the event that a red ball was selected on the first draw and R_2 denote the event that a red ball was selected on the second draw. Are the events R_1 and R_2 independent?

Solution to Exercise 6: We will compare $P(R_1 \text{ and } R_2)$ with $P(R_1)P(R_2)$. To calculate $P(R_1 \text{ and } R_2)$ we use Equation (16):

$$P(R_1 \text{ and } R_2) = P(R_2|R_1)P(R_1) = \frac{9}{14}\cdot\frac{10}{15} = \frac{3}{7}.$$

To calculate $P(R_2)$ we make use of the fact that R_2 occurs if one of two mutually exclusive events occur: either both balls drawn are red (denote this by F) or the first ball is white and the second ball is red (denote this by G). We calculate $P(R_2)$ as $P(F) + P(G)$ and use Equation (16) to obtain each of these terms. [We have already calculated $P(F)$.]

$$P(F) = P(R_1 \text{ and } R_2) = \frac{3}{7},$$

$$P(G) = P(R_1' \text{ and } R_2) = P(R_2 \mid R_1')P(R_1') = \frac{10}{14}\cdot\frac{5}{15} = \frac{5}{21},$$

$$P(R_2) = P(F) + P(G) = \frac{3}{7} + \frac{5}{21} = \frac{14}{21} = \frac{2}{3}.$$

Since $P(R_1) = 2/3$, we have $P(R_1)P(R_2) = 4/9$ while $P(R_1 \text{ and } R_2) = 3/7$. Since $P(R_1 \text{ and } R_2) \neq P(R_1)P(R_2)$, the events R_1 and R_2 are dependent.

Another way of investigating the independence of R_1 and R_2 is via Equation (16). We calculated above that $P(R_2|R_1) = 9/14$ and that $P(R_2) = 2/3$, so again we conclude R_1 and R_2 are dependent events.

SUMMARY AND GENERALIZATIONS

For ease of exposition the foregoing treatment of probability has focused on experiments in which there are a finite number of distinguishable, equally likely elementary outcomes. It turns out, however, that the results we obtained are valid in far more general situations. We summarize our results and stress their validity for a very broad definition of events and their probability.

(1) For any event E, the probability of E, $P(E)$, is a nonnegative number not exceeding one.

(2) If E' is the nonoccurrence of E, then

$$P(E') = 1 - P(E).$$

(3) Number of ways of selecting r items out of n:

	Selecting with replacement	*Selecting without replacement*
Ordered arrangements	n^r	$\frac{n!}{(n-r)!}$
Order not considered	$\binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!r!}$	$\binom{n}{r} = \frac{n!}{(n-r)!r!}$

(4) Two events E and F are mutually exclusive if $P(E \text{ and } F) = 0$. For mutually exclusive events, $P(E \text{ or } F) = P(E) + P(F)$.

(5) In general, for any pair of events E and F, $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.

(6) The conditional probability of an event E, given that an event F has occurred, is

$$P(E \mid F) = \frac{P(E \text{ and } F)}{P(F)}$$

provided $P(F) \neq 0$.

(7) Two events E and F are independent if $P(E|F) = P(E)$ or, equivalently, $P(E \text{ and } F) = P(E)P(F)$.

EXPECTED VALUE

The great value of probability theory resides in the fact that it provides us with a rational approach to uncertainty, and, particularly, to decision making under uncertainty, by furnishing certain numerical measures of uncertainty. One way of doing this is by means of *average* or *expected* values.

If in some situation in which we can realize exactly one of two outcomes, the first with probability p_1 and the second with probability p_2, and if the first yields a gain of v_1 units and the second a gain of v_2

units, we can assess the situation in terms of an average gain of $p_1v_1 + p_2v_2$. If there are N possible events (of which we can realize one) with respective probabilities of occurrence of $p_1, p_2, \ldots, p_N$ and associated gains of $v_1, v_2, \ldots, v_N$, we analogously assign an average gain of $p_1v_1 + p_2v_2 + \cdots + p_Nv_N$.

On the other hand, we may not be satisfied with use of the average. Thus we may rank the gains say $v_1 \leq v_2 \leq \cdots \leq v_{n-1} \leq v_N$, and compute the probability on one realization of achieving a gain of more than some specified number v. If

$$v_1 \leq v_2 \leq \cdots \leq v_k < v \leq v_{k+1} \leq \cdots \leq v_N.$$

we see that the probability that actual gain exceeds v is given by $p_{k+1} + p_{k+2} + \cdots + p_N$.

The foregoing enables us to make a decision when the outcome is not completely known. If one decision leads to an expected gain of 100, and another to an expected gain of 10, we have a rational basis for choosing the first action.

Unfortunately, the foregoing plausible procedure does not completely dispose of the problem of decision making under uncertainty. In many situations, such as those faced by insurance companies or by instructors assigning grades in large sections of several hundred students, average values are meaningful. In other situations, for example, a class of 17 students, the average grade may provide little information. In general, the choice of criterion to employ in decision making — whether average value or probability of achieving a desired level or some other measure — is one of great difficulty, as is also the task of assigning values to the outcomes.

These problems are not mathematical per se, but operational and psychological, requiring a deep understanding of the actual process. Probability theory is used, for want of a better tool, in situations involving ignorance; recall our earlier statement concerning the assumption of equally likely events. In all real situations there is always additional information available to supplement the results obtained from probabilistic calculations. However, it may be too expensive or time-consuming to obtain this supplementary data, and we may be forced to use the less complete results.

We have briefly pointed out the difficulties involved in deciding what measures of uncertainty to employ and the values to assign to outcomes. In some situations it is also difficult to make reasonable *a priori* estimates of the probabilities. To handle uncertainties of this type, we can conceive of nature as an opponent who is trying to choose the probabilities in such a way as to make it as difficult or as expensive as possible to discover the actual values. The problem of obtaining information, or of making appropriate decisions, can then be viewed as a game against nature. Many useful results can be obtained in this way, employing the Borel-von Neumann theory of games.[8]

In general, the game theory approach is too expensive to employ. It is not feasible to assume that everything is against us all the time, and the viewpoint is not one conducive to mental health. Different and more complex mathematical tools must be used, such as sequential analysis, dynamic programming, and adaptive control processes. References to some expository accounts will be found at the end of the chapter.

THE NUMBER OF SUCCESSES IN A SAMPLE

By a *sample of size n* we mean the realization of n independent outcomes of some given events. Thus a sample of size n may consist of the result of tossing n coins once, or of tossing one coin n times, or of drawing n cards from a deck (with or without replacement), or of the responses of a student to a multiple-choice exam with n questions, or the response of a class of n students to one question. The term *success* is used in the sense of an outcome of interest (to someone). For example, we may arbitrarily call the occurrence of a head in the toss of a coin a success, even though a person who has bet on a tail turning up may not consider it so. Similarly, success may be defined as an *incorrect* response to a question on a multiple-choice exam. The term is well established even though it may appear rather inappropriate

[8]See, for example, John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior.* (Princeton, N.J.: Princeton University Press, 1944), and Émile Borel, *Le Jeu, la Chance et Les Théories Scientifiques Modernes.* (Gallimard, 1941); other works are: Melvin Dresher, *Games of Strategy: Theory and Applications.* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1961) and J. D. Williams, *The Compleat Strategyst.* (New York: McGraw-Hill, Inc., 1954 and 1966).

at times. Let us consider it as a technical term and not worry about its meaning in ordinary parlance.

We now want to consider the probability that a certain number of successes occur in a given number of trials. For example, suppose a multiple-choice exam is given, each question having five responses listed with only one correct response. There are 100 questions, and 70 questions must be answered correctly in order to get a passing grade. If a student merely guesses the answer to each question, with no knowledge at all about the subject what is the probability that he will get a passing grade? We can answer this question by a series of simple steps which we now detail. First, we must model the process by which guesses are made. We *assume* that for each question one of the five responses is selected at random, that is, each response has the same probability of being picked, namely $\frac{1}{5}$. Next, we define *success* as the event of having the correct response on a question. Our model specifies that $P(\text{success}) = \frac{1}{5}$ for each question. Since a passing grade requires at least 70 correct responses, the event of getting a passing grade is composed of the events 70 successes or 71 successes or 72 successes or . . . or 100 successes. Since these latter events are mutually exclusive (verify this), the probability of getting a passing grade is the sum of the probability of these events (verify this also). We will obtain these probabilities by first considering the prototype of the calculations we must make here.

Consider n independent trials, each of which can result in only two outcomes, success (S) or failure (F). The probability of success on each trial is p. Consequently the probability of failure at each trial is $1 - p$. (Verify this.) What is the probability of exactly k successes in these n trials? Note that for one particular order in which the k successes (and consequently $n - k$ failures) occur, the probability is $p^k(1 - p)^{n-k}$. (Verify this, using the assumption that the n trials are independent.) Now, there are $\binom{n}{k}$ patterns of successes and failures with k successes in n trials and any two patterns are mutually exclusive. Therefore the probability of exactly k successes in n independent trials having probability of success equal to p at each trial, is given by $\binom{n}{k}p^k(1 - p)^{n-k}$.

We return now to the example. Based on the foregoing paragraph, we have

$$P\{\text{passing test}\} = P\{\text{at least 70 correct answers out of 100}\}$$
$$= P\{\text{70 correct out of 100}\} + P\{\text{71 correct out of 100}\} + \cdots + P\{\text{100 correct out of 100}\}$$
$$= \binom{100}{70}\left(\frac{1}{5}\right)^{70}\left(\frac{4}{5}\right)^{30} + \binom{100}{71}\left(\frac{1}{5}\right)^{71}\left(\frac{4}{5}\right)^{29} + \cdots + \binom{100}{100}\left(\frac{1}{5}\right)^{100}.$$

There are tables available listing the individual probabilities and the sum appearing in the preceding equation.[9] (They are called *binomial probabilities* and *cumulative binomial probabilities*, respectively.) The terms are exceedingly cumbersome to evaluate numerically, however. (Try, for example, to determine the first term,

$$\binom{100}{70}\left(\frac{1}{5}\right)^{70}\left(\frac{4}{5}\right)^{30},$$

to get an appreciation of this fact.) The task of obtaining $\binom{n}{k}p^k(1-p)^{n-k}$ becomes increasingly difficult as n increases. Fortunately, there is a good, easily obtainable approximation available for the binomial probabilities. Further, its validity increases as n and k increase in a fixed ratio. This is the *normal approximation to the binomial.* The normal distribution is the familiar bell-shaped curve pictured in Figure 1.1. It has the properties of being symmetric about its center line (at zero) and of having a unit area bounded between it and the horizontal axis. The table in Appendix B gives the shaded

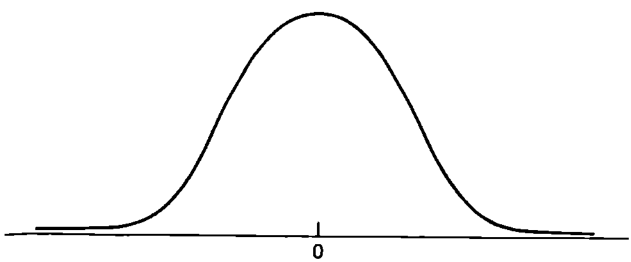

FIGURE 1.1

[9]For example, National Bureau of Standards, *Tables of the Binomial Probability Distribution*, Applied Math Series 6. U.S. Government Printing Office, 1950.

area of Figure 1.2; that is, $A(x)$ denotes the area bounded by the vertical line at zero, the vertical line at x, the curve, and the horizontal axis. By symmetry, the striped area bounded by the vertical line at

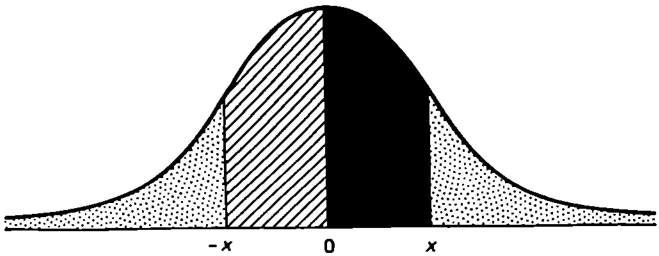

FIGURE 1.2

zero, the vertical line at $-x$, the curve, and the horizontal axis is also equal to $A(x)$. Using the symmetry and the fact that the entire area is one, it is easy to see that *each* of the dotted areas is equal to $\frac{1}{2} - A(x)$.

Let us accept the fact that the normal approximation to the binomial probability of *at least* k successes in n independent trials, where the probability of success at each trial is p, is given by the expression

$$\frac{1}{2} \pm A\left(\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right), \qquad \begin{array}{l} + \text{ if } k - \frac{1}{2} - np < 0 \\ - \text{ if } k - \frac{1}{2} - np > 0. \end{array}$$

Let us evaluate this numerically for the foregoing examination example. Here $n = 100$, $p = \frac{1}{5}$, $k = 70$; thus

$$k - \tfrac{1}{2} - np = 70 - \tfrac{1}{2} - 20 = 49.5,$$

$$\sqrt{np(1-p)} = \sqrt{16} = 4,$$

$$\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}} = \frac{49.5}{4} = 12.4,$$

and $A(12.4)$ is virtually .5, so that the probability of passing the test by guessing is essentially zero — as we might have predicted. Let us consider another example.

Sixty fair coins are tossed (a fair coin is one for which probability of heads equals probability of tails). What is the probability of at least 35 heads? The normal approximation yields

$$\frac{1}{2} - A\left(\frac{35 - \frac{1}{2} - 30}{\sqrt{60(\frac{1}{2})(\frac{1}{2})}}\right) = \frac{1}{2} - A\left(\frac{4.5}{\sqrt{15}}\right)$$

$$= \frac{1}{2} - A\left(\frac{4.5}{3.87}\right) = \frac{1}{2} - A(1.16) = \frac{1}{2} - .3770 = .1230.$$

This approximate value compares favorably with the exact value of .12253 obtained from tables of the binomial distribution.

EXERCISES

1. A multiple-choice exam containing 25 questions with five possible responses on each question is administered to a class of 30 students.
   (a) What is the probability that exactly four students miss question 1? State the assumptions you are making; that is specify the model you have set up to describe this situation.
   What is the probability that no more than four students miss question 1?
   (b) What is the probability that a particular student will get exactly 20 questions right? Again, state the assumptions you are making.
   What is the probability that this student will get at least 20 questions right?

2. A chest contains one each of 10 different pennants. Pennants are selected and arranged on a flag pole.
   (a) If each ordered arrangement constitutes a different pattern, how many patterns consisting of four pennants are there?
   (b) If order plays no role, but only which pennants are displayed is important, how many patterns are there?

3. It has been observed that each year 10 percent of school vice-principals leave the post for one reason or another (reassignment, promotion, retirement, and so on). An ambi-

tious teacher hopes to be promoted to a vice-principal's position at his school.
(a) What is the probability that a vacancy will occur this year?
(b) For the first time next year?
(c) For the first time two years hence?
(d) Within the next two years? State your assumptions carefully.

ANSWERS TO EXERCISES

p. 8: 1.

		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

2. (a) $\binom{7}{3}$ is the fourth entry in the eighth row, 35.

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

(b) $\binom{6}{4}$ is the fifth entry in the seventh row, 15.

$$\binom{6}{4} = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2 \cdot 1} = 15.$$

(c) $\binom{8}{3}$ is the fourth entry in the ninth row, 56.

$$\binom{8}{3} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$

p. 9: 1. $10^3 = 1000$
2. $26^4 = 456{,}976$
3. $20 \times 6 \times 20 = 2400$
4. $10^7 = 10{,}000{,}000$
5. $10^5 = 100{,}000$

p. 11:

2. Each student can pass if he answers at least three questions

correctly, that is if he answers three questions correctly or four questions correctly or all five questions correctly. These outcomes are mutually exclusive and can occur respectively in $\binom{5}{3}$, $\binom{5}{4}$, and $\binom{5}{5}$ ways. Thus each student can pass in $\binom{5}{3} + \binom{5}{4} + \binom{5}{5}$ ways, or $10 + 5 + 1 = 16$ ways. Thus both students can pass in $(16)^2 = 256$ ways.

p. 15: 1. $P(H_2 \mid H_1) = \dfrac{P(H_1 \text{ and } H_2)}{P(H_1)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$

$P(H_2) = \dfrac{1}{2}.$

2. $P(C \text{ and } D) = \dfrac{\#(C \text{ and } D)}{N} = \dfrac{4}{52 \cdot 6} = \dfrac{1}{78}$

$P(C)P(D) = \dfrac{4}{52} \cdot \dfrac{1}{6} = \dfrac{1}{78}.$

3. $P(A \text{ and } B) = \dfrac{1}{36}$

$P(A)P(B) = \dfrac{1}{6} \cdot \dfrac{1}{6} = \dfrac{1}{36}.$

5. A and B are mutually exclusive, so $P(A \text{ and } B) = 0$. But neither A nor B is the impossible event, so $P(A) \neq 0$ and $P(B) \neq 0$ and, consequently, $P(A)P(B) \neq 0$. If A and B were independent, $P(A \text{ and } B)$ would equal $P(A)P(B)$, which cannot be in this instance.

p. 23:

1. (a) Assumptions: (i) Students answer at random, that is, the probability of a correct answer to any question is $\frac{1}{5}$ so that the probability of an incorrect answer is $\frac{4}{5}$. (ii) Responses of any group of students are independent. P(exactly four students out of 30 students miss question 1) $= \binom{30}{4}\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^{26}.$

P(no more than four students miss question 1)

= P(zero or one or two or three or four students miss question 1)

= P(zero students miss question 1)

+ P(one student misses question 1)
+ P(two students miss question 1)
+ P(three students miss question 1)
+ P(four students miss question 1)

$$= \binom{30}{0}\left(\frac{4}{5}\right)^{0}\left(\frac{1}{5}\right)^{30} + \binom{30}{1}\left(\frac{4}{5}\right)^{1}\left(\frac{1}{5}\right)^{29} + \binom{30}{2}\left(\frac{4}{5}\right)^{2}\left(\frac{1}{5}\right)^{28}$$

$$= \binom{30}{3}\left(\frac{4}{5}\right)^{3}\left(\frac{1}{5}\right)^{27} + \binom{30}{4}\left(\frac{4}{5}\right)^{4}\left(\frac{1}{5}\right)^{26}.$$

(Note where the concept of mutual exclusivity was used.)
(b) Assumptions: (i) The student answers at random, that is, his probability of a correct answer to any question is $\frac{1}{5}$; (ii) his response to one question is independent of his response to any other question.

$$P(\text{exactly 20 correct answers}) = \binom{25}{20}\left(\frac{1}{5}\right)^{20}\left(\frac{4}{5}\right)^{5}$$

$$P(\text{at least 20 correct answers}) = \binom{25}{20}\left(\frac{1}{5}\right)^{20}\left(\frac{4}{5}\right)^{5}$$

$$+ \binom{25}{21}\left(\frac{1}{5}\right)^{21}\left(\frac{4}{5}\right)^{4} + \binom{25}{22}\left(\frac{1}{5}\right)^{22}\left(\frac{4}{5}\right)^{3} + \binom{25}{23}\left(\frac{1}{5}\right)^{23}\left(\frac{4}{5}\right)^{2}$$

$$+ \binom{25}{24}\left(\frac{1}{5}\right)^{24}\left(\frac{4}{5}\right)^{1} + \binom{25}{25}\left(\frac{1}{5}\right)^{25}\left(\frac{4}{5}\right)^{0}.$$

2. (a) $P(10,4) = \frac{10!}{6!} = 10\cdot9\cdot8\cdot7 = 5040.$

(b) $\binom{10}{4} = \frac{10!}{6!4!} = \frac{5040}{4\cdot3\cdot2\cdot1} = 210.$

3. (a) P(vacancy this year) = .10.
(b) P(vacancy for first time next year)
= P(no vacancy this year and a vacancy next year)

= (.9)(.1) = .09.

