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OPERATIONAL PROPERTIES OF LEGENDRE TRANSFORM AND ITS APPLICATIONS

by

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(received on 15.10.1966 and in revised form on 15.4.1967)

ABSTRACT

Certain operational properties of Legendre transform are obtained which are applied to evaluate a few integrals involving Legendre polynomials and Bessel functions with argument other than x.

INTRODUCTION

1. In the past years Bushman (1) and Tranter (6) have studied Legendre transform defined by Churchill (2) as

$$f(n) = T[\,F(x)\,] = \int_{-1}^{1} P_n(x)\, F(x)\, dx, \; n=0,1,2, \ldots\ldots \quad (1.1)$$

In this note two operational relations are proved which are given as lemmas in section 2. The proofs of lemmas have been omitted as they are very simple In sections 3 and 4 integrals have been evaluated by their applications which are general and yield known results as particular cases.

2. **Lemma 1.**—If $f(n)$ is the operational image of $F(x)$ in the Legendre's transform (1.1) and $-1<t<1$, then

$$\int_{-1}^{1} (1-2xt+t^2)^{-\frac{n+1}{2}} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right) F(x)dx$$

$$= \sum_{k=0}^{\infty} \frac{|\underline{(n+k)}}{|\underline{n}\;|\underline{k}}\, t^k f(n+k), \qquad (2.1)$$

where the series on the R.H.S. of (2.1) is convergent.

This is easily proved if we use (4, p. 169, R.7)

$$(1-2xt+t^2)^{-\frac{n+1}{2}} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right) = \sum_{k=o}^{\infty} \frac{\lfloor(n+k)}{\lfloor n \; \lfloor k} t^k P_{n+k}(x), \quad (2.2)$$

Multipy both sides by $F(x)$ and integrate w. r. t. x in $(-1, 1)$. Now interpreting the R.H.S. by (1.1) we obtain (2.1).

Lemma 2 :— If $f(n)$ is the operational image of $F(x)$ in the Legendre's transform (1.1) and $-1 < t < 1$, then

$$\int_{-1}^{1} e^{xt} J_o(t\sqrt{1-x^2}) F(x)\,dx = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} f(n) \quad \ldots (2.3)$$

provided that the series on the R.H S. of (2.3) is convergent.

Similarly this is readily obtained on using (4, p. 165, R. 5)

$$e^{xt} J_o(t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} P_n(x).$$

3. We now obtain some integrals using lemma 1.

(i) Let $F(x) = (1-x)^{\alpha-1}(1+x)^{\beta-1}$

then (4, p. 276, R. 6)

$$f(n) = 2^{\alpha+\beta-1} \Gamma\alpha\,\Gamma\beta\,\{\Gamma(\alpha+\beta)\}^{-1} {}_3F_2(-n, n+1, \alpha;\; 1, \alpha+\beta;\; 1)$$

where $R(\alpha)>0$, $R(\beta)>0$.

Now (2 1) gives

$$\int_{-1}^{1} (1-2xt+t^2)^{-(n+1)/2} (1-x)^{\alpha-1} (1+x)^{\beta-1} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right) dx$$

$$= 2^{\alpha+\beta-1} \Gamma\alpha\,\Gamma\beta\,\{\Gamma(\alpha+\beta)\}^{-1} \sum_{k=0}^{\infty} t^k \; {}_3F_2(-n-k, 1+n+k, \alpha;\; 1, \alpha+\beta;\; 1) \quad \ldots (3.1)$$

where $R(\alpha)>0$, $R(\beta)>0$, $|t|<1$.

Particular case :—When $t\to 1$, we obtain

$$\int_{-1}^{1} (1-x)^{\alpha-(n+3)/2} (1+x)^{\beta-1} P_n\left(\sqrt{\frac{1-x}{2}}\right) dx =$$

$$(-1)^{n+1} \Gamma\alpha\Gamma\beta \{\Gamma(\alpha+\beta)\}^{-1} 2^{\alpha+\beta+(n-1)/2} \sum_{k=0}^{\infty} \frac{\lfloor n+k}{\lfloor n \lfloor k} {}_3F_2(-n-k, 1+n+k, \alpha; 1, \alpha+\beta; 1) \tag{3.2}$$

where $R(\alpha)>(n+1)/2$, $R(\beta)>0$, the series on the R.H.S. of (3.2) is convergent.

(ii) Let $F(x) = (a^2+b^2-2abx)^{-\frac{1}{2}} \begin{matrix}\sin\\ \cos\end{matrix} (\lambda\sqrt{a^2+b^2-2abx})$

then (4, p. 277, R. 11, 12)

$$f(n) = \frac{\pi}{\sqrt{ab}} J_{n+\frac{1}{2}}(a\lambda) \begin{matrix} J_{n+\frac{1}{2}}(b\lambda) \\ Y_{n+\frac{1}{2}}(b\lambda) \end{matrix}$$

where $a, b>0$ or $0<a<b$ according as we take $J_{n+\frac{1}{2}}(b\lambda)$ or $Y_{n+\frac{1}{2}}(b\lambda)$.

Now (2.1) yields

$$\int_{-1}^{1} (1-2xt+t^2)^{-\frac{n+1}{2}} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right) \frac{\begin{matrix}\sin\\ \cos\end{matrix}(\lambda\sqrt{a^2+b^2-abx})}{\sqrt{a^2+b^2-2abx}} dx$$

$$= \frac{\pi}{\sqrt{ab}} \sum_{k=0}^{\infty} \frac{\lfloor n+k}{\lfloor n \lfloor k} t^k J_{n+k+\frac{1}{2}}(a\lambda) \begin{matrix} J_{n+k+\frac{1}{2}} \\ Y_{n+k+\frac{1}{2}} \end{matrix} (b\lambda) \tag{3.3}$$

where $|t|<1$, $a, b>0$ or $0<a<b$ according as we take $J_{n+k+\frac{1}{2}}(b\lambda)$ or $Y_{n+k+\frac{1}{2}}(b\lambda)$.

(iii) Take $F(x) = (t-x)^{-1} P_m(x)$

then (4. p. 278, R. 18)

$f(n) = 2 P_m(t) Q_n(t)$, $m = 0, 1, 2, \ldots\ldots$; $m \leqslant n$; t being in the cut plane along the real axis from -1 to $+1$.

Applying (2.1) we get

$$\int_{-1}^{1} (1-2xt+t^2)^{-\frac{n+1}{2}} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right) \frac{P_m(x)}{t-x} dx$$

$$= 2 P_m(t) \sum_{k=0}^{\infty} \frac{\lfloor n+k}{\lfloor n \lfloor k} t^k Q_{n+k}(t) \quad \ldots \tag{3.4}$$

where $|t| < 1$ m and n are positive integers.

Particular Case :—If $n \to 0$, $m \to 0$, (3.5) yields (3, p. 154)

$$\int_{-1}^{1} \frac{dx}{(t-x)\sqrt{1-2xt+t^2}} = 2 \sum_{k=0}^{\infty} t^k Q_k(t) \qquad \dots (3.5)$$

(iv) Take $F(x) = (1-x)^{\alpha-1} P_m(x)$

then (4, p. 278, R. 17)

$$f(n) = \frac{2^{\alpha} \Gamma\alpha \Gamma(1+n-\alpha)}{\Gamma(1-\alpha)\Gamma(1+n+\alpha)} \, {}_4F_3(-m, m+1, \alpha, \alpha ; 1, n+\alpha+1, \alpha-n ; 1)$$

where $R(\alpha)>0$, $m=0, 1, 2, \dots\dots$

applying (2.1) we obtain

$$\int_{-1}^{1} (1-x)^{\alpha-1} (1-2xt+t^2)^{-(n+1)/2} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right) P_m(x)\, dx$$

$$= \frac{2^{\alpha} \Gamma\alpha}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{\lfloor n+k}{\lfloor n \lfloor k} \frac{\Gamma(1+n+k-\alpha)}{\Gamma(1+n+k+\alpha)} t^k$$

$${}_4F_3(-m, m+1, \alpha, \alpha ; 1, 1+n+k+\alpha, \alpha-n-k ; 1) \qquad \dots (3.6)$$

$$= \frac{2^{\alpha} \Gamma\alpha}{\Gamma(1-\alpha)} \sum_{r=0}^{m} \frac{(-m)_r (m+1)_r (\alpha)_r (\alpha)_r \Gamma(1-\alpha+n-r)}{(\alpha-n)_r (1-\alpha+n)_r \Gamma(1+n+\alpha+r)}$$

$${}_2F_1(n+1, 1-\alpha+n-r ; 1+n+\alpha+r ; t) \qquad \dots (3.7)$$

where $R(\alpha)>0$, m and n are positive integers.

Particular Case :—If $m \to 0$, we get

$$\int_{-1}^{1} (1-x)^{\alpha-1} (1-2xt+t^2)^{-(n+1)/2} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right) dx$$

$$= \frac{2^{\alpha} \Gamma\alpha \Gamma(1+n-\alpha)}{\Gamma(1-\alpha)\Gamma(1+n+\alpha)} \, {}_2F_1(n+1, 1+n-\alpha ; 1+n+\alpha ; t) \qquad (3.8)$$

where $R(\alpha)>0$.

Further if $t \to 1$, we have

$$\int_{-1}^{1} (1-x)^{\alpha-(n+3)/2} P_n\left(\sqrt{\frac{1-x}{2}}\right) dx =$$

$$\frac{(-1)^n \, 2^{(n+1)/2+\alpha} \, \Gamma(2\alpha-n-1) \Gamma(1+n-\alpha)}{\Gamma(1-\alpha) \Gamma(2\alpha)} \qquad \dots (3\,9)$$

where $R(\alpha)>(n+1)/2$.

4. Following a similar procedure and using the operational pairs of $F(x)$ and $f(n)$ in lemma (2) exactly in the same order as used in section

(3), we obtain the following integrals :

$$\text{(i)} \int_{-1}^{1} e^{xt} (1-x)^{\alpha-1} (1+x)^{\beta-1} J_o (t\sqrt{1-x^2})\, dx$$

$$= \frac{2^{\alpha+\beta+1} \Gamma\alpha \Gamma\beta}{\Gamma(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} {}_3F_2 \left\{ \begin{matrix} -n, 1+n, \alpha \\ 1, \alpha+\beta \end{matrix} ; 1 \right\} \quad (4.1)$$

where $R(\alpha)>0$, $R(\beta)>0$.

Particular Case :—If $\alpha \to 1$, we get

$$\int_{-1}^{1} e^{xt} (1+x)^{\beta-1} J_o (t\sqrt{1-x^2})\, dx = \frac{2^\beta}{\beta} {}_1F_1 (-\beta ; 2+\beta ; t),$$

where $R(\beta)>0$.

$$\text{(ii)} \int_{-1}^{1} e^{xt} J_o (t\sqrt{1-x^2}) \frac{\begin{matrix}\sin\\ \cos\end{matrix} (\lambda\sqrt{a^2+b^2-2abx})}{\sqrt{a^2+b^2-2abx}} dx$$

$$= \frac{\pi}{\sqrt{ab}} \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} J_{n+\frac{1}{2}}^{(a\lambda)} \begin{matrix} J_{n+\frac{1}{2}} \\ Y_{n+\frac{1}{2}} \end{matrix} (b\lambda), \quad \ldots (4.2)$$

where $a, b>0$ or $a<b<0$

according as we take $J_{n+\frac{1}{2}}(b\lambda)$ or $Y_{n+\frac{1}{2}}(b\lambda)$.

$$\text{(iii)} \int_{-1}^{1} e^{xt} P_m(x) J_o (t\sqrt{1-x^2}) \frac{dx}{t-x} = 2P_m(t) \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} Q_n (t), \quad \ldots (4\text{-}3)$$

where m is a positive integer and the series on the R.H.S. of (4.6) is convergent.

$$\text{(iv)} \int_{-1}^{1} e^{xt} (1-x)^{\alpha-1} P_m(x) J_o (t\sqrt{1-x^2})\, dx$$

$$= 2^\alpha [\Gamma(\alpha)^2] \sum_{r=0}^{m} \frac{(-m)_r (1+m)_r (\alpha)_r (\alpha)_r}{\lfloor r\, \Gamma(1+\alpha+r)\, \Gamma(\alpha-r)}$$

$${}_1F_1 (1-\alpha-r; 1+\alpha+r ; t) \quad \ldots (4.4)$$

Particular Case :—If $m \to 0$, we get

$$\int_{-1}^{1} e^{xt} (1-x)^{\alpha-1} J_o (t\sqrt{1-x^2})\, dx = \frac{2^\alpha}{\alpha} {}_1F_1 (1-\alpha ; 1+\alpha ; t),$$

where $R(\alpha)>0$.

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INTEGRALS INVOLVING FOX H-FUNCTION

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ABSTRACT

Five Integrals involving Fox H–Function have been evaluated in terms of H–Function. The first two are further generalised to include a large number of integrals as particular cases.

1. Notations :—

(a_p, e_p) represents the set of parameters

$(a_1, e_1), (a_2, e_2) \ldots\ldots\ldots\ldots (a_p, e_p)$.

$$\Gamma(\lambda \pm \mu) = \Gamma(\lambda+\mu)\,\Gamma(\lambda-\mu)$$

$\lambda \pm \mu$ in H–function $= \lambda+\mu, \lambda-\mu$.

$$(1.1) \qquad \omega_1 = \sum_{j=1}^{u} e_j - \sum_{j=u+1}^{p} e_j + \sum_{j=1}^{l} f_j - \sum_{j=l+1}^{q} f_j$$

$$\omega_2 = \tfrac{1}{2}(p-q) + \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$$

2. Known results required in the sequel.

Fox H–function is denoted by Gupta[1] as

$$H_{p,q}^{l,u}\left[t \,\middle|\, \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{l} \Gamma(b_j - f_j s) \prod_{j=1}^{u} \Gamma(1-a_j+e_j s)}{\prod_{j=l+1}^{q} \Gamma(1-b_j+f_j s) \prod_{j=u+1}^{p} \Gamma(a_j - e_j s)} t^s \, ds \qquad \ldots (2.1)$$

where an empty product is interpreted as $1, 0 \leqslant 1 \leqslant q, 0 \leqslant u \leqslant p$, e's and f's are positive; L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - f_j s), j = 1, 2, \ldots\ldots l$, lie on the r. h. s. of the contour and those of $\Gamma(1 - a_j + e_j s), j = 1, 2 \ldots\ldots u$, lie on the l. h. s. Further the s-integral on the r.h.s. of (2.1) is absolutely convergent in at least one of the following cases :—

(i) $\omega_1 > o$, $|\arg Z| < \frac{1}{2}\omega_1 \pi$

(ii) $\omega_1 \geqslant o$, $|\arg Z| \leqslant \frac{1}{2}\omega_1 \pi$ and $R(\omega_2 + 1) < o$.

where ω_1 and ω_2 are given by (1.1).

Also (2)

If $R(\gamma \pm m - \frac{1}{2}) > 0$, then

$$\int_0^\infty t^{2\gamma - 1} W_{k,m}(t) W_{-k,m}(t)\, dt = \frac{\Gamma(2\gamma + 1)\Gamma(\gamma \pm m + \frac{1}{2})}{2\Gamma(\gamma \pm k + 1)} \quad \ldots(2.2)$$

If $R(\lambda \pm \mu \pm v) > 0$, then

$$\int_0^\infty t^{2\lambda - 1} K_{2\mu}(t) K_{2v}(t)\, dt = \frac{(2)^{2\lambda - 3}\, \Gamma(\lambda \pm \mu \pm v)}{\Gamma(2\lambda)} \quad \ldots(2.3)$$

If $R(\gamma) > 0$, $R(\alpha - \gamma + \sigma) > 0$, and $R(\beta - \gamma + \sigma) > 0$, then

$$\int_0^\infty t^{\gamma - 1} (1 + t)^{-\sigma}\, {}_2F_1(\alpha, \beta; \gamma; -t)\, dt =$$

$$\frac{\Gamma(\gamma)\Gamma(\alpha - \gamma + \sigma)\Gamma(\beta - \gamma + \sigma)}{\Gamma(\sigma)\Gamma(\alpha + \beta - \gamma + \sigma)} \quad \ldots(2.4)$$

If m is a positive integer (3)

$$\Gamma(mz) = (2\pi)^{\frac{1}{2} - \frac{m}{2}} (m)^{mz - \frac{1}{2}} \prod_{i=1}^{m} \Gamma\left(z + \frac{i - 1}{m}\right). \quad \ldots(2.5)$$

If $R(\gamma) > 0$, then (4)

$$\int_0^\pi (\sin\phi)^{\gamma - 1} e^{-\delta\phi}\, d\phi = \frac{\pi\,\Gamma(\gamma)\, e^{-\frac{\pi\delta}{2}}}{2^{\gamma - 1}\, \Gamma\left(\frac{\gamma \pm i\delta + 1}{2}\right)} \quad \ldots(2.6)$$

3. Integral I

If $R(\gamma \pm m - \frac{1}{2} - 2n\sigma) > 0$ and in at least one of the following cases (n is a positive integer)

(i) $\omega_1 > 0$, $|\arg Z| < \frac{1}{2}\omega_1\pi$,

(ii) $\omega_1 \geqslant 0$, $|\arg Z| \leqslant \frac{1}{2}\omega_1\pi$ and $R(\omega_2+1) < 0$,

where ω_1 and ω_2 are given by (1.1) we have

$$\int_0^\infty t^{2\gamma-1} W_{k,m}(t)\, W_{-k,m}(t)\, H^{l,\,u}_{p,\,q}\left[Zt^{-2n\sigma}\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right] dt.$$

$$=(2\pi)^{\frac{1}{2}-n}\,(2n)^{2\gamma-1}\, H^{l+4n,u}_{p+2n,\,q+4n}$$

$$\left[(2n)^{-2n\sigma} Z \left|\begin{matrix}(a_p, e_p), \{\Delta(n,1\pm k+\gamma),\sigma\}\\ \{\Delta(2n,2\gamma+1),\sigma\}, \{\Delta(n,\frac{1}{2}\pm m+\gamma),\sigma\}.(b_q, f_q)\end{matrix}\right.\right]$$

Proof :—Write the value of H–function in contour integration by (2.1), change the order of integration, use (2.2), (2.5) and interpret with the help of (2.1) we get the R.H.S.

Particular cases

(i) When $e_p=f_q=\sigma=1$ we get a result in Meijer G– function due to Verma (5)

(ii) When $e_p=f_q=\sigma=1$, $u=1$, $p=p+1$, $q=l$ we get a result in Mac-Robert E– Function due to Rathie (6)

Integral II

If $R\,(\lambda\pm\mu\pm\nu-2n\sigma)>0$, and in at least one of the following case (n is *a* positive integer),

(i) $\omega_1>0$, $|arg\ Z| \angle \frac{1}{2}\omega_1\pi$

(ii) $\omega_1\geqslant 0$, $|arg\ Z| \leqslant\frac{1}{2}\omega_1\pi$ and $R\,(\omega_2+1)<0$,

where ω_1 and ω_2 are given by (1.1), we have

$$\int_0^\infty t^{2\lambda\ \ 1} K_{2\mu}(t)\, K_{2\nu}(t)\, H^{l,\,u}_{p,\,q}\left[Zt^{-2n\sigma}\left|\begin{matrix}(a_p, e_p)\\(a_p, e_p)\end{matrix}\right.\right] dt.$$

$$=(\pi)^{\frac{3}{2}-n}\, 2^{-n-1}\, n^{2\lambda-\frac{3}{2}}$$

$$H^{l+4n,\,u}_{p+2n,\,q+4n}\left[n^{-2n\sigma} Z\left|\begin{matrix}(a_p, e_p), \{\Delta(2n, 2\lambda)\sigma\}\\ \{\Delta(n, \lambda\pm\mu\pm\nu), \sigma\}, (b_q, f_q)\end{matrix}\right.\right].$$

Proof :—Write the value of H–function in contour integration by (2.1), change the order of integration, use (2.3), (2.5) and interpret with the help of (2.1), we get the R.H.S.

Particular cases

(i) When $e_p=f_q=\sigma=1$ we, get a result in Meijer G–function due to Verma (5)

(ii) When $e_p=f_q=\sigma=1$, $p=p+1$, $u=1$, $q=l$ we get a result in Mac-Robert E–function due to Rathie (6)

Integral III

If $R(\gamma+2\lambda n)>0$, $R(\alpha-\gamma+\sigma)>2n\,R(\lambda-\mu)$, $R(\beta-\gamma+\sigma)>2n\,R(\lambda-\mu)$, and at least in one of the following cases (n is *a* positive integer).

(i) $\omega_1>0$, $|\arg Z|<\frac{1}{2}\omega_1\pi$

(ii) $\omega_1\geqslant0$, $|\arg Z|\leqslant\frac{1}{2}\omega_1\pi$ and $R(\omega_2+1)<0$.

where ω_1, ω_2 are given by (1.1), we have

$$\int_0^\infty t^{\gamma-1}(1+t)^{-\sigma}{}_2F_1(\alpha,\beta;\gamma;-t)\; H^{l,u}_{p,q}\left[Z\,t^{2\lambda n}(1+t)^{-2\mu n}\left|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right.\right]dt.$$

$$=(2\pi)^{\frac{1}{2}-n}(2n)^{-\frac{1}{2}}\,H^{l,\,u+6n}_{p+6n,\,q+4n}\left[Z\left|\begin{matrix}\{\Delta(2n,1-\gamma),\lambda\},\{\Delta(2n,1-\alpha+\gamma-\sigma),\mu-\lambda\},\\(b_q,f_q),\{\Delta(2n,1-\sigma),\mu\},\end{matrix}\right.\right.$$
$$\left.\begin{matrix}\{\Delta(2n,1-\beta+\gamma-\sigma),\mu-\lambda\},(a_p,e_p)\\\{\Delta(2n,1-\alpha-\beta+\gamma-\sigma),\mu-\lambda\}\end{matrix}\right]$$

Proof :—Write the value of H–function in contour integration by (2.1), change the order of integration, use (2.4), (2.5) and interpret with the help of (2.1) we get the R.H.S.

