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Numbers as Properties of Objects: Frege and the Nyāya

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In ordinary language, number-words have a predominantly adjectival function. We say that the table has wooden legs, but equally well that it has four legs, or that the planets are nine, just as we say that they are spherical. The adjectival role of number-words in sentences like these encourages a view of number-words as functioning within a natural language as first-level predicates, and hence as standing for properties of the thing or things which the sentence is about. I would like to examine this natural and intuitive account of numbers, and the logical form of the sentences which ascribe them. It is a view which was advocated in the west by, amongst others, Mill and Husserl, and it is a view which has been developed in some detail by the Indian Nyāya-Vaiśeṣika school. However, what we might call the 'adjectival' account of number strikes many people nowadays as naive and implausible. This rejection of the account derives largely from Frege's sustained blistering attack in his *The Foundations of Arithmetic*, where he sets down a series of arguments all designed to show that number attributions cannot have the same logical form as the attribution of properties such as colour or shape. The main aim of this paper is to show that Frege's arguments against at least some versions of the adjectival account are unsound. In particular, I will try to demonstrate that they do not refute the later Nyāya theory, which appears to treat number-words as many-place relational expressions. A corollary is that the attempt to read the Nyāya theory as an anticipation of Frege's own is mistaken.[1]

The Nyāya Theory

As is well known, the philosophical outlook of the Nyāya is *realist*, in almost every sense of that term. They believed, for example, in the external existence and mind-independence, not just of the ordinary objects of daily experience, but also of the universals or properties which reside in those objects. They combined this 'metaphysical realism' with a correspondingly realist or externalist approach towards cognitive and semantic content. The contribution of a word to the content of a sentence, they say, is just the external entity for which it stands, and the content of a sentence is a sort of relational complex or Russellian proposition.[2] For example, to a sentence like "The book is on the table" there would be assigned a triple of objects <a, R, b>, where 'a' stands for the book, 'b' for the table, and 'R' for the relation of contact.

Analogously, the sentence "Mars is spherical" is construed as saying something like 'Sphericity inheres in Mars' or, more accurately, 'Mars is the locus of sphericity by the inherence relation', where here inherence is a logically primitive two-place relation holding between objects and their properties.

A problem arose, however, when this model of analysis was applied to number attributions. For, as the Naiyāyikas were to observe, while ordinary properties inhere in each of their loci, numbers do not. As Matilal puts it:

> [I]f for example duality inheres (i.e. resides by inherence) in two things counted as two then it must inhere also in either of them separately, and hence we must tolerate the oddity of saying that, of a pair, say the sky and the earth, the sky has duality.[3]

To put it another way, we cannot infer from "The table's legs are four" to "Each of the table's legs is four", although we can infer from "The table's legs are wooden" to "Each of the table's legs is wooden". The crudest form of the adjectival theory, in which numbers are entirely analogous to other attributes, offers no explanation of the inadmissibility of such inferences, and hence, considered as a proposal about the logical form of number attributions, must be mistaken.

The Nyāya solution to this problem was simple enough. They claimed that there must exist another logically primitive relation, which they called the 'completion' or '*paryāpti*' relation. Raghunātha, the first to discuss it, remarks that

> the 'completion' relation, whose existence is indicated by contructions such as "This is one pot" and "These are two", is a special kind of self-linking relation.[4]

His commentator Jagadīśa adds,

> It might be thought that the 'completion' relation is [in fact] nothing but inherence... So Raghunātha states that 'completion' [is

a special kind of self-linking relation].... In a sentence like "This is one pot", 'completion' relates the property pot-hood by delimiting it as a property which resides in only one pot, but in a sentence like "These are two pots", 'completion' relates the property two-hood by delimiting it as a property which resides in both pots. Otherwise, it would follow that there is no difference between saying "These are two" and "Each one possesses two-hood".[5]

The idea seems to be that 'completion' is a one-to-many relation which obtains between numbers on the one hand, and objects on the other. It relates numbers to pluralities of objects, but not to objects taken individually. Suppose we let 'I' stand for inherence, and 'C' for completion. Inherence, I suggest, should be thought of as sharing the formal properties of the class-membership or class-inclusion relations. Thus the analysis of a sentence like