(c) P(vacancy for the first time two years hence)

P(no vacancy this year and no vacancy next year and a vacancy two years hence)

$= (.9)(.9)(.1) = .081.$

(d) P(vacancy within the next two years)

$= P$(vacancy this year or vacancy for the first time next year or vacancy for the first time two years hence)

$= P$(vacancy this year) $+ P$(vacancy for the first time next year) $+ P$(vacancy for the first time two years hence)

$= .10 + .09 + .081 = .271.$

(Note that this could also have been calculated as:

P(vacancy within the next two years)

$= 1 - P$(no vacancy within the next two years)

$= 1 - P$(no vacancy this year or next year or two years hence)

$= 1 - P$(no vacancy this year) P(no vacancy next year) P(no vacancy two years hence)

$= 1 - (.9)^3 = 1 - .729 = .271.$

Assumption: Vacancies in different years are independent events.

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*Elementary.
**Advanced.

Chapter Two

PROGRAMMED STATISTICS

INSTRUCTIONS

Read these Instructions carefully

Your success in mastering the contents of this chapter will depend on how carefully you follow these instructions.

> On the following pages is a Pre-Test. Take this test *before* you read the chapter. Do not be concerned if you miss some of the questions on the Pre-Test. This is expected.
>
> At the end of the chapter is a Post-Test. Take this test *after* you have read the chapter. Do not look at the Post-Test until you have finished studying the chapter.

Turn to the next page of the text and cover the right-hand column of the page with a slip of paper. Read the statement at the top of the page. Select the best answer to each of the multiple-choice questions that appear on the page. Then check your answer with the correct answer which appears in the right-hand column of the page. Continue this procedure as you read through the chapter.

PRE-TEST

1. If the teacher returns a paper marked with a raw score of 26 and says nothing further about the other grades on the examination, this means:

   (a) you failed the examination
   (b) you are in the 26th percentile
   (c) the raw score is not meaningful without more information — (c)

2. The measures of central tendency listed below are:

   (a) the raw score
   (b) the mean
   (c) the median
   (d) standard deviation — (b), (c)

3. The middle point in a set of scores is called the ________. — Median

4. Scores that differ greatly from the measure of central tendency are called:

   (a) raw scores
   (b) the best scores
   (c) extreme scores
   (d) *Z*-scores — (c)

5. The ________ is sensitive to extreme scores, while the ________ is not. — Mean, Median

6. Arithmetical average signifies:

   (a) the mean
   (b) the median
   (c) neither of these — (a)

7. The effect that extreme scores have on the mean is to pull it in their direction.

   ( ) True
   ( ) False — True

8. The elimination of all low extreme scores from a set has the effect of:

   (a) lowering the mean
   (b) raising the mean
   (c) no effect on the mean (b)

9. If all the scores on examination cluster around the mean, the dispersion is said to be:

   (a) small
   (b) large
   (c) normal (a)

10. One measure of dispersion is called: ______________ ______________. Standard Deviation

11 If the dispersion is small, the standard deviation is:

   (a) large
   (b) small
   (c) zero (b)

12. A ______________ score is more useful than a raw score because it gives information about how the score relates to other scores. Derived

13. A percentile gives the percentage of all scores that are located ______________ it. Below

14. Extreme scores have more effect on the mean than on the median.

   ( ) True
   ( ) False True

15. The mean is calculated by adding all scores in the set and dividing by the number of scores using what formula? $M = \frac{\Sigma X}{N}$

16. A correlation coefficient can range from ______ to ______. -1 to $+1$

This is the end of the Pre-Test. Continue on the next page and read the text of the chapter, using the same procedure for checking your answers to the multiple-choice questions.

PROGRAMMED MATERIAL

2.1 A score on an examination is more meaningful if you know how it compares with all the other scores on the same examination.

If a teacher returns your paper marked with the *raw score* 26 and says nothing about the other grades on the examination, this means:

(a) you failed the examination
(b) you are in the 26th quartile
(c) the raw score is not meaningful without more information (c)

2.2 The *raw score* on an examination is the number obtained by applying the scoring key to the test paper. It alone does not tell very much about how you did on the test. The raw score is usually the count of the number of correct answers.

On a test containing 100 questions, suppose that the professor says you got 60 percent of the answers correct. Your raw score on this test would be:

(a) 60
(b) 40
(c) 30
(d) none of these (a)

2.3 But sometimes on true-false tests the test will be scored by subtracting the number of wrong answers from the number of correct answers to obtain the raw score. In that case the raw score for the test in the previous example would be:

(a) 60
(b) 40
(c) 20
(d) none of these (c)

2.4 For a raw score on an examination to be meaningful, you must know its *relation* with the other scores on the same examination. You may want to compare your score with some measure of the *central tendency* of the group. The measurement of central tendency is sometimes the *average* of all the scores on the examination.

Thus, if the average score for the examination is 30, a measure of central tendency would be:

(a) 26
(b) 30
(c) neither of these (b)

2.5 The arithmetic average (or the *mean*) of a sample or set of scores is not the only measure of central tendency. The *median* is also sometimes used for this purpose.

The mean of an examination is 69, the median is 68, and the standard deviation is 2. The measures of central tendency for this examination are:

(a) .26
(b) 69
(c) 68
(d) 2 (b), (c)

2.6 Two measures of central tendency for a set of examination scores are the *median* and the *mean*.

( ) True
( ) False True

2.7 The word *median* signifies the *middle* point in a set of ordered scores. The median is obtained by arranging all the scores in order from the highest to the lowest, and then counting down the list half way.

What is the median for this set of scores:

60, 51, 40, 29, 11 40

2.8 Where there is an *odd number* of scores in the list, the middle point or median will be an actual score, as in the previous example. But where there is an *even number* of scores in the list, the median is taken to be the point half way between the two scores located in the middle of the list.

What is the median for this set of scores:

90, 88, 70, 60, 55, 44 — 65

2.9 The word *mean* signifies the arithmetical average of the set of scores. It is obtained by adding all the scores together and then dividing by the number of scores added up.

Calculate the mean for the following set of scores:

100, 80, 70, 40, 20 — $\frac{310}{5} = 62$

2.10 What is the median in the previous example? — 70

2.11

$$\frac{\Sigma X}{N} = M$$

The formula given above is used for calculating the sample *mean*. It signifies:

$$\frac{\text{Sum of the scores}}{\text{Number of scores}} = \text{Mean.}$$

Apply the formula to the following set of scores and calculate the mean:

80, 40, 35, 30, 20 — $\frac{205}{5} = 41$

2.12 For the set of scores in the previous example, identify the median, mean, and mode. — Median = 35
Mean = 41
No mode

2.13 The *median* is the middle point in a set of ordered scores, and the *mean* is the arithmetical average of the set of scores. These two measures of central tendency do not always coincide. Nor is the mean necessarily equal to one of the observations. Identify the median and the mean for the following set of scores:

100, 80, 60, 10, 10 — = 60 Median
= 52 Mean

2.14 Scores that differ greatly from the mean or the median are known as *extreme scores*. These are the very high scores at the top of the list and the very low scores at the bottom of the list.

Identify the extreme scores in the following list:

100, 40 = mean, 30 = median, 20, 10 — 100, 10

2.15 The mean is sensitive to extreme scores, while the median is not. That is to say, changing a few scores by making them larger or smaller may have a noticeable effect on the mean or arithmetical average of all the scores, but this need not affect the median or middle point of the set of scores.

The measure of central tendency that is most sensitive to extreme scores is:

(a) the mean
(b) the median
(c) both of these — (a)

2.16 The effect that *extreme scores* have on the *mean* is to pull it in their direction. Thus a few extreme scores at the top of the set will raise the mean. Likewise, a few extreme scores at the bottom of the set will lower the mean.

Calculate the median and the mean for the three sets of scores shown below.

(a) 100, 60, 50, 30, 10
(b) 100, 60, 50, 40, 30
(c) 70, 60, 50, 40, 10

	Median	Mean
(a)	50	50
(b)	50	56
(c)	50	46

2.17 The elimination of extreme scores at the bottom of the set has the effect of:

(a) lowering the mean
(b) raising the mean
(c) no effect (b)

2.18 The elimination of extreme scores at the top of the set has the effect of:

(a) lowering the mean
(b) raising the mean
(c) no effect (a)

2.19 Extreme scores will have the following effect on the median of an examination:

(a) they may tend to raise it
(b) they may tend to lower it
(c) they may have no effect on it (c)

2.20 The *mode* is defined as the item in a set of scores that occurs most often. It is the point of greatest frequency or density. A mode exists if there are two or more papers with the same score.

Which score is the mode in the following set?

16, 15, 14, 12, 12, 11 12

2.21 Sometimes there is more than one mode in a set of scores. Any score for which there are two or more papers will constitute a mode.

Identify the modes in the following set of scores:

16, 16, 15, 14, 12, 12, 10, 9, 2 — 12 and 16

2.22 Extreme scores on an examination have the following effect on the mode:

(a) they tend to raise it
(b) they tend to lower it
(c) they have no effect on it — (c)

2.23 The most sensitive measure of central tendency, because it is influenced by every score in the set, is:

(a) the mode
(b) the mean
(c) the median — (b)

2.24 Which of the following statements is/are always true?

(a) The mean has an effect on extreme scores.
(b) The median has an effect on extreme scores.
(c) Extreme scores have an effect on the mean.
(d) Extreme scores have an effect on the median. — (c)

2.25 Suppose you have a set of examination papers that contains many scores below 50 and only a few scores above 50. The preferred measure of central tendency in this case would be:

(a) the mean
(b) the median
(c) need more information to tell — (c)

2.26 Which of the following statements is true?

(a) Some sets of scores have a median, others have a mean, but no set has both of these.

(b) The median is always the same as the mean.
(c) The median is never the same as the mean.
(d) For a score to be meaningful, you must know something more than the median or the mean. (d)

2.27 In this set of scores (40, 30, 29, 29, 20, 10), the *frequency* of score 29 is:

(a) 1
(b) 2
(c) 3
(d) 4 (b)

2.28 When we say that a score has a *frequency* of 16, we mean how many students got this score?

(a) 160 percent of the students
(b) 16 students
(c) 10 students
(d) .016 percent of the students (b)

2.29 The frequency of a score in a set of scores is the same as the number of students who got that score.

( ) True
( ) False True

2.30 Each score in a set of scores may be represented graphically by a small rectangle, as in Figure 2.1.

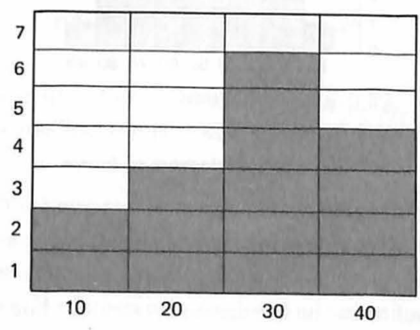

FIGURE 2.1

At the bottom are shown the scores the students could make, while the squares above the numbers represent the ________________ (or number) of students who made that score. — Frequency

2.31 In Figure 2.1, the frequency of the score 20 is:

(a) 5
(b) 3
(c) 2
(d) 7 — (b)

2.32 When there are many scores, the range of possible scores is often subdivided into equal intervals and the frequency of scores in each interval shown by the height of the graph above that interval. This graph is called a *histogram*. (See Figure 2.2.)

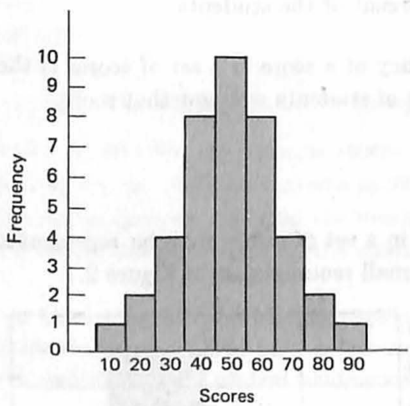

FIGURE 2.2 Histogram of Scores

2.33 Another way of representing these data is via a *frequency polygon*. It is formed by joining adjacent centers of the tops of the line in the histogram by straight line segments. The process of constructing a frequency polygon and the result are shown in Figures 2.3 and 2.4.

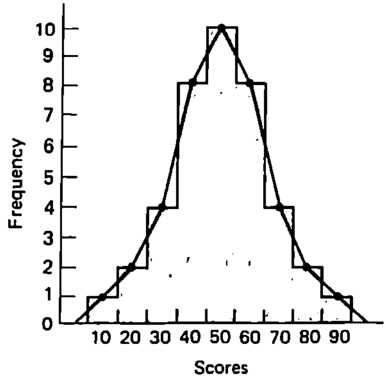

FIGURE 2.3 Frequency Polygon Superimposed on Histogram

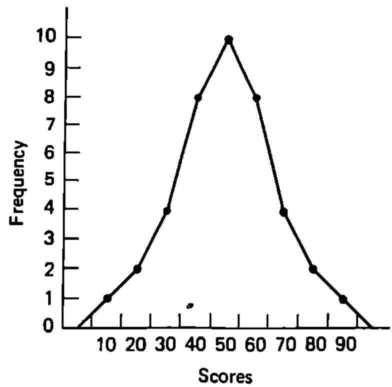

FIGURE 2.4 Frequency Polygon

2.34 In the frequency distribution of Frame 2.32, which interval of scores has the largest frequency? 40 to 50

2.35 Suppose that your raw score on an examination was 55, and the professor tells you that the mean for the examination was 40, you will still need more information about the examination scores in order for your score to really be meaningful to you. You should ask about the *dispersion* or *variability* of the scores on the examination.

In order for you to interpret your score on an examination, you must know:

(a) the measure of central tendency
(b) the measure of dispersion or variability
(c) the raw score
(d) all of these — (d)

2.36 A measure of *dispersion* gives information about how the scores on an examination *spread out;* it tells whether all the scores cluster around the measure of central tendency (mean or median) or whether there is a small or large spread of the scores away from it. The dispersion is said to be *small* if the scores cluster around the mean; it is *large* if they spread out from the mean.

How would you describe the dispersion of this set of scores?

99, 98, 97 (Mean), 96, 95 — Small

2.37 How would you describe the dispersion of this set of scores:

100, 80, 75, 30, 13, 2 — Large

2.38 If many of the scores on the examination cluster around the mean:

(a) the dispersion is small
(b) the dispersion is large
(c) need more information to tell — (c)

2.39 If the teacher says of the examination, "There was *no* dispersion," what does he mean? — All grades on the examination were exactly the same

2.40 If there are many extreme scores on an examination, the dispersion is:

(a) large
(b) small
(c) normal (a)

2.41 One measure of dispersion in a sample is called the sample *standard deviation* and is denoted by s. It is defined by the formula:

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - M)^2},$$

where: n is the number of examination scores (or other measurements)

$x_1, x_2, \ldots, x_n$ are the n examination scores (or other measurements)

M is the arithmetic mean of $x_1, \ldots, x_n$ defined in Frame 2.11

and

$\sqrt{\ }$ denotes the square root.

For purposes of computation, the following formula is often more convenient than the foregoing definition.

$$s = \sqrt{\frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right]}$$

2.42 An examination raw score can be converted to a new measure in which the unit is the standard deviation, and the origin (that is, zero point) of the scale is the mean of the raw scores. If a measurement in the new scale is denoted Y and the corresponding raw score by X, their relation is $Y = \dfrac{X - \bar{x}}{s}$. Negative values of Y correspond to raw scores below the mean, positive values to raw scores above the mean. The numerical value of Y measures the distance the score is from the mean in units of the standard deviation. Generally speaking, Y values as large as $+3$ (small as -3) indicate raw scores considerably above (below) the mean.

2.43 Calculate the standard deviation for the following set of scores, indicating the mean:

100, 20, 90, 30, 80, 40, 70, 50, 60

Using the formula in Frame 2.41, one has:

$$s^2 = \frac{1}{8}[(100^2 + 20^2 + 90^2 + 30^2 + 80^2 + 40^2 + 70^2 + 50^2 + 60^2)$$
$$- \frac{1}{9}(100 + 20 + 90 + 30 + 80 + 40 + 70 + 50 + 60)^2]$$
$$= \frac{100}{8}[(100 + 4 + 81 + 9 + 64 + 16 + 49 + 25 + 36)$$
$$- \frac{1}{9}(10 + 2 + 9 + 3 + 8 + 4 + 7 + 5 + 6)^2]$$
$$= \frac{100}{8}\left[384 - \frac{1}{9}(54)^2\right]$$
$$= \frac{100}{8}[384 - 324]$$
$$= \frac{100}{8}[60] = \frac{6000}{8} = 750; \; s = 27.4.$$

The mean is calculated to be $M = 540/9 = 60$.

2.44 How would you describe the dispersion for the set of scores given in the previous example? Large

2.45 The scores of students on an examination are influenced by many factors, e.g., knowledge of the subject, loss of sleep, misunderstanding of an examination question, breaking a pencil point during an examination, the kind of examination, and so on. If the outcome of a set of events such as the scores on an examination is determined by many factors, a frequency distribution representing the events often has a ________________ appearance. Bell-shaped

2.46 This is called:

(a) a bell-shaped frequency distribution
(b) an abnormal curve
(c) none of these (a)

2.47 Figure 2.5 shows the spread of grades that can be expected in a normal distribution. The curve is *bell-shaped.* *Z*-scores, *T*-scores, ETS (Educational Testing Service)-scores, the WISC (Wechsler Intelligence Scale for Children), IQ, and percentiles are also shown on the chart. Study this chart, and then answer the questions that appear below.

A normal distribution can be illustrated by a ______ ______ curve. Bell-shaped

2.48

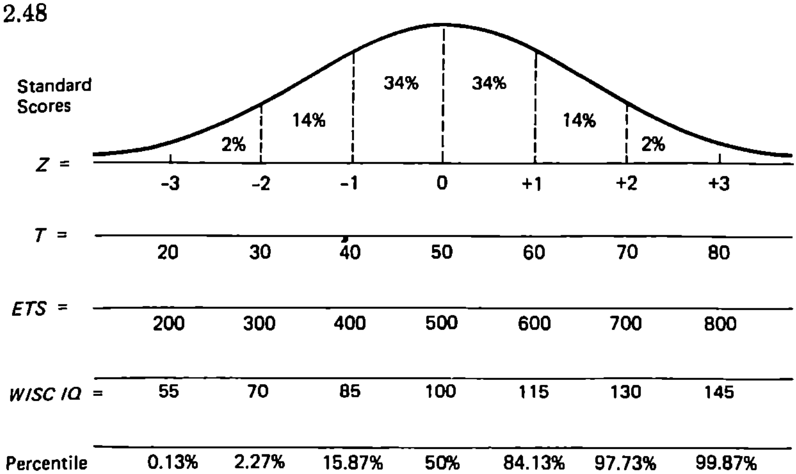

FIGURE 2.5 Normal Probability Distribution

2.49 There are many factors that determine how many times the number seven will come up if two dice are rolled, for example, the manner in which the dice are held in the hand before the roll, how hard they are tossed, the size of the dice, and so on.