Particular cases

(i) When $e_p=f_q=\lambda=\mu=1$ we get a result in Meijer G–function

(ii) When $e_p=f_q=\lambda=\mu=1$, $p=p+1$, $u=1$, $q=l$ we get a corresponding result in Mac-Robert E–function.

Integral IV

If $R(\gamma - 2n\lambda) > 0$ n is a positive integer and in at least one of the following cases :

(i) $\omega_1 > 0$, $|arg\ Z| < \frac{1}{2}\omega_1\pi$

(ii) $\omega_1 \geqslant 0$, $|org\ Z| \leqslant \frac{1}{2}\omega_1\pi$ and $R(\omega_2+1) < 0$

where ω_1 and ω_2 are given by (1.1) then we have

$$\int_0^{\pi} (\sin\phi)^{\gamma-1} e^{-\delta\phi} H^{l,u}_{p,q}\left[(\sin\phi)^{-2n\lambda} Z \middle| \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] d\phi.$$

$$= e^{-\frac{\pi\delta}{2}} \left(\frac{\pi}{n}\right)^{\frac{1}{2}} H^{l+2n.\ u}_{p+2n.\ q+2n}$$

$$\left[Z \middle| \begin{matrix}(a_p, e_p), \left\{\Delta(n, \frac{\gamma+i\delta+1}{2}, \lambda\right\}, \left\{\Delta(n, \frac{\gamma-i\delta+1}{2}), \lambda\right\}\\ \{\Delta(2n, \gamma), \lambda\}, (b_q, f_q)\end{matrix}\right]$$

Proof :—Write the value of H–function in contour integration by (2.1), change the order of integration, use (2.5), (2.7) and interpret with the help of (2.1), we get the R.H.S.

Particular cases :

(i) when $e_p = f_q = \lambda = 1$ we get a result in Meijer G–function.

(ii) When $e_p = f_q = \lambda = 1$, $p = p+1$, $u = 1$, $q = l$. We get a corresponding result in Mac-Robert E-function by Ragab (7).

Integral V.

If $R(\gamma + 2\lambda n) > 0$, n is a positive integer, and in at least one of the following cases :

(i) $\omega_1 > 0$, $|\arg Z| < \frac{1}{2}\omega_1\pi$

(ii) $\omega_1 \geqslant 0$, $|\arg Z| \leqslant \frac{1}{2}\omega_1\pi$ and $R(\omega_2+1) < 0$.

where ω_1 and ω_2 are given by (1.1) then we have

$$\int_0^{\pi} (\sin\phi)^{\gamma-1} e^{-\delta\phi} H^{l,u}_{p,q}\left[(\mathrm{Sin}\ \phi)^{2n\lambda} Z \middle| \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] d\phi.$$

$$= \left(\frac{\pi}{n}\right)^{\frac{1}{2}} e^{-\frac{\pi\delta}{2}} H^{l,\ u+2n}_{p+2n,\ q+2n}\left[Z \middle| \begin{matrix}\{\Delta(2n, 1-\gamma), \lambda\}, (a_p, e_p)\\ (b_q, f_q), \ \{\Delta(n, \frac{1-\gamma\pm i\delta}{2}), \lambda\}\end{matrix}\right]$$

Proof :—Write the value of H–function in contour integration by (2.1), change the order of integration, use (2.5), (2.6) and interpret with the help of (2.1) we get the R.H.S.

Particulars cases :-

(i) When $e_p = f_q = \lambda = 1$, we get a result in Meiger G-function.

(ii) When $e_p = f_q = \lambda = 1$, $u = 1$, $p = p+1$, $q = l$ we get a result in Mac—Robert E–function which by use of reduces to a result due to Ragab (7).

4. Generalisation of integrals (1) & (2).

Consider a function $(m_1 + n_2 \leqslant n_1 + m_2)$

$$\phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\alpha n_1 + s)\, \Gamma(\beta n_2 - s)}{\Gamma(\delta m_1 + s)\, \Gamma(\eta m_2 - s)}\, t^{-s}\, ds. \qquad \ldots\ldots (4\cdot1)$$

which is in fact, the sums of n_1 generalised hypergeometric series of the type $_{m_1+n_2}F_{m_2+n_1-1}$, then by Mellin inversion formula

$$\frac{\Gamma(\alpha n_1 + s)\, \Gamma(\beta n_2 - s)}{\Gamma(\delta m_1 + s)\, \Gamma(\eta m_2 - s)} = \int_0^\infty \phi(t)\, t^{s-1}\, dt \ldots\ldots\ldots\ldots (4.2).$$

Therefore the generalised integral to be evaluated is

$$I = \int_0^\infty t^{2\gamma - 1}\, \phi(t)\, H_{p,q}^{l,u}\left[Z\, t^{-2n\sigma} \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dt.$$

Write the value of the H–function by (2.1), change the order of integration, simplify, use (2.5) and interpret with the help of (2.1) to obtain

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{l} \Gamma(b_j - f_j s) \prod_{j=1}^{u} \Gamma(1 - a_j + e_j s)}{\prod_{j=l+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=u+1}^{p} \Gamma(a_j - e_j s)} \times$$

$$\frac{\Gamma(\alpha n_1 + 2\gamma - 2n\sigma s)\, \Gamma(\beta n_2 - 2\gamma + 2n\sigma s)}{\Gamma(\delta m_1 + 2\gamma - 2n\sigma s)\, \Gamma(\eta m_2 - 2\gamma + 2n\sigma s)}\, z^s\, ds.$$

$$= (2\pi)^{(\frac{1}{2}-n)(n_1+n_2-m_1-m_2)} (2n)^{\sum_{r=1}^{n_1}(\alpha_r)+\sum_{r=1}^{n_2}(\beta_r)-\sum_{r=1}^{m_1}(\delta_r)-\sum_{r=1}^{m_2}(\eta_r)+2\gamma(n_1-n_2+m_1-m_2)} \times (2n)^{-\frac{1}{2}(n_1+n_2-m_1-m_2)}$$

$$H^{l+2nn_1,\, u+2nn_2}_{2nn_2+p+2nm_1,\, 2nn_1+q+2nm_2}\left[(2n)^{2n\sigma(n_2-n_1+m_1-m_2)} z \,\middle|\, \begin{matrix}\{\triangle(2n, 1-\beta n_2+2\gamma),\sigma\}, (a_p, e_p), \{\triangle(2n, \delta m_1+2\gamma),\sigma\} \\ \{\triangle(2n, \alpha n_1+2\gamma),\sigma\}, (b_q, f_q). \{\triangle(2n, 1-n_{m_2}+2\gamma),\sigma\}\end{matrix}\right].$$

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ON SELF RECIPROCAL FUNCTION

by

M. A. PATHAN

(Recd. on 15–7–67)

Fox [1., p. 408] has shown that a function H(x) defined by

$$(x) \equiv H^{q,\ p}_{2p,\ 2q}\left[x \begin{array}{l} \{1-a_p,\ e_p\},\{a_p-e_p,\ e_p\} \\ \{b_q,\ c_q\},\{1-b_q-c_q,\ c_q\} \end{array}\right]$$

$$= \frac{1}{2\lambda i}\int_L \frac{\prod_{m=1}^{q}\Gamma(b_m+c_m s)\prod_{n=1}^{p}\Gamma(a_n-e_n s)}{\prod_{m=1}^{q}\Gamma(b_m+c_m-c_m s)\prod_{n=1}^{p}\Gamma(a_n-e_n+e_n s)}\, x^{-s}\, ds$$

is a symmetrical Fourier kernel, then it can be shown easily that

$$K(x) \equiv n\left(\frac{m}{n}\right)^{\frac{n}{2}} x^{\frac{n-1}{2}} H^{q\ ,\ p}_{2p,2q}\left[\left(\frac{m}{n}x\right)^{n} \begin{array}{l} \{1-a_p,e_p\},\{a_p-e_p,e_p\} \\ \{b_q,c_q\},\{1-b_q-c_q,c_q\} \end{array}\right] \tag{1}$$

is also a symmetric Fourier kernel.

If $f(x)$ satisfies the equation

$$f(x) = \int_0^\infty K(xy)\, f(y)\, dy. \quad \ldots \quad \ldots \tag{2}$$

then $f(x)$ is said to be self reciprocal for the kernel $K(x)$ and shall be denoted by

$$R^m_n\left[\begin{array}{l} \{1-a_p,\ e_p\}\ ,\ \{a_p-e_p,\ e_p\} \\ \{b_q,\ c_q\}\ ,\ \{1-b_q-c_q,\ c_q\} \end{array}\right]$$

Since the kernel $K(x)$ given by (1) is a kernel of very general form, it can be reduced to another kernels as follows :—

(i) If we take e's and c's equal to unity, we get a new kernel

$$K_1(x) \equiv n\left(\frac{m}{n}\right)^{\frac{n}{2}} x^{\frac{n-1}{2}}$$

$$G^{q,\,p}_{2p,2q}\left[\left(\frac{m}{n}x\right)^n \middle| \begin{matrix} 1-a_1,\dots 1-a_p,\ a_1-1,\dots a_p-1 \\ b_1,\dots b_q,\ -b_1,\dots -b_q \end{matrix}\right] \quad \dots(3)$$

A function which is a self reciprocal for this kernel $K_1(x)$ shall be denoted by

$$S^m_n\left[\begin{matrix} (1-a_p),\ (a_p-1) \\ (b_q),\ (-b_q) \end{matrix}\right]$$

(ii) On replacing a_p, e_p, c_q by $1-a_p$, $\frac{1}{2}e_p$, $\frac{1}{2}c_q$ and taking $m=\frac{1}{2}$, $n=1$, e's and c's equal to unity in (1), we get a kernel

$$K_2(x) \equiv \sqrt{2}\ G^{q,\,p}_{2p,2q}\left[\frac{x^2}{4} \middle| \begin{matrix} a_1,\dots a_p,\ \frac{1}{2}-a_1,\dots \frac{1}{2}-a_p \\ b_1,\dots b_q,\ \frac{1}{2}-b_1,\dots \frac{1}{2}-b_q \end{matrix}\right] \quad \dots(4)$$

given by Roop Narain [4., p. 298]

(iii) Again on taking $n=2$, $m=1$, $q=2$, $p=1$ all e's, c's equal to unity and replacing a_1, b_1 and b_2 by $\frac{3}{2}+\frac{v}{2}+m-k$, $\frac{v}{2}$ and $\frac{v}{2}+2m$, we get

$$K_3(x) \equiv x^{\frac{1}{2}+v}\, 2^{-v}\, X_{v,k,m}\left(\frac{x^2}{4}\right) = x^{\frac{1}{2}}$$

$$G^{2,\,1}_{2,\,4}\left[\frac{x^2}{4} \middle| \begin{matrix} k-m-\frac{v}{2}-\frac{1}{2},\ \frac{v}{2}+m-k+\frac{1}{2} \\ \frac{v}{2},\ \frac{v}{2}+2m,\ -\frac{v}{2},\ -\frac{v}{2}-2m \end{matrix}\right]$$

with this kernel integral equation (2) is transformed to

$$f(x) = \frac{1}{2^v}\int_0^\infty (xy)^{v+\frac{1}{2}}\, X_{v,k,m}\left(\frac{x^2y^2}{4}\right) f(y)\, dy. \quad \dots(5)$$

Transform (5) is the generalisation of Hankel transform given by Roop Narain [3., p. 270] and a function self reciprocal for this kernel $K_3(x)$ shall be denoted by $R_v(k,m)$.

The object of this paper is to obtain certain theorems which are the generalisation of the theorem recently obtained by S. P. Singh and Roop Narain. These theorems help us in finding out certain new kernels. Later on a number of kernels have been deduced from these as particular cases.

We shall need the following lemma obtained by the author [2] which we shall require in proving the theorems.

Leema :

If

(i) $\int_0^x f(x)\,dx = \int_0^\infty f(y)\,\frac{k'(xy)}{y}\,dy,\quad k'(x)=\int_0^x k(x)\,dx.$

(ii) $C_j>0,\ j=1, \ldots q,\ e_j>0,\ j=1, \ldots p$ and

$$D=2\left(\sum_{j=1}^{q} C_j - \sum_{j=1}^{p} e_j\right)>0$$

(iii) $Re\,(b_j) > -\tfrac{1}{2} c_j,\ j=1,\ldots q$

$Re\,(a_j) > \tfrac{1}{2} e_j,\ j=1,\ldots p.$

(iv) $E\,(\tfrac{1}{2}-s)$ is an even function of s.

and

(v) $Q(s)\,E(s) \in L_2\,(\tfrac{1}{2}-i\infty,\ \tfrac{1}{2}+i\infty)$

then

$$f(x)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Q(s).\ E(s)\,x^{-s}\,ds. \qquad \ldots (6)$$

where

$$Q(s)=\frac{\left(\frac{m}{n}\right)^{-\frac{s}{2}} \prod_{j=1}^{q} \Gamma\left[\left(b_j+c_j\left(\frac{s}{n}+\frac{1}{2}-\frac{1}{2n}\right)\right]\right.}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{s}{n}+\frac{1}{2}-\frac{1}{2n}\right)\right]}$$

Theorem I :

If $f(x)$ is $R_n^m \left[\begin{matrix} \{1-a_p, e_p\}, \{a_p - e_p, e_p\} \\ \{b_q, c_q\}, \{1-b_q - c_q, c_q\} \end{matrix} \right]$ and

$$P(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\prod_{j=1}^{q} \Gamma\left[d_j + \alpha_j \left(\frac{1}{2} + \frac{1}{2n} - \frac{s}{n} \right) \right]}{\prod_{j=1}^{p} \Gamma\left[h_j - k_j + k_j \left(\frac{1}{2} + \frac{1}{2n} - \frac{s}{n} \right) \right]} \times$$

$$\frac{\prod_{j=1}^{q} \Gamma\left[b_j + c_j \left(\frac{1}{2} - \frac{1}{2n} + \frac{s}{n} \right) \right]}{\prod_{j=1}^{q} \Gamma\left[a_j - e_j + e_j \left(\frac{1}{2} - \frac{1}{2n} + \frac{s}{n} \right) \right]} \times$$

$$\omega(s)\, e^{sx}\, dx, \quad x>0$$

$$= o \qquad , \qquad x<o \qquad \ldots (1.1)$$

and $\omega(s)$ satisfies the functional equation $\omega(s)=\omega(1-s)$,

then

$$g(x) = \frac{1}{x} \int_o^x P\left[log\left(\frac{x}{y}\right) \right] f(y)\, dy \qquad \ldots (1.2)$$

is $R_n^m \left[\begin{matrix} \{1-h_p, k_p\}, \{h_p - k_p, k_p\} \\ \{d_q, l_q\}, \{1-d_q - l_q, l_q\} \end{matrix} \right]$

Proof :

Since $f(x)$ is $R_n^m \left[\begin{matrix} \{1-a_p, e_p\}, \{a_p - e_p, e_p\} \\ \{b_q, c_q,\}, \{1-b_q - c_q, c_q\} \end{matrix} \right]$

we have by above lemma

$$f(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}$$

$$\frac{\left(\frac{m}{n}\right)^{-\frac{s}{2}} \prod_{j=1}^{q} \Gamma\left[b_j + c_j \left(\frac{s}{n} + \frac{1}{2} - \frac{1}{2n} \right) \right]}{\prod_{i=1}^{p} \Gamma\left[a_j - e_j + e_j \left(\frac{s}{n} + \frac{1}{2n} - \frac{1}{2n} \right) \right]} E(s)\, x^{-s}\, ds$$

and

$$g(x) = \frac{1}{x} \int_o^x P\left[\log\left(\frac{x}{y}\right) \right] dy \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \left(\frac{m}{n}\right)^{-\frac{s}{2}} \times$$

$$\frac{\prod_{i=1}^{q} \Gamma\left[b_j+c_j\left(\frac{s}{n}+\tfrac{1}{2}-\frac{1}{2n}\right)\right]}{\prod_{i=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{s}{n}+\frac{1}{2}-\frac{1}{2n}\right)\right]} E(s).\ y^{-s}\,ds.$$

On changing order of integration, we get

$$g(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \left(\frac{m}{n}\right)^{-\frac{s}{2}} \frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}$$

$$E(s).\ x^{-s}\,ds\int_0^{\infty} e^{(s-1)u}\,P(u)\,du.$$

Now using inversion formula for Laplace transform, we get from (1.1)

$$\int_0^{\infty} e^{-su}\,P(u)\,du = \frac{\prod_{j=1}^{q} \Gamma\left[d_j+\alpha_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}$$

$$\frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}\omega(s).$$

On writing $1-s$ for s

$$\int_0^{\infty} e^{(s-1)u}\,P(u)\,du = \frac{\prod_{j=1}^{q} \Gamma\left[d_j+\alpha_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}$$

$$\frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}\omega(s).$$

since $\omega(1-s)=\omega(s)$.

Therefore,

$$g(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} \frac{\prod_{j=1}^{q} \Gamma\left[d_j+\alpha_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} \times$$

$$\left(\frac{m}{n}\right)^{-\frac{s}{2}} \times \frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]} \omega(s)\, E(s)\, x^{-s}\, ds$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\left(\frac{m}{n}\right)^{-\frac{s}{2}} \prod_{j=1}^{q} \Gamma\left[d_j+\alpha_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} \phi(s)\ x^{-s}\, ds.$$

Where

$$\phi(s) = \frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} \frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]} \omega(s)\ E(s)$$

and $\phi(s)$ satisfies the relation $\phi(s)=\phi(1-s)$, then from the lemma $g(x)$ is

$$R_n^m\left[\begin{matrix}\{1-h_p, k_p\}, \{h_p-k_p, k_p\}\\ \{d_q, \alpha_q\}, \{1-d_q-\alpha_q, \alpha_q\}\end{matrix}\right]$$

COROLLARY I

If we take e'^s, c'^s, α'^s and k'^s equal to unity in the above theorem we get a corollary

If $f(x)$ is $S_n^m\left[\begin{matrix}(1-a_p), (a_p-1)\\ (b_q), (-b_q)\end{matrix}\right]$ and

$$P(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\prod_{j=1}^{q}\Gamma\left[d_j+\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right]\prod_{j=1}^{q}\Gamma\left[b_j+\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right]}{\prod_{j=1}^{p}\Gamma\left[h_j+\frac{1}{2n}-\frac{1}{2}-\frac{s}{n}\right]\prod_{j=1}^{p}\Gamma\left[a_j-\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right]} \times \omega(s)\, e^{sx}\, ds\ ,\ x>0.$$

$$= 0\ ,\ x<0$$

and $\omega(s)$ satisfies the functional equation $\omega(s)=\omega(1-s)$ then

$$g(x) = \frac{1}{x}\int_0^x P\left[\log\left(\frac{x}{y}\right)\right] f(y)\, dy$$

is

$$S_n^m\left[\begin{matrix}(1-h_p), (h_p-1)\\ (d_q), (-d_q)\end{matrix}\right]$$

COROLLARY II

On taking $m=1$, $n=2$, $q=2$, $p=1$ and replacing a_1, b_1, b_2, h_1, d_1 and d_2 by $\frac{3}{2}+\frac{v}{2}+m-k$, $\frac{v}{2}$, $\frac{v}{2}+2m$, $\frac{3}{2}+\frac{\mu}{2}+n-l$, $\frac{\mu}{2}$ and $\frac{\mu}{2}+2n$ in the above corollary, we get the theorem given by S. P. Singh [5., p. 338].

THEOREM II

If $f(x)$ is $R_n^m\left[\begin{matrix}\{1-a_p, e_p\}, \{a_p-e_p, e_p\}\\ \{b_q, c_q\}, \{1-b_q-c_q, c_q\}\end{matrix}\right]$ and

$$g(x) = \int_0^\infty P(xy)\, f(y)\, dy \qquad \cdots \qquad \cdots \qquad (2.1)$$

is $R_n^m\left[\begin{matrix}\{1-h_p, k_p\}, \{h_p-k_p, k_p\}\\ \{d_q, \alpha_q\}, \{1-d_q-\alpha_q, \alpha_q\}\end{matrix}\right]$

then

$$P(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\prod_{j=1}^{q} \Gamma\left[d_j+\alpha_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} \frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}\left(\frac{m}{n}\right)^{-s} \omega(s).\, x^{-s}\, ds, \quad x>0$$

$$= 0 \quad , \quad x<0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.2)$$

where $\omega(s) = \omega(1-s)$. ... ... (2.3)

PROOF—

Multiplying (2.1) by x^{s-1} and integrating between the limits zero and infinity, we get

$$\int_0^\infty g(x).\, x^{s-1}\, dx = \int_0^\infty x^{s-1}\, dx \int_0^\infty f(y).\, P(xy)\, dy.$$

Changing the order of integration

$$\int_0^\infty x^{s-1}\, g(x)\, dx = \int_0^\infty f(y)\, dy \int_0^\infty x^{s-1}\, P(xy)\, dx.$$

$$= \int_0^\infty y^{-s} f(y)\, dy \int_0^\infty u^{s-1}\, P(u)\, du \quad \ldots (2.4)$$

Now since $f(x)$ is $\mathcal{R}_n^m\left[\begin{matrix}\{1-a_p,\ e_p\},\ \{a_p-e_p,\ e_p\}\\ \{b_q,\ c_q\},\ \{1-b_q-c_q,\ c_q\}\end{matrix}\right]$

then from the lemma, we have

$$f(x)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\left(\frac{m}{n}\right)^{-\frac{s}{2}} \prod_{i=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} E(s).\, x^{-s}\, ds.$$

where $E(s)$ satisfies $E(s) = E(1-s)$.