(1) Venus and Mars are planets

would be

I(planethood, Venus) & I(planethood, Mars),

which might be read as asserting that Venus is a planet and so is Mars. From this it trivially follows that each of Venus and Mars is a planet. Similarly, the sentence "The table's legs are wooden" says the class comprising the legs of the table is a sub-set of the class of wooden things. From this it follows that each of the legs of the table is wooden. Introducing a new logical relation of 'completion', enables the Nyāya to analyse a sentence like

(2) Venus and Mars are two,

as

C(2; Venus, Mars),

which, by hypothesis, does not entail that the number 2 resides in each individually. Similarly, "There are two pots" asserts that the number 2 is related by the completion relation with the two pots, not that the class of pots is a sub-set of the class of objects numbered 2, whatever that might be. In effect, what the Nyāya claim here is that there is an ambiguity in the "are" of predication for sentences with plural subjects. As well as standing for the class-inclusion relation, it can also stand for a relation which relates a property jointly with a multiplicity of objects.

I think we can simplify this proposal a little without losing its essential structure. For rather than saying that the property planethood resides in Mars by the inherence relation, we would now say that the predicate "... is a planet" is true of Mars, building so to speak the inherence relation into the predicate. In an entirely analogous way, we can build the completion relation into the number-predicate, which then becomes, if the number is n, an n-place relation. Thus the sentence "Venus and Mars are two" asserts of Venus and Mars that they stand in a certain 2-place relation, the relation which is the number **2**. The Nyāya idea, then, is that number-adjectives are n-place relational predicates, and that numbers are n-place relations holding jointly between n distinct objects. It in no way follows from the statement that the relation **2** holds between Venus and Mars, that it holds just with Venus, any more than it follows from the statement that X is between Y and Z, that X is between Y, full stop. On the Nyāya proposal, then, it looks as though the troublesome inference is blocked because its conclusion is not even well formed, since the phrase "Venus is two", like the phrase "X is between Y", is an incomplete or unsaturated expression.

Frege's Arguments

I would like now to review Frege's arguments (in *The Foundations of Arithmetic*) against adjectival theories of number, and to see whether any of them constitutes a good objection to the Nyāya approach. I think that we can discern in Frege four distinct arguments. In the first, Frege criticises the idea that numbers can be classified along with colours as attributes of external things, because number attributions and colour attributions have different properties;

> Is it not in a totally different sense that we speak of a tree as having 1000 leaves and again as having green leaves? The green colour we ascribe to each single leaf, but not the number 1000. If we call all the leaves of a tree taken together its foliage, then the foliage too is green, but it is not 1000. To what then does the property 1000 belong? It almost looks as though it belongs neither to any single one of the leaves, nor to the totality of them all; is it possible that it does not really belong to things in the external world at all? (§22).

Now, as various authors have noted, Frege's argument here is linked with an important distinction in the domain of sentences which have plural subjects. This is the distinction between those which have *distributive*, and those which have *collective* predicates. A predicate like '...is green' is distributive, in the sense that it holds of a collection or plurality of objects only if it holds individually of each member in the plurality. Thus, the sentence "The leaves are green" is true only if each of the leaves is green. On the other hand, in a sentence like "The birds darkened the sky", the predicate applies collectively, since it is not the case that any individual birds darkened the sky, but only the flock as a whole.

Frege here observes, in effect, that if number-words are construed adjectively, then they must stand for collective predicates. But this does not in itself constitute an argument against the adjectival theory, for it is hardly suprising that number adjectives do not share all formal properties with adjectives like 'green'. For his argument to succeed, it must be also be shown that there can be no account of collective predicates which analyses them as ordinary level-1 predicates, that is as being true of objects or pluralities of objects. If there is such an account, then Frege is wrong to assert that collective predicates cannot be of level-I, and that the sentence "The tree has 1000 leaves" must be analysed as a level-2 predication, that is, as asserting of the concept '...is a leaf of the tree' that it has 1000 instances.