If the sum of the faces of a pair of dice is recorded in a frequency distribution, a ________________ curve would be the expected result. Bell-shaped

2.50 With more tosses of the dice, the curve of the frequency distribution representing the number of each toss would become:

(a) more irregular
(b) more smooth and regular
(c) flat and straight
(d) none of these (b)

2.51 If the dice were rolled many times, this would lead to an almost perfect bell-shaped frequency distribution similar to a curve mathematicians call a normal curve.

( ) True
( ) False True

2.52 Earlier we spoke of the mean and standard deviation of a sample (or set) of examination scores. These values are estimates of corresponding quantities in the (perhaps idealized) *population* of all examination scores. Often the distribution of examination scores in the population is adequately described by the normal distribution with a particular mean and standard deviation. To distinguish the mean and standard deviation of the population from the mean and standard deviation of the sample, the former are often denoted μ and σ, respectively. (Recall that the sample mean is denoted by M and the sample standard deviation by s.) The *parameters* μ and σ are not defined in terms of finite sums as in Frames 2.11 and 2.41, respectively, but through the process of *integration*, a concept from calculus which is outside the scope of this book.

In the remainder of this chapter the terms "mean" and "standard deviation" refer to the *population mean* and *population standard deviation*. When you encounter these terms elsewhere, however, be sure to ascertain how they are used — as descriptors of the sample or of the population. Careful authors will observe the distinction.

2.53 Suppose that the standard deviation of a normal distribution is taken as a unit of dispersion. The horizontal axis is then measured off in these standard score units. The mean of the distribution is taken as a starting point, and measurements go in both directions. If the mean corresponds to a standard score of zero, then a point 3 standard deviations to the right of the mean has a standard score of +3, while a point 2 standard deviations to the left of the mean has a standard score of:

(a) +2
(b) −3
(c) −2
(d) +3 (c)

2.54 A point 1.5 standard deviations to the left of the mean would have a standard score of:

(a) −1.5
(b) +1.5
(c) −3.5
(d) +2.5 (a)

2.55 A point 3 standard deviations to the right of the mean would have a standard score of:

(a) +3
(b) +2
(c) −3
(d) −1 (a)

2.56 Of the scores in any normal distribution, almost ______ percent of them are within 3 standard deviations from the mean of the distribution?

(a) 50 percent
(b) 80 percent
(c) 100 percent
(d) none of these (c)

2.57 About ______ percent of all the scores in a normal distribution are within 3 standard deviations to the right of the mean?

(a) 90 percent
(b) 80 percent
(c) 50 percent
(d) 20 percent (c)

2.58 Almost 50 percent of all the scores in a normal distribution lie between a standard score of zero and −3.

( ) True
( ) False True

2.59 A *normal distribution* is symmetrical. The percentages of scores in terms of standard score units is the same to the left of the mean as to the right of the mean.

( ) True
( ) False True

2.60 The percentage of all the scores in a normal distribution included between a standard score of zero and −2 is:

(a) 2 percent
(b) 14 percent
(c) 34 percent
(d) 48 percent (d)

2.61 Between a standard score of −1 and +1 the percentage of all the scores in a normal distribution is:

(a) 34 percent
(b) 17 percent
(c) 68 percent
(d) none of these (c)

2.62 In a normal distribution the percentage of grades that can be expected to fall between −1 and +2 standard deviations from the mean is:

(a) 14 percent
(b) 34 percent
(c) 82 percent
(d) 84 percent (c)

2.63 A person whose ETS-score (Educational Testing Service) is 700 is how many standard deviations above or below the mean, given that the mean is 500 and the standard deviation is 100?

(a) +1
(b) −2
(c) +3
(d) +2 (d)

2.64 A person whose ETS-score is 650 is ______ standard deviations from the mean. +1.5

2.65 A derived score differs from a raw score in that a *raw score* gives no information about what the score means and says nothing about the relation between the raw score and the other scores in the distribution, while a *derived score* does give such information and is therefore meaningful.

( ) True
( ) False True

2.66 A *standard score* gives some information about the relative standing of the student with respect to other students in the population.

( ) True
( ) False True

2.67 *T-scores* are shown in Figure 2.5 on the second line below the Normal Probability Distribution graph. The T-score is also a derived score. To convert a standard score, sometimes also called *Z-score*, to a corresponding T-score, multiply the Z-score by 10 and add 50.

If the Z-score is +3, what is the corresponding T-score? 80

2.68 If the Z-score is 0, what is the corresponding T-score? 50

2.69 If the Z-score is -3, what is the corresponding T-score? 20

2.70 In Frame 2.69, the -3 is multiplied by 10 to get -30, after which 50 is added to -30 making a T-score of 20.

If the Z-score is -1, what is the corresponding T-score? 40

2.71 A Z-score that contains a decimal is likewise changed to a T-score by multiplying the Z-score by 10 and adding 50.

If the Z-score is 2.5, what is the corresponding T-score? 75

2.72 Percentiles are shown on the Normal Probability Distribution graph in Figure 2.5. Percentiles represent the percentage of scores in the distribution that are located to the left of the number shown.

Would a score that falls at the 40th percentile on the Normal Probability Distribution graph be to the left or to the right of the figure 50 percent? Left

2.73 The population *median* is located at the 50th percentile, which means that half of the scores are to the left of it and half to the right of it.

What percentage of the scores in a distribution are to the right of the 50th percentile? 50 percent

2.74 What percentages of the scores in a distribution are located between the 50th and 98th percentiles? 48 percent

2.75 What percentage of the scores in a distribution are located to the left of the 16th percentile? 16 percent

2.76 A distribution may be *skewed* either to the right or to the left. If the large bulk of scores in the distribution are low scores, with fewer and fewer scores to the right of the central tendency, the distribution is *skewed to the right* (the longer tail of the distribution extends toward the right). But if the large bulk of the scores are high scores and there are fewer scores toward the left of the central tendency, the distribution is *skewed to the left* (the longer tail of the distribution extends towards the left).

2.77

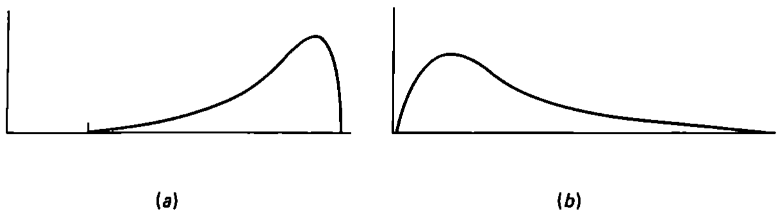

FIGURE 2.6

The distribution shown in Figure 2.6 (*b*) is skewed to the ______ Right

The distribution shown in Figure 2.6 (*a*) is skewed to the ______ Left

2.78 When a distribution is skewed to the left, we say that it is *negatively skewed*, and when a distribution is skewed to the right, we say it is *positively skewed.*

2.79 The distribution shown in Figure 2.6 (*a*) is:

(a) negatively skewed
(b) positively skewed (a)

2.80 The distribution shown in Figure 2.6 (*b*) is:

(a) negatively skewed
(b) positively skewed (b)

2.81 The *difficulty* of a test question is measured by the proportion of students who get the question right. This proportion is *inversely related* to the difficulty of the question.

That is to say, the more students that get the question right, the less difficult (or the easier) the question is said to be.

The *difficulty factor*, $D = R/n$, is determined simply by dividing the number of students who got the question right (R) by the total number of students (n) who answered the question.

If 10 students answered the question and 8 answered it correctly, what is the difficulty factor for this question? .8

2.82 We sometimes say that the *average* score received on a test indicates how difficult the test is. Likewise, the notion of a difficulty factor for determining the difficulty of a question on a test is analogous to this — it is the average score on the question.

The proportion or percentage of students that answer a question correctly is ____________ related to the difficulty of that question. Inversely

2.83 A difficulty factor of .8 would signify:

(a) the question is difficult
(b) the question is not difficult
(c) more than half the students answered the question wrong
(d) more information is needed (b)

2.84 A difficulty factor of .5 would signify:

(a) the question is difficult
(b) the question is not difficult
(c) half of the students answered the question right
(d) more information is needed (c)

2.85 A difficulty factor of .2 signifies that many students could not answer the question correctly and the question is therefore presumed to be difficult.

( ) True
( ) False True

2.86 All the questions on an examination might involve about .5 difficulty, or there might be a rather wide range with some questions being significantly more difficult, but with the average of all items being about .5.

If you want to motivate the poorest students and to challenge the best students, what difficulty factor would you use for your test questions?

(a) .5 on all questions
(b) .2 on all questions
(c) .8 on all questions
(d) wide range of difficulty values (d)

2.87 *Correlation coefficient* is a decimal, signifying the amount of association between two *variables*. Each variable must actually vary or change in order for the pair to be correlated.

If one item is fixed and unchangeable and the other item varies, can you measure their correlation?

( ) Yes
( ) No No

2.88 When *two variables tend to vary together* — that is, when the high scores in one variable are associated with the high scores in the other variable and when the low scores in one are likewise associated with the low scores in the other — these variables are said to be *positively correlated.*

Variables that tend to vary together directly result in a ____________ correlation. Positive

2.89 But if the high scores in one of the variables tend to go with the low scores in the other variable, or vice versa, then these variables are said to be *negatively correlated.*

Variables whose high and low scores vary inversely to one another result in a ____________ correlation. Negative

2.90 If the variables are totally unrelated, so that a high score in one is equally likely to be associated with a high score or a low score (or even a medium score) in the other, then these variables are said to be *uncorrelated.*

If the figure +1 signifies a *perfect positive correlation* and the figure −1 signifies a *perfect negative correlation,* then the figure 0 signifies:

(a) a perfect correlation
(b) uncorrelated variables
(c) not significant (b)

2.91 *Correlation* is a decimal, not a percentage. It is measured on a scale, as shown in Figure 2.7, which extends from −1 through zero to +1.

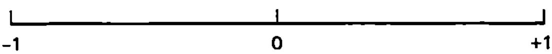

FIGURE 2.7

There is no such thing as a correlation greater than +1 or less than −1.

( ) True
( ) False True

2.92 A perfect correlation is signified by:

(a) 0
(b) +1
(c) −1
(d) +2 (b) or (c)

2.93 Suppose that the correlation between age and mistakes on an automobile driving test is −.3.

This statement means:

(a) as the age of the driver increases, the number of questions he misses on the test tends also to increase
(b) as the age of the driver increases, the number of questions he misses on the test tends to decrease
(c) a perfect negative correlation exists, so there is no relation between age and the test questions missed (b)

2.94 A measure of association between two variables is called *correlation*. Since height and weight tend to be closely related, that is, short people generally weigh less than tall people, we can say height and weight are positively *correlated* variables.

( ) True
( ) False True

2.95 Variables are said to be correlated if information concerning one of them gives information concerning another variable.

If we find that the average number of science books read by a student affects his grade in nonscience courses, we can say that there is a correlation between reading books on science and the grade received in nonscience courses.

( ) True
( ) False True

2.96 If a history professor tells his class: "All those who got 100 on the history test got 20 on the mathematics test, and all those that got 100 on the mathematics test got 20 on the history test," he is saying that the correlation between the history test and the mathematics is:

(a) negative
(b) positive
(c) uncorrelated variables (a)

2.97 Although correlation does not necessarily imply causation, a correlation coefficient is useful in making *predictions*. Thus, if the scores of students on one test are correlated with their scores on another test, a student's score on one test can be used to predict his score on the other test.

If two events are *perfectly correlated*, this means that the one event is the *cause* of the other event.

( ) True
( ) False — False

One way two events can be highly correlated is for both to be produced by a third event.

2.98 The *reliability* of a test signifies dependability of the results obtained from using that test.

The same test is given three times to a student. The three resulting scores differ widely. The test is probably not reliable.

( ) True
( ) False — True

2.99 A test is said to be valid if it measures whatever it was designed to measure.

Which statement is false?

(a) A test can be valid without being reliable, but it cannot be reliable unless it is valid.
(b) A test can be reliable without being valid, but it cannot be valid unless it is reliable. — (a)

2.100 You have now completed this chapter and are ready to take the Post-Test, which appears on the following pages.

Go now to the Post-Test and answer the questions. Check your answers against the correct ones, which appear in the right-hand column of the page.

POST-TEST

1. If the professor returns your paper marked with a raw score of 26 and tells you that the mean for the examination was 20, what information do you still need in order for your score to be meaningful?

   (a) the median for the examination
   (b) the dispersion of the scores
   (c) the difficulty factor
   (d) none of these (b)

2. Name two measures of central tendency. Median, Mean

3. The measure of central tendency most sensitive to extreme scores is the median.

   ( ) True
   ( ) False False

4. Which of these difficulty factors signifies an easy question:

   (a) .3
   (b) .5
   (c) .7
   (d) .9 (d)

5. If the scores on an examination spread out far from the mean, the dispersion is:

   (a) large
   (b) small
   (c) normal (a)

6. In a normal distribution, what percentage of the grades can be expected to fall between −2 and +2 standard deviations from the mean? 96 percent

7. How many standard deviations above or below the mean is a person whose ETS-score is 700?

   (a) +3
   (b) +2
   (c) −2
   (d) −3 (b)

8. A normal distribution is represented by a ________ ________ curve. Bell-shaped

9. A person with an ETS-score of 600 would be presumed to have an IQ of:

   (a) 100
   (b) 115
   (c) 130 (b)

10. A very few large scores (or small scores) will have an effect upon the mean.

    ( ) True
    ( ) False True

11. The middle point in a set of scores is called:

    (a) the mode
    (b) the mean
    (c) the median (c)

12. The last step in calculating standard deviation involves finding ________________. Square root

13. Scores that differ greatly from the measure of central tendency are called:

    (a) raw scores
    (b) extreme scores
    (c) *Z*-scores (b)

14. The ________________ is a point or score that separates the bottom half of the set from the top half. Median

15. The ________________ is not pulled away from the bulk of scores by a few scores that are extreme. Median

16. Dispersion (or variability) for a set of scores signifies how spread out the scores are.

    ( ) True
    ( ) False True

17. ________________ scores are expressed in decimals and negative numbers. Standard

18. On a graph, the ________________ of an interval of scores is represented by the height of a curve above that interval on the horizontal axis. Frequency

19. A frequency distribution with many average scores and fewer extreme scores is often ________________. Bell-shaped

20. We say that a test is ________________ if the same or similar results are obtained every time it is used. Reliable

21. We say that a test is ______________ if it measures whatever it is supposed to measure. Valid

22. If a test is not a reliable or dependable measuring instrument, we say it has low ______________.

    Reliability

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Chapter Three

THE DIGITAL COMPUTER

Let us now present a brief and elementary discussion of one of the major scientific developments of the twentieth century, the digital computer. This is important for three reasons. In the first place, it is essential to understand why the existence of the digital computer makes it easy to apply simple statistical ideas in a number of significant ways. Second, the material is needed as a background to our subsequent discussion of "programmed instruction." Third, the computer has become an integral part of our culture, which means that every educated person must comprehend certain of its fundamental aspects.

There are three principle types of computer: *digital, analog,* and *hybrid.* A *digital computer* is a machine for doing arithmetic: addition, subtraction, multiplication, and division. The greater part of contemporary computers operate according to electronic principles. An *analog computer* is a device for obtaining numerical answers to certain types of problems by converting the original problem into that of observing the behavior of a specific physical system. For example, we can solve systems of linear algebraic equations by the use of electrical or mechanical networks. A *hybrid computer* combines both digital and analog aspects.

A digital computer does arithmetic in the scale of two, a "yes-no" system. The binary scale corresponds to the sequence of "heads" and "tails" discussed in Chapter 1. Zeros and ones, like heads and tails are essentially ON and OFF symbols. (See Figure 3.1.) This is why we can use vacuum tube circuits, transistor circuits, solid state devices, and so forth, in computers.

Decimal	Binary								
1	1	9	1001	17	10001	25	11001	33	100001
2	10	10	1010	18	10010	26	11010	34	100010
3	11	11	1011	19	10011	27	11011	35	100011
4	100	12	1100	20	10100	28	11100	36	100100
5	101	13	1101	21	10101	29	11101	37	100101
6	110	14	1110	22	10110	30	11110	38	100110
7	111	15	1111	23	10111	31	11111	39	100111
8	1000	16	10000	24	11000	32	100000	40	101000

FIGURE 3.1 The Binary Number System

Since multiplication can be carried out in terms of addition, and division in terms of multiplication and addition, *all* of the fundamental operations in the digital computer are done in terms of addition. Currently, commercially available computers require about one microsecond to add two 10-digit numbers and about 10 times as long to multiply two 10-digit numbers. In other words, in *one* second the computer can perform 100,000 multiplications of this formidable nature. This fantastic ability to do rapid arithmetic, and by simple extension, rapid symbol manipulation, is what has revolutionized science and society.

A computer contains devices for five main functions as pictured in Figure 3.2. *Input* signifies the process by which information enters the computer, while *Output* is the process whereby the results are taken out of the computer. *Arithmetic* indicates the part of the computer that performs the basic arithmetic operations on the data held in *Storage*. *Control* signifies parts of the computer that dictate the functions to be performed by all the other parts.

The information as to the operations to be carried out by the computer is contained in a *computer program*, which may be written in any of a number of special computer languages, such as FORTRAN, BASIC, COBOL, ALGOL, or PL1. The logical organization of the sequence of operations is contained in a *flow chart*. To illustrate the

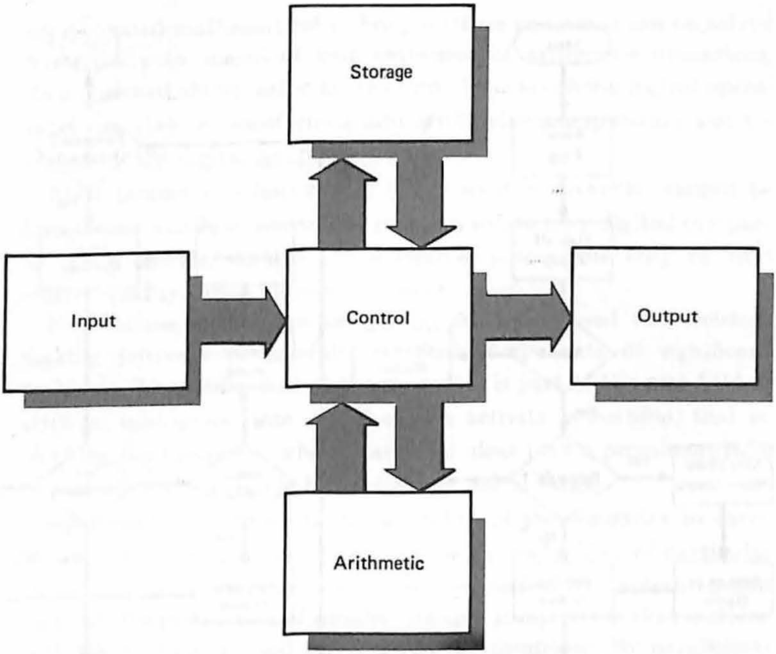

FIGURE 3.2 The Logical Elements of a Computer

basic idea, consider a flow chart for getting a reluctant student to school in the morning (Figure 3.3).