Applying Mellin inversion formula, we have

$$\int_0^\infty x^{s-1} f(x)\,dx = \frac{\left(\frac{m}{n}\right)^{-\frac{s}{2}} \prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} E(s).$$

Writing $1-s$ for s, we obtain

$$\int_0^\infty x^{-s} f(x)\,dx = \frac{\left(\frac{m}{n}\right)^{\frac{s}{2}-\frac{1}{2}} \prod_{j=1}^{q} \Gamma\ b_j+c_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)}{\prod_{j=1}^{p} \Gamma\ a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}-\frac{s}{n}\right)} E(s)$$

Similarly, if $g(x)$ is $R_n^m\left[\begin{matrix}\{1-h_p, k_p\}, \{h_p-k_p, k_p\}\\ \{d_q, \alpha_q\}, \{1-d_q-\alpha_q, \alpha_q\}\end{matrix}\right]$

we get with the help of lemma

$$g(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\left(\frac{m}{n}\right)^{-\frac{s}{2}} \prod_{j=1}^{q} \Gamma\left[b_j+\alpha_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p} \Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} \phi(s)\, x^{-s}\, ds.$$

where $\phi(s)$ satisfiies $\phi(s) = \phi(1-s)$

Therefore form (2.4), we get

$$\int_o^\infty u^{s-1} P(u)du = \frac{\left(\frac{m}{n}\right)^{-s} \prod_{j=1}^{q} \Gamma\left[d_j+\alpha_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}$$

$$\frac{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{q} \Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]} \omega(s)$$

where

$$\omega(s) = \frac{\left(\frac{m}{n}\right)^{\frac{1}{2}} \prod_{j=1}^{p} \Gamma\left[a_j - e_j + e_j\left(\frac{1}{2} - \frac{1}{2n} + \frac{s}{n}\right)\right]}{\prod_{j=1}^{q} \Gamma\left[b_j + c_j\left(\frac{1}{2} - \frac{1}{2n} + \frac{s}{n}\right)\right]} \cdot \frac{\prod_{j=1}^{p} \Gamma\left[a_j - e_j + e_j\left(\frac{1}{2} + \frac{1}{2n} - \frac{s}{n}\right)\right]}{\prod_{j=1}^{q} \Gamma\left[b_j + c_j\left(\frac{1}{2} + \frac{1}{2n} - \frac{s}{n}\right)\right]} \cdot \frac{\phi(s)}{E(s)}$$

and $\omega(s)$ satisfies $\omega(s)=\omega(1-s)$.

Then by Mellin inversion formula, we get (2 2).

COROLLARY I

On taking e's, c's, k's, α's equal to unity in the above theorem, we get

If $f(x)$ is $S_n^m\left[\begin{matrix}(1-a_p), (a_p-1)\\(b_q), (-b_q)\end{matrix}\right]$ and

$$g(x) = \int_0^\infty P(xy)\, f(y)dy$$

is $S_n^m\left[\begin{matrix}(1-h_p), (h_p-1)\\(d_q), (-d_q)\end{matrix}\right]$

then

$$P(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\left(\frac{m}{n}\right)^{-s} \prod_{j=1}^{q} \Gamma\left(d_j + \frac{1}{2} - \frac{1}{2n} + \frac{s}{n}\right)}{\prod_{j=1}^{p} \Gamma\left(h_j - \frac{1}{2} - \frac{1}{2n} + \frac{s}{n}\right)} \cdot \frac{\prod_{j=1}^{q} \Gamma\left(b_j + \frac{1}{2} - \frac{1}{2n} + \frac{s}{n}\right)}{\prod_{j=1}^{p} \Gamma\left(a_j - \frac{1}{2} - \frac{1}{2n} + \frac{s}{n}\right)} \omega(s)\, x^{-s}\, ds,\ x>0.$$

$= 0,\ x < 0$

Where $\omega(s)$ satisfies $\omega(s) = \omega(1-s)$.

COROLLARY II

In above corollary, if we take $m=p=1$, $n=q=2$ and replace a, b_1, b_2, h, d_1 and d_2 by $\frac{3}{2}+\frac{\nu}{2}+m-k, \frac{\nu}{2}, \frac{\nu}{2}+2m, \frac{3}{2}+\frac{\nu}{2}+n-l, \frac{\mu}{2}$ and $\frac{\mu}{2}+2n$, we get the theorem by S. P. Singh [4., p. 342]

Converse of this corollary was given by Roop Narain [3., p. 287].

THEOREM III

If $f(x)$ is $\mathbb{R}_n^m\begin{bmatrix}\{1-a_p, e_p\}, \{a_p-e_p, e_p\}\\ \{b_p, c_q\}, \{1-b_q-c_q, c_q\}\end{bmatrix}$ and

$$g(x)=\frac{1}{x}\int_0^\infty f(y)\, K(y/x)\, dy \qquad \ldots \qquad \ldots \qquad (3.1)$$

is

$$\mathbb{R}_n^m\begin{bmatrix}\{1-h_p, k_p\}, \{h_p-k_p, k_p\}\\ \{d_q, \alpha_q\}, \{1-d_q-\alpha_q, \alpha_q\}\end{bmatrix}$$

then

$$P(x)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\prod_{j=1}^{q}\Gamma\left[d_j+\alpha_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}{\prod_{j=1}^{p}\Gamma\left[h_j-k_j+k_j\left(\frac{1}{2}+\frac{1}{2n}-\frac{s}{n}\right)\right]}\,\frac{\prod_{j=1}^{q}\Gamma\left[b_j+c_j\left(\frac{1}{2}-\frac{1}{2n}+\frac{s}{n}\right)\right]}{\prod_{j=1}^{p}\Gamma\left[a_j-e_j+e_j\left(\frac{1}{2}-\frac{1}{2n}-\frac{s}{n}\right)\right]}\,\omega(s)$$

$$\times x^{-s}\, ds,\ x>0$$

$$=0,\ x<0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3.2)$$

where $\omega(s)=\omega(1-s)$

The proof follows from the above theorem II.

COROLLARY I

With the similar substitutions as in corollary II of theorem II, we get result due to S. P. Singh [5., p. 344] and converse of this corollary was given by Roop Narain [3., p. 289].

EXAMPLE

If we take

$$\omega(s)=\frac{\prod_{j=1}^{l}\Gamma\left(1-y_j-z_j+z_js\right)\prod_{j=1}^{l}\Gamma\left(1-y_j-z_js\right)}{\prod_{j=l+1}^{r}\Gamma\left(y_j+z_j-z_js\right)\prod_{j=l+1}^{r}\Gamma\left(y_j+z_js\right)}$$

which satisfies $\omega(s)=\omega(1-s)$, then with the help of the theorem II, $P(x)$ is given by

$$H_{r+2p,\,r+2q}^{l+2q,\,l}\left[\frac{m}{n}x\left|\begin{array}{l}\{y_r,z_r\},\left\{h_p-k_p+k_p\left(\frac{1}{2}-\frac{1}{2n}\right),\frac{k_p}{n}\right\},\\ \left\{d_q,\alpha_q\left(\frac{1}{2}-\frac{1}{2n}\right),\frac{\alpha_q}{n}\right\},\left\{b_q+c_q\left(\frac{1}{2}-\frac{1}{2n}\right)\frac{e_q}{n}\right\},\end{array}\right.\begin{array}{l}\left\{a_p-e_p+e_p\left(\frac{1}{2}-\frac{1}{2n}\right),\frac{e_p}{n}\right\}\\ \left\{1-y_r-z_r,z_r\right\}\end{array}\right] \quad \ldots (4.1)$$

(4.1) is a kernel which transforms

$$R_n^m\left[\begin{array}{l}\{1-a_p,e_p\},\ \{a_p-e_p,e_p\}\\ \{b_q,c_q\},\{1-b_q-c_q,c_q\}\end{array}\right]$$

in to

$$R_n^m\left[\begin{array}{l}\{1-h_p,k_p\},\{h_p-k_p,k_p\}\\ \{d_q,\alpha_q\},\{1-d_q-\alpha_q,\alpha_q\}\end{array}\right]$$

and vice versa in accordance with the equation (2.1).

Particular cases : Giving suitable values to the parameters in (4.1) we can deduce a number of kernels.

(i)

$$G_{\gamma+2,\,\gamma+4}^{l+4,\,l}\left[\frac{x^2}{4}\left|\begin{array}{l}(y_1,\ldots y_\gamma),\frac{\mu}{2}+n-l+\frac{3}{4},\\ \frac{\mu}{2}+\frac{1}{4},\frac{\mu}{2}+2n+\frac{1}{4},\frac{\nu}{2}+\frac{1}{4},\end{array}\right.\begin{array}{l}\frac{\nu}{2}+m-k+\frac{3}{4}\\ \frac{\nu}{2}+2m+\frac{1}{4},(-y_1,\ldots-y_\gamma)\end{array}\right] \quad \ldots (4.2)$$

which changes $R_{\nu}(k, m)$ in to $R_{\mu}(l, n)$ in accordance with the equations (2.1)

(ii)

$$2\pi^{-\frac{1}{2}}\left(\frac{m}{n}x\right)^{a} K_{b+c}\left[\left(\frac{m}{n}x\right)^{\frac{1}{2}}\right] K_{b-c}\left[\left(\frac{m}{n}x\right)^{\frac{1}{2}}\right] \quad (4.3)$$

is a kernel transforming

$$S_{n}^{m}\left[\begin{matrix}-a-\frac{1}{2n},\ a+\frac{1}{2n} \\ a-c-\frac{1}{2}+\frac{1}{2n},\ a-b-\frac{1}{2}+\frac{1}{2n},\ -a+c+\frac{1}{2}-\frac{1}{2n},\ -a+b+\frac{1}{2}-\frac{1}{2n}\end{matrix}\right]$$

in to

$$S_{n}^{m}\left[\begin{matrix}\frac{1}{2}-a-\frac{1}{2n},\ a+\frac{1}{2n}-\frac{1}{2} \\ a+b-\frac{1}{2}+\frac{1}{2n},\ a+c-\frac{1}{2}+\frac{1}{2n},\ -a-b+\frac{1}{2}-\frac{1}{2n},\ -a-c+\frac{1}{2}-\frac{1}{2n}\end{matrix}\right]$$

(iii)

$$\pi^{\frac{1}{2}}\left(\frac{m}{n}x\right)^{-\frac{1}{2}} W_{a,b}\left[2\left(\frac{m}{n}x\right)^{\frac{1}{2}}\right] W_{-a,b}\left[2\left(\frac{m}{n}x\right)^{\frac{1}{2}}\right] \quad ..(4.4)$$

is a kernel transforming

$$S_{n}^{m}\left[\begin{matrix}a-\frac{1}{2n},\ \frac{1}{2n}-a \\ b-\frac{1}{2}+\frac{1}{2n},\ -b-\frac{1}{2}+\frac{1}{2n},\ \frac{1}{2}-b-\frac{1}{2n},\ b+\frac{1}{2}-\frac{1}{2n}\end{matrix}\right]$$

in to

$$S_{n}^{m}\left[\begin{matrix}-a-\frac{1}{2n},\ a+\frac{1}{2n} \\ \frac{1}{2n}-\frac{1}{2},\ \frac{1}{2n},\ \frac{1}{2}-\frac{1}{2n},\ -\frac{1}{2n}\end{matrix}\right]$$

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SERIES OF MEIJER G–FUNCTION OF TWO VARIABLES (PART I)

By

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Abstract :—The sum of finite series involving Meijer's G-function of two variables defined by Agarwal (1) have been given in Meijer's G-function of two variables.

INTRODUCTION

Meijer's G-function of two variables defined by Agarwal (1) is

$$G^{n_1, n_2, n_3, m_1, m_2}_{p_1, q_1, p_2, q_2}\left[\begin{matrix} x \\ y \end{matrix} \middle| \left\{ \left(a_{p_1}\right); \left(d_{p_2}\right) \right\} : \left\{\left(b_{q_1}\right); \left(c_{q_1}\right)\right\} : \left\{ \left(e_{q_2}\right); \left(f_{q_2}\right) \right\} \right]$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \phi\,(s+t)\ \psi\,(s, t)\ x^s\, y^t\, ds\, dt \qquad (1.0)$$

where
$$\phi\,(s+t) = \frac{\prod_{j=1}^{n_1} \Gamma\left(1-a_j+s+t\right)}{\prod_{j=1+n_1}^{p_1} \Gamma\left(a_j-s-t\right) \prod_{j=1}^{p_2} \Gamma\left(d_j+s+t\right)} \qquad (1.1)$$

$$\psi\,(s, t) = \frac{\prod_{j=1}^{m_1} \Gamma\left(e_j-s\right) \prod_{j=1}^{n_2} \Gamma\left(b_j+s\right)}{\prod_{j=1+m_1}^{q_2} \Gamma\left(1-e_j+s\right) \prod_{j=1+n_2}^{q_1} \Gamma\left(1-b_j-s\right)}$$

$$\frac{\prod_{j=1}^{m_2} \Gamma\left(f_j-t\right) \prod_{j=1}^{n_3} \Gamma\left(c_j+t\right)}{\prod_{j=1+m_2}^{q_2} \Gamma\left(1-f_j+t\right) \prod_{j=1+n_3}^{q_1} \Gamma\left(1-c_j-t\right)} \qquad (1.2)$$

$$0 \leqslant m_1 \leqslant q_2,\ 0 \leqslant m_2 \leqslant q_2,\ 0 \leqslant n_2 \leqslant q_1,\ 0 \leqslant n_3 \leqslant q_1,\ 0 \leqslant n_1 \leqslant p_1 \tag{1.3}$$

$$(\alpha_p) = \alpha_1, \alpha_2, \ldots \alpha_p \text{ and } (\alpha_{m,p}) = \alpha_m, \alpha_{m+1}, \ldots \alpha_p \tag{1.4}$$

$$\left.\begin{array}{l} \omega < 2\bar{\omega},\ |\arg x| < \pi\,(\bar{\omega} - \tfrac{1}{2}\omega) \\ \omega < 2\bar{\omega}_1,\ |\arg y| < \pi\,(\omega_1 - \tfrac{1}{2}\omega) \end{array}\right\} \tag{1.5}$$

$$\omega = p_1 + q_1 + p_2 + q_2,\ \bar{\omega} = m_1 + n_2 + n_1,\ \bar{\omega}_1 = m_2 + n_3 + n_1 \tag{1.6}$$

Also the sequence of parameters (e_{m_1}), (f_{m_2}), (b_{n_2}), (c_{n_3}) and (a_{n_1}) are such that none of the poles of the integrand coincide. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(e_j - s)$, $j = 1,2,\ldots\ldots m_1$ and $\Gamma(f_k - t)$, $k = 1,2,\ldots\ldots m_2$ lie to the right and those of $\Gamma(b_j + s)$, $j = 1,2,\ldots\ldots n_2$, and $\Gamma(c_k + t)$, $k = 1;2,\ldots\ldots n_3$ and $\Gamma(1 - a_j + s + t)$, $j = 1,2,\ldots\ n_1$ lie to the left of the imaginary axis.

The following well known results will also be required in the proof of the sequel

$$2F_1(a, b; c; 1) = \frac{\Gamma(c)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)},\quad R(c-n-b) > 0 \tag{1.7}$$

$$\Gamma(1 - a_p + n + s)\left[\Gamma(1 - a_p + s)\right]^{-1}$$

$$= (-1)^n\ \Gamma(a_p - s)\left[\Gamma(a_p - s - n)\right]^{-1} \tag{1.8}$$

Also, if the parameters are given as in the l.h.s. of (1.0), we shall abbreviate it as $G(x,y)$ and in case any one of them is different, only the defferent one is mentioned e.g.

$$G^{n_1, n_2, n_3, m_1, m_2}_{p_1, q_1, p_2, q_2}\left[\begin{array}{c} x \\ y \end{array}\middle| \left\{(a_{p_1}); (d_{p_2})\right\} : \left\{(b_{q_1}); (c_{q_1})\right\} : \left\{e_1 + r, (e_{2,q_2}); f_1 + r_1, (f_{2,q_2})\right\}\right]$$

will be written as

$$G\left[\begin{array}{c} x \\ y \end{array}\middle| \left\{\quad\right\} : \left\{\quad\right\} : \left\{e_1 + r, (e_{2,q_2}); f_1 + r_1, (f_{2,q_2})\right\}\right]$$

here after.

Theorem I

If the following set of conditions are satisfied

(i) $\omega < 2\bar{\omega}$, $|\arg x| < \pi(\bar{\omega} - \frac{1}{2}\omega)$

(ii) $\omega < 2\bar{\omega}_1$, $|\arg y| < \pi(\bar{\omega}_1 - \frac{1}{2}\omega)$

(iii) $R(b_{q_1} + n) > 0$, $R(c_{q_1} + m) > 0$

(iv) m and n are positive integers

then

$$\sum_{r=0}^{n} \sum_{r_1=0}^{m} \frac{(-1)^{m+n+r+r_1}\, {}^{n}c_{r}\, {}^{m}c_{r_1}}{\Gamma(e_1 + b_{q_1} + r)\, \Gamma(f_1 + c_{q_1} + r_1)}$$

$$G\left[\begin{matrix} x \\ y \end{matrix}\middle\| \{\ \} : \{\ \} : \{ e_1 + r, (e_{2,q_2}); f_1 + r_1, (f_{2,q_2}) \}\right]$$

$$= \frac{\left[\Gamma(f_1 + c_{q_1} + m)\right]^{-1}}{\Gamma(e_1 + b_{q_1} + n)}\, G\left[\begin{matrix} x \\ y \end{matrix}\middle\| \{\ \} :\right.$$

$$\left. \{ (b_{q_1 - 1}), b_{q_1} + n ; (c_{q_1 - 1}), c_{q_1} + m \} : \{\ \} \right] \tag{1.9}$$

where ω, $\bar{\omega}_1$, $\bar{\omega}$ are given by (1.2)

Proof :—In the l.h.s. of (1.9), write down the value of G (x, y) in contour form by (1.0), change the order of integration and summation, which is justified under condition (iii) of the theorem, simplify, use (1.7) along with (1.8), interpret with the help of (1.0), the r.h.s. of (1.9) follows.

Theorem II :

If the following set of conditions are satisfied

(i) $\omega < 2\bar{\omega}$, $|\arg x| < \pi(\bar{\omega} - \frac{1}{2}\omega)$

(ii) $\omega < 2\bar{\omega}_1$, $|\arg y| < \pi(\omega_1 - \frac{1}{2}\omega)$

(iii) $R(b_{q_1} + e_1) < n + 1$, $R(c_{q_1} + f_1) < m + 1$

(iv) m and n are positive integers

then

$$\sum_{r=0}^{n} \sum_{r_1=0}^{m} (-1)^{m+n+r+r_1} \, {}^{n}c_{r} \, {}^{m}c_{r_1}$$

$$G\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\middle\| \left\{ \right\} ; \left\{ \left(b_{q_1-1} \right), b_{q_1} - r ; \left(c_{q_1-1} \right), c_{q_1} - r_1 \right\} : \left\{ e_1 + r, \left(e_{2,q_2} \right) ; f_1 + r_1, \left(f_{2,q_2} \right) \right\} \right]$$

$$= \frac{\Gamma(b_{q_1}+e_1)\,\Gamma(c_{q_1}+f_1)}{\Gamma(b_{q_1}+e_1+n)\,\Gamma(c_{q_1}+f_1+m)} \, G\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\middle\| \left\{ \right\} ; \left\{ \left(b_{q_1-1} \right), b_{q_1} - n ; \left(c_{q_1-1} \right), c_{q_1} - m \right\} : \left\{ \right\} \right] \tag{2.0}$$

where ω, $\bar{\omega}$, $\bar{\omega}_1$, are given by (1.2).

Proof :—Begin with (2.0) and follow a similar scheme as for the proof of theorem I.

Theorem III

If the following set of conditions are satisfied

(i) $\omega < 2\bar{\omega}$, $|\arg x| < \pi(\bar{\omega} - \frac{1}{2}\omega)$

(ii) $\omega < 2\omega_1$, $|\arg y| < \pi(\bar{\omega}_1 - \frac{1}{2}\omega)$

(iii) $R(b_{q_1} + e_{q_2}) < n+1$, $R(c_{q_1} + f_{q_2}) < m+1$

(iv) m and n are positive integers

then

$$\sum_{r=0}^{n} \sum_{r_1=0}^{m} {}^{n}c_{r} \cdot {}^{m}c_{r_1} \, G\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\middle\| \left\{ \right\} ; \left\{ \left(b_{q_1-1} \right), b_{q_1} - r ; \left(c_{q_1-1} \right), c_{q_1} - r_1 \right\} : \left\{ \left(e_{q_2-1} \right), e_{q_2} + r ; \left(f_{q_2-1} \right), f_{q_2} + r_1 \right\} \right]$$

$$= \frac{\Gamma(1-b_{q_1}-e_{q_2}+n)\,\Gamma(1-c_{q_1}-f_{q_2}+m)}{\Gamma(1-e_{q_2}-b_{q_1})\,\Gamma(1-f_{q_2}-c_{q_1})} \, G\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\middle\| \left\{ \right\} : \left\{ \left(b_{q_1-1} \right), b_{q_1} - n ; \left(c_{q_1-1} \right), c_{q_1} - m \right\} : \left\{ \right\} \right] \tag{2.1}$$

where ω, $\bar{\omega}$, $\bar{\omega}_1$ are given by (1.2).