I have noted that the Nyāya introduces two relations, the inherence and completion relations. Their motive, we can now see, was exactly to account for the distinction between collective and distributive properties. For the recognition that the inference from "These are two pots" to "Each pot is two" is invalid is just the recognition that the predicate two does not distribute over plural subjects. The Nyāya's idea is to analyse collective predicates like '...are two', not as one-place predicates of aggregates or sets, but as n-place relational predicates, true of n-objects jointly. Since such relational predicates still take objects as subjects, this indeed shows that recognising the distinction between distributive and collective predicates does not force us to abandon the adjectival view. The Nyāya, indeed, have a term for collective properties: they call them *vyāsajya-vṛtti-dharma* or 'properties which occur jointly'.

The logical form of a sentence is that aspect of its structure by which can be explained the validity of those formally valid inferences into which the sentence enters. Frege's first argument bears upon the logical form of number attributions, for it reveals the asymmetry between the following pair of "plural-dropping" inferences:

The tree's leaves are green	The tree's leaves are 1000
Each tree leaf is green	*Each tree leaf is 1000

The first of these inferences is valid. If sentences with distributive predicates have the logical form of universally quantified constructions, its validity is explicable as a case of universal-elimination. The second inference is invalid. If sentences with numerical predicates have the logical form of many-place predications, its invalidity is apparent explicable for the conclusion is ill-formed.

Frege's second argument again seeks to demonstrate that numbers must be predicates of concepts, and not of objects. In a famous passage, he says:

> If I give someone a stone with the words: Find the weight of this, I have given him precisely the object he is to investigate. But if I place a pile of cards in his hands with the words: Find the number of these, this does not tell him whether I wish to know the number of cards, or of complete packs of cards, or even of honour cards at skat. To have given him the pile in his hands is not yet to have given him completely the object he is to investigate; I must add some further word-cards, or packs, or honours (§22).

Another example makes the same point:

> While I am not able, simply by thinking of it differently, to alter the colour or hardness of a thing in the slightest, I am able to think of the Iliad either as one poem, or as 24 books, or as some large number of verses (§22).

I think that the force of this argument is clearest when it is construed as an objection to the view of Mill and Husserl. For them, a number is an attribute of what they call an "aggregate", where an aggregate is to be thought of as a single but complex or composite object. The sentence, "The table has four legs", for example, is to be construed as predicating the number four to an aggregate entity which is made up from the four legs, i.e. has the four legs as proper parts. Frege's point now is just that the parts of the aggregate themselves have parts, and since the part-whole relation is transitive, there is no determinate answer to the question "How many parts does the aggregate have?". "There are various ways in which an agglomeration can be separated into parts", he says, "and we cannot say that one alone would be characteristic." Since it is precisely the function of a concept to slice up the world in a determinate way, Frege concludes that numbers are attributes, not of aggregates of objects, but of concepts.

Nevertheless, there is a mistake in Frege's argument, and it occurs right at the very end of this line of thought. For what Frege has shown is that number attributions *depend on* or *involve* concepts; it does not at all follow that numbers are attributes of concepts. There is, after all, a class of quite ordinary adjectives which depend for their application on a concept without applying to concepts, namely, the class of adjectives like "large", "heavy", etc. To say of an object that it is large makes no sense until we add something like "for an animal" or "for an elephant", since the very same object might be small for an elephant but large for an animal. But it is nevertheless the *object* to which the adjective applies, not the concept which we have supplied. In the case of number attributions, it is enough to observe that, if the plurality of objects to which the number is applied is picked out under a demonstrative, a concept is needed to resolve the familiar indeterminacy in acts of ostension.

Once again, Frege's argument bears upon the logical form of number sentences, this time via the "substantive-dropping" inferences:

These are paper cards	These are 52 cards
These are paper	*These are 52

The Nyāya, however, go one step further. For in taking as well-formed sentences like "Venus and Mars are two", they implicitly reject the Fregean claim that *every* number attribution can be rendered in the form "There are n Fs". There is, in this sentence, no concept which can serve as the subject of a numerical predication; rather, two objects are picked out directly by name. The Nyāya analysis of sentences in which the plurality of objects is picked out by name rather than by description, is to regard them as ordinary n-place predications. Frege concludes from the above argument that a numeral is a 2nd-order predicate of the concept denoted by the substantive. However, if that were right, then sentences like "Mars and Venus are two", in which the plural subject is a conjunction of names, would not be well-formed. This is not a problem for the relational account, which parses such constructions thus:

"The packs are two" $(\exists x)\ (\exists y)\ [\text{Pack}(x)\ \&\ \text{Pack}(y)\ \&\ \mathbf{2}\ (x, y)]$

"Mars and Venus are two" $\mathbf{2}$ (Mars, Venus)

Although he does not discuss such cases in connection with the sentence "Solon is one", Frege says that "in isolation "one" cannot be a predicate", and in a footnote adds that "usages do occur which appear to contradict this; but if we look more closely we shall find that some general term has to be supplied, or else that "one" is not being used as a number word" (§29). The suggestion that we can simply supply a general term, however, will not stand up, not least because there is no determinate way to decide which general term to supply. And the idea that numerals are ambiguous does not commend itself either, for as a general rule, a semantic proposal which can explain all the linguistic phenomena without postulating ambiguities is to be preferred over one which cannot.

The Nyāya account has the consequence that "Mars and Venus are three" is not well-formed, even though both "Mars and Venus are two" and "Mars and Venus are not two" are. Further, and perhaps more seriously, it treats " Hepserus and Phosphorus are one" as ill-formed, when intuitively it seems to be true. One possible solution might be after all to agree with Frege that "one" is ambiguous, and to read the above sentence as "Hesperus is one with (i.e. identical to) Phosphorus".

Frege's third and fourth arguments concern respectively the numbers one and zero. Frege claims that the adjectival view of numbers runs into a particular difficulty with regard to the property one:

> It must strike us immediately as remarkable that every single thing should possess this property. It would be incomprehensible why we should still ascribe it expressly to a thing at all. It is only in virtue of the possibility of something not being wise that it makes sense to say "Solon is wise". The content of a concept diminishes as its extension increases; if its extension becomes all-embracing, its content must vanish altogether. It is not easy to imagine how language could have come to invent a word for a property which could not be of the slightest use for adding to the description of any object whatsoever (§29).

Let us call a property whose extension is all-embracing, an ever-present property. Ironically, the study of such properties was an important theme in Nyāya logical theory, and the claim that they exist was something they defended at great length. Note, first of all, that ever-present terms are very closely related to empty terms. Indeed, if 'P' is a predicate whose extension includes everything, then the complement of 'P', i.e. '–P', is an empty term, a predicate which applies to nothing. It follows, therefore, that if ever-present predicates lack content, then so do empty terms. Now Frege's desire to purge language of empty terms is well-known, and his reason, that such terms can have no function in a proper scientific language, significantly echoes the passage just quoted. Frege's hostility to ever-present concepts is linked too with the fact that at the time of writing the *Foundations*, he had not yet introduced the distinction between sense and reference. Once this distinction is in place, it becomes possible to say that the content of an ever-present concept resides in the particular way in which it presents the universal class. The Naiyāyikas claim that "nameability" and even "knowability" are ever-present terms, but that they differ in meaning because of the intensionality of the properties with which they are associated. We might think that the predicate "one" is also associated with a particular intensionally individuated property, or that it presents its extension in a particular manner. His argument here, that a predicate acquires content only through its classificatory capacity, is refuted by the acceptance of a notion of content as manner of presentation.

Frege's final argument is that true statements involving the word "zero" cannot be accommodated by the adjectival hypothesis, since there is nothing for zero, conceived of as a property, to be attributed to:

> If I say "Venus has no moons", there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the concept "moon of Venus", namely that of including nothing under it (§46).

This argument, I grant, has some force as directed against the aggregate theory of Mill and Husserl, for there is no such thing as an aggregate of no things. Bell, in his book on Husserl, acknowledges this point, but thinks that there is a reply. Since "There are zero moons of Venus" means the same as "There does not exist any moon of Venus", we might regard the word "zero", not as a name at all, but as standing for the negation particle. This of course implies something Husserl explicitly said, that zero is not a number, but is otherwise an acceptable reply.