The successful application of statistical techniques to most problems of importance requires extensive amounts of data handling: storage and retrieval of data and arithmetic processing of this data. There are a number of major obstacles: space, time, accuracy, and display. The huge mass of data required often overwhelms even the largest of contemporary storage capacities; the time required to carry out the calculations is often prohibitively long; the input data are often of limited accuracy, which means that large numbers of arithmetical operations, and consequent round-off, produce unacceptable errors; the final results are so extensive that it is difficult to communicate them to the user in any feasible fashion. The general principle to to be emphasized is that the use of a computer to extract information from data is seldom routine.

Many major problems in the physical and social sciences can be translated into complex mathematical equations. With the use of

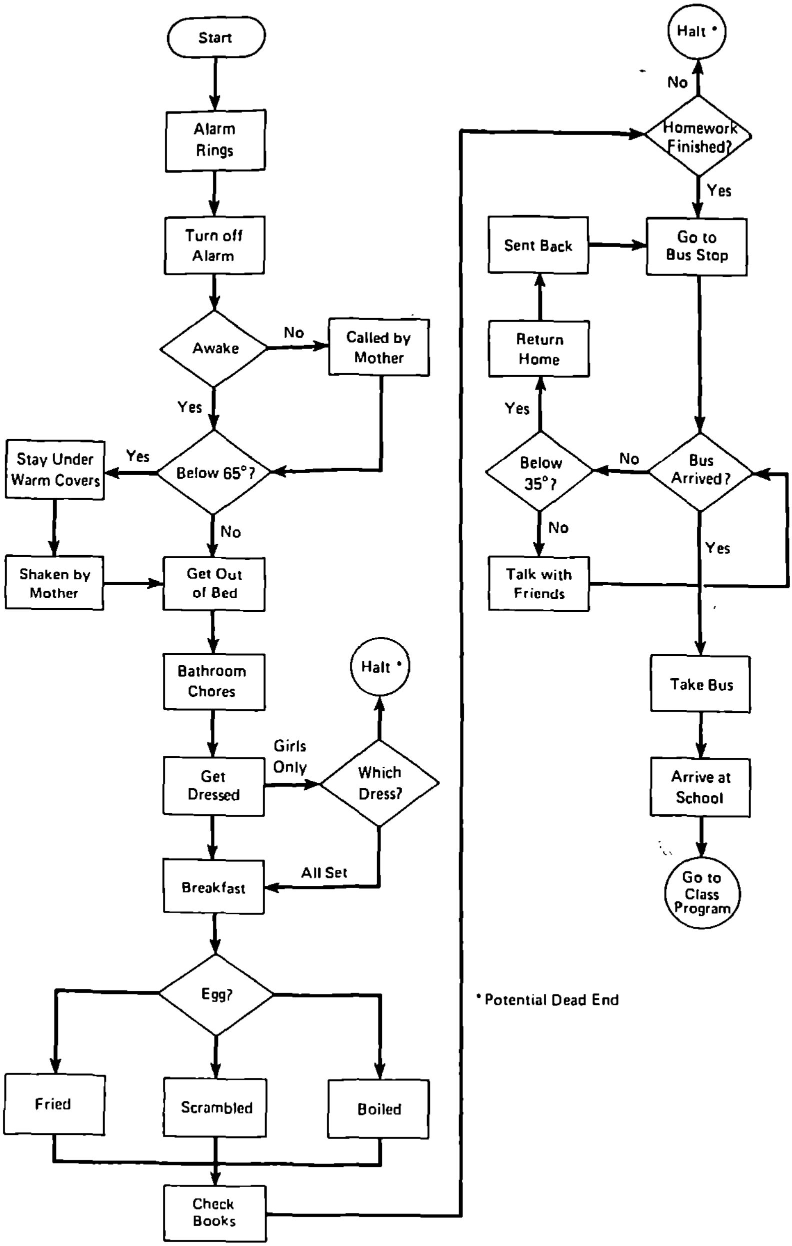

FIGURE 3.3 A Sample Flow Chart (Reprinted with permission from: Vincent S. Darnowski, *Computers — Theory and Uses*. National Science Teachers Association, Washington, D.C., 1964, p. 45)

sophisticated mathematical techniques these equations can be solved numerically by means of long sequences of arithmetic operations. As mentioned above, other types of problems involving logical operations can also be transformed into arithmetic manipulation and resolved by the digital computer.

Most problems of importance in our society, however, cannot be formulated totally in terms that permit a solution by digitial computer. They contain various imponderables susceptible only to that elusive quality called human judgment.

Nonetheless, the computer can be profitably used for decision-making purposes in carefully selected components of significant problems. This utilization of the computer is part of the new field of *artificial intelligence.* One aspect of this activity is *learning;* that is, teaching the computer, where statistical ideas play a prominent role, to perform certain simple but extremely useful tasks.

Significant improvements in the ability of the computer to carry out its many functions are constantly being made. Two of particular importance are *miniaturization* and *parallelization.* By miniaturization we mean the production of smaller, cheaper components that increase both the availability and the speed of the computer. By parallelization we mean the ability of the computer to perform different operations simultaneously.

It is now generally recognized that the most efficient utilization of a digital computer to solve complex problems requires a man-machine partnership, with each member of the team contributing particular talents and with man occupying the principal role.

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Chapter Four

PROGRAMMED LEARNING

One of the interesting ways in which the flexibility of the digital computer may be seen is in its use for training purposes. Many operations of our society, as mentioned in Chapter 3, cannot be expressed in arithmetic terms. They can, however, be broken down into sequences of simple operations, often requiring choices at various stages. Consider, for example, the use of a digital computer as a teaching aid, a teaching machine.

Suppose that we wish to drill a student in a language, say Latin. One function of the teacher is to describe the general structural features of the language, and to explain certain basic concepts. Others are far more complex, involving subtle combinations of psychological methods and assessment of the abilities of the student. Drilling, on the whole, however, is a low-level activity that can be safely left to animate or inanimate assistants. Let us see how we might employ a digital computer for this purpose.

The basic point is that all possible questions and answers that we allow are enumerated ahead of time and labeled with numbers. Hence the instruction to display a specific question is an instruction to display a number. Suppose that the first question is the following:

What is the first person singular of "to love?"

Let us suppose that we want to use a multiple-choice format and display four possible answers for the student:

lavo	amo	amat	none of these
1	2	3	4

If the student chooses the right answer, *amo*, which is to say if he pushes the second button, or types in "2," the computer displays a second question and choice of answers; for example:

What is the infinitive of "to love?"

amare	amatos	amabo	none of these
1	2	3	4

The process continues in this fashion.

There are a number of possibilities that we can explore if the student chooses the wrong answer:

(1) The computer can flash WRONG; CHOOSE ANOTHER ANSWER.

(2) An explanation of the mistake can be provided, or a number of possible explanations.

(3) The student can be referred to the appropriate place in the text where the correct answer is given.

With the aid of a digital computer programmed in the foregoing fashion, on the basis of cooperation between an experienced teacher in the particular subject and a programmer, we readily can provide drill in any part of the subject when both the questions and the answers can be simply stated. We cannot expect a teaching machine to be of any use in providing fundamental concepts, nor in any genuinely complex area requiring subtlety. It is designed to supplement, but not replace, the teacher. Note that this book is designed in part as a teaching machine.[1]

If we do not approve of multiple-choice questions, we can ask the student to type in his answer in certain standard format. In the near future we can expect to use two-way verbal communication with the computer as long as the questions and answers are both of simple form and prescribed ahead of time. The idea is again a simple one. The sound impulses are transformed into electrical impulses and proceed as before. We cannot expect anything like ordinary conversation. Once again, this is where the teacher plays a paramount role.

Programmed instruction materials can be used as self-instruction if carefully prepared on the basis of extensive experience. The point is that a student can go into a booth and a computer acting like a teacher can instruct and test various subjects using different techniques. Programmed learning at a slightly more sophisticated level is also possible with the computer. Thus:

> Computer assisted instruction (CAI) can take many forms. At the most rudimentary level the trainee-machine interaction is minimal. The computer presents instructional material via a display such as a teleprinter or cathode-ray tube (CRT); the trainee scans the presentation and indicates when he is ready to go on by means of a switch or push-button. The computer then may give further information, or it may present questions whose answers are to be recorded in a notebook or programmed textbook. Again the trainee notifies the computer when he is ready to proceed . . . and so on to the end of the lesson.
>
> Here the level of interaction is limited to a relatively primitive sequence without evaluation, interpretation or variation on the part of the computer. As refinement increases, the computer is brought into play with growing subtlety and flexibility.
>
> First the relationship between trainee and machine is solidified by doing away with the text or notebook. The computer then presents all instructional material and records the trainee's responses. At a later

[1]A medieval teaching machine used to train knights was known as a "quintain," which consisted of an object (usually a shield) attached to a movable crossbar mounted on a post and used as a target in the sport of tilting. The appropriate response was for the knight to strike the shield directly in the center with his lance; if struck off center, the device would deliver feedback by striking the horseman a blow as he rode by. A "quintain" is pictured in *The Random House Dictionary of the English Language.* (New York: Random House, Inc., 1966), p. 1181.

> stage the computer actively scores the trainee and presents the results to him; by this time the displays could include pictorial matter that changes to show evolving relationships. Ultimately, as CAI approaches its full development, the computer continuously evaluates the trainee's responses and leads him through remedial material if he fails to demonstrate understanding of key points in the main presentation. At this level displays can be as simple or as complex as the course material demands; and, if necessary, trainee responses can be in graphic form, conveyed to the computer by 'drawing' on the face of the scope with a light pen.[2]

The main point is *self-instruction*, whether it is provided by a programmed textbook that the student holds in his hand or by an elaborate computer-assisted educational system; these devices can be used in classroom situations involving both slow and fast learners. When there is a small body of knowledge to be learned by rote, such as the multiplication tables, a programmed book leads the student in small steps through a small amount of information until he has mastered the task. Computers will free classroom teachers from these low level activities and will give them more time for instruction. Teachers will have time to evaluate each student's work and to determine what is the right match between the use of programmed learning materials and individual instruction. A teacher may find it desirable to assign several students to work that involves programmed instructional materials while she concentrates the greater part of her efforts on one or two other children in another subject. Once again emphasis is upon a man-machine mixture, in which machines do what they can do very well (which is low-level rote work) while people do what they can do best (which is imaginative and creative thinking).

The following idea has often been suggested. A small box is located in a small room with a television screen in front of a desk where there is a typewriter. Programmed learning instruction materials are fed by a technician into a computer, a student pulls or pushes a knob; a report is sent electronically to the computer, where it is recorded on tape; and the whole operation during the period of instruction takes place without a teacher in the room. It is a mistake to think that the educational process will someday become completely automated;

[2] "Command/Control & CAI," *SDC Magazine*, Volume 10, Number 2, February 1967, pp. 4–5.

instead of taking away from the teacher's function, the computer will actually enhance it. For the first time she will actually be able to handle classes with large numbers of students. The versatility of programmed instruction when assisted by computers should be emphasized: not only will it benefit slow learners and fast learners; it will be a way of keeping every student gainfully employed in the classroom.

It may be true that we are about to see a revolution in our teaching methods, and that 20 years from now much of our education will be presented by automated computers and teaching machines; but whatever the cost of these devices, it must be included in the current costs of public and private school operations. The most affluent schools and districts will, naturally, be the first to adopt the new technologies.

EXAMPLES OF PROGRAMMED INSTRUCTION

The Linear Program

A *linear* program follows a sequential development of the material, with each student proceeding in the same order regardless of his response. (See Figure 4.1.) Despite any errors, the student proceeds

FIGURE 4.1

from item to item, each of which is normally short, and proceeds through a gradual development. He is informed of the correctness or incorrectness of his replies.

The Branching Program

A *branching* program, through interaction with the student, presents instructions based on his previous responses. (See Figure 4.2.) Item 1 presents information, followed by a test question on the material. A correct response refers the student to Item 5, the next unit of information; while an incorrect response refers him to either Item 9 or 13 for additional information before taking the test question again. It continues this way at Item 5 and through all subsequent items in the program.

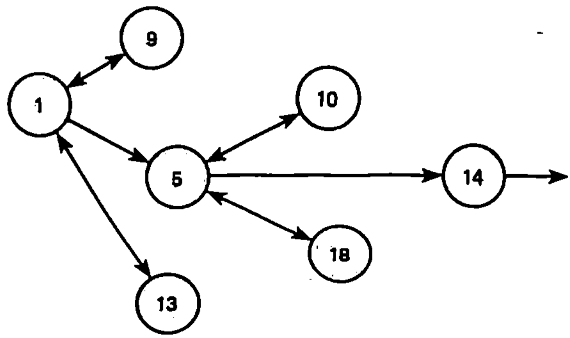

FIGURE 4.2

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Chapter Five

SELECTING AT RANDOM

By selecting an element at random from a collection of elements we mean simply that every element of the collection has the same probability of being selected. Specifically, if one item is to be drawn at random from a collection containing a finite number, n, of items, the probability of drawing any specified element is $1/n$.

Similarly, if k items are to be drawn at random from a collection of n elements, each possible subset of size k must have the same probability of selection. (Just what this probability is, in terms of n and k, depends on whether the drawings are made with or without replacement and whether the order in which the elements appear is important. Once the mode of sampling has been determined, the appropriate probabilities can be found in Chapter 1.)

How can random selection be achieved? The items comprising the collection can be numbered, these numbers written down on slips of paper or counters of some sort, these slips or counters put into a container and thoroughly mixed, and then drawings can be made from the container. The number chosen determines the item to be selected from the collection. Whether this results in random selection depends on the physical process of mixing and drawing.

There is available a ten-sided die — actually a cylindrically shaped object whose cross section is a regular decagon — whose faces are labeled 0–9. A roll of this device produces a random decimal digit; two rolls yield two random decimal digits that can be taken to be one of the 100 numbers 0–99; three rolls provide a random one of the 1000 numbers 0–999, and so on.

Another procedure for selecting at random is to use specially prepared tables of random numbers. Such a table is an array of numbers produced in such a way that each digit 0 through 9 appears with frequency 1/10, every pair of digits 00 through 99 appears with frequency 1/100, and so on. Perhaps the most famous table of random numbers is the set produced by the RAND Corporation.[1] An excerpt from this table appears at the end of this chapter.

To illustrate the process of random selection, consider drawing 11 items at random without replacement from a collection of 73 items. Only the composition of the sample, not the order in which it is drawn, is of importance.

For the first method discussed one would put slips of paper numbered from 1 to 73 into a container, stir, and draw 11 slips.

Using the 10-sided die, one would make a series of pairs of rolls (or, equivalently, make a series of rolls of two such dice) and select the first 11 distinct numbers between 1 and 73. The repetition of a number already rolled, the occurrence of a number larger than 73 or of 00 is ignored and another pair of rolls is made.

To use a table of random numbers, one selects a page and some starting point on the page[2] and reads off pairs of numbers from left to right beginning at the starting point. Suppose that one had selected p. 103 of the RAND Corporation tables (the reproduced extract) and the upper-left-hand corner of the page as the starting point. (Note that the first column of five digits merely number the rows of the table and are not random digits.) Reading off pairs of numbers, one obtains 12✓, 65✓, 16✓, 46✓, 11✓, 76, 97, 51✓, 09✓, 86, 99, 69✓, 76, 69, 25✓, 75✓, 73✓, 25, 35, ... Notice that 20 pairs of numbers had to be read off

[1]The RAND Corporation, *A Million Random Digits with 100,000 Normal Deviates.* (Glencoe, Illinois: The Free Press, 1955).

[2]The introduction to the RAND Corporation tables suggests that these be randomly selected also and that one take steps to assure that on different occasions, different portions of the tables be used.

before 11 suitable numbers (that is, between 1 and 73) were found. One had to ignore repetitions (16, 69, 25) and numbers larger than 73 (76, 97, 86, 99, 76, 75) in the selection process.

The ideas inherent in the preceding discussion should be sufficiently clear to let the reader construct procedures for sampling with replacement as well as for obtaining ordered samples.

Table of Random Digits

05100	12651	61646	11769	75109	86996	97669	25757	32535	07122	76763
05101	81769	74436	02630	72310	45049	18029	07469	42341	98173	79260
05102	36737	98863	77240	76251	00654	64688	09343	70278	67331	98729
05103	82861	54371	76610	94934	72748	44124	05610	53750	95938	01485
05104	21325	15732	24127	37431	09723	63529	73977	95218	96074	42138
05105	74146	47887	62463	23045	41490	07954	22597	60012	98866	90959
05106	90759	64410	54179	66075	61051	75385	51378	08360	95946	95547
05107	55683	98078	02238	91540	21219	17720	87817	41705	95785	12563
05108	79686	17969	76061	83748	55920	83612	41540	86492	06447	60568
05109	70333	00201	86201	69716	78185	62154	77930	67663	29529	75116
05110	14042	53536	07779	04157	41172	36473	42123	43929	50533	33437
05111	59911	08256	06596	48416	69770	68797	56080	14223	59199	30162
05112	62368	62623	62742	14891	39247	52242	98832	69533	91174	57979
05113	57529	97751	54976	48957	74599	08759	78494	52785	68526	64618
05114	15469	90574	78033	66885	13936	42117	71831	22961	94225	31816
05115	18625	23674	53850	32827	81647	80820	00420	63555	74489	80141
05116	74626	68394	88562	70745	23701	45630	65891	58220	35442	60414
05117	11119	16519	27384	90199	79210	76965	99546	30323	31664	22845
05118	41101	17336	48951	53674	17880	45260	08575	49321	36191	17095
05119	32123	91576	84221	78902	82010	30847	62329	63898	23268	74283
05120	26091	68409	69704	82267	14751	13151	93115	01437	56945	89661
05121	67680	79790	48462	59278	44185	29616	76531	19589	83139	28454
05122	15184	19260	14073	07026	25264	08388	27182	22557	61501	67481
05123	58010	45039	57181	10238	36874	28546	37444	80824	63981	39942
05124	56425	53996	86245	32623	78858	08143	60377	42925	42815	11159

05125	82630	84066	13592	60642	17904	99718	63432	88642	37858	25431
05126	14927	40909	23900	48761	44860	92467	31742	87142	03607	32059
05127	23740	22505	07489	85986	74420	21744	97711	36648	35620	97949
05128	32990	97446	03711	63824	07953	85965	87089	11687	92414	67257
05129	05310	24058	91946	78437	34365	82469	12430	84754	19354	72745
05130	21839	39937	27534	88913	49055	19218	47712	67677	51889	70926
05131	08833	42549	93981	94051	28382	83725	72643	64233	97252	17133
05132	58336	11139	47479	00931	91560	95372	97642	33856	54825	55680
05133	62032	91144	75478	47431	52737	30289	42411	91886	51818	78292
05134	45171	30557	53116	04118	58301	24375	65609	85810	18620	49198
05135	91611	62656	60128	35609	63698	78356	50682	22505	01692	36291
05136	55472	63819	86314	49174	93582	73604	78614	78849	23096	72825
05137	18573	09729	74091	53994	10970	86557	65661	41854	26037	53296
05138	60866	02955	90288	82136	83644	94455	06560	78029	98768	71296
05139	45043	55608	82767	60890	74646	79485	13619	98868	40857	19415
05140	17831	09737	79473	75945	28394	79334	70577	38048	03607	06932
05141	40137	03981	07585	18128	11178	32601	27994	05641	22600	86064
05142	77776	31343	14576	97706	16039	47517	43300	59080	80392	63189
05143	69605	44104	40103	95635	05635	81673	68657	09559	23510	95875
05144	19916	52934	26499	09821	87331	80993	61299	36979	73599	35055
05145	02606	58552	07678	56619	65325	30705	99582	53390	46357	13244
05146	65183	73160	87131	35530	47946	09854	18080	02321	05809	04898
05147	10740	98914	44916	11322	89717	88189	30143	52687	19420	60061
05148	98642	89822	71691	51573	83666	61642	46683	33761	47542	23551
05149	60139	25601	93663	25547	02654	94829	48672	28736	84994	13071

Reprinted with permission from: The RAND Corporation, *A Million Random Digits With 100,000 Normal Deviates*. (Glencoe, Illinois: The Free Press, 1955), p. 103.