Proof :—Begin with l.h.s. of (2.1) and follow a similar routine as for the proof of theorem I to arrive at the r.h.s. of (2.1).

Theorem IV :

If the following set of conditions are satisfied

(i) $\omega < 2\bar{\omega}$. $|\arg x| < \pi(\bar{\omega} - \frac{1}{2}\omega)$

(ii) $\omega < 2\bar{\omega}_1$ $|\arg y| < \pi(\omega_1 - \frac{1}{2}\omega)$

(iii) $R(e_{q_2}+n) > 0$, $R(f_{q_2}+m) > 0$

(iv) m and n are positive integers

then

$$\sum_{r=0}^{n}\sum_{r_1=0}^{m} \frac{(-1)^{m+n+r+r_1}\ {}^{n}c_{r}\ .\ {}^{m}c_{r_1}}{\Gamma(b_1+e_{q_2}+r)\,\Gamma(c_1+f_{q_2}+r_1)}$$

$$G\left[\begin{matrix}x\\y\end{matrix}\middle|\left\{\ \right\}:\left\{b_1+r,\left(b_{2,q_1}\right);c_1+r_1,\left(c_{2,q_1}\right)\right\}:\left\{\ \right\}\right]$$

$$=\left[\Gamma(b_1+e_{q_2}+n)\,\Gamma(c_1+f_{q_2}+m)\right]^{-1}$$

$$G\left[\begin{matrix}x\\y\end{matrix}\middle|\left\{\ \right\}:\left\{\ \right\}:\left\{\left(e_{q_2-1}\right),e_{q_2}+n;\left(f_{q_2-1}\right),f_{q_2}+m\right\}\right] \tag{2.2}$$

Proof :—Begin with the l.h.s of (2.2) and follow a process similar to the proof of theorem I to arrive at the r.h.s. of (2.2).

Particular cases :

We give here interesting particular cases for theorem I. Following the same scheme, similar particular cases for the remaining theorems can be easily jolted out.

(i) When $m=n=1$, we have the recurrence relation

$$\left(e_1+b_{q_1}\right)\left(f_1+c_{q_1}\right)G\left[\begin{matrix}x\\y\end{matrix}\right]+G\left[\begin{matrix}x\\y\end{matrix}\middle|\left\{\ \right\}:\left\{\ \right\}:\right.$$

$$\left.\left\{e_1+1,\left(e_{2,q_2}\right);f_1+1,\left(f_{2,q_2}\right)\right\}\right]$$

$$=G\left[\begin{matrix}x\\y\end{matrix}\middle|\left\{\ \right\}:\left\{\left(b_{q_1-1}\right),b_{q_1}+1;\left(c_{q_1-1}\right),c_{q_1}+1\right\}:\left\{\ \right\}\right]$$

$$+ \left(e_1+b_{q_1}\right) G\left[\begin{matrix}x\\y\end{matrix}\middle\| \left\{\ \right\}:\left\{\ \right\}:\left\{\left(e_{q_2}\right); f_1+1, \left(f_{2,q_2}\right)\right\}\right]$$

$$+ \left(f_1+c_{q_1}\right) G\left[\begin{matrix}x\\y\end{matrix}\middle\| \left\{\ \right\}:\left\{\ \right\}:\left\{e_1+1, \left(e_{2,q_2}\right); \left(f_{q_2}\right)\right\}\right] \tag{2.3}$$

(ii) $m=n=1$, $n_1=n_2=n_3=m_1=m_2=p_1=p_2=q_1=q_2 = 1$, then
$\theta\ (\theta+1)\ F_1\ (\lambda;\ \mu,\ v;\ \theta;\ -x,-y) - y\ (\theta+1)\ F_1\ (1+\lambda;\ \mu,\ v+1;\ 1+\theta;\ -x,-y) - x\ (\theta+1)\ F_1\ (1+\lambda;\ \mu+1,\ v;\ 1+\theta; -x,-y) - F_1\ (\lambda;\ \mu+1,\ v+1;\ \theta;\ -x,-y) + x\ y\ F_1\ (2+\lambda;\ \mu+1,\ v+1;\ 2+\theta;\ -x,+y)=0$ (2.4)

where $\lambda=1-a_1+e_1+f_1$; $\mu=b_1+e_1$; $v=c_1+f_1$ and $\theta=d_1+e_1+f_1$.

(iii) When $m=n=1$, $n_1=n_2=m_1=m_2=p_1=q_1=n_3=1$, $p_2=0$. $q_2=2$, then we have
$ab\ F_2(\lambda;\ \mu,\ v;\ a,\ b;\ -x,\ -y) - a\lambda y\ F_2(\lambda+1;\ \mu,\ v+1;\ a,\ b+1;\ -x,\ -y) - b\lambda x\ F_2(1+\lambda;\ \mu+1,\ v;\ 1+a,\ b;\ -x,\ -y) + \lambda(\lambda+1)\ xy\ F_2(2+\lambda;\ \mu+1,\ v+1;\ 1+a,\ 1+b;\ -x,\ -y) = ab\ F_2(\lambda;\ \mu+1,\ v+1;\ a,\ b;\ -x,\ -y)$ (2.5)

where $a=1+e_1-e_2$; $b=1+f_1-f_2$.

(iv) When $m=n=1$, $n_1=p_1=0$, $n_2=n_3=q_1=2$, $m_8=m_2=p_2=q_2=1$, then
$F_3(\mu,\ v,\ \psi,\ \xi;\ \lambda;\ -x,\ -y) - \mu(\lambda+1)\ x\ F_3(\mu+1,\ v,\ \psi+1,\ \xi;\ \lambda;\ -x,\ -y) - v(\lambda+1)\ y\ F_3(\mu,\ v+1,\ \psi,\ \xi+1;\ \lambda+1;\ -x,\ -y) + \mu v xy\ F_3(\mu+1,\ v+1,\ \psi+1,\ \xi+1;\ \lambda+2;\ -x,\ -y) = \lambda(\lambda+1)\ F_3(\mu,\ v,\ \psi+1,\ \xi+1;\ \lambda;\ -x\ -y)$

where $\psi=b_2+e_1$, $\xi=c_2+f_1$ (2.6)

(v) When $m=n=1$, $n_1=p_1=q_2=2$, $n_2=n_3=q_1=p_2=0$, $m_1=m_2=1$, then
$(e_1f_1-1)\ ab\ F_4\ (\lambda,\ \eta;\ a,\ b;\ \ x,\ -y) - e_1a\lambda\eta y\ F_4\ (1+\lambda,\ 1+\eta;\ a,\ b+1;\ -x,\ -y) + bf_1\lambda\eta x\ F_4\ (1+\lambda,\ 1+\eta;\ 1+a,\ b;\ -x,\ -y) + xy\lambda\eta\ (1+\lambda)(1+\eta)\times F_4(2+\lambda,\ 2+\eta;\ 1+a,\ 1+b;\ -x,\ -y) = 0$

where $\eta=1-a_2+e_1+f_1$. (2.7)

(vi) When $m=0=n_2=n_3=q_1=p_2$, $m_2=1$, $e_1=b_1$, $b_{q_1}=1-a_{p_1}$ and if we take the limit as y tends to zero, we get a result in Meijers G-funciion due to Bhise (2) by the property (Agarwal 1)

$$\underset{\substack{y\to 0\\(p_1\leqslant q_2)}}{dt}\; G^{n_1,0,0,m_1,1}_{p_1,0,0,q_2}\left(\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix}(a_{p_1})\\ \vdots\;\vdots \\ (e_{q_2}),(0)\end{matrix}\right) = G^{m_1,n_1}_{p_1,q_2}\left(x\,\middle|\,\begin{matrix}(a_{p_1})\\(e_{q_2})\end{matrix}\right)$$

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A GENERALIZED FUNCTION OF TWO VARIABLES—I

by

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This paper deals with a generalization of the Fox's H-function in two variables. The generalized function yields as special cases the generalized G-function defined by Agarwal *(1)*, S-function defined by Sharma *(5)*, A-function defined by Goyal *(4)* and the Kampe de Feriet's function of higher orders.

1. INTRODUCTION

C. Fox *(3)* has defined H-function as

(1) $$H_{p\,q}^{m\,n}\left[x\,\middle|\,\begin{matrix}(a_1, e_1),\ldots\ldots, (a_p, e_p)\\(b_1, f_1),\ldots\ldots, (b_q, f_q)\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{i=1}^{m}\Gamma(b_i-f_i\,s)\prod_{i=1}^{n}\Gamma(1-a_i+e_i\,s)}{\prod_{i=m+1}^{q}\Gamma(1-b_i+f_i\,s)\prod_{i=n+1}^{p}\Gamma(a_i-e_i\,s)}x^s\,ds,$$

where an empty product is interpreted as 1, $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$; e's and f's are all positive; L is a suitable Contour of Barne's type such that the poles of $\Gamma(b_i-f_i\,s)$, $i=1,2,3,\ldots\ldots,m$ lie on the R.H.S. of the Contour and those of $\Gamma(1-a_i+e_i\,s)$, $i=1, 2,\ldots\ldots n$ lie on the L. H. S. of the Contour. Also the parameters are so restricted that the integral on the right of (1) is convergent.

The aim of this paper is to define a generalized H-function of two variables which includes G-function of two variables, hypergeometric functions of higher orders etc.

The following notations are used throughout this paper :

(i) $(a)_m=\Gamma(a+m)/\Gamma(a)$

(ii) The symbol $\{a, e, e' : p\}$ denotes the sequence (a_1, e_1, e_1') (a_2, e_2, e_2'), ..., (a_p, e_p, e_p').

Also that $\{ a, 1, 1 : p \}$ is represented by $\{ a_p \}$ which denotes the sequence $a_1, a_2, \ldots\ldots a_p$.

(iii) The symbol $\triangle(a_1 : p)$ stands for $\triangle(a_1), \triangle(a_2), \ldots\ldots$, $\triangle(a_p)$ and the symbol $\triangle(a_{m+1}. e : p)$ stands for $\triangle(a_{m+1}, e), \triangle(a_{m+2}, e), \ldots\ldots, \triangle(a_p, e)$.

2. The $H^{m_1, n_1; m_2, n_2; m_3, n_3}_{p_1, q_1; p_2, q_2; p_3, q_3}\left[\begin{matrix} x \\ y \end{matrix}\right]$ function

Consider the double Contour integral

$$(2) \quad I = \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \phi(s+t) f(s) \psi(t) x^s y^t \, ds \, dt,$$

where

$$\phi(s+t) = \frac{\prod_{i=1}^{n_1} \Gamma(1-a_i+e_i s+e_i' t) \prod_{i=1}^{m_1} \Gamma(a_i'-f_i s-f_i' t)}{\prod_{i=n_1+1}^{p_1} \Gamma(a_i-e_i s-e_i' t) \prod_{i=m_1+1}^{q_1} \Gamma(1-a_i'+f_i s+f_i' t)}$$

$$f(s) = \frac{\prod_{i=1}^{n_2} \Gamma(1-b_i+g_i s) \prod_{i=1}^{m_2} \Gamma(b_i'-k_i s)}{\prod_{i=n_2+1}^{p_2} \Gamma(b_i-g_i s) \prod_{i=m_2+1}^{q_2} \Gamma(1-b_i'+k_i s)}$$

$$\psi(t) = \frac{\prod_{i=1}^{n_3} \Gamma(1-c_i+r_i t) \prod_{i=1}^{m_3} \Gamma(b_i'-h_i t)}{\prod_{i=n_3+1}^{p_3} \Gamma(c_i-r_i t) \prod_{i=m_3+1}^{q_3} \Gamma(1-c_i'+h_i t)}$$

and where

(a) L_1 and L_2 are two suitable Contours such that

(i) L_1 is in the s-plane extending from $-i\infty$ to $+i\infty$, with loops, if necessary to ensure that the poles of $\Gamma(b_i'-k_i s)$, $i=1,2,\ldots\ldots\ldots,m_2$

and $\Gamma\,(a_i'-f_i\,s-f_i'\,t)$ $i=1,2,\ldots\ldots\ldots,m_1$ lie to the right and those of $\Gamma\,(1-b_i+g_i\,s)$, $i=1,2,\ldots\ldots\ldots$, n_2 and $\Gamma\,(1-a_i+e_i\,s+e_i'\,t)$, $i=1,2,\ldots\ldots\ldots$, n_1 lie to the left of the Contour;

(ii) L_2 is in the t-plane extending from $-i\infty$ to $+i\infty$ such that the poles of $\Gamma\,(c_i'-h_i t)$, $i=1,2,\ldots,m_3$ and $\Gamma\,(a_i'-f_i s-f_i' t)$, $i=1,2,\ldots$, m_1 lie to the right and those of $\Gamma\,(1-c_i+r_i t)$, $i=1,2,\ldots\ldots,n_3$ and $\Gamma\,(1-a_i+e_i s+e_i' t)$ $i=1,2,\ldots,n_1$ lie to the left of the Contour.

(b) The positive indices p_i,q_i,m_i,n_i $(i=1,2,3)$ satisfy the inequlities

$q_i\geqslant 1,\ p_i\geqslant n_i\geqslant 0;\ q_i\geqslant m_i\geqslant 0;$

$ep_1+gp_2\leqslant fq_1+kq_2;\ ep_1+rp_3\leqslant fq_1+hq_3$

(c) The values $x=0$ and $y=0$ are excluded.

(d) All e's, f's, g's, k's, r's and h's are real and positive.

(e) The investigations into the convergence of the double integral on the right of (2) show that it is absolutely convergent if

$$\text{(i)}\quad |\arg x| < \left\{\frac{e+e'}{4}p_1+\frac{f+f'}{4}q_1+\frac{gp_2}{2}+\frac{kq_2}{2}-\frac{e+e'}{2}n_1-\frac{f+f'}{2}m_1 -\frac{gn_2}{2}-\frac{km_2}{2}\right\}\wedge$$

and $(e+e')p_1+(f+f')q_1+2(gp_2+kg_2)<2(a+e')n_1+2(f+f')m_1+4gn_2+4km_2$

$$\text{(ii)}\quad |\arg y| < \left\{\frac{e+e'}{4}p_1+\frac{f+f'}{4}q_1+\frac{rp_3}{2}+\frac{hq_3}{2}-\frac{e+e'}{2}n_1-\frac{f+f'}{2}m_1 -\frac{rn_3}{2}-\frac{hm_3}{2}\right\}\wedge$$

and $(e+e')\,p_1+(f+f')q_1+2(rp_3+hq_3)<2(e+e')n_1+2(f+f')m_1+4rn_3+4hm_3$

where $e=\max.(e_i)$, $e'=\max.(e_i')$ and so on.

The R.H.S. of (2) will be henceforth represented symbolically as

$$(3)\quad H^{m_1,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} \{a,\,e,\,e' : p_1\} : \{b,g : p_2\} : \{c,r : p_3\} \\ \{a',f,f' : q_1\} : \{b',k : q_2\} : \{c',h : q_3\} \end{matrix}\right]$$

or $\mathsf{H}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} \{a, e, e' : p_1\} : \{b, g : p_2\} : \{c, r : p_3\} \\ \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right]$

or simply

(4) $\mathsf{H}^{m_1, n_1; m_2, n_2; m_3, n_3}_{p_1, q_1; p_2, q_2; p_3, q_3}\left[\begin{matrix} x \\ y \end{matrix}\right]$ or $\mathsf{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$

in the absence of any ambiguity of indices and parameters.

3. Certain Properties of $\mathsf{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$

In this section some simple properties of $\mathsf{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$ are stated; the proofs being very simple, are omitted.

(5)
$$\mathsf{H}^{m_1+1, n_1; m_2, n_2; m_3, n_3}_{p_1+1, q_1+1; p_2, q_2; p_3, q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (\alpha, \beta, \gamma), \{a, e, e' : p_1\} : \{b, g : p_2\} : \{c, r : p_3\} \\ \{a', f, f' : q_1\}, (\alpha, \beta, \gamma) : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right] = \mathsf{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$$

(6)
$$x^{\sigma} y^{\rho} \mathsf{H}\left[\begin{matrix} x \\ y \end{matrix}\right] = \mathsf{H}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} \{a-\sigma e-\rho e', e, e' : p_1\} : \{b+\sigma g, g : p_2\} : \{c+\rho r, r : p_3\} \\ \{a'-\sigma f-\rho f', f, f' : q_1\} : \{b'+\sigma k, k : q_2\} : \{c'_1+\rho h, h : q_3\} \end{matrix}\right]$$

(7)
$$\mathsf{H}\left[\begin{matrix} x^{-1} \\ y^{-1} \end{matrix}\right] = \mathsf{H}^{n_1, m_1; n_2, m_2; n_3, m_3}_{q_1, p_1; q_2, p_2; q_3, p_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} \{1-a, e, e' : p_1\} : \{1-b, g : p_2\} : \{1-c, r : p_3\} \\ \{1-a', f, f' : q_1\} : \{1-b', k : q_2\} : \{1-c', h : q_3\} \end{matrix}\right]$$

(8) If e, f, g, k, r and h are positive integers, then

$$\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} \{a_{p_1}, e, e\} : \{b_{p_2}, g\} : \{c_{p_3}, r\} \\ \{a'_{q_1}, f, f\} : \{b'_{q_2}, k\} : \{c'_{q_3}, h\} \end{matrix}\right]$$

$$= \frac{f^{\sum_{i=1}^{q_1} (a'_i - \frac{1}{2})} \; k^{\sum_{i=1}^{q_2} (b'_i - \frac{1}{2})} \; h^{\sum_{i=1}^{q_3} (c'_i - \frac{1}{2})}}{e^{\sum_{i=1}^{p_1} (a_i - \frac{1}{2})} \; g^{\sum_{i=1}^{p_2} (b_i - \frac{1}{2})} \; r^{\sum_{i=1}^{p_3} (c_i - \frac{1}{2})}}$$

$$\times \left|\!-\!\right| \begin{matrix} m_1 f,\ n_1 e;\ m_2 k,\ n_2 g;\ m_3 h,\ n_3 r \\ p_1 e,\ q_1 f;\ p_2 g,\ q_2 k;\ p_3 r,\ q_3 h \end{matrix}$$

$$\left[\begin{matrix} Lx \\ \\ My \end{matrix} \middle| \begin{matrix} \Delta(a_1 - R,\ e : n_1),\ \Delta(a_{n_1+1} + R,\ e : p_1),\ \Delta(b_1 + R,\ g : n_2), \\ \Delta b_{n_2+1} + R,\ g : p_2),\ \Delta(c_1 - R,\ r : n_3),\ \Delta(c_{n_3+1} + R, r : p_3) \\ \Delta(a_1' - R, f : m_1),\ \Delta(a'_{m_1+1} + R, f : q_1),\ \Delta(b_1' - R,\ k : m_2), \\ \Delta(b'_{m_2+1} + R, k : q_2),\ \Delta(c_1' - R, h : m_3),\ \Delta(c'_{m_3+1} + R, h : q_3) \end{matrix}\right]$$

where

$$\Delta(A \pm t, \delta) = \frac{A}{\delta}, \frac{A \pm 1}{\delta}, \ldots, \frac{A \pm \overline{t-1}}{\delta}, t = 0, 1.2 \ldots, \delta - 1 \text{ and}$$

$$L = e^{ep_1} g^{gp_2} f^{-fq_1} k^{-kq_2}, \quad M = e^{ep_1} r^{rp_3} f^{-fq_1} h^{-hq_3}$$

(9) $g_1 x \dfrac{\partial}{\partial x} \left|\!-\!\right| \begin{bmatrix} x \\ y \end{bmatrix} =$

$$\left|\!-\!\right| \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} \{a, e, e' : p_1\} : (b_1 - 1, g_1), (b_2, g_2 : p_2) : \{c, r : p_3\} \\ \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right] - (1 - b_1) \left|\!-\!\right| \begin{bmatrix} x \\ y \end{bmatrix}$$

where $p_2 \geqslant 1$. A similar result can be obtained for $r_1 y \dfrac{\partial}{\partial y} \left|\!-\!\right| \begin{bmatrix} x \\ y \end{bmatrix}$.

(10) $\left\{\dfrac{1 - b_1}{g_1} + \dfrac{b_1'}{k_1}\right\} \left|\!-\!\right| \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \frac{1}{g_1} \left|\!-\!\right| \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} \{a, e, e' : p_1\} : (b_1 - 1, g_1), (b_2, g_2 : p_2) : \{c, r : p_3\} \\ \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right]$$

$$+ \frac{1}{k_1} \left|\!-\!\right| \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} \{a, e, e' : p_1\} : \{b, g : p_2\} : \{c, r : p_3\} \\ \{a', f, f' : q_1\} : (b_1' + 1, k_1), (b_2', k_2 : q_2) : \{c' h : q_3\} \end{matrix}\right]$$

where $p_2 \geqslant 1$, $q_2 \geqslant 1$.