The Nyāya theory, however, has a radically non-Fregean account of the semantics of "zero". I should say that I am aware of no Nyāya text in which the problem of zero is discussed. Yet if the reconstruction of the Nyāya theory I have offered is correct, it is clear what they would say. If the number n is an n-place relation, then the number zero is a zero-place relation, or, in other words, a complete proposition. But let us note that if we take "zero" as standing for any self-contradictory proposition, then a sentence like "Venus has zero moons" can be read as equivalent to "If there is a moon of Venus, then zero [i.e. a contradiction]". The point is that, since "if p, then contradiction" is logically equivalent to "not p", by taking the term "zero" as standing for any self-contradictory proposition, we can recover the equivalence of zero-sentences and negative existentials. Zero, then, is a n-place relation which never obtains. The advantage of this over the Bell-Husserl approach is just that it retains a univocal account of all number-sentences; the word "zero", as with all other number-adjectives, is analysed as standing for a relation. It is now true, of course, that zero-sentences are conditional constructions, while all other number-sentences are relational predications, but we might compare this with the standard reparsing of "Some men are mortal" and "All men are mortal", the former being treated as conjunctive while the latter conditional.

Summing up

The Nyāya view differs in one very important respect from comparable adjectival theories in the west. The latter all seek some new *object*, which number properties can inhere in, whether it be a complex aggregate as in Mill and Husserl, or a concept as with Frege. The Nyāya prefer not to expand their ontology in such an extravagant way; instead, they introduce a new kind of relation, the 'completion' relation, which is a one-many relation between numbers and objects. Numbers *are* different from properties like colours or shapes, but this is not because they inhere in different types of object; it is because they occur in objects collectively and plurally.

I would like to observe a point of similarity between the Nyāya theory and Russell's definition of the number n as the class of all classes of n objects. If, as I have argued, we can interpret the Nyāya as claiming that the number n is that n-place relation which holds between any n objects, then the extension of the number is the class of all ordered n-tuples. But since the relation is symmetric in all its places, this reduces to Russell's definition. The Nyāya approach, however, has one virtue over Russell, which is due to the fact that numbers are relations taken in *intension*, not in extension. This means that the Nyāya has no need for Russell's 'axiom of infinity', the postulate that there are an infinite number of objects in the universe. Russell introduced the axiom because, if there is a finite number, m say, of objects in the universe, then the numbers m+1, m+2, etc. would all have as their extension the empty set. However, if such numbers are intensionally individuated relations, then their co-extensiveness will not entail their identity.

The Nyāya argue that numbers are relations, but which relations are they? This is not a question of logical form, but of the meaning analysis of number-words. Bigelow[6] arrives at the same conclusion from the opposite direction, via a meaning analysis. He notes that, since "The planets are (at least) three" is 'logically equivalent' to

$$(\exists x)\ (\exists y)\ (\exists z)\ [\text{Planet}(x)\ \&\ \text{Planet}(y)\ \&\ \text{Planet}(z)\ \&\ x \neq y\ \&\ y \neq z\ \&\ z \neq x]$$

numbers are "n-fold relations of mutual distinctness". Although the Nyāya do not discuss this point, Bigelow's suggestion is one they may well endorse.

NOTES

1. Among those who have attempted just this are Ingalls (1951:77) and Shaw (1982). However, Ingalls, in his Preface to Guha (1968), expresses reservation about his earlier claim.

2. I ignore here another level of semantic content recognised by the Nyāya, that which has to do with the delimiting features (*avacchedakas*) under which the objects are presented. See Ganeri (forthcoming) for a fuller discussion.

3. See Matilal, 1985.

4. *paryāptiś ca ayam eko ghaṭaḥ imau dvau iti pratītisākṣakaḥ svarūpasaṃbandha-viśeṣa eva.* In Jagadīśa (38).

5. *nanu paryāptiḥ samavāyaḥ ... paryāptiśceti. ayam eko ghaṭa iti ekamātra-vṛttidharmāvacchedena ghaṭatvasya paryāptim darśayituṃ, imau dvau ity ubhayamātravṛttidharmāvacchedena dvitvasya paryāptiṃ darśayiham. anyathā dvau dvitvavān iti pratītyor aviśeṣaprasaṃgād iti bhāvaḥ.* Jagadīśa (38-9).

6. 1988.

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