Chapter Six

NEW COMPUTATIONAL AIDS FOR EDUCATORS

JOSS[1]

JOSS is a personalized computing service developed at The RAND Corporation that allows the user — student, teacher, or researcher — to interact directly with a central high-speed computer by means of a typewriter console. (See Figure 6.1.) Unlike most other systems designed for computer specialists, no technical "language" or programming techniques need be learned to operate JOSS. The student communicates with JOSS in simple imperative English language sentences that follow the standard rules for spelling, capitalization, punctuation, spacing, and so on. (See Figure 6.2.) Because of the ease with which JOSS can be learned, elementary and high-school students can use it to solve problems in basic arithmetic, algebra, trigonometry, and logic. Professional mathematicians, engineers, and others also use it in their research work to solve more sophisticated numerical problems.

[1]JOSS is the trademark and service mark of The RAND Corporation for its computer program and services using that program.

A unique feature is the dialogue that is possible between the student and JOSS; whenever a user of the console violates a mathematical convention or a standard rule of English grammar, JOSS will type for him an "error message" — an instruction that helps him decide what he did wrong so that he can correct his error and continue.[2]

FIGURE 6.1 The JOSS Console, Consisting of a Modified IBM Selectric Typewriter and a Control Box

JOSS is ideally suited for solving numerical problems; the student is able to approach his console with only a partially formed idea of his problem and yet come away in a few minutes or hours with the correct answer. He must be able to specify "all data relevant to describing his problem and the algorithm for its solution, but need only provide a minimum of detail regarding how his problem is to be

[2]There is also a users' manual of operation entitled *The JOSS Primer*, coauthored by S. L. Marks and G. W. Armerding (RAND Memorandum RM-5220-PR) which the student can consult for assistance when he is in trouble at the console.

solved. . . ."[3] The JOSS service is estimated to be about ten times faster than the usual approach used in solving problems by a computer. The result has been that many problems that formerly were not worth the effort are being solved with ease by the use of JOSS.

The JOSS service is *time-shared;* that is, it provides simultaneous

```
Type 2+2.
          2+2  =        4
Type "ok" if 500· 3*6≤1000.
ok
type 2+2
Eh?
Set x=3.
Type x.
            x  =        3
Type x+2, x-2, x·2, x/2, x*2.
          x+2  =        5
          x-2  =        1
          x·2  =        6
          x/2  =        1.5
          x*2  =        9
Type [(|x-5|·3+4)·2-15]·3+10.
[(|x-5|·3+4)·2-15]·3+10 = 25
x=7
Type (x-7|·3+4)·2 ·
Type [(|x-5|·3+4)·2-15]·3+10.
[(|x-5|·3+4)·2-15]·3+10 = 25
Type sqrt (3),_, sqrt (4).
       sqrt (3) =        1.73205081

       sqrt (4) =        2
Type sqrt (-1).
I have a negative argument for sqrt.
Type sin (.5), cos (.5).
       sin (.5)  =       .479425539
       cos (.5)  =       .877582562
Type exp (0), exp (1), exp (20).
       exp (0)   =       1
       exp (1)   =       2.71828183
      exp (20)   =       4.85165195·10*8
y = -1.23456·10*2
Type y, ip(y), fp(y), dp(y), xp(y).
            y  =         -123.456
        ip(y)  =         -123
        fp(y)  =            -.456
        dp(y)  =           -1.23456
        xp(y)  =            2
N=100
Type sum [i=1(1)N:i*2].
sum [i=1(1)N:i*2]  =  338350
Type prod [N,N+1,2·N+1]/6.
prod [N,N+1,2·N+1]/6  =  338350
```

FIGURE 6.2 An Example of JOSS, Showing the Actual Typewriter Output

[3]G. E. Bryan, *JOSS: Introduction to the System Implementation* (RAND Paper P-3486, November 1966), p. 1.

service to a number of different users at the same time, each at their own individual typewriter consoles, which are connected by telephone lines to a central computer.

The name JOSS stands for "JOHNNIAC Open Shop System." Work was first started on JOSS in 1960. The system was originally implemented on the JOHNNIAC computer (which is now displayed at the Los Angeles County Museum and was named in honor of John Von Neumann) by J. C. Shaw of RAND, to whom goes the bulk of the credit for the design.

THE RAND TABLET[4]

The RAND Tablet is a graphical input device that enables the student, using a special stylus, to input graphic information into a digital computer, and when accompanied by an appropriate computer

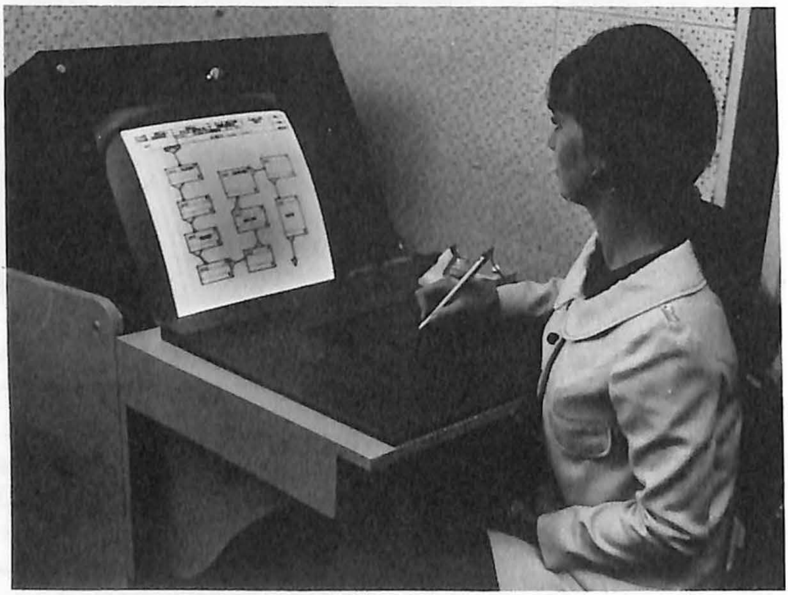


FIGURE 6.3

[4]T. O. Ellis and M. R. Davis, "Digital Computer and Graphic Input System." (United States Patent No. 3,399,401, issued August 27, 1968).

program, it can be used to solve mathematical and other problems in a natural way. (See Figure 6.3.)

The Tablet, invented by T. O. Ellis and M. R. Davis at The RAND Corporation, consists of a 10 in. × 10 in. horizontal surface on which the student writes or sketches with a stylus ("electronic pen"). His manipulations of the stylus — handwritten text (mathematical equations, formulas, and so on) or hand drawings (curves, sketches, and so on) — serve as inputs to the computer and also to a cathode-ray tube (CRT) display. The computer through appropriate programming (GRAIL,[5] and so on) interprets the manipulations of the stylus in such a way that as the student,

> . . . draws, moves, erases, and connects the various graphic symbols, the system responds with a variety of feedback. The primary response when the physical stylus is pressed lightly against the Tablet (closing a microswitch in the tip) is the appearance of display 'ink.' The ink follows the virtual stylus movement until the stylus is raised and the switch opens. The ink track is processed to determine whether the symbol and its position are appropriate. If everything checks, the track is replaced by a stylized symbol. Otherwise, the ink merely disappears.[6]

The fact that the student is "on-line" with the computer makes it possible to express himself directly and naturally, and he has the ability to control the activities of the computer and the information it presents to him.

The RAND Tablet has potential in such applications as digitizing map information and as a working tool for mathematicians or engineers in those areas where graphical languages are applied to man-machine interactions. For example, one of its uses has been in the area of building design. . . . A scaled floor plan of a school building is drawn on the writing surface of the Tablet and then viewed on the CRT display during study and analysis; the data thus generated can then be transmitted to the digital computer for comparison of optimum floor space allowances, classroom positioning, student flow patterns, and so on. This technique is useful in the design of optimum patterns for school or college libraries.

[5]GRAIL—*Gr*aphic *I*nput *L*anguage, which is under development at The RAND Corporation.

[6]T. O. Ellis and W. L. Sibley, *On the Problem of Directness in Computer Graphics.* (RAND Paper P-3697, March 1968), p. 12.

Other educational applications of the RAND Tablet exist in the areas of mathematical models, psychological testing, automated teaching, and situations involving the use of teacher-student questionnaires.

REFERENCES

JOSS

Baker, C. L., *JOSS: Introduction to a Helpful Assistant.* (RAND Memorandum RM-5058-PR, July 1966).

> This is a description of the capability of the JOSS system, presented through a step-by-step demonstration of the process, with illustrative material taken from the actual JOSS output.

Bryan, G. E., *JOSS: Introduction to the System Implementation.* (RAND Paper P-3486, November 1966).

> This paper provides an overview of the JOSS system, its history, a description of the hardware, and the JOSS language.

Bryan, G. E., and Smith, J. W., *JOSS Language.* (RAND Memorandum RM-5377-PR, August 1967).

> This memorandum is designed to serve as an introduction to JOSS for people with some programming experience. It presents summaries of the actions that can be requested of JOSS and of the language for requesting these actions.

Gimble, E. P., *JOSS: Problem Solving for Engineers.* (RAND Memorandum RM-5322-PR, May 1967).

> This memorandum, written from the engineer's point of view, introduces the basic principle of JOSS operation, in a sequence designed to enable the engineer to solve successively more involved scientific problems. It contains an overview of the JOSS system, including descriptions of the console and the language.

Marks, S. L., and Armerding, G. W., *The JOSS Primer.* (RAND Memorandum RM-5220-PR, August 1967).

> This is the users' manual of operation. It is designed to introduce JOSS to the beginning user by means of examples that can be followed by a reader without any programming experience.

THE RAND TABLET

Davis, M. R., and Ellis, T. O., *The RAND Tablet: A Man-Machine Graphical Communication Device*. (RAND Memorandum RM-4122-ARPA, August 1964).

This is the original report describing the RAND Tablet which is believed to be the first such man-machine graphical communication device that is digital.

Ellis, T. O., and Sibley, W. L., *On the Problem of Directness in Computer Graphics*. (RAND Paper P-3697, March 1968).

This report describes on-going work in the field of "Computer Graphics" and also discusses the history of the RAND Tablet.

Chapter Seven

THE NEXT DECADE IN EDUCATION

In this book, we have addressed ourselves to the latest developments in education in the fields of programmed learning texts, computer-assisted instruction, new computational aids for learning, and so on. A word should now be added about how these developments will apply to schools and colleges in the future.

Our country has entered the 1970s in a state of dramatic social change which is affecting all of our institutions, especially education. This occurs at a time when student enrollments in colleges and in elementary and secondary schools, both public and private, have increased steadily and sharply since the 1950s; in 1969 it has been estimated that there were about 60 million students in public and private schools in grades K–12 and another 7 million attending colleges and universities. If our extrapolations of the trend are borne out, we can expect to see in the next few decades an eventual total school and college enrollment of over 100 million students.[1] (See Figure 7.1.)

[1]These extrapolations were made using a linear least-squares fit to available data from 1950 in the case of public schools (K–12), from 1940 in the case of nonpublic schools (K–12), and from 1950 for higher education. As with all extrapolations, the present ones must be cautiously interpreted. Their accuracy is subject to many unknowable contingencies, such as worldwide and national economic, social, and political developments, trends in family planning (the "pill"), and so on.

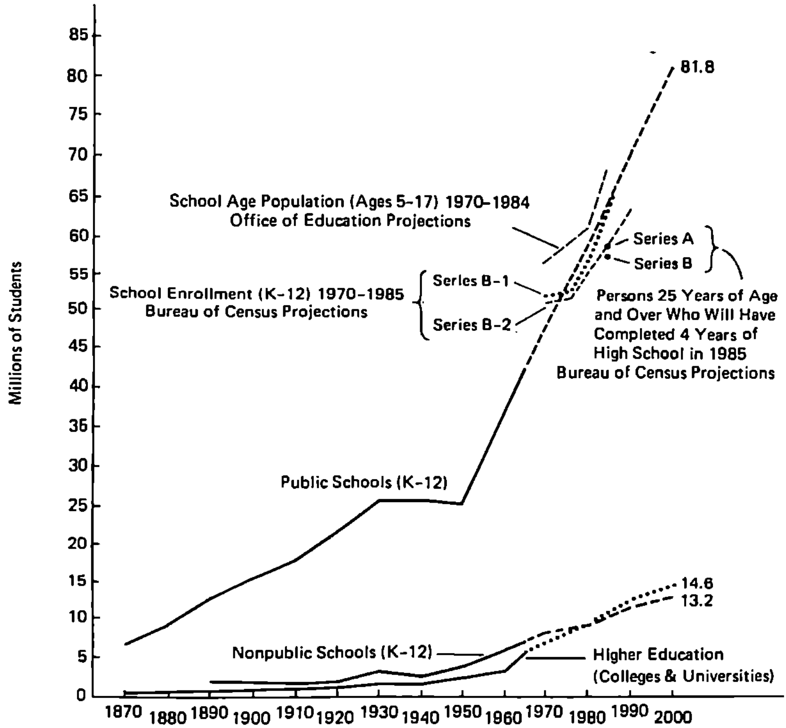


FIGURE 7.1 School and College Enrollments from 1870–2000

SOURCES: *Historical Statistics of the United States, Colonial Times to 1957*, Supplement 1962
Statistical Abstract of the United States, 1969
Digest of Educational Statistics, 1968
Projections of Educational Statistics to 1975–1976

The sheer number of students will make use in the classroom of educational television, computer-assisted instruction, programmed learning devices, and so on, very attractive. Indeed, these innovations are already now in limited use.

It is becoming generally recognized that *accountability* in our public schools and colleges is overdue; far too long we have been "wasteful" in education in the United States. Classroom space is going unused at certain hours and days of the week; teachers are not always performing at their capacities (due in part to some weaknesses in the tenure system); grading, testing, and evaluation methods used in our schools and colleges are in need of much study and improvement, and so on. The rising costs of teachers' salaries, mounting student enrollments, increased costs of new building construction, faculty and student unrest, all these are factors which point to *accountability* as a key word in education in the 1970s and beyond. As a result, parents, taxpayers, legislators, and the general public will be seeking educational reforms in such areas as teacher tenure, faculty promotion practices, obsolete staffing procedures, credential and Ph.D. requirements to teach, and so on. Even the "compulsory attendance" law, which requires attendance at school until either a certain age or the receipt of a diploma, must be considered for possible change. (In some districts today, this law is not rigidly enforced because of insufficient truant officers, on the one hand, and because some school administrators realize that their overall discipline problems are eased by the absence of certain students, on the other.)

Public education in the past has been a monopoly; now this is changing and on the horizon is *competition*, not just competition from nonpublic (private) schools but from business itself. The manufacturers of the new educational devices, the "outsiders" who have stood by and seen their materials and machines unused or misused, can be expected in the next decade to move into the teaching field themselves. This has, for some time, been the case with post-secondary trade schools. Certain computer manufacturing firms (for example, Control Data Corporation) and other private companies, are now offering commercial courses of study in computer programming and in computer maintenance. Already, some school districts have contracted out to private firms some of the functions which traditionally were the domains of the public school, (e.g., the San Diego City Schools recently talked with two private companies which will "guarantee" reading improvement of minority-group students or face a financial penalty. The two firms are Educational Development Laboratories of New York, a subsidiary of the McGraw-Hill Publishing Company, and Science Research Associates of Chicago, a subsidiary of IBM). When the results of these, and other, experimental programs become known, state legislatures will surely be making certain changes in the education law. Program budgeting, cost analysis, and statistical techniques (many of which are far more sophisticated than those described in the second chapter of this book)[2] are a necessary part of the evaluation of the affectiveness (and effectiveness) of these new programs.

There is much evidence, therefore, that needed educational reforms will occur, at all levels, in our public schools and colleges during the next decade and that some of the ideas described in this book will play an important role in these reforms.

[2]A fact which suggests that one should pursue his study of statistics beyond the present book.