A similar result for $\left\{\dfrac{1 - c_1}{r_1} + \dfrac{c_1'}{h_8}\right\} \left|\!-\!\right| \begin{bmatrix} x \\ y \end{bmatrix}$ can also be easily deduced.

(11) If $e_1 = e_1'$ and $f = f'$ only, then

$$\left\{\frac{1 - a_1}{e_1} + \frac{a_1'}{f_1}\right\} \left|\!-\!\right| \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{e_1} \left|\!-\!\right| \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a_1 - 1, e_1, e_1),\ (a_2, e_2, e_2' : p_3) : \{b, g, : p_2\} : \{c, r : p_3\} \\ (a_1', f_1, f_1),\ (a_1', f_2, f_2' : q_1) : \{b', k : q_2\} : \{c', h : q_1\} \end{matrix}\right]$$

$$+ \frac{1}{f_1} \left|\!-\!\right| \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a_1, e_1, e_1),\ (a_2, e_2, e_2' : p_1) : \{b, g : p_2\} : \{c, r : p_3\} \\ (a_1' + 1, f_1, f_1),\ (a_2', f_2, f_2' : q_1) : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right]$$

where $p_1 \geqslant 1$, $q_1 \geqslant 1$.

(12) If $e_1 = e_1'$ and $p_1, p_2, p_3 \geqslant 1$, then

$$\left\{\frac{1-b_1}{g_1} + \frac{1-c_1}{r_1} - \frac{1-a_1}{e_1}\right\} \mathrm{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$$

$$= \frac{1}{g_1} \mathrm{H}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} \{a, e, e' : p_1\} : (b_1 - 1, g_1), (b_2, g_2 : \{c, r : p_3\} \\ \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right]$$

$$+ \frac{1}{r_1} \mathrm{H}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} \{a, e, e' : p_1\} : \{b, g : p_2\} : (c_1 - 1, r_1), (c_2, r_2 : p_3) \\ \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right]$$

$$- \frac{1}{e_1} \mathrm{H}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_1 - 1, e_1, e_1), (a_2, e_2, e_2' : p_1) : \{b, g : p_2\} : \{c, r : p_3\} \\ \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\} \end{matrix}\right]$$

Similarly the result for $\left\{\frac{b_1'}{k_1} + \frac{c_1'}{h_1} - \frac{a_1'}{f_1}\right\} \mathrm{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$ can be deduced if $f_1 = f_1'$ and $q_1, q_2, q_3 \geqslant 1$.

The above recurrence relations are only a few specimen of a large no. of results which can be derived. It is clear from the fact that the function $\mathrm{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$ contains many parameters.

4. Derivatives of $\mathrm{H}\left[\begin{matrix} x \\ y \end{matrix}\right]$

Lemma :

If $f(x, y) = \sum_{u=\lambda_1}^{\lambda_2} \sum_{v=\mu_1}^{\mu_2} A\, x^u y^v$ where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are non-negative integers, then

$$(13) \quad \frac{\partial^{m+n}}{\partial x^m\, \partial y^n}\left\{ f(x, y)\, \mathrm{H}\left[\begin{matrix} x^{\alpha} \\ y^{\beta} \end{matrix}\right]\right\} = \sum_{u=\lambda_1}^{\lambda_2} \sum_{v=\mu_1}^{\mu_2} (-1)^{m+n} A\, x^{u-m} y^{v-n}\, \mathrm{H}_1\left[\begin{matrix} x^{\alpha} \\ y^{\beta} \end{matrix}\right]$$

where

$$\mathrm{H}_1\left[\begin{matrix} x^{\alpha} \\ y^{\beta} \end{matrix}\right] = \mathrm{H}^{m_1, n_1;\, m_2+1, n_2;\, m_3+1, n_3}_{p_1, q_1;\, p_2+1, q_2+1;\, p_3+1, q_3+1}$$

$$\left[\begin{matrix} x^{\alpha} \\ y^{\beta} \end{matrix}\middle| \begin{matrix} \{a, e, e' : p_1\} : \{b, g : p_2\}, (-u, \alpha) : \{c, r : p_3\}, (-v, \beta) \\ \{a', f, f' : q_1\} : (-u-m, \alpha), \{b', k : q_2\} : (-v-n, \beta), \{c', h : q_3\} \end{matrix}\right]$$

$u > 0,\ v > 0.$

$$(14)\ \frac{\partial^{m+n}}{\partial x^m\,\partial y^n}\left\{ f(x,y)\ \mathsf{H}\begin{bmatrix} x^{\alpha} \\ y^{\beta}\end{bmatrix}\right\}$$

$$= \sum_{u=\lambda_1}^{\lambda_2}\ \sum_{v=\mu_1}^{\mu_2} (-1)^{m+n} A\, x^{u-m}\, y^{v-n}\ \mathsf{H}_2\begin{bmatrix} x^{\alpha} \\ y^{\beta}\end{bmatrix}$$

where

$$\mathsf{H}_2\begin{bmatrix} x^{\alpha} \\ y^{\beta}\end{bmatrix} = \mathsf{H}\begin{matrix} m_1,\ n_1;\ m_2,\ n_2+1;\ m_3,\ n_3+1 \\ p_1,\ q_1;\ p_2+1,\ q_2+1;\ p_3+1,\ q_3+1\end{matrix}$$

$$\left[\begin{matrix} x^{\alpha} \\ y^{\beta}\end{matrix}\ \middle|\ \begin{matrix} \{a,e,e' : p_1\} : (1+u-m,-\alpha),\ \{b,g : p_2\} : (1+v-n,-\beta),\ \{c,r : p_3\} \\ \{a',f,f' : q_1\} : \{b',k : q_2\},\ (1+u,-\alpha) : \{c',h : q_3\},\ (1+v,-\beta)\end{matrix}\right]$$

$u<0,\ v<0.$

Corollary 1. If $f(x,y)=x^u\, y^v,\ u>0,\ v>0,$ then

$$(15)\ \frac{\partial^{m+n}}{\partial x^m\,\partial y^n}\left\{ x^u\, y^v\ \mathsf{H}\begin{bmatrix} x^{\alpha} \\ y^{\beta}\end{bmatrix}\right\} = (-1)^{m+n}\, x^{u-m}\, y^{v-n}\ \mathsf{H}_1\begin{bmatrix} x^{\alpha} \\ y^{\beta}\end{bmatrix}$$

Corollary 2. If $f(x,y)=x^u\, y^v;\ u<0,\ v<0,$ then

$$(16)\ \frac{\partial^{m+n}}{\partial x^m\,\partial y^n}\left\{ x^u\, y^v\ \mathsf{H}\begin{bmatrix} x^{\alpha} \\ y^{\beta}\end{bmatrix}\right\} = (-1)^{m+n}\, x^{u-m}\, y^{v-n}\ \mathsf{H}_2\begin{bmatrix} x^{\alpha} \\ y^{\beta}\end{bmatrix}$$

5. Evaluation of $\mathsf{H}\begin{bmatrix} (\lambda x)^{\xi} \\ y\end{bmatrix}$ or $\mathsf{H}\begin{bmatrix} x \\ (\mu y)^{\eta}\end{bmatrix}$

Lemma :

If $f(\lambda x)$ is a function which can be expanded and if it has all derivatives and the expansion is differentiable term by term any number of times, then by Taylor's theorem, we have

$$(17)\ f(\lambda x) = \sum_{n=0}^{\infty} \frac{(\lambda-1)^n}{\underline{|n}}\, x^n\, \frac{d^n}{dx^n} f(x)$$

$$(18)\ F(\mu y) = \sum_{n=0}^{\infty} \frac{(\mu-1)^n}{\underline{|n}}\, y^n\, \frac{d^n}{dy^n} F(y)$$

Now if we use the results obtained in section 4, we easily establish that

$$(19)\quad \lambda^{-b_1-1}\, H\left[\begin{matrix}(\lambda x)^{-g_1}\\ y\end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(1-\lambda)^n}{\lfloor n}$$

$$H\left[\begin{matrix} x^{-g_1} & \{a, e, e' : p_1\} : (b_1-n, g_1), (b_2, g_2 : p_2) : \{c, r : p_3\}\\ y & \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\}\end{matrix}\right]$$

$$(20)\quad \lambda^{c_1-1}\, H\left[\begin{matrix} x\\ (\lambda g)^{-r_1}\end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(1-\lambda)^n}{\lfloor n}$$

$$H\left[\begin{matrix} x & \{a, e, e' : p_1\} : \{b, g : p_2\} : (c_1-n, r_1), (c_2, r_2 : p_3)\\ y^{-r_1} & \{a', f, f' : q_1\} : \{b', k : q_2\} : \{c', h : q_3\}\end{matrix}\right]$$

$$(21)\quad \lambda^{-b'_1}\, H\left[\begin{matrix}(\lambda x)^{k_1}\\ y\end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(1-\lambda)^n}{\lfloor n}$$

$$H\left[\begin{matrix} x^{k_1} & \{a, e, e' : p_1\} : \{b, g : p_2\} : \{c, r : p_3\}\\ y & \{a', f, f' : q_1\} : (b'_1+n, k_1), (b'_2, k_2 : q_2) : \{c', h : q_3\}\end{matrix}\right]$$

$$(22)\quad \lambda^{-c'_1}\, H\left[\begin{matrix} x\\ (\lambda y)^{h_1}\end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(1-\lambda)^n}{\lfloor n}$$

$$H\left[\begin{matrix} x & \{a, e, e' : p_1\} : \{b, g : p_2\} : \{c, r : p_3\}\\ y^{h_1} & \{a', f, f' : q_1\} : \{b', k : q_2\} : (c'_1+n, h_1), (c'_1. h_2 : q_3)\end{matrix}\right]$$

Likewise many more results can be derived,

6. Particular cases of $H\left[\begin{matrix}x\\ y\end{matrix}\right]$

$$(23)\quad H^{0,\, n;\, m_1,\, v_1;\, m_2,\, v_2}_{p,\, s;\, t,\, q;\, t,\, q}\left[\begin{matrix} x & \{\epsilon_p\} : \{1-\gamma_t\} : \{1-\gamma'_t\}\\ y & \{1-\delta_s\} : \{\beta_q\} : \{\beta'_q\}\end{matrix}\right]$$

$$= G^{m_1,\, m_2,\, n,\, v_1,\, v_2}_{p,\, q,\, s,\, t}\left[\begin{matrix}x\\ y\end{matrix}\right] = G\left[\begin{matrix}x\\ y\end{matrix}\right]$$

where $\mathfrak{G}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is an extension of Meijer G-function in two variables given by Agarwal, R. P. (*1*).

$$(24)\quad H^{0,\, m_1;\, n_2,\, m_2;\, n_3,\, m_3}_{p_1-m_1, q_1;\, p_2-m_2,\, q_2-n_2;\, p_3-m_3,\, q_3-n_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} \{a_{p_1}\}:\{c_{p_2}\}:\{e_{p_3}\} \\ \{b_{q_1}\}:\{d_{q_2}\}:\{f_{q_3}\} \end{matrix}\right.\right]$$

$$= S\left[\begin{matrix} \left[\begin{matrix} m_1,\, 0 \\ p_1-m_1,\, q_1 \end{matrix}\right] \\ \left(\begin{matrix} m_2,\, n_2 \\ p_2-m_2,\, q_2-n_2 \end{matrix}\right) \\ \left(\begin{matrix} m_3,\, n_3 \\ p_3-m_3,\, q_3-n_3 \end{matrix}\right) \end{matrix}\left|\begin{matrix} \{1-a_{p_1}\};\{1-b_{q_1}\} \\ \{c_{p_2}\};\{d_{q_2}\} \\ \{e_{p_3}\};\{f_{q_3}\} \end{matrix}\right|\, x,\, y\right] = S\,(x, y).$$

where $S\,(x, y)$ is another generalized Meijer G-function in two variables studied by Sharma, B. L. (*5*).

$$(25)\quad H^{0,\, m_1;\, n_2,\, m_2;\, n_3,\, m_3}_{p_1-m_1,\, q_1;\, p_2-m_2,\, q_2-n_2;\, n_2-m_3,\, q_3-n_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} \{a_{p_1},\alpha_{p_1}\}:\{c_{p_2},\gamma_{p_2}\}:\{e_{p_3},\lambda_{p_3}\} \\ \{b_{q_1},\beta_{q_1}\}:\{d_{q_2},\delta_{q_2}\}:\{f_{q_3},\mu_{q_3}\} \end{matrix}\right.\right]$$

$$= A\left[\begin{matrix} \left[\begin{matrix} m_1,\, 0 \\ p_1-m_1,\, q_1 \end{matrix}\right] \\ \left(\begin{matrix} m_2,\, n_2 \\ p_2-m_2,\, q_2-n_2 \end{matrix}\right) \\ \left(\begin{matrix} m_3,\, n_3 \\ p_3-m_3,\, q_3-n_3 \end{matrix}\right) \end{matrix}\left|\begin{matrix} \{a_{p_1},\alpha_{p_1}\};\{b_{q_1},\beta_{q_1}\} \\ \{c_{p_2},\gamma_{p_2}\};\{d_{q_2},\delta_{q_2}\} \\ \{e_{p_3},\lambda_{p_3}\}:\{f_{q_3},\mu_{q_3}\} \end{matrix}\right|\, x,\, y\right] = A\,(x, y)$$

where $A\,(x, y)$ is a generalized S-function given by Goyal, A. N. (*4*).

$$(26)\quad H^{0,0\,;\, m_2, n_2\,;\, m_3, n_3}_{0,0\,;\, p_2,\, q_2\,;\, p_3,\, q_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} -:\{b, g : p_2\}:\{c, \gamma : p_3\} \\ -:\{b', k, q_2\}:\{c', h : q_3\} \end{matrix}\right.\right]$$

$$= H^{m_2\, n_2}_{p_2\, q_2}\left[x\left|\begin{matrix} \{b, g : p_2\} \\ \{b'.k : q_2\} \end{matrix}\right.\right] H^{m_3\, n_3}_{p_3\, q_3}\left[y\left|\begin{matrix} \{c, \gamma : p_3\} \\ \{c', h : q_3\} \end{matrix}\right.\right].$$

If $m_2=m_3=1$, $n_2=n_3=0=p_2=p_3$, $q_2=q_3=2$, $b'_1=c'_1=0$, $k_1=h_1=1$, $b'_2=-v$, $k_2=\mu$, $c'_2=-\lambda$, $h_2=\delta$, then

$$(27)\quad H\begin{matrix}0.0\,;\,1,0\,;\,1,0\\0,0\,;\,0,2\,;\,0,2\end{matrix}\left[\begin{matrix}x\\y\end{matrix}\;\begin{matrix}:\text{——}:\\\text{——}:(0,1):(-v,\mu):(0,1):(-\lambda,\delta)\end{matrix}\right]$$

$$= J_{v}^{\mu}(x)\, J_{\delta}^{\lambda}(y),$$

where $J_{v}^{\mu}(x)$ is Maitland's Bessel function.

$$(28)\quad Lim_{y\to 0}\; H\begin{matrix}0,0\,;\,0,0\,;\,m_3,n_3\\0,0\,;\,0,0\,;\,p_3,\,q_3\end{matrix}\left[\begin{matrix}x\\y\end{matrix}\;\begin{matrix}:\text{——}:\{c,r:p_3\}\\:\qquad:\{c',h:q_3\}\end{matrix}\right]$$

$$= H\begin{matrix}m_3\;n_3\\p_3\;q_3\end{matrix}\left[x\;\middle|\;\begin{matrix}\{c,\,r:p_3\}\\\{c',h:q_3\}\end{matrix}\right]$$

$$(29)\quad H\begin{matrix}0,1\,;\,m_2,n_2\,;\,1,0\\1,0\,;\,p_2+1,q_2\,;\,0,1\end{matrix}\left[\begin{matrix}x^{-1}\\y/x\end{matrix}\;\middle|\;\begin{matrix}(1,1,1):\{b,g:p_2\},(1,1):(0,1)\\\text{——}:\{b',k\,,\,q_2\}:\qquad\text{—}\end{matrix}\right]$$

$$= H\begin{matrix}m_2\;n_2\\p_2\;q_2\end{matrix}\left[\frac{1}{x+y}\;\middle|\;\begin{matrix}\{b,\,g:p_2\}\\\{b',k:q_2\}\end{matrix}\right].$$

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A NOTE ON GENERALISED KONTOROVITCH-LEBDEV TRANSFORM

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1. Introduction :

Jet Wimp has given the generalisation of Kontorovitch Lebdev transform pair [(3) ; p. 173], in various forms, in his paper, 'A Class of Integral Transform', [(4) ; p. 37 ; (4.9) and (4.10)]. In this paper we have discussed one of these integral transform pairs, in which, the kernel is a Whittaker function, written below, slightly in a different form, viz :

$$f(x) = \left(\frac{\pi}{ax}\right)^{1/2} \int_0^\infty W_{k,it}(ax)\, g(t)\, dt, \tag{1.1}$$

and

$$g(x) = \frac{a}{\pi^{5/2}}\, x \sinh(2\pi x)\, \Gamma(\tfrac{1}{2}-k\pm ix) \int_0^\infty (at)^{-3/2}\, W_{k,ix}(at) f(t) dt. \tag{1.2}$$

In this paper we have obtained a theorem which gives a relation between (1.1) and the well known Laplace transform :

$$\phi(p) = \int_0^\infty e^{-pt} f(t)\, dt. \tag{1.3}$$

In the integral transform (1.1), integration is performed with respect to the parameters, involved in the kernel, where as in the inversion formula (1.2), it is performed with respect to the argument. Known special cases of the results are Kontorovitch-Lebdev transform pair, namely :

$$f(x) = \int_0^\infty K_{it}(x)\, g(t)\, dt \tag{1.4}$$

and

$$g(x) = \frac{2}{\pi^2}\, x \sinh(\pi x) \int_0^\infty t^{-1} K_{ix}(t) f(t)\, dt \tag{1.5}$$

which is obtained when we put $k=0$ and $a=2$, in (1.1) and (1.2) respectively, by virtue of the identity :

$$W_{o,m}(x) = \left(\frac{x}{\pi}\right)^{\frac{1}{2}} K_m\left(\frac{x}{2}\right) \tag{1.6}$$

We shall denote (1.1), (1.4) and (1.3) in symbolic expressions as :

$$f(x) = \frac{W}{K}\, g(t), \tag{1.7}$$

$$f(x) = \frac{K}{L}\, g(t) \tag{1.8}$$

and

$$\phi(p) \doteq f(t). \tag{1.9}$$

Moreover, throughout this paper, we will write

$$G^{m,n}_{h,q}\left(\alpha \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_h \\ b_1, b_2, \ldots, b_q \end{matrix}\right) \text{ and } \Gamma(a+b)\,\Gamma(a-b)$$

as

$$G^{m,n}_{h,q}\left(\alpha \,\middle|\, \begin{matrix} \{a_h\} \\ \{b_q\} \end{matrix}\right) \text{ and } \Gamma(a \pm b) \text{ respectively.} \tag{1.10}$$

Section 2 contains the main result and its derivation. Section 3, contains the three corollaries of the main result. Two examples are given in the Section 4 to illustrate the theorem.

2. The main result :

If

$$\phi(p) \doteq f(t),$$

and

$$e^{\alpha p}\, p^{\frac{1}{2}-l} f(p) \frac{W}{K}\, g(t),$$

then

$$(\pi)^{-\frac{1}{2}} \Gamma(l-k+1) \; (p+\alpha+{}^{a}/_{2})^{l+\frac{1}{2}} \phi(p)$$

$$=\int_{0}^{\infty} \Gamma(l \pm ix+\tfrac{1}{2}) \left(\frac{a}{p+\alpha+{}^{a}/_{2}}\right)^{ix} {}_2F_1\left(\begin{matrix} ix+l+\frac{1}{2},\, ix-k+\frac{1}{2} \\ l-k+1 \end{matrix}; \frac{p+\alpha-{}^{a}/_{2}}{p+\alpha+{}^{a}/_{2}}\right) g(x)\, dx \qquad (2.1)$$

provided $R(l+\frac{1}{2})>0$, $R(p+\alpha+{}^{a}/_{2})>0$, $R(e)>0$, $R(p-A)>0$ and $g(x) \in L(0,\infty)$ where $f(t) = 0(t^{e-1})$ for small values of t and $f(t)=0(t^{\sigma-1} e^{At})$ for large values of t.

Proof :

By definition (1.1)

$$e^{\alpha p} \; p^{\frac{1}{2}-l} \; f(p)=\left(\frac{\pi}{ap}\right)^{\frac{1}{2}} \int_{0}^{\infty} W_{k,it}\,(ap)\, g(t)\, dt. \qquad (2.2)$$

Substituting the value of $f(t)$ from (2.2) in (1.3), we have

$$\phi(p)=\left(\frac{\pi}{a}\right)^{\frac{1}{2}} \int_{0}^{\infty} e^{-pt} \; e^{-\alpha t} \; t^{l-t} \int_{0}^{\infty} W_{k,ix}\,(at)\, g(x)\, dx\, dt. \qquad (2.3)$$

Changing the order of integration and evaluating the inner integral with the help of [(2); p. 216; (16)], viz :

$$t^{v-1} \; W_{k,\mu}(at) \doteq \frac{\Gamma(v \pm \mu+\frac{1}{2}).\; a^{\mu+\frac{1}{2}}}{\Gamma(v-k+1)(p+{}^{a}/_{2})^{v+\mu+\frac{1}{2}}} \; {}_2F_1\left(\begin{matrix} \mu+v+\frac{1}{2},\, \mu-k+\frac{1}{2} \\ v-k+1 \end{matrix}; \frac{p-{}^{a}/_{2}}{p+{}^{a}/_{2}}\right) \qquad (2.4)$$

where $R(v \pm \mu)>-\frac{1}{2}$ and $R(p+{}^{a}/_{2})>0$, we have :

$$\phi(p)=\left(\frac{\pi}{a}\right)^{\frac{1}{2}} \int_{0}^{\infty} g(x) \; \frac{\Gamma(l \pm ix+\frac{1}{2})\; a^{ix+\frac{1}{2}}}{\Gamma(l-k+1)(p+\alpha+{}^{a}/_{2})^{ix+l+\frac{1}{2}}} \times {}_2F_1\left(\begin{matrix} ix+l+\frac{1}{2},\, ix-k+\frac{1}{2} \\ l-k+1 \end{matrix}; \frac{p+\alpha-{}^{a}/_{2}}{p+\alpha+{}^{a}/_{2}}\right) dx \qquad (2.5)$$

and the result (2.1) follows from (2.5), after re-adjusting the terms.

The above proof involves the change of order of integration. To justify the same, we observe that since by [(1); p. 264; (5)] and [(1); p. 183; (1)].

$$W_{k,ix}(at) = e^{-at/2} (at)^k \, {}_2F_0\left(\tfrac{1}{2} - k \pm ix \, ; \, -\frac{1}{at}\right)$$

$$= e^{-at/2} (at)^k \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} - k \pm ix + n)}{(n) \, ! \, \Gamma(\frac{1}{2} - k \pm ix)} \left(-\frac{1}{at}\right)^n \qquad (2.6)$$

If we use [(1); p. 47; (6)], viz.

$$\lim_{|y| \to \infty} \left| \Gamma(x \pm iy) \right| = (2\pi)^{\frac{1}{2}} e^{\frac{1}{2}\pi |y|} |y|^{\frac{1}{2} - x} \; ; x, y \text{ real} \quad (2.7)$$

wc have :

$$\lim_{x \to \infty} W_{k,ix}(at) = \lim_{x \to \infty} e^{-at/2} . (at)^k . e^{-\frac{1}{at |x^2|}}$$

Therefore, $W_{k,ix}(at)$ is bounded for large values of x and for finite a and t.

Hence, the x integral in the equation (2.3), is absolutely convergent if $g(x) \varepsilon L(0, \infty)$ as $W_{k,ix}(at)$ shall be bounded for large values of x; and the integrand in the t – integral in the same equation (2.3), for large values of t is comparable with $| e^{-(p+\alpha+a/2)t} \, t^{l+k-\frac{1}{2}} |$ and for small values of t, it is comparable with $\{t^{l-1} (at)^{\frac{1}{2} \pm ix}\}$, since :

(i) $W_{k,m}(t) \backsim A e^{-\frac{1}{2}t} \, t^k$; for large values of t,

(ii) $W_{k,m}(t) \backsim A t^{\frac{1}{2}+m} + B t^{\frac{1}{2}-m}$; for small values of t.

Therefore the t—integral in (2.3) is absolutely convergent when $R(p+\alpha+a/2)>0$ and $R(l+\frac{1}{2})>0$. Also the resulting integral on the left of (2.1), is comparable with $| e^{-(p-A)t} \, t^{\sigma} |$ for large values of t, and it is comparable with $| t^{e-1} |$ for small values of t, since $f(t)=0(t^{e-1})$ for small values of t and $f(t)=0(t^{\sigma-1} e^{At})$ for large values of t. Hence, it will be convergent if $R(e)>0$ and $R(p-A)>0$.

3. Corollaries :

(i) When we put $k=0$ and $a=2$, in the main result it becomes :

If

$$\phi(p) \doteq f(t), \tag{3.1}$$

and

$$e^{\alpha p}\, p^{\frac{1}{2}-l}\, f(p) \frac{K}{\overline{\overline{L}}}\, g(t), \tag{3.2}$$

then

$$(\pi)^{-\frac{1}{2}}\, \Gamma(l+1)\,(p+\alpha-1)^{l+\frac{1}{2}}\, \phi(p)$$

$$= \int_0^\infty \Gamma(l \pm ix + \tfrac{1}{2}) \left(\frac{2}{p+\alpha+1}\right)^{ix} {}_2F_1\left(\begin{matrix} ix+l+\frac{1}{2},\ ix+\frac{1}{2} \\ l+1 \end{matrix};\frac{p+\alpha-1}{p+\alpha+1}\right) g(x)\, dx \tag{3.3}$$

provided $R(l+\frac{1}{2})>0$, $R(p+\alpha+1)>0$, $R(\rho)>0$, $R(p-A)>0$ and $g(x)\varepsilon L(0,\infty)$ where $f(t)=0(t^{\rho-1})$ for small values of t and $f(t)=0(t^{\sigma-1}\, e^{At})$ for large values of t.

(ii) When we put $\alpha=1$ and $l=1/2$, in the main result, it takes the form :

If

$$\phi(\mathrm{p}) \doteq f(t), \tag{3.4}$$

and

$$e^{p} f(p) \frac{W}{\overline{\overline{k}}}\, g(t), \tag{3.5}$$

then

$$(\pi)^{-\frac{1}{2}}\, \Gamma(\tfrac{3}{2}-k)\,(p+1+{}^{a}/_{2})\, \phi(p)$$

$$= \int_0^\infty \Gamma(\tfrac{3}{2} \pm ix) \left(\frac{a}{p+1+{}^{a}/_{2}}\right)^{ix} {}_2F_1\left(\begin{matrix} ix+\frac{3}{2},\ ix-k+\frac{1}{2} \\ \frac{3}{2}-k \end{matrix};\frac{p+1-{}^{a}/_{2}}{p+1+{}^{a}/_{2}}\right) g(x)dx \tag{3.6}$$

provided $R(p+1+{}^{a}/_{2})>o$, $R(e_{j}>o$, $R(p-A)>o$ and $g(x)\ \varepsilon\ L(o,\infty)$ where $f(t)=o(t^{e-1})$ for small values of t and $f(t)=o(t^{\sigma-1}\, e^{At})$ for large values of t.

(iii) If we put $\alpha={}^{a}/_{2}$, in the main result, it can be written in the form :

If

$$\phi(p) \doteq f(t) \tag{3.7}$$

and

$$e^{ap/2}\ p^{\frac{1}{2}-l}\ f(p)\ \frac{W}{\overline{k}}\ g(t) \tag{3.8}$$

then

$$(\pi)^{-\frac{1}{2}}\ \Gamma(l-k+1)\ (p+a)^{l+\frac{1}{2}}\ \phi(p)$$

$$=\int_0^\infty \Gamma\ (l\pm ix+\tfrac{1}{2}) \left(\frac{a}{p+a}\right)^{ix} {}_2F_1\left(\begin{matrix} ix+l+\frac{1}{2},\ ix-k+\frac{1}{2} \\ l-k+1 \end{matrix} ;\ \frac{p}{p+a}\right) g(x)\ dx \tag{3.9}$$

provided $R\ (l+\frac{1}{2}) > o$, $R\ (p+a) > o$, $R\ (e) > o$, $R\ (p-A) > o$ and $g(x)$ $\varepsilon\ L\ (o,\infty)$ where $f(t)=o\ (t^{e-1})$ for small values of t, and $(t)=o$ $(t^{\sigma-1}\ e^{At})$ for large values of t.

4. Applications :

(i) Let us start from [(2); p. 144; (3)]:

$$f(t)=t^{\mu-1}\ e^{-\beta t}$$

$$\doteq \Gamma(\mu)\quad (p+\beta)^{-\mu}=\phi(p) \tag{4.1}$$

where $R(\mu)>o$ and $R(p)>-R(\beta)$, therefore we have from (1.1), (1.2) and the result [(2); p. 216; (16)]:

$$e^{\alpha p}\ p^{\frac{1}{2}-l}\ f(p)=e^{\alpha p}\ p^{\mu-l-\frac{1}{2}}\ e^{-\beta p}$$

$$\frac{W}{\overline{k}}\ \pi^{-5/2}.\ x \sinh\ (2\pi x)\,\Gamma(\tfrac{1}{2}-k\pm ix).\ \frac{\Gamma(\mu-l-\frac{1}{2}\pm ix)}{\Gamma(\mu-l-k)}$$

$$\times \frac{a^{ix}}{(\beta-\alpha+a/2)^{ix+\mu-l-\frac{1}{2}}}\ {}_2F_1\left(\begin{matrix} ix+\mu-l-\frac{1}{2},\ ix-k+\frac{1}{2} \\ \mu-l-k \end{matrix} ;\ \frac{\beta-\alpha-a/2}{\beta-\alpha+a/2}\right)$$

$$=g(x) \tag{4.2}$$

where :

$$R(\mu)>R(l+\tfrac{1}{2}) \text{ and } R(\beta-\alpha+a/2)>o.$$

Substituting the values of $\phi\ (p)$ and $g(x)$ from (4.1) and (4.2), in the result (2.1), we get :

$$\int_0^\infty x \sinh\ (2\pi x)\,\Gamma(l\pm ix+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-k\pm ix)\,\Gamma(\mu-l\pm ix-\tfrac{1}{2})$$

$$\left[\frac{a^2}{(p+\alpha+a/2)\ (\beta-\alpha+a/2)}\right]^{ix}$$

$$\times {}_2F_1\left(\begin{matrix} ix+l+\frac{1}{2},\ ix-k+\frac{1}{2} \\ l-k+1 \end{matrix} ;\ \frac{p+\alpha-a/2}{p+\alpha+a/2}\right)$$

$$ {}_2F_1\left(\begin{matrix} ix+\mu-l-\frac{1}{2},\ ix-k+\frac{1}{2} \\ \mu-l-k \end{matrix} ;\ \frac{\beta-\alpha-a/2}{\beta-\alpha+a/2}\right)\ dx$$

$$=\pi^2\ \Gamma(\mu)\ \Gamma(l-k+1)\ \Gamma(\mu-l-k)\ (p+\alpha+a/2)^{l+\frac{1}{2}}$$

$$(\beta-\alpha+a/2)^{\mu-l-\frac{1}{2}}\ (p+\beta)^{-\mu} \qquad (4.3)$$

where :

$R(p)>-R(\beta)$, $R(\mu)>R(l+\frac{1}{2})>o$, $R(p+\alpha+a/2)>o$ and $R(\beta-\alpha+a/2)>o$.

By virtue of the result [(1); p. 105; (2)], we also have :

$$\int_{-\infty}^{+\infty} x \sinh(2\pi x)\ \Gamma(l\pm ix+\tfrac{1}{2})\ \Gamma(\tfrac{1}{2}-k\pm ix)\ \Gamma(\mu-l\pm ix-\tfrac{1}{2})$$

$$\left[\frac{a^2}{(p+\alpha+a/2)(\beta-\alpha+a/2)}\right]^{ix}$$

$$\times {}_2F_1\left(\begin{matrix} ix+l+\frac{1}{2},\ ix-k+\frac{1}{2} \\ l-k+1 \end{matrix} ;\ \frac{p+\alpha-a/2}{p+\alpha+a/2}\right)$$

$$ {}_2F_1\left(\begin{matrix} ix+\mu-l-\frac{1}{2},\ ix-k+\frac{1}{2} \\ \mu-l-k \end{matrix} ;\ \frac{\beta-\alpha-a/2}{\beta-\alpha+a/2}\right)\ dx$$

$$=2\pi^2\ \Gamma(\mu)\ \Gamma(l-k+1)\ \Gamma(\mu-l-k)\ (p+\alpha+a/2)^{l+\frac{1}{2}}$$

$$(\beta-\alpha+a/2)^{\mu-l-\frac{1}{2}}\ (p+\beta)^{-\mu} \qquad (4.4)$$

where :

$R(p)>-R(\beta)$, $R(\mu)>R(l+\frac{1}{2})>0$, $R(p+\alpha+a/2)>0$, $R(\beta-\alpha+a/2)>0$.

(ii) Let us start with [(2); p. 222; (34)] :

$$f(t)=t^{-\delta}\ G^{m,n}_{h,q}\left(\beta t\ \middle|\ \begin{matrix}\{a_h\}\\ \{b_q\}\end{matrix}\right)$$

$$\doteq p^{\delta-1}\ G^{m,n+1}_{h+1,q}\left(\frac{\beta}{p}\ \middle|\ \begin{matrix}\delta,\ \{a_h\}\\ \{b_q\}\end{matrix}\right)=\phi(p), \quad (4.5)$$

where $h+q<2(m+n)$, $|\arg\beta|<(m+n-\frac{1}{2}h-\frac{1}{2}q)\pi$, $R(e)>0$ and $R(\delta)<R(b_j+1)$; $j=1, 2, \ldots\ldots, m$,

and therefore we have from (1.1), (1.2) and using the results [(1); p. 216; ; (8)] and [(3) ; p. 422; (14)], after replacing α by $^{a}/_{2}$

$$e^{ap/2}\, p^{\frac{1}{2}-l}\, f(p) = e^{ap/2}\, p^{\frac{1}{2}-\delta-l}\, G^{m,n}_{h,q}\left(\beta p \,\middle|\, \begin{matrix}\{a_h\}\\ \{b_q\}\end{matrix}\right)$$

$$\overset{w}{\underset{K}{=}}\ \pi^{-5/2}\, a^{\delta+l-\frac{1}{2}}\, x \sinh\,(2\pi x)$$

$$G^{m+1,\,n+2}_{h+2,\,q+1}\left(\frac{\beta}{a} \,\middle|\, \begin{matrix}\delta+l+\frac{1}{2}\pm ix,\ \{a_h\}\\ \delta+1-k,\ \{b_q\}\end{matrix}\right)$$

$$= g(x) \tag{4.6}$$

with the conditions :

$h+q<2(m+n)$, $|\arg \beta| <(m+n-\frac{1}{2}h-\frac{1}{2}q)\pi$, $R(a)>o$,

$R(k-\delta-l+a_j-1)<o$, $j=1,\ldots,n$; and $R(\frac{1}{2}-\delta-l+b_i)>o, i=1,\ldots,m$.

Substituting the value of $\phi(p)$ and $g(x)$ from (4.5) and (4.6), in the result (2.1), we get :

$$\int_0^\infty x \sinh\,(2\pi x)\ \Gamma\,(l\pm ix+\tfrac{1}{2}) \left(\frac{a}{p+a}\right)^{ix}$$

$$\times\ {}_2F_1\left(\begin{matrix}ix+l+\frac{1}{2},\ ix-k+\frac{1}{2}\\ l-k+1\end{matrix};\ \frac{a}{p+a}\right)$$

$$G^{m+1,\,n+2}_{h+2,\,q+1}\left(\beta/a \,\middle|\, \begin{matrix}\delta+l\pm\frac{1}{2}\pm ix,\ \{a_h\}\\ \delta+l-k,\ \{b_q\}\end{matrix}\right) dx$$

$$=\pi^2 .\ 2^{\frac{1}{2}-\delta-l}\ \Gamma\,(l-k+1)\ (p+a)^{l+\frac{1}{2}}\ p^{\delta-1}$$

$$G^{m,\,n+1}_{h+1,\,q}\left(\beta/p \,\middle|\, \begin{matrix}\delta,\ \{a_h\}\\ \{b_q\}\end{matrix}\right) \tag{4.7}$$

where $R(p)>o$, $R(a)>o$, $h+q<2(m+n)$, $-\frac{1}{2}<R(l)<\frac{1}{2}$,

$|\arg \beta| <(m+n-\frac{1}{2}h-\frac{1}{2}q)\pi$, $R(\delta)<R(b_j)+1$, $j=1,\ldots\ldots, m$;

and $R(k-\delta-l+a_k-1)<o$, $k=1,\ldots\ldots, n$.

And by virtue of the result [(1); p. 105; (2)], we get :

$$\int_{-\infty}^{+\infty} x \sinh (2\pi x)\, \Gamma(l \pm ix + \tfrac{1}{2}) \left(\frac{a}{p+a}\right)^{ix}$$

$$\times\ {}_2F_1\left(\begin{matrix} ix+l+\frac{1}{2},\ ix-k+\frac{1}{2} \\ l-k+1 \end{matrix};\ \frac{p}{p+a}\right)$$

$$G^{m+1,\,n+2}_{h+2,\,q+1}\left(\frac{\beta}{a}\,\middle|\,\begin{matrix} \delta+l+\frac{1}{2}\pm ix,\ \{a_h\} \\ \delta+l-k,\ \{b_q\} \end{matrix}\right)dx$$

$$=2\pi^2.2^{\frac{1}{2}-\delta-l}\ \Gamma(l-k+1)\,(p+a)^{l+\frac{1}{2}}\ p^{\delta-1}$$

$$G^{m,\,n+1}_{h+1,\,q}\left(\frac{\beta}{p}\,\middle|\,\begin{matrix} \delta,\ \{a_h\} \\ \{b_q\} \end{matrix}\right) \tag{9.8}$$

where :

$R(p)>o,\ R(a)>o,\ h+q<2(m+n),\ -\frac{1}{2}<R(l)<\frac{1}{2},$

$|\arg\beta|<(m+n-\frac{1}{2}h-\frac{1}{2}q)\pi,\ R(\delta)<R(b_j)+1,\ j=1,\ \ldots,\ m;$

and $R(k-\delta-l+a_k-1)<o,\ k=1,\ldots\ldots,\ n.$

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FORM-FUNCTION OF A PARTIALLY INHIBITED CYLINDRICAL CHARGE WITH THE DISTRIBUTION OF HOLES OVER n RINGS IN THE 4-FOLD AXIS OF SYMMETRY

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Summary :

In this paper the form-function of a cylindrical charge with holes distributed over n rings in the 4-fold axis of symmetry has been studied. The rings are similar and similarly situated squares and the holes have been distributed along their sides in such a manner that at the end of the first stage of combustion, every hole touches all its neighbouring holes. That part of the cross-section of the charge which lies outside the outermost ring at the end of the first stage of combustion has been inhibited.

Introduction :

It has been proved in an earlier paper (Patni, Girraj Prasad and Jain, 1966) that symmetrical distribution of points in the 2-dimensional space is possible about 1-fold, 2-fold, 3-fold, 4-fold and 6-fold axes of symmetry. Consequently, the centres of the holes in a cylindrical charge fulfilling the conditions of symmetrical distribution are also to be arranged accordingly. The cases of 1-fold and 2-fold axes of symmetry are trivial, and hence the only axes about which distribution of holes may be considered are 3-fold, 4-fold and 6-fold. The form-function of a partially inhibited cylindrical charge in which the holes are distributed about a 3-fold axis of symmetry has also been studied in the paper referred to above.

In the present paper, we have studied and discussed the form-function of a partially inhibited cylindrical charge with its holes distributed along n rings about the 4-fold axis of symmetry. The rings are similar and similarly situated squares and the holes lie along their sides. Further, the centres of these holes are situated on the vertices of exactly alike squares. The portion of the charge which lies outside the outer most ring at the end of the first phase of combustion has been inhibited and this inhibition reduces the stages of burning to two.

Notations :

D = The exterior diameter of the charge.

d = The diameter of the holes of the charge.

L = The length of the charge.

e_r = The web size, i.e., the distance between any two adjacent holes or between any exterior hole and the curved surface of the grain.

m = The ratio of the diameter of the charge to the diameter of a hole $=\frac{D}{d}$

ρ = The ratio of the length of the charge to its diameter $=\frac{L}{D}$.

V_o = The initial volume of the charge.

V = The volume of the charge at any instant t.

S_o = The initial surface of the charge.

S = The surface of the charge at any instant t.

It can be seen that the number of holes on a side of the innermost square can be 2 or 3 as shown in the figure 1.

Fig. 1

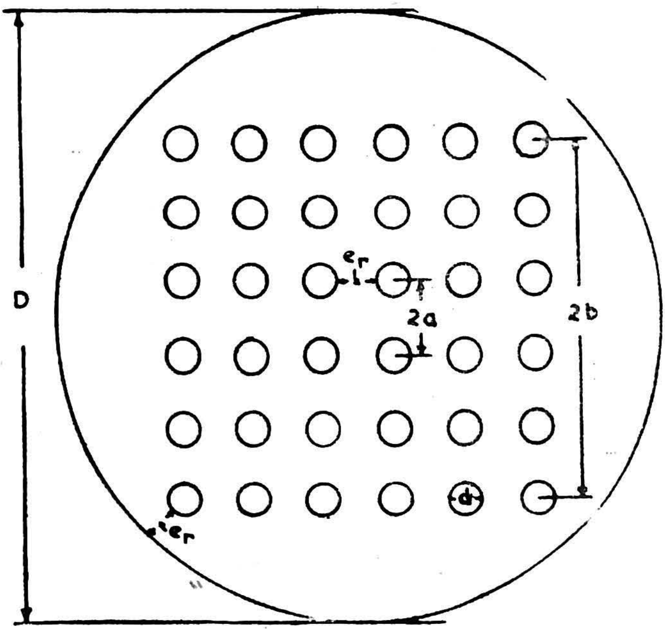

Fig. 2—Showing the section of the charge at the beginning of the combustion, for $n=3$ and $r=1$.