APPENDIXES

APPENDIX A

TABLE OF SQUARES AND SQUARE ROOTS

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
1	1	1.00 000	3.16 228
2	4	1.41 421	4.47 214
3	9	1.73 205	5.47 723
4	16	2.00 000	6.32 456
5	25	2.23 607	7.07 107
6	36	2.44 949	7.74 597
7	49	2.64 575	8.36 660
8	64	2.82 843	8.94 427
9	81	3.00 000	9.48 683
10	100	3.16 228	10.00 00
11	121	3.31 662	10.48 81
12	144	3.46 410	10.95 45
13	169	3.60 555	11.40 18
14	196	3.74 166	11.83 22
15	225	3.87 298	12.24 74
16	256	4.00 000	12.64 91
17	289	4.12 311	13.03 84
18	324	4.24 264	13.41 64
19	361	4.35 890	13.78 40
20	400	4.47 214	14.14 21
21	441	4.58 258	14.49 14
22	484	4.69 042	14.83 24
23	529	4.79 583	15.16 58
24	576	4.89 898	15.49 19
25	625	5.00 000	15.81 14
26	676	5.09 902	16.12 45
27	729	5.19 615	16.43 17
28	784	5.29 150	16.73 32
29	841	5.38 516	17.02 94
30	900	5.47 723	17.32 05
31	961	5.56 776	17.60 68
32	1 024	5.65 685	17.88 85
33	1 089	5.74 456	18.16 59
34	1 156	5.83 095	18.43 91
35	1 225	5.91 608	18.70 83
36	1 296	6.00 000	18.97 37
37	1 369	6.08 276	19.23 54
38	1 444	6.16 441	19.49 36
39	1 521	6.24 500	19.74 84
40	1 600	6.32 456	20.00 00
41	1 681	6.40 312	20.24 85
42	1 764	6.48 074	20.49 39
43	1 849	6.55 744	20.73 64
44	1 936	6.63 325	20.97 62
45	2 025	6.70 820	21.21 32
46	2 116	6.78 233	21.44 76
47	2 209	6.85 565	21.67 95
48	2 304	6.92 820	21.90 89
49	2 401	7.00 000	22.13 59
50	2 500	7.07 107	22.36 07
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
50	2 500	7.07 107	22.36 07
51	2 601	7.14 143	22.58 32
52	2 704	7.21 110	22.80 35
53	2 809	7.28 011	23.02 17
54	2 916	7.34 847	23.23 79
55	3 025	7.41 620	23.45 21
56	3 136	7.48 331	23.66 43
57	3 249	7.54 983	23.87 47
58	3 364	7.61 577	24.08 32
59	3 481	7.68 115	24.28 99
60	3 600	7.74 597	24.49 49
61	3 721	7.81 025	24.69 82
62	3 844	7.87 401	24.89 98
63	3 969	7.93 725	25.09 98
64	4 096	8.00 000	25.29 82
65	4 225	8.06 226	25.49 51
66	4 356	8.12 404	25.69 05
67	4 489	8.18 535	25.88 44
68	4 624	8.24 621	26.07 68
69	4 761	8.30 662	26.26 79
70	4 900	8.36 660	26.45 75
71	5 041	8.42 615	26.64 58
72	5 184	8.48 528	26.83 28
73	5 329	8.54 400	27.01 85
74	5 476	8.60 233	27.20 29
75	5 625	8.66 025	27.38 61
76	5 776	8.71 780	27.56 81
77	5 929	8.77 496	27.74 89
78	6 084	8.83 176	27.92 85
79	6 241	8.88 819	28.10 69
80	6 400	8.94 427	28.28 43
81	6 561	9.00 000	28.46 05
82	6 724	9.05 539	28.63 56
83	6 889	9.11 043	28.80 97
84	7 056	9.16 515	28.98 28
85	7 225	9.21 954	29.15 48
86	7 396	9.27 362	29.32 58
87	7 569	9.32 738	29.49 58
88	7 744	9.38 083	29.66 48
89	7 921	9.43 398	29.83 29
90	8 100	9.48 683	30.00 00
91	8 281	9.53 939	30.16 62
92	8 464	9.59 166	30.33 15
93	8 649	9.64 365	30.49 59
94	8 836	9.69 536	30.65 94
95	9 025	9.74 679	30.82 21
96	9 216	9.79 796	30.98 39
97	9 409	9.84 886	31.14 48
98	9 604	9.89 949	31.30 50
99	9 801	9.94 987	31.46 43
100	10 000	10.00 000	31.62 28
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
100	10 000	10.00 00	31.62 28
101	10 201	10.04 99	31.78 05
102	10 404	10.09 95	31.93 74
103	10 609	10.14 89	32.09 36
104	10 816	10.19 80	32.24 90
105	11 025	10.24 70	32.40 37
106	11 236	10.29 56	32.55 76
107	11 449	10.34 41	32.71 09
108	11 664	10.39 23	32.86 34
109	11 881	10.44 03	33.01 51
110	12 100	10.48 81	33.16 62
111	12 321	10.53 57	33.31 67
112	12 544	10.58 30	33.46 64
113	12 769	10.63 01	33.61 55
114	12 996	10.67 71	33.76 39
115	13 225	10.72 38	33.91 16
116	13 456	10.77 03	34.05 88
117	13 689	10.81 67	34.20 53
118	13 924	10.86 28	34.35 11
119	14 161	10.90 87	34.49 64
120	14 400	10.95 45	34.64 10
121	14 641	11.00 00	34.78 51
122	14 884	11.04 54	34.92 85
123	15 129	11.09 05	35.07 14
124	15 376	11.13 55	35.21 36
125	15 625	11.18 03	35.35 53
126	15 876	11.22 50	35.49 65
127	16 129	11.26 94	35.63 71
128	16 384	11.31 37	35.77 71
129	16 641	11.35 78	35.91 66
130	16 900	11.40 18	36.05 55
131	17 161	11.44 55	36.19 39
132	17 424	11.48 91	36.33 18
133	17 689	11.53 26	36.46 92
134	17 956	11.57 58	36.60 60
135	18 225	11.61 90	36.74 23
136	18 496	11.66 19	36.87 82
137	18 769	11.70 47	37.01 35
138	19 044	11.74 73	37.14 84
139	19 321	11.78 98	37.28 27
140	19 600	11.83 22	37.41 66
141	19 881	11.87 43	37.55 00
142	20 164	11.91 64	37.68 29
143	20 449	11.95 83	37.81 53
144	20 736	12.00 00	37.94 73
145	21 025	12.04 16	38.07 89
146	21 316	12.08 30	38.20 99
147	21 609	12.12 44	38.34 06
148	21 904	12.16 55	38.47 08
149	22 201	12.20 66	38.60 05
150	22 500	12.24 74	38.72 98
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
150	22 500	12.24 74	38.72 98
151	22 801	12.28 82	38.85 87
152	23 104	12.32 88	38.98 72
153	23 409	12.36 93	39.11 52
154	23 716	12.40 97	39.24 28
155	24 025	12.44 99	39.37 00
156	24 336	12.49 00	39.49 68
157	24 649	12.53 00	39.62 32
158	24 964	12.56 98	39.74 92
159	25 281	12.60 95	39.87 48
160	25 600	12.64 91	40.00 00
161	25 921	12.68 86	40.12 48
162	26 244	12.72 79	40.24 92
163	26 569	12.76 71	40.37 33
164	26 896	12.80 62	40.49 69
165	27 225	12.84 52	40.62 02
166	27 556	12.88 41	40.74 31
167	27 889	12.92 28	40.86 56
168	28 224	12.96 15	40.98 78
169	28 561	13.00 00	41.10 96
170	28 900	13.03 84	41.23 11
171	29 241	13.07 67	41.35 21
172	29 584	13.11 49	41.47 29
173	29 929	13.15 29	41.59 33
174	30 276	13.19 09	41.71 33
175	30 625	13.22 88	41.83 30
176	30 976	13.26 65	41.95 24
177	31 329	13.30 41	42.07 14
178	31 684	13.34 17	42.19 00
179	32 041	13.37 91	42.30 84
180	32 400	13.41 64	42.42 64
181	32 761	13.45 36	42.54 41
182	33 124	13.49 07	42.66 15
183	33 489	13.52 77	42.77 85
184	33 856	13.56 47	42.89 52
185	34 225	13.60 15	43.01 16
186	34 596	13.63 82	43.12 77
187	34 969	13.67 48	43.24 35
188	35 344	13.71 13	43.35 90
189	35 721	13.74 77	43.47 41
190	36 100	13.78 40	43.58 90
191	36 481	13.82 03	43.70 35
192	36 864	13.85 64	43.81 78
193	37 249	13.89 24	43.93 18
194	37 636	13.92 84	44.04 54
195	38 025	13.96 42	44.15 88
196	38 416	14.00 00	44.27 19
197	38 809	14.03 57	44.38 47
198	39 204	14.07 12	44.49 72
199	39 601	14.10 67	44.60 94
200	40 000	14.14 21	44.72 14
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
200	40 000	14.14 21	44.72 14
201	40 401	14.17 74	44.83 30
202	40 804	14.21 27	44.94 44
203	41 209	14.24 78	45.05 55
204	41 616	14.28 29	45.16 64
205	42 025	14.31 78	45.27 69
206	42 436	14.35 27	45.38 72
207	42 849	14.38 75	45.49 73
208	43 264	14.42 22	45.60 70
209	43 681	14.45 68	45.71 65
210	44 100	14.49 14	45.82 58
211	44 521	14.52 58	45.93 47
212	44 944	14.56 02	46.04 35
213	45 369	14.59 45	46.15 19
214	45 796	14.62 87	46.26 01
215	46 225	14.66 29	46.36 81
216	46 656	14.69 69	46.47 58
217	47 089	14.73 09	46.58 33
218	47 524	14.76 48	46.69 05
219	47 961	14.79 86	46.79 74
220	48 400	14.83 24	46.90 42
221	48 841	14.86 61	47.01 06
222	49 284	14.89 97	47.11 69
223	49 729	14.93 32	47.22 29
224	50 176	14.96 66	47.32 86
225	50 625	15.00 00	47.43 42
226	51 076	15.03 33	47.53 95
227	51 529	15.06 65	47.64 45
228	51 984	15.09 97	47.74 93
229	52 441	15.13 27	47.85 39
230	52 900	15.16 58	47.95 83
231	53 361	15.19 87	48.06 25
232	53 824	15.23 15	48.16 64
233	54 289	15.26 43	48.27 01
234	54 756	15.29 71	48.37 35
235	55 225	15.32 97	48.47 68
236	55 696	15.36 23	48.57 98
237	56 169	15.39 48	48.68 26
238	56 644	15.42 72	48.78 52
239	57 121	15.45 96	48.88 76
240	57 600	15.49 19	48.98 98
241	58 081	15.52 42	49.09 18
242	58 564	15.55 63	49.19 35
243	59 049	15.58 85	49.29 50
244	59 536	15.62 05	49.39 64
245	60 025	15.65 25	49.49 75
246	60 516	15.68 44	49.59 84
247	61 009	15.71 62	49.69 91
248	61 504	15.74 80	49.79 96
249	62 001	15.77 97	49.89 99
250	62 500	15.81 14	50.00 00
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
250	62 500	15.81 14	50.00 00
251	63 001	15.84 30	50.09 99
252	63 504	15.87 45	50.19 96
253	64 009	15.90 60	50.29 91
254	64 516	15.93 74	50.39 84
255	65 025	15.96 87	50.49 75
256	65 536	16.00 00	50.59 64
257	66 049	16.03 12	50.69 52
258	66 564	16.06 24	50.79 37
259	67 081	16.09 35	50.89 20
260	67 600	16.12 45	50.99 02
261	68 121	16.15 55	51.08 82
262	68 644	16.18 64	51.18 59
263	69 169	16.21 73	51.28 35
264	69 696	16.24 81	51.38 09
265	70 225	16.27 88	51.47 82
266	70 756	16.30 95	51.57 52
267	71 289	16.34 01	51.67 20
268	71 824	16.37 07	51.76 87
269	72 361	16.40 12	51.86 52
270	72 900	16.43 17	51.96 15
271	73 441	16.46 21	52.05 77
272	73 984	16.49 24	52.15 36
273	74 529	16.52 27	52.24 94
274	75 076	16.55 29	52.34 50
275	75 625	16.58 31	52.44 04
276	76 176	16.61 32	52.53 57
277	76 729	16.64 33	52.63 08
278	77 284	16.67 33	52.72 57
279	77 841	16.70 33	52.82 05
280	78 400	16.73 32	52.91 50
281	78 961	16.76 31	53.00 94
282	79 524	16.79 29	53.10 37
283	80 089	16.82 26	53.19 77
284	80 656	16.85 23	53.29 17
285	81 225	16.88 19	53.38 54
286	81 796	16.91 15	53.47 90
287	82 369	16.94 11	53.57 24
288	82 944	16.97 06	53.66 56
289	83 521	17.00 00	53.75 87
290	84 100	17.02 94	53.85 16
291	84 681	17.05 87	53.94 44
292	85 264	17.08 80	54.03 70
293	85 849	17.11 72	54.12 95
294	86 436	17.14 64	54.22 18
295	87 025	17.17 56	54.31 39
296	87 616	17.20 47	54.40 59
297	88 209	17.23 37	54.49 77
298	88 804	17.26 27	54.58 94
299	89 401	17.29 16	54.68 09
300	90 000	17.32 05	54.77 23
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
300	90 000	17.32 05	54.77 23
301	90 601	17.34 94	54.86 35
302	91 204	17.37 81	54.95 45
303	91 809	17.40 69	55.04 54
304	92 416	17.43 56	55.13 62
305	93 025	17.46 42	55.22 68
306	93 636	17.49 29	55.31 73
307	94 249	17.52 14	55.40 76
308	94 864	17.54 99	55.49 77
309	95 481	17.57 84	55.58 78
310	96 100	17.60 68	55.67 76
311	96 721	17.63 52	55.76 74
312	97 344	17.66 35	55.85 70
313	97 969	17.69 18	55.94 64
314	98 596	17.72 00	56.03 57
315	99 225	17.74 82	56.12 49
316	99 856	17.77 64	56.21 39
317	100 489	17.80 45	56.30 28
318	101 124	17.83 26	56.39 15
319	101 761	17.86 06	56.48 01
320	102 400	17.88 85	56.56 85
321	103 041	17.91 65	56.65 69
322	103 684	17.94 44	56.74 50
323	104 329	17.97 22	56.83 31
324	104 976	18.00 00	56.92 10
325	105 625	18.02 78	57.00 88
326	106 276	18.05 55	57.09 64
327	106 929	18.08 31	57.18 39
328	107 584	18.11 08	57.27 13
329	108 241	18.13 84	57.35 85
330	108 900	18.16 59	57.44 56
331	109 561	18.19 34	57.53 26
332	110 224	18.22 09	57.61 94
333	110 889	18.24 83	57.70 62
334	111 556	18.27 57	57.79 27
335	112 225	18.30 30	57.87 92
336	112 896	18.33 03	57.96 55
337	113 569	18.35 76	58.05 17
338	114 244	18.38 48	58.13 78
339	114 921	18.41 20	58.22 37
340	115 600	18 43 91	58.30 95
341	116 281	18.46 62	58.39 52
342	116 964	18.49 32	58.48 08
343	117 649	18.52 03	58.56 62
344	118 336	18.54 72	58.65 15
345	119 025	18.57 42	58.73 67
346	119 716	18.60 11	58.82 18
347	120 409	18.62 79	58.90 67
348	121 104	18.65 48	58.99 15
349	121 801	18.68 15	59.07 62
350	122 500	18.70 83	59.16 08
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
350	122 500	18.70 83	59.16 08
351	123 201	18.73 50	59.24 53
352	123 904	18.76 17	59.32 96
353	124 609	18.78 83	59.41 38
354	125 316	18.81 49	59.49 79
355	126 025	18.84 14	59.58 19
356	126 736	18.86 80	59.66 57
357	127 449	18.89 44	59.74 95
358	128 164	18.92 09	59.83 31
359	128 881	18.94 73	59.91 66
360	129 600	18.97 37	60.00 00
361	130 321	19.00 00	60.08 33
362	131 044	19.02 63	60.16 64
363	131 769	19.05 26	60.24 95
364	132 496	19.07 88	60.33 24
365	133 225	19.10 50	60.41 52
366	133 956	19.13 11	60.49 79
367	134 689	19.15 72	60.58 05
368	135 424	19.18 33	60.66 30
369	136 161	19.20 94	60.74 54
370	136 900	19.23 54	60.82 76
371	137 641	19.26 14	60.90 98
372	138 384	19.28 73	60.99 18
373	139 129	19.31 32	61.07 37
374	139 876	19.33 91	61.15 55
375	140 625	19.36 49	61.23 72
376	141 376	19.39 07	61.31 88
377	142 129	19.41 65	61.40 03
378	142 884	19.44 22	61.48 17
379	143 641	19.46 79	61.56 30
380	144 400	19.49 36	61.64 41
381	145 161	19.51 92	61.72 52
382	145 924	19.54 48	61.80 61
383	146 689	19.57 04	61.88 70
384	147 456	19.59 59	61.96 77
385	148 225	19.62 14	62.04 84
386	148 996	19.64 69	62.12 89
387	149 769	19.67 23	62.20 93
388	150 544	19.69 77	62.28 96
389	151 321	19.72 31	62.36 99
390	152 100	19.74 84	62.45 00
391	152 881	19.77 37	62.53 00
392	153 664	19.79 90	62.60 99
393	154 449	19.82 42	62.68 97
394	155 236	19.84 94	62.76 94
395	156 025	19.87 46	62.84 90
396	156 816	19.89 97	62.92 85
397	157 609	19.92 49	63.00 79
398	158 404	19.94 99	63.08 72
399	159 201	19.97 50	63.16 64
400	160 000	20.00 00	63.24 56
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
400	160 000	20.00 00	63.24 56
401	160 801	20.02 50	63.32 46
402	161 604	20.04 99	63.40 35
403	162 409	20.07 49	63.48 23
404	163 216	20.09 98	63.56 10
405	164 025	20.12 46	63.63 96
406	164 836	20.14 94	63.71 81
407	165 649	20.17 42	63.79 66
408	166 464	20.19 90	63.87 49
409	167 281	20.22 37	63.95 31
410	168 100	20.24 85	64.03 12
411	168 921	20.27 31	64.10 93
412	169 744	20.29 78	64.18 72
413	170 569	20.32 24	64.26 51
414	171 396	20.34 70	64.34 28
415	172 225	20.37 15	64.42 05
416	173 056	20.39 61	64.49 81
417	173 889	20.42 06	64.57 55
418	174 724	20.44 50	64.65 29
419	175 561	20.46 95	64.73 02
420	176 400	20.49 39	64.80 74
421	177 241	20.51 83	64.88 45
422	178 084	20.54 26	64.96 15
423	178 929	20.56 70	65.03 85
424	179 776	20.59 13	65.11 53
425	180 625	20.61 55	65.19 20
426	181 476	20.63 98	65.26 87
427	182 329	20.66 40	65.34 52
428	183 184	20.68 82	65.42 17
429	184 041	20.71 23	65.49 81
430	184 900	20.73 64	65.57 44
431	185 761	20.76 05	65.65 06
432	186 624	20.78 46	65.72 67
433	187 489	20.80 87	65.80 27
434	188 356	20.83 27	65.87 87
435	189 225	20.85 67	65.95 45
436	190 096	20.88 06	66.03 03
437	190 969	20.90 45	66.10 60
438	191 844	20.92 84	66.18 16
439	192 721	20.95 23	66.25 71
440	193 600	20.97 62	66.33 25
441	194 481	21.00 00	66.40 78
442	195 364	21.02 38	66.48 31
443	196 249	21.04 76	66.55 82
444	197 136	21.07 13	66.63 33
445	198 025	21.09 50	66.70 83
446	198 916	21.11 87	66.78 32
447	199 809	21.14 24	66.85 81
448	200 704	21.16 60	66.93 28
449	201 601	21.18 96	67.00 75
450	202 500	21.21 32	67.08 20
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
450	202 500	21.21 32	67.08 20
451	203 401	21.23 68	67.15 65
452	204 304	21.26 03	67.23 09
453	205 209	21.28 38	67.30 53
454	206 116	21.30 73	67.37 95
455	207 025	21.33 07	67.45 37
456	207 936	21.35 42	67.52 78
457	208 849	21.37 76	67.60 18
458	209 764	21.40 09	67.67 57
459	210 681	21.42 43	67.74 95
460	211 600	21.44 76	67.82 33
461	212 521	21.47 09	67.89 70
462	213 444	21.49 42	67.97 06
463	214 369	21.51 74	68.04 41
464	215 296	21.54 07	68.11 75
465	216 225	21.56 39	68.19 09
466	217 156	21.58 70	68.26 42
467	218 089	21.61 02	68.33 74
468	219 024	21.63 33	68.41 05
469	219 961	21.65 64	68.48 36
470	220 900	21.67 95	68.55 65
471	221 841	21.70 25	68.62 94
472	222 784	21.72 56	68.70 23
473	223 729	21.74 86	68.77 50
474	224 676	21.77 15	68.84 77
475	225 625	21.79 45	68.92 02
476	226 576	21.81 74	68.99 28
477	227 529	21.84 03	69.06 52
478	228 484	21.86 32	69.13 75
479	229 441	21.88 61	69.20 98
480	230 400	21.90 89	69.28 20
481	231 361	21.93 17	69.35 42
482	232 324	21.95 45	69.42 62
483	233 289	21.97 73	69.49 82
484	234 256	22.00 00	69.57 01
485	235 225	22.02 27	69.64 19
486	236 196	22.04 54	69.71 37
487	237 169	22.06 81	69.78 54
488	238 144	22.09 07	69.85 70
489	239 121	22.11 33	69.92 85
490	240 100	22.13 59	70.00 00
491	241 081	22.15 85	70.07 14
492	242 064	22.18 11	70.14 27
493	243 049	22.20 36	70.21 40
494	244 036	22.22 61	70.28 51
495	245 025	22.24 86	70.35 62
496	246 016	22.27 11	70.42 73
497	247 009	22.29 35	70.49 82
498	248 004	22.31 59	70.56 91
499	249 001	22.33 83	70.63 99
500	250 000	22.36 07	70.71 07
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N²	√N	√10N
500	250 000	22.36 07	70.71 07
501	251 001	22.38 30	70.78 14
502	252 004	22.40 54	70.85 20
503	253 009	22.42 77	70.92 25
504	254 016	22.44 99	70.99 30
505	255 025	22.47 22	71.06 34
506	256 036	22.49 44	71.13 37
507	257 049	22.51 67	71.20 39
508	258 064	22.53 89	71.27 41
509	259 081	22.56 10	71.34 42
510	260 100	22.58 32	71.41 43
511	261 121	22.60 53	71.48 43
512	262 144	22.62 74	71.55 42
513	263 169	22.64 95	71.62 40
514	264 196	22.67 16	71.69 38
515	265 225	22.69 36	71.76 35
516	266 256	22.71 56	71.83 31
517	267 289	22.73 76	71.90 27
518	268 324	22.75 96	71.97 22
519	269 361	22.78 16	72.04 17
520	270 400	22.80 35	72.11 10
521	271 441	22.82 54	72.18 03
522	272 484	22.84 73	72.24 96
523	273 529	22.86 92	72.31 87
524	274 576	22.89 10	72.38 78
525	275 625	22.91 29	72.45 69
526	276 676	22.93 47	72.52 59
527	277 729	22.95 65	72.59 48
528	278 784	22.97 83	72.66 36
529	279 841	23.00 00	72.73 24
530	280 900	23.02 17	72.80 11
531	281 961	23.04 34	72.86 97
532	283 024	23.06 51	72.93 83
533	284 089	23.08 68	73.00 68
534	285 156	23.10 84	73.07 53
535	286 225	23.13 01	73.14 37
536	287 296	23.15 17	73.21 20
537	288 369	23.17 33	73.28 03
538	289 444	23.19 48	73.34 85
539	290 521	23.21 64	73.41 66
540	291 600	23.23 79	73.48 47
541	292 681	23.25 94	73.55 27
542	293 764	23.28 09	73.62 06
543	294 849	23.30 24	73.68 85
544	295 936	23.32 38	73.75 64
545	297 025	23.34 52	73.82 41
546	298 116	23.36 66	73.89 18
547	299 209	23.38 80	73.95 94
548	300 304	23.40 94	74.02 70
549	301 401	23.43 07	74.09 45
550	302 500	23.45 21	74.16 20
N	N²	√N	√10N

N	N²	√N	√10N
550	302 500	23.45 21	74.16 20
551	303 601	23.47 34	74.22 94
552	304 704	23.49 47	74.29 67
553	305 809	23.51 60	74.36 40
554	306 916	23.53 72	74.43 12
555	308 025	23.55 84	74.49 83
556	309 136	23.57 97	74.56 54
557	310 249	23.60 08	74.63 24
558	311 364	23.62 20	74.69 94
559	312 481	23.64 32	74.76 63
560	313 600	23.66 43	74.83 31
561	314 721	23.68 54	74.89 99
562	315 844	23.70 65	74.96 67
563	316 969	23.72 76	75.03 33
564	318 096	23.74 87	75.09 99
565	319 225	23.76 97	75.16 65
566	320 356	23.79 08	75.23 30
567	321 489	23.81 18	75.29 94
568	322 624	23.83 28	75.36 58
569	323 761	23.85 37	75.43 21
570	324 900	23.87 47	75.49 83
571	326 041	23.89 56	75.56 45
572	327 184	23.91 65	75.63 07
573	328 329	23.93 74	75.69 68
574	329 476	23.95 83	75.76 28
575	330 625	23.97 92	75.82 88
576	331 776	24.00 00	75.89 47
577	332 929	24.02 08	75.96 05
578	334 084	24.04 16	76.02 63
579	335 241	24.06 24	76.09 20
580	336 400	24.08 32	76.15 77
581	337 561	24.10 39	76.22 34
582	338 724	24.12 47	76.28 89
583	339 889	24.14 54	76.35 44
584	341 056	24.16 61	76.41 99
585	342 225	24.18 68	76.48 53
586	343 396	24.20 74	76.55 06
587	344 569	24.22 81	76.61 59
588	345 744	24.24 87	76.68 12
589	346 921	24.26 93	76.74 63
590	348 100	24.28 99	76.81 15
591	349 281	24.31 05	76.87 65
592	350 464	24.33 11	76.94 15
593	351 649	24.35 16	77.00 65
594	352 836	24.37 21	77.07 14
595	354 025	24.39 26	77.13 62
596	355 216	24.41 31	77.20 10
597	356 409	24.43 36	77.26 58
598	357 604	24.45 40	77.33 05
599	358 801	24.47 45	77.39 51
600	360 000	24.49 49	77.45 97
N	N²	√N	√10N

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
600	360 000	24.49 49	77.45 97
601	361 201	24.51 53	77.52 42
602	362 404	24.53 57	77.58 87
603	363 609	24.55 61	77.65 31
604	364 816	24.57 64	77.71 74
605	366 025	24.59 67	77.78 17
606	367 236	24.61 71	77.84 60
607	368 449	24.63 74	77.91 02
608	369 664	24.65 77	77.97 44
609	370 881	24.67 79	78.03 85
610	372 100	24.69 82	78.10 25
611	373 321	24.71 84	78.16 65
612	374 544	24.73 86	78.23 04
613	375 769	24.75 88	78.29 43
614	376 996	24.77 90	78.35 82
615	378 225	24.79 92	78.42 19
616	379 456	24.81 93	78.48 57
617	380 689	24.83 95	78.54 93
618	381 924	24.85 96	78.61 30
619	383 161	24.87 97	78.67 66
620	384 400	24.89 98	78.74 01
621	385 641	24.91 99	78.80 36
622	386 884	24.93 99	78.86 70
623	388 129	24.96 00	78.93 03
624	389 376	24.98 00	78.99 37
625	390 625	25.00 00	79.05 69
626	391 876	25.02 00	79.12 02
627	393 129	25.04 00	79.18 33
628	394 384	25.05 99	79.24 65
629	395 641	25.07 99	79.30 95
630	396 900	25.09 98	79.37 25
631	398 161	25.11 97	79.43 55
632	399 424	25.13 96	79.49 84
633	400 689	25.15 95	79.56 13
634	401 956	25.17 94	79.62 41
635	403 225	25.19 92	79.68 69
636	404 496	25.21 90	79.74 96
637	405 769	25.23 89	79.81 23
638	407 044	25.25 87	79.87 49
639	408 321	25.27 84	79.93 75
640	409 600	25.29 82	80.00 00
641	410 881	25.31 80	80.06 25
642	412 164	25.33 77	80.12 49
643	413 449	25.35 74	80.18 73
644	414 736	25.37 72	80.24 96
645	416 025	25.39 69	80.31 19
646	417 316	25.41 65	80.37 41
647	418 609	25.43 62	80.43 63
648	419 904	25.45 58	80.49 84
649	421 201	25.47 55	80.56 05
650	422 500	25.49 51	80.62 26
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
650	422 500	25.49 51	80.62 26
651	423 801	25.51 47	80.68 46
652	425 104	25.53 43	80.74 65
653	426 409	25.55 39	80.80 84
654	427 716	25.57 34	80.87 03
655	429 025	25.59 30	80.93 21
656	430 336	25.61 25	80.99 38
657	431 649	25.63 20	81.05 55
658	432 964	25.65 15	81.11 72
659	434 281	25.67 10	81.17 88
660	435 600	25.69 05	81.24 04
661	436 921	25.70 99	81.30 19
662	438 244	25.72 94	81.36 34
663	439 569	25.74 88	81.42 48
664	440 896	25.76 82	81.48 62
665	442 225	25.78 76	81.54 75
666	443 556	25.80 70	81.60 88
667	444 889	25.82 63	81.67 01
668	446 224	25.84 57	81.73 13
669	447 561	25.86 50	81.79 24
670	448 900	25.88 44	81.85 35
671	450 241	25.90 37	81.91 46
672	451 584	25.92 30	81.97 56
673	452 929	25.94 22	82.03 66
674	454 276	25.96 15	82.09 75
675	455 625	25.98 08	82.15 84
676	456 976	26.00 00	82.21 92
677	458 329	26.01 92	82.28 00
678	459 684	26.03 84	82.34 08
679	461 041	26.05 76	82.40 15
680	462 400	26.07 68	82.46 21
681	463 761	26.09 60	82.52 27
682	465 124	26.11 51	82.58 33
683	466 489	26.13 43	82.64 38
684	467 856	26.15 34	82.70 43
685	469 225	26.17 25	82.76 47
686	470 596	26.19 16	82.82 51
687	471 969	26.21 07	82.88 55
688	473 344	26.22 98	82.94 58
689	474 721	26.24 88	83.00 60
690	476 100	26.26 79	83.06 62
691	477 481	26.28 69	83.12 64
692	478 864	26.30 59	83.18 65
693	480 249	26.32 49	83.24 66
694	481 636	26.34 39	83.30 67
695	483 025	26.36 29	83.36 67
696	484 416	26.38 18	83.42 66
697	485 809	26.40 08	83.48 65
698	487 204	26.41 97	83.54 64
699	488 601	26.43 86	83.60 62
700	490 000	26.45 75	83.66 60
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
700	490 000	26.45 75	83.66 60
701	491 401	26.47 64	83.72 57
702	492 804	26.49 53	83.78 54
703	494 209	26.51 41	83.84 51
704	495 616	26.53 30	83.90 47
705	497 025	26.55 18	83.96 43
706	498 436	26.57 07	84.02 38
707	499 849	26.58 95	84.08 33
708	501 264	26.60 83	84.14 27
709	502 681	26.62 71	84.20 21
710	504 100	26.64 58	84.26 15
711	505 521	26.66 46	84.32 08
712	506 944	26.68 33	84.38 01
713	508 369	26.70 21	84.43 93
714	509 796	26.72 08	84.49 85
715	511 225	26.73 95	84.55 77
716	512 656	26.75 82	84.61 68
717	514 089	26.77 69	84.67 59
718	515 524	26.79 55	84.73 49
719	516 961	26.81 42	84.79 39
720	518 400	26.83 28	84.85 28
721	519 841	26.85 14	84.91 17
722	521 284	26.87 01	84.97 06
723	522 729	26.88 87	85.02 94
724	524 176	26.90 72	85.08 82
725	525 625	26.92 58	85.14 69
726	527 076	26.94 44	85.20 56
727	528 529	26.96 29	85.26 43
728	529 984	26.98 15	85.32 29
729	531 441	27.00 00	85.38 15
730	532 900	27.01 85	85.44 00
731	534 361	27.03 70	85.49 85
732	535 824	27.05 55	85.55 70
733	537 289	27.07 40	85.61 54
734	538 756	27.09 24	85.67 38
735	540 225	27.11 09	85.73 21
736	541 696	27.12 93	85.79 04
737	543 169	27.14 77	85.84 87
738	544 644	27.16 62	85.90 69
739	546 121	27.18 46	85.96 51
740	547 600	27.20 29	86.02 33
741	549 081	27.22 13	86.08 14
742	550 564	27.23 97	86.13 94
743	552 049	27.25 80	86.19 74
744	553 536	27.27 64	86.25 54
745	555 025	27.29 47	86.31 34
746	556 516	27.31 30	86.37 13
747	558 009	27.33 13	86.42 92
748	559 504	27.34 96	86.48 70
749	561 001	27.36 79	86.54 48
750	562 500	27.38 61	86.60 25
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
750	562 500	27.38 61	86.60 25
751	564 001	27.40 44	86.66 03
752	565 504	27.42 26	86.71 79
753	567 009	27.44 08	86.77 56
754	568 516	27.45 91	86.83 32
755	570 025	27.47 73	86.89 07
756	571 536	27.49 55	86.94 83
757	573 049	27.51 36	87.00 57
758	574 564	27.53 18	87.06 32
759	576 081	27.55 00	87.12 06
760	577 600	27.56 81	87.17 80
761	579 121	27.58 62	87.23 53
762	580 644	27.60 43	87.29 26
763	582 169	27.62 25	87.34 99
764	583 696	27.64 05	87.40 71
765	585 225	27.65 86	87.46 43
766	586 756	27.67 67	87.52 14
767	588 289	27.69 48	87.57 85
768	589 824	27.71 28	87.63 56
769	591 361	27.73 08	87.69 26
770	592 900	27.74 89	87.74 96
771	594 441	27.76 69	87.80 66
772	595 984	27.78 49	87.86 35
773	597 529	27.80 29	87.92 04
774	599 076	27.82 09	87.97 73
775	600 625	27.83 88	88.03 41
776	602 176	27.85 68	88.09 09
777	603 729	27.87 47	88.14 76
778	605 284	27.89 27	88.20 43
779	606 841	27.91 06	88.26 10
780	608 400	27.92 85	88.31 76
781	609 961	27.94 64	88.37 42
782	611 524	27.96 43	88.43 08
783	613 089	27.98 21	88.48 73
784	614 656	28.00 00	88.54 38
785	616 225	28.01 79	88.60 02
786	617 796	28.03 57	88.65 66
787	619 369	28.05 36	88.71 30
788	620 944	28.07 13	88.76 94
789	622 521	28.08 91	88.82 57
790	624 100	28.10 69	88.88 19
791	625 681	28.12 47	88.93 82
792	627 264	28.14 25	88.99 44
793	628 849	28.16 03	89.05 05
794	630 436	28.17 80	89.10 67
795	632 025	28.19 57	89.16 28
796	633 616	28.21 35	89.21 88
797	635 209	28.23 12	89.27 49
798	636 804	28.24 89	89.33 08
799	638 401	28.26 66	89.38 68
800	640 000	28.28 43	89.44 27
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$
800	640 000	28.28 43	89.44 27
801	641 601	28.30 19	89.49 86
802	643 204	28.31 96	89.55 45
803	644 809	28.33 73	89.61 03
804	646 416	28.35 49	89.66 60
805	648 025	28.37 25	89.72 18
806	649 636	28.39 01	89.77 75
807	651 249	28.40 77	89.83 32
808	652 864	28.42 53	89.88 88
809	654 481	28.44 29	89.94 44
810	656 100	28.46 05	90.00 00
811	657 721	28.47 81	90.05 55
812	659 344	28.49 56	90.11 10
813	660 969	28.51 32	90.16 65
814	662 596	28.53 07	90.22 19
815	664 225	28.54 82	90.27 74
816	665 856	28.56 57	90.33 27
817	667 489	28.58 32	90.38 81
818	669 124	28.60 07	90.44 34
819	670 761	28.61 82	90.49 86
820	672 400	28.63 56	90.55 39
821	674 041	28.65 31	90.60 91
822	675 684	28.67 05	90.66 42
823	677 329	28.68 80	90.71 93
824	678 976	28.70 54	90.77 44
825	680 625	28.72 28	90.82 95
826	682 276	28.74 02	90.88 45
827	683 929	28.75 76	90.93 95
828	685 584	28.77 50	90.99 45
829	687 241	28.79 24	91.04 94
830	688 900	28.80 97	91.10 43
831	690 561	28.82 71	91.15 92
832	692 224	28.84 44	91.21 40
833	693 889	28.86 17	91.26 88
834	695 556	28.87 91	91.32 36
835	697 225	28.89 64	91.37 83
836	698 896	28.91 37	91.43 30
837	700 569	28.93 10	91.48 77
838	702 244	28.94 82	91.54 23
839	703 921	28.96 55	91.59 69
840	705 600	28.98 28	91.65 15
841	707 281	29.00 00	91.70 61
842	708 964	29.01 72	91.76 06
843	710 649	29.03 45	91.81 50
844	712 336	29.05 17	91.86 95
845	714 025	29.06 89	91.92 39
846	715 716	29.08 61	91.97 83
847	717 409	29.10 33	92.03 26
848	719 104	29.12 04	92.08 69
849	720 801	29.13 76	92.14 12
850	722 500	29.15 48	92.19 54
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