These arrangements (fig. 1) will be called category I and II respectively. In the case of the category II, there will be an extra hole at the centre of the innermost square.

The number of holes on a side of the nth ring

$$=(2n+r-1)=\mu_r+1 \qquad \ldots \qquad (1)$$

where $$\mu_r=2n+r-2 \qquad \ldots \qquad (2)$$

and $r=1, 2$ for the two categories,

so that the total number of holes in the charge is given by

$$N_r=(2n+r-1)^2=(\mu_r+1)^2. \qquad \ldots \qquad (3)$$

Let $2a$ be the side of the innermost square (i.e. the 1st ring) and $2b$ the side of the outermost square (i.e. the nth ring). Then

$$b=\frac{\mu_r}{r}a \qquad \ldots \qquad (4)$$

Now $$D=\frac{2\sqrt{2}(\mu_r+\sqrt{2})a}{r}-d=md$$

$\therefore$ $$a=\frac{r\sqrt{2}\,(m+1)\,d}{4\,(\mu_r+\sqrt{2})}. \qquad \ldots \qquad (5)$$

Also $$e_r=\frac{1}{r}(2a-rd)=\frac{[\sqrt{2}(m-1)-2\mu_r]d}{2\,(\mu_r+\sqrt{2})} \qquad \ldots \qquad (6)$$

Clearly $$m \geqslant (\sqrt{2}\,\mu_r+1) \qquad \ldots \qquad (6A)$$

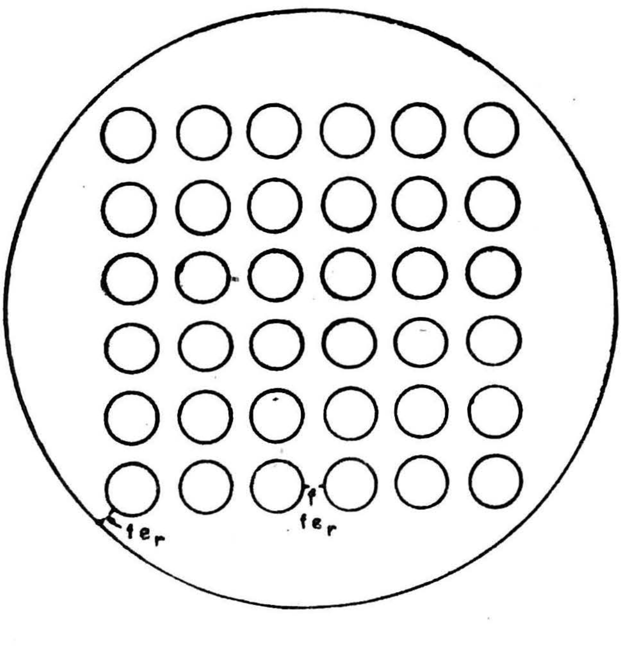

Fig. 3. Showing the section of the charge during the first phase of combustion, for $n=3$ and $r=1$.

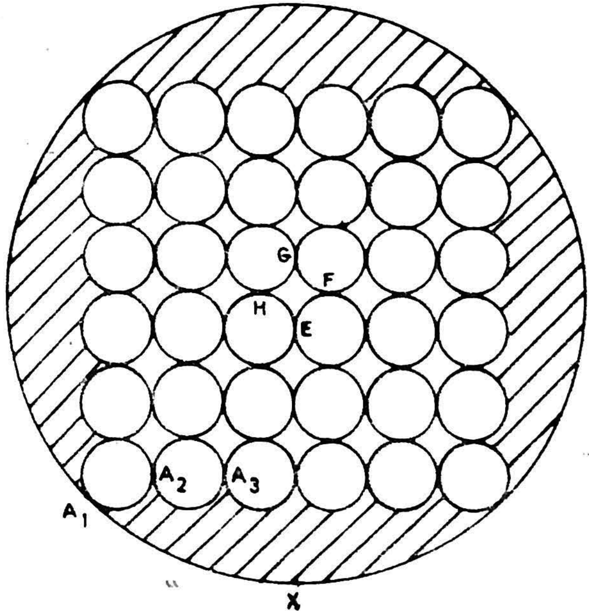

Fig. 4. Showing the section of the charge at the end of the first phase of combustion for $n=3$ and $r=1$. The portion inhibited has been shaded.

At the end of the first phase of combustion, every hole touches all the holes adjacent to it and the charge consists of

(i) μ_r^2 exactly alike prisms whose bases are curvilinear squares like EFGH;

(ii) four exactly alike prisms having their bases Curvilinear figures like $A_1 A_2 \ldots\ldots A_{2n+1} X$, and lying outside the nth ring.

At this instant the diameter of the charge

$$= 2b \sec \frac{\pi}{4} + \frac{2a}{r}$$

$$= \frac{2a}{r} (\sqrt{2}\ \mu_r + 1) = D', \text{(say)} \qquad \ldots \qquad (7)$$

and the diameter of a hole

$$=\frac{2a}{r}=d', \text{ (say)} \qquad \ldots \qquad (8)$$

so that the area of the curvilinear base EFGH $=\left(\frac{2a}{r}\right)^2-\frac{\pi d'^2}{4}$

and the area of the bases of the four outer prisms

$$=\frac{\pi D'^2}{4}-\mu_r^2 \text{ Area } EFGH-N_r\cdot\frac{\pi d'^2}{4}$$

$$=\frac{\mu_r (m+1)^2 \triangle_r}{4(\mu_r+\sqrt{2})^2}d^2 \qquad (9)$$

where $\triangle_r=(\pi-2)\ \mu_r+(\sqrt{2}-1)\pi.$ (10)

Inhibiting these four outer prisms, we have

$$V_o=\left[\frac{\pi D^2}{4}-N_r\frac{\pi d^2}{4}-\frac{\mu_r (m+1)^2\triangle_r}{4\,(\mu_r+\sqrt{2})^2}\,d^2\right]L$$

$$=\frac{\rho m d^3}{4(\mu_r+\sqrt{2})^2}\left[\pi\,(m^2-N_r)\,(\mu_r+\sqrt{2})^2-(m+1)^2\,\mu_r\,\triangle_r\right] \quad (11)$$

and

$$V=\left[\pi\left\{\frac{D}{2}-\frac{e_r\,(1-f)}{2}\right\}^2-N_r\,\pi\left\{\frac{d}{2}+\frac{e_r\,(1-f)}{2}\right\}^2-\frac{\mu_r\,(m+1)^2\triangle_r}{4(\mu_r+\sqrt{2})^2}\,d^2\right]\times[L-e_r\,(1-f)]$$

$$=\frac{d^3}{4\,(\mu r+\sqrt{2})^2}\Big[\pi\,(\mu_r+\sqrt{2})^2\,(m^2-N_r)-\mu_r(m+1)^2\,\triangle_r$$
$$-4\pi\,(\mu_r+\sqrt{2})\,\{\sqrt{2}\,(m-1)-2\mu_r\}\,(m+N_r)$$
$$(1-f)-\pi\,\{\sqrt{2}\,(m-1)-2\mu_r\}\,(N_r-1)\,(1-f)^2\Big]$$
$$\left[m\rho-\frac{\{\sqrt{2}\,(m-1)-2\mu_r\}\,(1-f)}{2\,(\mu_r+\sqrt{2})}\right] \qquad (12)$$

where f is the fraction of the web size remaining at any instant t.

$$\therefore \quad z=\frac{V_0\delta-V\delta}{V_0\delta}=1-\frac{V}{V_0}$$

$$=(1-f)(A-Bf-Cf^2) \tag{13}$$

where

$$A=\{\sqrt{2}(m-1)-2\mu_r\}[2\pi(\mu_r+\sqrt{2})\{4\mu_r+\sqrt{2}(N_r+3)\}m^2\rho$$
$$+2\pi(\mu_r+\sqrt{2})\{N_r(2\mu_r+3\sqrt{2})+(2\mu_r+\sqrt{2})\}m\rho$$
$$+\{(2\mu_r+\sqrt{2})^2-2N_r)\pi-4\mu_r\Delta_r\}(m+1)^2]\div 8m\rho(\mu_r+\sqrt{2})$$
$$[\pi(m^2-N_r)(\mu_r+\sqrt{2})^2-(m+1)^2\mu_r\Delta_r] \tag{14}$$

$$B=2\pi\{\sqrt{2}(m-1)-2\mu_r\}^2[(\mu_r+\sqrt{2})(N_r-1)m\rho-(2\mu_r+\sqrt{2}N_r$$
$$+\sqrt{2})(m+1)]\div 8m\rho(\mu_r+\sqrt{2})[\pi(m^2-N_r)(\mu_r+\sqrt{2})^2-$$
$$(m+1)^2\mu_r\Delta_r] \tag{15}$$

$$C=\frac{\pi(N_r-1)\{\sqrt{2}(m-1)-2\mu_r\}^3}{8m\rho(\mu_r+\sqrt{2})[\pi(m^2-N_r)(\mu_r+\sqrt{2})^2-(m+1)^2\mu_r\Delta_r]} \tag{16}$$

Now $\qquad L=\rho D=\rho m d$

$$=\frac{2(\mu_r+\sqrt{2})m\rho}{\sqrt{2}(m-1)-2\mu_r}\cdot er. \tag{17}$$

In order that all-burnt position of the charge may not occur before or coincide with the rupture of the grain, we must have

$$L>e_r \tag{18}$$

or

$$\rho>\frac{\sqrt{2}(m-1)-2\mu_r}{2(\mu_r+\sqrt{2})m}=\rho min \quad \text{, say} \tag{19}$$

Now (14) can be written as

$$A = A_0 \,(m, N_r) + \frac{A_1\,(m, N_r)}{\rho} \tag{20}$$

where

$$A_0\,(m,N_r) = \{\sqrt{2}\,(m-1) - 2\mu_r\}\,[2\pi(\mu_r + \sqrt{2})\,\{4\mu_r + \sqrt{2}\,(N_r+3)\}m + 2\pi\,(\mu_r + \sqrt{2})\,\{N_r\,(2\mu_r + 3\sqrt{2}) + (2\mu_r + \sqrt{2})\}\,] \div 8\,(\mu_r + \sqrt{2})\,[\pi\,(m^2 - N_r)\,(\mu_r + \sqrt{2})^2 - (m+1)^2\,\mu_r \triangle_r] \tag{21}$$

and

$$A_1\,(m,N_r) = \frac{\{\sqrt{2}\,(m-1) - 2\mu_r\}[\{(2\mu_r + \sqrt{2})^2 - 2N_r\}\pi - 4\mu_r \triangle_r](m+1)^2}{8\,(\mu_r + \sqrt{2})\,[\pi\,(m^2 - N_r)\,(\mu_r + \sqrt{2})^2 - (m+1)^2\,\mu_r \triangle_r]m} \tag{22}$$

Hence for a given value of N_r, we have

$$\begin{aligned} Amax &= A_0\,(m,N_r) + \frac{A_1(m,N_r)}{\rho min} \\ &= 1, \text{ a value independent of } m \text{ and } N_r. \end{aligned} \tag{23}$$

Also for given values of m and N_r, A is minimum when $\rho = +\infty$, hence from (19)

$$A\,\min = A_0\,(m, Nr). \tag{24}$$

Further if $m = \dfrac{\sqrt{2} + 2\mu r}{\sqrt{2}}$, $A_0\,(m, Nr) = 0$

and if $m = \infty$, $A_0\,(m, Nr)$

$$= \frac{\pi[2\sqrt{2}\,\mu_r + N_r + 3]}{4[\pi(\mu_r + \sqrt{2} - \mu_r \triangle_r]}.$$

Now $$\frac{S}{S_0} = \frac{dz}{df} \Big/ \left(\frac{dz}{df}\right)_{f=1} \tag{25}$$

which with the help of (13) gives

$$\frac{S}{S_0}=\alpha-\beta f-\gamma f^2 \tag{26}$$

where

$$\alpha=\frac{A+B}{A-B-C}$$

$$=(m+1)\,[4\pi\,(\mu_r+\sqrt{2})\,(2\mu_r+\sqrt{2}\,N_r+\sqrt{2})\,m\rho+2\pi\,(2\mu_r^2-3N_r-1)\,m+2\pi\,(6\mu_r^2+2\sqrt{2}\,\mu_r N_r+6\sqrt{2}\,\mu_r+N_r+3)-4\mu_r\Delta_r(m+1)]\div 4[(\mu_r+\sqrt{2})^2\,(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\Delta_r(m+1)^2] \tag{27}$$

$$\beta=\frac{2(B-C)}{A-B-C}$$

$$=2\pi\,[\sqrt{2}\,(m-1)-2\mu_r]\,[2(\mu_r+\sqrt{2})(N_r-1)m\rho-(4\mu_r+3\sqrt{2}\,N_r+\sqrt{2})m-(6\mu_r+\sqrt{2}\,N_r-2\mu_r N_r+3\sqrt{2})]\div 4\,[\pi(\mu_r+\sqrt{2})^2\,(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\Delta_r(m+1)^2] \tag{28}$$

$$\gamma=\frac{3C}{A-B-C}$$

$$=\frac{4\pi(N_r-1)\,\{\sqrt{2}\,(m-1)-2\mu_r\}^2}{4[\pi(\mu_r+\sqrt{2})^2\,(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\Delta_r(m+1)^2]} \tag{29}$$

Equation (26) gives a relation between $\frac{S}{S_o}$ and f for the first phase of combustion. Putting $f=0$, we get

$$\left[\frac{S}{S_o}\right]_{f=0}=\propto$$

which is the ratio of the surface at the end of the first phase of combustion to the initial surface.

From (26), we have

$$\frac{d}{df}\left(\frac{S}{S_o}\right)=-\beta-2\gamma f \tag{30}$$

and

$$\frac{d^2}{df^2}\left(\frac{S}{S_o}\right)=-2\gamma \tag{31}$$

It is clear from (29) that γ is always positive, hence $\frac{d^2}{df^2}\left(\frac{S}{S_o}\right)$ is always negative. Thus $\frac{S}{S_o}$ can have only a maximum value for some value of f. Now, for a maximum

$$\frac{d}{df}\left(\frac{S}{S_o}\right) = -\beta - 2\gamma f = 0, \text{ so that}$$

$$f = -\frac{\beta}{2\gamma}. \tag{32}$$

Since $1 \geqslant f \geqslant 0$, we have

$$-\frac{\beta}{2\gamma} \leqslant 1 \tag{33}$$

and

$$-\frac{\beta}{2\gamma} \geqslant 0. \tag{34}$$

From (28) and (29), we get

$$-\frac{\beta}{2\gamma} = [(6\mu_r + \sqrt{2}N_r - 2\mu_r N_r + 3\sqrt{2}) + (4\mu_r + 3\sqrt{2}N_r + \sqrt{2})m - 2(\mu_r + \sqrt{2})(N_r - 1)m\rho] \div 3(N_r - 1)[\sqrt{2}(m-1) - 2\mu_r] \tag{35}$$

so that (33) gives

$$[(6\mu_r + \sqrt{2}N_r - 2\mu_r N_r + 3\sqrt{2}) + (4\mu_r + 3\sqrt{2}N_r + \sqrt{2})m - 2(\mu_r + \sqrt{2})(N_r - 1)m\rho] \div [3(N_r - 1)\{\sqrt{2}(m-1) - 2\mu_r\} \leqslant 1$$

or

$$\rho \geqslant \frac{2(m+N_r)}{(N_r - 1)m} = \rho_1, \text{ (say)}. \tag{36}$$

Again (34) gives

$$[(6\mu_r + \sqrt{2}N_r - 2\mu_r N_r + 3\sqrt{2}) + (4\mu_r + 3\sqrt{2}N_r + \sqrt{2})m - 2(\mu_r + \sqrt{2})(N_r - 1)m\rho] \div [3(N_r - 1)\{\sqrt{2}(m-1) - \mu_r\} \leqslant 0$$

$$\text{or } \rho \leqslant \frac{(6\mu_r+\sqrt{2}N_r-2\mu_r N_r+3\sqrt{2})+(4\mu_r+3\sqrt{2}N_r+\sqrt{2})m}{2(\mu_r+\sqrt{2})(N_r-1)m}=\rho_2,$$

(say). (37)

(36) and (37) give the least and the greatest value of ρ for a maximum value of $\frac{S}{S_o}$. If $\rho=\rho_1$, the maximum of $\frac{S}{S_0}$ occurs at the beginning of the combustion and if $\rho=\rho_2$, it occurs at the end of the first phase. In order that this happens during the first phase, ρ should lie between ρ_1 and ρ_2.

Using (19), (35), (36) and (37) in

$$f=-\frac{\beta}{2\gamma}$$

we have

$$f=1-\frac{\rho-\rho_1}{\rho_{\min}}=\frac{\rho_2-\rho}{\rho_{\min}} \tag{38}$$

which gives the value of f for a maximum of $\frac{S}{S_o}$.

With the help of (28), (29) and (36), we get from (30)

$$\left[\frac{d}{df}\left(\frac{S}{S_o}\right)\right]_{f=1}=-\beta-2\gamma$$

$$=\frac{4\eta(\mu_r+\sqrt{2})\{\sqrt{2}(m-1)-2\mu_r\}(N_r-1)m(\rho_1-\rho)}{4[\eta(\mu_r+\sqrt{2})^2(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\Delta_r(m+1)^2]} \tag{39}$$

and (30) gives with the help of (28) and (37)

$$\left[\frac{d}{df}\left(\frac{S}{S_o}\right)\right]_{f=0}=-\beta$$

$$=\frac{4\eta(\mu_r+\sqrt{2})\{\sqrt{2}(m-1)-2\mu_r\}(N_r-1)m(\rho_2-\rho)}{4[\eta(\mu_r+\sqrt{2})^2(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\Delta_r(m+1)^2]} \tag{40}$$

Hence in general, for any given value of N_r, if

(i) $\rho_{\min}\leqslant\rho\leqslant\rho_1$, $\frac{d}{df}\left(\frac{S}{S_o}\right)$ is always positive right from the beginning and the charge is throughout degressive.

(ii) $\rho_1 < \rho \leqslant \rho_2$, $\frac{d}{df}\left(\frac{S}{S_o}\right)$ is negative in the beginning and then positive. Hence the charge is first progressive and then degressive.

(iii) $\rho > \rho_2$, $\frac{d}{df}\left(\frac{S}{S_o}\right)$ is always negative and the charge is throughout progressive.

These results can be summarised as follows :

	$\rho_{min} \leqslant \rho \leqslant \rho_1$	$\rho_1 < \rho \leqslant \rho_2$	$\rho > \rho_2$
$\frac{S}{S_o}$	Decreasing function of f and the charge is throughout degressive.	Increasing function of f in the beginning and then decreasing function. The charge is first progressive and then degressive.	Increasing function of f throughout. The charge is always progressive.

Formfunction for the second phase of combustion.

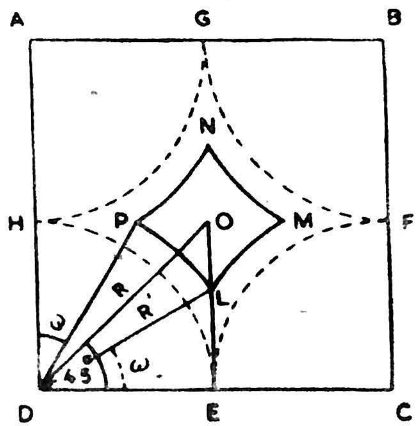

Fig. 5. Showing the burning of the section of a sliver during the second phase of combustion.

At the begining of the second phase of combustion, the charge consists of μ_r^2 exactly alike prisms having curvilinear squares like *EFGH* as bases and the length of the charge at this instant

$$=L-e_r=\frac{[2(\mu_r+\sqrt{2})m\rho-\{\sqrt{2}\,(m-1)-2\mu_r\}]d}{2(\mu_r+\sqrt{2})} \qquad (41)$$

while the radius of the arc like $EH=\frac{a}{r}$ (42)

The side of a square like $ABCD=\frac{2a}{r}$. (43)

Let R be the circum-radius of a square like $ABCD$ and R' the radius of the arc of a curvilinear square like $LMNP$, to which the curvilinear square $EFGH$ shrinks in time t during the second phase of combustion. If the bounding radii of the arc $L\rho$ make angle ω each with the sides DA and DC of the square $ABCD$, we have

$$R\cos\frac{\pi}{4}=R'\cos\omega=\frac{a}{r}$$

$$\therefore\quad R=\frac{\sqrt{2}\,a}{r} \text{ and } R'=\frac{a}{r}\sec\omega. \tag{44}$$

For the complete combustion of the prisms, $R=R'$ so that from (44), we get

$$\cos\omega=\frac{1}{\sqrt{2}}$$

$$\therefore\quad \omega=\frac{\pi}{4}. \tag{45}$$

This being independent of r, the complete combustion of both the categories takes place when $\omega=\frac{\pi}{4}$. Now, at any time t during the second stage of buring, when the curvilinear square $EFGH$ shrinks to $LMNP$, the length of the grain is given by

$$L^1=L—e_r-2(R'-\frac{a}{r})$$

$$=\left[(m\rho+1)-\frac{\sqrt{2}\,(m+1)}{2(\mu_r+\sqrt{2})}\sec\omega\right]d \tag{46}$$

This should remain positive when complete combustion takes place, i. e. when $\omega=\frac{\pi}{4}$.