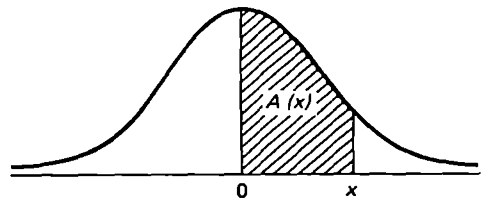
N	N^2	$\sqrt{N}$	$\sqrt{10N}$
850	722 500	29.15 48	92.19 54
851	724 201	29.17 19	92.24 97
852	725 904	29.18 90	92.30 38
853	727 609	29.20 62	92.35 80
854	729 316	29.22 33	92.41 21
855	731 025	29.24 04	92.46 62
856	732 736	29.25 75	92.52 03
857	734 449	29.27 46	92.57 43
858	736 164	29.29 16	92.62 83
859	737 881	29.30 87	92.68 23
860	739 600	29.32 58	92.73 62
861	741 321	29.34 28	92.79 01
862	743 044	29.35 98	92.84 40
863	744 769	29.37 69	92.89 78
864	746 496	29.39 39	92.95 16
865	748 225	29.41 09	93.00 54
866	749 956	29.42 79	93.05 91
867	751 689	29.44 49	93.11 28
868	753 424	29.46 18	93.16 65
869	755 161	29.47 88	93.22 02
870	756 900	29.49 58	93.27 38
871	758 641	29.51 27	93.32 74
872	760 384	29.52 96	93.38 09
873	762 129	29.54 66	93.43 45
874	763 876	29.56 35	93.48 80
875	765 625	29.58 04	93.54 14
876	767 376	29.59 73	93.59 49
877	769 129	29.61 42	93.64 83
878	770 884	29.63 11	93.70 17
879	772 641	29.64 79	93.75 50
880	774 400	29.66 48	93.80 83
881	776 161	29.68 16	93.86 16
882	777 924	29.69 85	93.91 49
883	779 689	29.71 53	93.96 81
884	781 456	29.73 21	94.02 13
885	783 225	29.74 89	94.07 44
886	784 996	29.76 58	94.12 76
887	786 769	29.78 25	94.18 07
888	788 544	29.79 93	94.23 38
889	790 321	29.81 61	94.28 68
890	792 100	29.83 29	94.33 98
891	793 881	29.84 96	94.39 28
892	795 664	29.86 64	94.44 58
893	797 449	29.88 31	94.49 87
894	799 236	29.89 98	94.55 16
895	801 025	29.91 66	94.60 44
896	802 816	29.93 33	94.65 73
897	804 609	29.95 00	94.71 01
898	806 404	29.96 66	94.76 29
899	808 201	29.98 33	94.81 56
900	810 000	30.00 00	94.86 83
N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N²	√N	√10N
900	810 000	30.00 00	94.86 83
901	811 801	30.01 67	94.92 10
902	813 604	30.03 33	94.97 37
903	815 409	30.05 00	95.02 63
904	817 216	30.06 66	95.07 89
905	819 025	30.08 32	95.13 15
906	820 836	30.09 98	95.18 40
907	822 649	30.11 64	95.23 65
908	824 464	30.13 30	95.28 90
909	826 281	30.14 96	95.34 15
910	828 100	30.16 62	95.39 39
911	829 921	30.18 28	95.44 63
912	831 744	30.19 93	95.49 87
913	833 569	30.21 59	95.55 10
914	835 396	30.23 24	95.60 33
915	837 225	30.24 90	95.65 56
916	839 056	30.26 55	95.70 79
917	840 889	30.28 20	95.76 01
918	842 724	30.29 85	95.81 23
919	844 561	30.31 50	95.86 45
920	846 400	30.33 15	95.91 66
921	848 241	30.34 80	95.96 87
922	850 084	30.36 45	96.02 08
923	851 929	30.38 09	96.07 29
924	853 776	30.39 74	96.12 49
925	855 625	30.41 38	96.17 69
926	857 476	30.43 02	96.22 89
927	859 329	30.44 67	96.28 08
928	861 184	30.46 31	96.33 28
929	863 041	30.47 95	96.38 46
930	864 900	30.49 59	96.43 65
931	866 761	30.51 23	96.48 83
932	868 624	30.52 87	96.54 01
933	870 489	30.54 50	96.59 19
934	872 356	30.56 14	96.64 37
935	874 225	30.57 78	96.69 54
936	876 096	30.59 41	96.74 71
937	877 969	30.61 05	96.79 88
938	879 844	30.62 68	96.85 04
939	881 721	30.64 31	96.90 20
940	883 600	30.65 94	96.95 36
941	885 481	30.67 57	97.00 52
942	887 364	30.69 20	97.05 67
943	889 249	30.70 83	97.10 82
944	891 136	30.72 46	97.15 97
945	893 025	30.74 09	97.21 11
946	894 916	30.75 71	97.26 25
947	896 809	30.77 34	97.31 39
948	898 704	30.78 96	97.36 53
949	900 601	30.80 58	97.41 66
950	902 500	30.82 21	97.46 79
N	N²	√N	√10N

N	N²	√N	√10N
950	902 500	30.82 21	97.46 79
951	904 401	30.83 83	97.51 92
952	906 304	30.85 45	97.57 05
953	908 209	30.87 07	97.62 17
954	910 116	30.88 69	97.67 29
955	912 025	30.90 31	97.72 41
956	913 936	30.91 92	97.77 53
957	915 849	30.93 54	97.82 64
958	917 764	30.95 16	97.87 75
959	919 681	30.96 77	97.92 85
960	921 600	30.98 39	97.97 96
961	923 521	31.00 00	98.03 06
962	925 444	31.01 61	98.08 16
963	927 369	31.03 22	98.13 26
964	929 296	31.04 83	98.18 35
965	931 225	31.06 44	98.23 44
966	933 156	31.08 05	98.28 53
967	935 089	31.09 66	98.33 62
968	937 024	31.11 27	98.38 70
969	938 961	31.12 88	98.43 78
970	940 900	31.14 48	98.48 86
971	942 841	31.16 09	98.53 93
972	944 784	31.17 69	98.59 01
973	946 729	31.19 29	98.64 08
974	948 676	31.20 90	98.69 14
975	950 625	31.22 50	98.74 21
976	952 576	[illegible].24 10	98.79 27
977	954 529	31.25 70	98.84 33
978	956 484	31.27 30	98.89 39
979	958 441	31.28 90	98.94 44
980	960 400	31.30 50	98.99 49
981	962 361	31.32 09	99.04 54
982	964 324	31.33 69	99.09 59
983	966 289	31.35 28	99.14 64
984	968 256	31.36 88	99.19 68
985	970 225	31.38 47	99.24 72
986	972 196	31.40 06	99.29 75
987	974 169	31.41 66	99.34 79
988	976 144	31.43 25	99.39 82
989	978 121	31.44 84	99.44 85
990	980 100	31.46 43	99.49 87
991	982 081	31.48 02	99.54 90
992	984 064	31.49 60	99.59 92
993	986 049	31.51 19	99.64 94
994	988 036	31.52 78	99.69 95
995	990 025	31.54 36	99.74 97
996	992 016	31.55 95	99.79 98
997	994 009	31.57 53	99.84 99
998	996 004	31.59 11	99.89 99
999	998 001	31.60 70	99.95 00
1000	1000 000	31.62 28	100.00 00
N	N²	√N	√10N

APPENDIX B

NORMAL CURVE AREAS



x	$A(x)$	x	$A(x)$	x	$A(x)$	x	$A(x)$	x	$A(x)$
.00	.0000	.10	.0398	.20	.0793	.30	.1179	.40	.1554
.01	.0040	.11	.0438	.21	.0832	.31	.1217	.41	.1591
.02	.0080	.12	.0478	.22	.0871	.32	.1255	.42	.1628
.03	.0120	.13	.0517	.23	.0910	.33	.1293	.43	.1664
.04	.0160	.14	.0557	.24	.0948	.34	.1331	.44	.1700
.05	.0199	.15	.0596	.25	.0987	.35	.1368	.45	.1736
.06	.0239	.16	.0636	.26	.1026	.36	.1406	.46	.1772
.07	.0279	.17	.0675	.27	.1064	.37	.1443	.47	.1808
.08	.0319	.18	.0714	.28	.1103	.38	.1480	.48	.1844
.09	.0359	.19	.0754	.29	.1141	.39	.1517	.49	.1879

.50	.1915	.75	.2734	1.00	.3413	1.25	.3944	1.50	.4332
.51	.1950	.76	.2764	1.01	.3438	1.26	.3962	1.51	.4345
.52	.1985	.77	.2794	1.02	.3461	1.27	.3980	1.52	.4357
.53	.2019	.78	.2823	1.03	.3485	1.28	.3997	1.53	.4370
.54	.2054	.79	.2852	1.04	.3508	1.29	.4015	1.54	.4382
.55	.2088	.80	.2881	1.05	.3531	1.30	.4032	1.55	.4394
.56	.2123	.81	.2910	1.06	.3554	1.31	.4049	1.56	.4406
.57	.2157	.82	.2939	1.07	.3577	1.32	.4066	1.57	.4418
.58	.2190	.83	.2967	1.08	.3599	1.33	.4082	1.58	.4430
.59	.2224	.84	.2996	1.09	.3621	1.34	.4099	1.59	.4441
.60	.2258	.85	.3023	1.10	.3643	1.35	.4115	1.60	.4452
.61	.2291	.86	.3051	1.11	.3665	1.36	.4131	1.61	.4463
.62	.2324	.87	.3079	1.12	.3686	1.37	.4147	1.62	.4474
.63	.2357	.88	.3106	1.13	.3708	1.38	.4162	1.63	.4485
.64	.2389	.89	.3133	1.14	.3729	1.39	.4177	1.64	.4495
.65	.2422	.90	.3159	1.15	.3749	1.40	.4192	1.65	.4505
.66	.2454	.91	.3186	1.16	.3770	1.41	.4207	1.66	.4515
.67	.2486	.92	.3212	1.17	.3790	1.42	.4222	1.67	.4525
.68	.2518	.93	.3238	1.18	.3810	1.43	.4236	1.68	.4535
.69	.2549	.94	.3264	1.19	.3830	1.44	.4251	1.69	.4545
.70	.2580	.95	.3289	1.20	.3849	1.45	.4265	1.70	.4554
.71	.2612	.96	.3315	1.21	.3869	1.46	.4279	1.71	.4564
.72	.2642	.97	.3340	1.22	.3888	1.47	.4292	1.72	.4573
.73	.2673	.98	.3365	1.23	.3907	1.48	.4306	1.73	.4582
.74	.2704	.99	.3389	1.24	.3925	1.49	.4319	1.74	.4591

Normal Curve Areas (*continued*)

x	$A(x)$	x	$A(x)$	x	$A(x)$	x	$A(x)$	x	$A(x)$
1.75	.4599	1.95	.4744	2.15	.4842	2.35	.4906	2.55	.4946
1.76	.4608	1.96	.4750	2.16	.4846	2.36	.4909	2.56	.4948
1.77	.4616	1.97	.4756	2.17	.4850	2.37	.4911	2.57	.4949
1.78	.4625	1.98	.4762	2.18	.4854	2.38	.4913	2.58	.4951
1.79	.4633	1.99	.4767	2.19	.4857	2.39	.4916	2.59	.4952
1.80	.4641	2.00	.4773	2.20	.4861	2.40	.4918	2.60	.4953
1.81	.4649	2.01	.4778	2.21	.4865	2.41	.4920	2.61	.4955
1.82	.4656	2.02	.4783	2.22	.4868	2.42	.4922	2.62	.4956
1.83	.4664	2.03	.4788	2.23	.4871	2.43	.4925	2.63	.4957
1.84	.4671	2.04	.4793	2.24	.4875	2.44	.4927	2.64	.4959
1.85	.4678	2.05	.4798	2.25	.4878	2.45	.4929	2.65	.4960
1.86	.4686	2.06	.4803	2.26	.4881	2.46	.4931	2.66	.4961
1.87	.4693	2.07	.4808	2.27	.4884	2.47	.4932	2.67	.4962
1.88	.4700	2.08	.4812	2.28	.4887	2.48	.4934	2.68	.4963
1.89	.4706	2.09	.4817	2.29	.4890	2.49	.4936	2.69	.4964
1.90	.4713	2.10	.4821	2.30	.4893	2.50	.4938	2.70	.4965
1.91	.4719	2.11	.4826	2.31	.4896	2.51	.4940	2.71	.4966
1.92	.4726	2.12	.4830	2.32	.4898	2.52	.4941	2.72	.4967
1.93	.4732	2.13	.4834	2.33	.4901	2.53	.4943	2.73	.4968
1.94	.4738	2.14	.4838	2.34	.4904	2.54	.4945	2.74	.4969

2.75	.4970	3.00	.4987	3.25	.4994	3.50	.4998	3.75	.4999
2.76	.4971	3.01	.4987	3.26	.4994	3.51	.4998	3.76	.4999
2.77	.4972	3.02	.4987	3.27	.4995	3.52	.4998	3.77	.4999
2.78	.4973	3.03	.4988	3.28	.4995	3.53	.4998	3.78	.4999
2.79	.4974	3.04	.4988	3.29	.4995	3.54	.4998	3.79	.4999
2.80	.4974	3.05	.4989	3.30	.4995	3.55	.4998	3.80	.4999
2.81	.4975	3.06	.4989	3.31	.4995	3.56	.4998	3.81	.4999
2.82	.4976	3.07	.4989	3.32	.4996	3.57	.4998	3.82	.4999
2.83	.4977	3.08	.4990	3.33	.4996	3.58	.4998	3.83	.4999
2.84	.4777	3.09	.4990	3.34	.4996	3.59	.4998	3.84	.4999
2.85	.4978	3.10	.4990	3.35	.4996	3.60	.4998	3.85	.4999
2.86	.4979	3.11	.4991	3.36	.4996	3.61	.4999	3.86	.4999
2.87	.4980	3.12	.4991	3.37	.4996	3.62	.4999	3.87	.5000
2.88	.4980	3.13	.4991	3.38	.4996	3.63	.4999	3.88	.5000
2.89	.4981	3.14	.4992	3.39	.4997	3.64	.4999	3.89	.5000
2.90	.4981	3.15	.4992	3.40	.4997	3.65	.4999	3.90	.5000
2.91	.4982	3.16	.4992	3.41	.4997	3.66	.4999	3.91	.5000
2.92	.4983	3.17	.4992	3.42	.4997	3.67	.4999	3.92	.5000
2.93	.4983	3.18	.4993	3.43	.4997	3.68	.4999	3.93	.5000
2.94	.4984	3.19	.4993	3.44	.4997	3.69	.4999	3.94	.5000
2.95	.4984	3.20	.4993	3.45	.4997	3.70	.4999	3.95	.5000
2.96	.4985	3.21	.4993	3.46	.4997	3.71	.4999	3.96	.5000
2.97	.4985	3.22	.4994	3.47	.4997	3.72	.4999	3.97	.5000
2.98	.4986	3.23	.4994	3.48	.4998	3.73	.4999	3.98	.5000
2.99	.4986	3.24	.4994	3.49	.4998	3.74	.4999	3.99	.5000

APPENDIX C

GLOSSARY OF COMPUTER AND PROGRAMMED INSTRUCTION TERMS

ACCESS TIME The time required to transfer information from storage to where it is going to be used.

ADAPTIVE TEACHING MACHINES Teaching machines that automatically alter the instructional presentation sequence as a function of the pupil's performance. Example: The machine may shift to a smaller step size if the pupil is making more than four incorrect responses out of every ten frames.

ADAPTIVITY The capacity of the teaching machine and its associated program to adjust in one or more ways, on the basis of the learner's responses, to his specific needs.

ADDRESS A label, usually a number, identifying a place in storage where a piece of information may be stored.

ALGORITHM A step-by-step routine for computation.

ANALOG COMPUTER A computer that represents numbers by actual physical changes; contrasting with digital.

BINARY ARITHMETIC A number system based on only two choices, 0 and 1.

BRANCHING A style of programming in which selection of the next frame to be presented depends on the response given in the current frame.

CODING The processing of representing rules for handling the processing of information in a synthetic or computer language.

COLLATOR Component of a teaching machine that measures and records the learning process by collecting and recording data such as the number of errors, the type of error, time intervals required for response, and so on, in such a way that each item is collated with the part of the program to which it pertains.

COMPARATOR Component of a teaching machine that judges the correctness of the pupil's response. This evaluation is then transmitted, depending on the mode of operation, to the pupil, the reinforcement dispenser, the collator, and/or the sequence control unit.

COMPILER A special set of instructions contained in a computer to translate a source program into machine language.

CONTROL Part of the computer that effects an orderly sequence of operation of the other parts of the computer.

DISPLAY MECHANISM The unit of a teaching machine that presents the content material in a series of frames.

FEEDBACK The function of a teaching machine that consists of providing the pupil with knowledge of results.

FLOW CHART A diagram or graphic representation of a plan for the sequence of operations in solving a problem on a computer.

FORTRAN *For*mula *Tran*slation, a scientific code usable in many computers for computer operations.

FRAME A unit of a program: the segment displayed at each step in the sequence. Usually the unit that requires a response.

HARDWARE The mechanical, electrical, and magnetic devices and materials from which an automatic computer system is constructed.

HYBRID COMPUTER A machine having different functions and representing a cross between two types of computers, as the analog and digital.

INPUT Information that is transferred from the outside to the inside of a computer for the purpose of processing; also refers to machinery used to bring information into the computer.

ITEM Any single unit of a test or experiment; that is a single question on a test or a single nonsense syllable in a list of syllables.

LOGICAL OPERATION An operation dealing with the validity of thought in an arithmetic computation, or other activities such as comparing or selecting information.

MACHINE LANGUAGE The coded operations that control information and addresses in a digital computer.

MACHINE WORD A set of characters occupying one storage location and treated as a unit. May be of fixed or variable length.

MICROSECOND One millionth part of a second (.000001 sec).

MILLISECOND One thousandth part of a second (.001 sec).

NANOSECOND One billionth part of a second (.000000001 sec).

OUTPUT Information transferred out of any part of a computer as a result of data processing.

PROGRAM A plan of detailed instructions for solving the machine problem in a digital computer.

PROGRAMMER One who prepares the sets of instructions.

PROGRAMMING The process of arranging the material to be learned into a series of sequential steps; usually moves the student from a familiar background into a complex and new set of concepts, principles, and understandings; also refers to preparation of instructions for a computer.

PROMPT Programming techniques designed to insure the desired response to a frame.

PUNCHED CARDS Cards containing information expressed by means of specially coded holes.

REAL TIME Computer operation simultaneous with the occurrence of the event that supplies the material for the input.

ROUTINE The set of coded instruction necessary for performing an operation in a digital computer.

SEQUENCING Arranging the frames of a program in an order that provides the most efficient situation for learning.

SIMULATION The representation of physical systems and phenomena by computers, in which the processing done by the computer represents the process itself.

SOCRATIC METHOD A method of instruction that consists of a conversational quiz in which a tutor asks questions, the student replies, and the tutor confirms or denies the student by a series of questions to the correct response.

STEP The increment in subject matter level to be learned with each succeeding item or frame in the program.

STICK, HICKORY A primitive teaching machine.

STORAGE A part of the computer that holds material received for future use.

APPENDIX D

LIST OF MATHEMATICAL SYMBOLS

N	The number of distinguishable, equally likely outcomes of an experiment; also used to denote the number of scores (or other measures) in a set. (n is also used for this.)
$P(E)$	The probability of the event E.
$\#(E)$	The number of elementary outcomes comprising E.
E'	The complementary event to E; the event E' occurs if the event E does not occur.
n_1,n_2	Number of outcomes favorable for a first and second condition, respectively. (Used in the statement of the First Basic Combinatorial Principle.)
$n!$	Read "n factorial." Defined for positive integers n by: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. By convention $0! = 1$.
$P(n,r)$	The number of permutations of n things taken r at a time. $P(n,r) = n(n-1)\cdots(n-r+1)$. Alternatively, $P(n,r) = \frac{n!}{(n-r)!}$.
$\binom{n}{r}$	Read "n binomial r." A binomial coefficient. The number of combinations of n things taken r at a time. $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$
$P(A\|B)$	The conditional probability of A given B. $P(A\|B) = P(A \text{ and } B)/P(B)$ provided $P(B) \neq 0$.
p_1,p_2, and so on	Probabilities of various events.

v_1, v_2, and so on — Gains realized if certain events occur.

$<$ — Less than.

$\leq$ — Less than or equal to.

$>$ — Greater than.

$\geq$ — Greater than or equal to.

$A(x)$ — The area under the standard normal curve enclosed by vertical lines at zero and at x, and the horizontal axis.

$\sqrt{\ }$ — Square root symbol.

X — A score or other measurement.

$\sum$ — Greek capital letter sigma. Used to denote summation.

M — The mean or arithmetic average of a sample. $M = \Sigma X/N$.

s — The sample standard deviation,

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_1 - M)^2}.$$

Alternatively,

$$s = \sqrt{\frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2\right]}.$$

n — See N.

Y — A score converted from a raw score.

μ — The population mean.

σ — The population standard deviation.

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