Hence $$(m\rho+1)-\frac{\sqrt{2}(m+1)\sqrt{2}}{2(\mu_r+\sqrt{2})}\geqslant 0$$

or $$\rho\geqslant\left[\frac{(m+1)}{\mu r+\sqrt{2}}-1\right]\frac{1}{m}=\rho_3$$

where $$\rho_3=\left[\frac{(m+1)}{\mu_r+\sqrt{2}}-1\right]\frac{1}{m} \tag{47}$$

Now the area of curvilinear square $LMNP$

= Area of the square $ABCD-8\,\triangle\,DLE-4$ sector DLP

$$=\frac{a^2}{r^2}\left[4-4\tan\omega-(\pi-4\omega)\sec^2\omega\right]$$

Hence area of all the curvilinear squares

$$=\frac{\mu_r^2(m+1)^2d^2}{8(\mu_r+\sqrt{2})^2}\left\{4(1-\tan\omega)-(\pi-4\omega)\sec^2\omega\right\}=\frac{\mu_r^2(m+1)^2d^2}{8(\mu_r+\sqrt{2})^2}F(\omega) \tag{48}$$

where $$F(\omega)=4(1-\tan\omega)-(\pi-4\omega)\sec^2\omega. \tag{49}$$

Denoting the volume of a prism having a base $LMNP$ at any instant during the second phase of combustion by $V(LMNP)$, the volume of the charge at this instant

$$=\mu_r^2\,V(LMNP)$$

$$=\frac{\mu_r^2(m+1)^2d^3}{8(\mu_r+\sqrt{2})^2}F(\omega)\left[(m\rho+1)-\frac{\sqrt{2}(m+1)}{2(\mu_r+\sqrt{2})}\sec\omega\right]$$

$$=\frac{\mu_r^2(m+1)^2d^3}{16(\mu_r+\sqrt{2})^3}G(\omega) \tag{50}$$

where $$G(\omega)=[2(\mu_r+\sqrt{2})(m\rho+1)-\sqrt{2}(m+1)\sec\omega]F(\omega). \tag{51}$$

$$\therefore z = 1 - \frac{\mu_r^2 V(LMNP)}{V_o}$$

$$= 1 - \frac{\mu_r^2 (m+1)^2 G(\omega)}{4(\mu_r + \sqrt{2})[\pi(m^2 - N_r)(\mu_r + \sqrt{2})^2 - (m+1)^2 \mu_r \Delta_r] m\rho}. \tag{52-}$$

Initially $\omega = 0$,

$$F(\omega) = 4 - \pi \tag{53}$$

and $G(\omega) = [2(\mu_r + \sqrt{2})(m\rho + 1) - \sqrt{2}(m+1)](4 - \pi)$. (54)

So that with the help of (54), (52) gives the value of z at the beginning of the second phase of combustion as

$$z = 1 - \frac{\mu_r^2 (m+1)^2 (4-\pi)[2(\mu_r + \sqrt{2})(m\rho + 1) - \sqrt{2}(m+1)]}{4(\mu_r + \sqrt{2})[\pi(m^2 - N_r)(\mu_r + \sqrt{2})^2 - (m+1)^2 \mu_r \Delta_r] m\rho}$$

$$= \{\sqrt{2}(m-1) - 2\mu_r\}] \, 2\pi(\mu_r + \sqrt{2}) \{4\mu_r + \sqrt{2}(N_r + 3)\} m^2\rho + 2\pi(\mu_r + \sqrt{2})\{N_r(2\mu_r + 3\sqrt{2}) + (2\mu_r + \sqrt{2})\} m\rho + (m+1)^2 \{(2\mu_r + \sqrt{2})^2 - 2N_r)\pi - 4\mu_r\Delta_r\}) \div 8m\rho(\mu_r + \sqrt{2})[\pi(m^2 - N_r)(\mu_r + \sqrt{2})^2 - (m+1)^2 \times \mu_r \Delta_r]$$

$$= A \tag{55}$$

which is the value of z at the end of the first phase of combustion. At the time of complete combustion of the grain

$$\omega = \frac{\pi}{4}, \; G(\omega) = 0,$$

so that $z = 1$, a value which z should attain at the instant of complete combustion.

Let f be defined as the ratio of the distance receded (from the begi ı ning of the second phase of combustion up to the instant considered) to the initial thickness.

Then $$f=\frac{2\left(\frac{a}{r}-R'\right)}{e_r}$$

$$=\frac{\sqrt{2}(m+1)}{\sqrt{2}(m-1)-2\mu_r}\left(1-\frac{1}{\cos\omega}\right). \quad (56)$$

Initially $\omega=0$ and $f=0$.

For $\omega=\frac{\pi}{4}$, $$\left[f\right]_{\omega=\frac{\pi}{4}}=\frac{\sqrt{2}(m+1)}{\sqrt{2}(m-1)-2\mu_r}\left(1-\sqrt{2}\right). \quad (57)$$

Using (51) and (56) in (52), we get

$$z=1-\frac{\mu_r(m+1)^2[2(\mu_r+\sqrt{2})m\rho-(\sqrt{2}m-\sqrt{2}-2\mu_r)(1-f)]F(\omega)}{4[\pi(m^2-N_r)(\mu_r+\sqrt{2})^2-(m+1)^2\mu_r\Delta_r](\mu_r+\sqrt{2})m\rho} \quad (58)$$

We shall now find out $\frac{S}{S_0}$ for the second phase of combustion in terms of ω and N_r.

$$\frac{dz}{df}=\frac{dz/d\omega}{df/d\omega}$$

$$=\mu_r^2(m+1)\{\sqrt{2}(m-1)-2\mu_r\}[4(4\omega-\pi)(\mu_r+\sqrt{2})(m\rho+1)\sec\omega-\sqrt{2}(m+1)\{(12\omega-3\pi)\sec^2\omega+4(1-\tan\omega)\}]\div 4\sqrt{2}(\mu_r+\sqrt{2})[\pi(m^2-N_r)(\mu_r+\sqrt{2})^2-(m+1)^2\mu_r\Delta_r]m\rho \quad (59)$$

and

$$\left[\frac{dz}{df}\right]_{f=1}=-(A-B-C)$$

$$=\frac{4\{\sqrt{2}(m-1)-2\mu_r\}[\mu_r\Delta_r(m+1)^2-\pi(\mu_r+\sqrt{2})^2(2m^2\rho+2m\rho N_r+m^2-N_r)]}{8m\rho(\mu_r+\sqrt{2})[\pi(m^2-N_r)(\mu_r+\sqrt{2})^2-(m+1)^2\mu_r\Delta_r]} \quad (60)$$

Hence

$$\frac{S}{S_0}=\frac{dz/df}{\left(dz/df\right)_{f=1}}$$

$$=\frac{\mu_r^2(m+1)[4(4\omega-\pi)(\mu_r+\sqrt{2})(m\rho+1)\sec\omega-\sqrt{2}(m+1)H(\omega)]}{2\sqrt{2}[(m+1)^2\mu_r\Delta_r-\pi(\mu_r+\sqrt{2})^2(2m^2\rho+2m\rho N_r+m^2-N_r)]}$$

(61)

where $H(\omega)=(12\omega-3\pi)\sec^2\omega+4(1-\tan\omega)$. (62)

Initially for $\omega=0$, $H(\omega)=-3\pi+4$.

and

$$\frac{S}{S_0}=\frac{\mu_r^2(m+1)[4\pi(\mu_r+\sqrt{2})(m\rho+1)+\sqrt{2}(m+1)(4-3\pi)]}{2\sqrt{2}[\pi(\mu_r+\sqrt{2})^2(2m^2\rho+2m\rho N_r+m^2-N_r)-(m+1)^2\mu_r\Delta_r]}.$$

(63)

At the instant of complete combustion $\omega=\frac{\pi}{4}$, $H(\omega)=0$, so that from (63)

$$\frac{S}{S_0}=0.$$

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ON A GENERALIZED LAPLACE TRANSFORM OF TWO VARIABLES

by

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(Recd. on 26–3–1969)

1. Introduction

The Laplace transform in its classical form is :

$$(1.1) \qquad f(s) = \int_{0}^{\infty} e^{-st} \theta(t)\, dt.$$

Verma [3] has given a generalization of (1.1) in the form :

$$(1.2) \qquad f(s) = \int_{0}^{\infty} e^{-\frac{1}{2}st} (st)^{m-\frac{1}{2}} W_{k,\,m} (st)\, \theta\, (t)\, dt$$

which reduces to (1.1) when $k+m=\frac{1}{2}$. It can also be represented in the form

$$(1.3) \qquad f(s) = \int_{0}^{\infty} e^{-st} (st)^{2m} \psi\, (\tfrac{1}{2}-k+m,\, 2m+1\, ;\, st)\, \theta\, (t)\, dt$$

where ψ denotes Tricomi's Confluent Hypergeometric function given by the relation

$$(1.4) \qquad W_{k,m}(z) = e^{-\frac{1}{2}z} z^{m+\frac{1}{2}} \psi\, (\tfrac{1}{2}-k+m,\, 2m+1;\, z).$$

Ditkin and Prudnikov [1] have defined Laplace transform of two variables as

$$(1.5) \qquad F(p,q) = \int_{0}^{\infty}\int_{0}^{\infty} e^{-px-qy} f(x,y)\, dx\, dy.$$

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We now define a generalized Laplace transform of two variables in the form

$$(1.6) \qquad F(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy} (pqxy)^{2m} \psi(\tfrac{1}{2}-k+m, 2m+1; px) \times \psi(\tfrac{1}{2}-k+m, 2m+1; qy) f(x,y)\, dx\, dy$$

where ψ is Tricomi's Hypergeometric function [2, p. 257].

The object of this note is to obtain a convergence theorem and some rules and results in operational calculus for (1.6).

2. The Generalization

Let $f(x,y)$ be a complex function of two variables defined on the region R $(0 \leqslant x < \infty, 0 \leqslant y < \infty)$ and integrable in the sense of Lebesgue over an arbitrary rectangle $R_{a,b}$ $(0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b)$.

Let us consider the integral

$$(2.1) \qquad F(p,q;a,b) = \int_0^a \int_0^b e^{-px-qy} (pqxy)^{2m} \psi(\tfrac{1}{2}-k+m, 2m+1; px) \times \psi(\tfrac{1}{2}-k+m, 2m+1; qy) f(x,y)\, dx\, dy$$

where $p=\sigma+i\mu$ and $q=\gamma+iv$ are complex parameters determining a a point (p,q) in the plane of two complex dimensions. Let S be the class of functions such that the following conditions are satisfied for at least one point (p,q).

A. The integral (2.1) is bounded at the point (p,q) which respect to the variables a and b, i.e.

$$|F(p,q;a,b)| < M(p,q)$$

in all $a \geqslant 0$, $b \geqslant 0$ where $M(p,q)$ is a constant depending upon p, q but independent of a and b.

B. At the point (p,q)

$$\lim_{\substack{a\to\infty \\ b\to\infty}} F(p,q;a,b) = F(p,q)$$

exists. We denote this limit by

$$(2.2) \qquad F(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy} (pqxy)^{2m} \psi(\tfrac{1}{2}-k+m, 2m+1;xp) \times \psi(\tfrac{1}{2}-k+m, 2m+1;yq) f(x,y)\, dx\, dy.$$

If the conditions A and B are satisfied simultaneously, we say that the integral (2.2) converges for at least one point (p, q). The integral (2.2) can be called a generalised two dimensional Laplace transform. The function $f(x,y)$ is known as original and $F(p,q)$ as image.

To save space we denote

$$(2.3) \qquad e^{-px} (px)^{2m} \psi(\tfrac{1}{2}-k+m, 2m+1; px) = G(px)$$

where no ambiguity arises.

3. Convergence Theorem

THEOREM : If the integral (2.2) converges boundedly at the point (p_o, q_o), then it converges at all points (p, q) provided that

$Re\,(p) > Re\,(p_o)$, $Re\,(q) > Re\,(q_o)$ and

(i) $\tfrac{1}{2}-k\pm m \neq 0, -1, -2, \ldots$

or

(ii) $\tfrac{1}{2}-k+m = 0$, $Re(m)>0$.

Proof : Using the notation (2.3) we can write (2.1) as :

$$F(p,q;a,b) = \int_o^a \int_o^b G(px)\, G(qy)\, f(x,y)\, dx\, dy$$

$$= \int_o^a G(px)\, dx \int_o^b \frac{G(qy)}{G(q_o y)} \frac{\delta}{\delta y} \int_o^y G(q_o,\eta)\, f(x,\eta)\, d\eta.$$

Integrating by parts

$$= \int_o^a G(px) \left[\frac{G(qb)}{G(q_o,b)} \int_o^b G(q_o,\eta)\, f(x,\eta)\, d\eta - \int_o^b \frac{d}{dy}\left(\frac{G(q_o,y)}{G(q_o,y)}\right) dy \int_o^y G(q_o,\eta)\, f(x,\eta)\, d\eta \right]$$

$$= \int_0^a \frac{G(px)}{G(p_o x)} \frac{\partial}{\partial x} \int_0^x G(p_o \xi) d\xi \left[\frac{G(qb)}{G(q_o b)} \int_0^b G(q_o \eta)\, f(\xi,\eta)\, d\eta \right.$$

$$\left. - \int_0^b \frac{d}{dy}\left(\frac{G(qy)}{G(q_o y)} \right) dy \int_0^y G(q_o \eta)\, f(\xi,\eta)\, d\eta \right],$$

$$= \frac{G(pa)}{G(p_o a)} \int_0^a G(p_o \xi)\, d\xi \left[\frac{G(qb)}{G(q_o b)} \int_0^b G(q_o \eta)\, f(\xi,\eta)\, d\eta \right.$$

$$- \int_0^x \frac{d}{dx}\left(\frac{G(px)}{G(p_o x)} \right) dx \int_0^x G(p_o \xi)\, d\xi \left[\frac{G(qb)}{G(q_o b)} \int_0^b G(q_o \eta)\, f(\xi,\eta)\, d\eta \right.$$

$$\left. - \int_0^b \frac{d}{dy}\left(\frac{G(qy)}{G(q_o y)} \right) dy \int_0^y G(q_o \eta)\, f(\xi,\eta)\, dn \right].$$

Let

$$\theta(x,y) = \int_0^x G(p_o \xi)\, d\xi \int_0^y G(q_o \eta)\, d\eta. \tag{3.1}$$

Therefore

$$F(p,q;a,b) = \frac{G(pa)\, G(qb)}{G(p_o a)\, G(q_o b)} \theta(a,b) - \frac{G(pa)}{G(p_o a)} \int_0^b \frac{d}{dy}\left(\frac{G(qy)}{G(q_o y)} \right) \tag{3.2}$$

$$\theta(a,y)\, dy - \frac{G(qb)}{G(q_o b)} \int_0^a \frac{d}{dx}\left(\frac{G(px)}{G(p_o x)} \right) \theta(x,b)\, dx + \int_0^a \frac{d}{dx}\left(\frac{G(px)}{G(p_o x)} \right) dx$$

$$\int_0^b \frac{d}{dy}\left(\frac{G(qy)}{G(q_o y)} \right) \theta(x,y)\, dy;$$

Since (2.2) converges at the point (p_o, q_o), therefore

$$| \theta(x,y) | < M(p_o,q_o) = M. \tag{3.3}$$

Now

$$(3.4)\qquad (s/s_o)^{2m}\frac{e^{-sR}\,\psi(\tfrac{1}{2}-k+m,2m+1;sR)}{e^{-s_oR}\,\psi(\tfrac{1}{2}-k+m,2m+1;s_oR)}\infty(s/s_o)^{m+k-\frac{1}{2}}$$

$$e^{-R(s-s_o)},\ (R\to\infty) = o(1),\ (R\to\infty);$$

since [2, p. 278] :

$$(3.5)\qquad \psi(\tfrac{1}{2}-k+m,2m+1;z) \infty z^{k-m-\frac{1}{2}},\ (z\to\infty).$$

Again let us set

$$Q = \psi(\tfrac{1}{2}-k+m,2m+1;x)$$

then by [2, p. 262] :

(i) If $\frac{1}{2}-k\pm m \neq 0$ or a negative o integer and $x\to 0$, then

$$(3.6)\qquad Q \infty \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k+m)}+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}x^{-2m}$$

or

(ii) If $\frac{1}{2}-k\pm m \neq 0$ or a negative integer, $m = 0$ or a positive integer and x tends to zero, then

$$(3.7)\qquad Q \infty \frac{(-1)^{2m+1}}{(2m)\,!\,\Gamma(\frac{1}{2}-k+m)}\Big[\log x+\psi(\tfrac{1}{2}-k+m)-\psi(2m+1)$$

$$-\psi(1)\Big]+\frac{(2m+1)\,!}{\Gamma(\frac{1}{2}-k+m)}x^{-2m};$$

or

(iii) If $\frac{1}{2}-k\pm m \neq 0$ or a negative integer, $2m$ is a negative integer and x tends to zero, then

$$(3.8)\qquad Q \infty \frac{(-1)^{-2m-1}\,x^{-2m}}{(2m)\,!\,\Gamma(\frac{1}{2}-k+m)}\Big[\log x+\psi(\tfrac{1}{2}-k+m)-$$

$$\psi(-2m+1)-\psi(1)\Big]+\frac{(-2m-1)\,!}{\Gamma(\frac{1}{2}-k+m)}$$

or

(iv) If $\frac{1}{2}-k+m \neq 0$, $m>0$ and x tends to zero, then

$$(3.9)\qquad Q \infty x^{-2m}.$$

Using the results (3.6), (3.7), (3.8) and (3.9), we have

$$\lim_{t\to 0}\frac{e^{-st}\,\psi(\tfrac{1}{2}-k+m,2m+1;st)}{e^{-s_ot}\,\psi(\tfrac{1}{2}-k+m.2m+1;s_ot)} = (s/s_o)^{-2m} = 1$$

also

$$\left| \int_0^R \frac{d}{dx}\left(\frac{G(px)}{G(p_o x)}\right) \phi(x,y)\, dx \right| \leqq M \left| \int_0^R \frac{d}{dx}\left(\frac{G(px)}{G(p_o x)}\right) dx \right|$$

$$= M \left| \left[\frac{G(px)}{G(p_o x)}\right]_{x=0}^{x=R} \right| .$$

Therefore

$$\lim_{\substack{a \to \infty \\ b \to \infty}} F(p,q;a,b) = \lim_{\substack{a \to \infty \\ b \to \infty}} \int_0^a \frac{d}{dx}\left(\frac{G(px)}{G(p_o x)}\right) dx \int_0^b \frac{d}{dy}\left(\frac{G(qy)}{G(q_o y)}\right) \theta(x,y)\, dy,$$

from which it follows that the condition B is satisfied and the theorem is proved.

4. Some Rules :

Now we give some rules in operational calculus for (2.2). We, however, consider the following more general form

$$(4.1)\quad F(p,q) = pq \int_0^\infty \int_0^\infty e^{-px-qy} (px)^{2m} (qy)^{2m'} \psi(\tfrac{1}{2}-k+m, 2m+1; px)$$
$$\times \psi(\tfrac{1}{2}-k'+m', 2m'+1; qy)\, f(x,y)\, dx\, dy$$

which we denote symbolically as

$$F(p,q) \overset{m,k}{\underset{m',k'}{\rightrightarrows}} f(x,y).$$

We further use the notation

$$F(p) \overset{m,k}{\longrightarrow} f(y)$$

to denote the generalized Laplace transform

$$(4.2)\quad F(p) = p \int_0^\infty e^{-py} (py)^{2m} \psi(\tfrac{1}{2}-k+m, 2m+1; py)\, f(y)\, dy.$$

Rules : It is plain that if

$$f(x,y) = f_1(x)\, f_2(y)$$

and if

$$F_1(p) \overset{m,k}{\longrightarrow} f_1(y)$$

$$F_2(q) \overset{m',k'}{\longrightarrow} f_2(y)$$

then

$$F(p,q) = F_1(p)\, F_2(q).$$

Similarly, it is easily seen that if

$$F_1(p,q) \underset{m',k'}{\overset{m,k}{\rightrightarrows}} f_1(x,y)$$

$$F_2(p,q) \underset{m',k'}{\overset{m,k}{\rightrightarrows}} f_2(x,y)$$

....

....

then

$$\sum_{r=1}^{n} F_r(p,q) \underset{m',k'}{\overset{m,k}{\rightrightarrows}} \sum_{r=1}^{n} f_r(x,y).$$

If in (4.1) we replace p by p/a and q by q/b where a and b are constants, we get the so called 'Similarity Rule', i.e.

(4.3) $$F(p/a,q/b) \underset{m',k'}{\overset{m,k}{\rightrightarrows}} f(ax,by).$$

On differentiating the relation (4.3) with respect to a and b partially and then putting a and b both equal to unity, we obtain

$$p\frac{\partial}{\partial p} F(p,q) \underset{m',k'}{\overset{m,k}{\rightrightarrows}} x\frac{\partial}{\partial x} f(x,y)$$

$$q\frac{\partial}{\partial q} F(p,q) \underset{m',k'}{\overset{m,k}{\rightrightarrows}} y\frac{\partial}{\partial y} f(x,y).$$

If after dividing (4.3) by a and b respectively and integrate between $(0,\infty)$, we get

$$\int_{o}^{\infty} a^{-1} F(a,q)\, da \underset{m',q'}{\overset{m,k}{\rightleftarrows}} \int_{o}^{\infty} a^{-1} f(a,y)\, da$$

$$\int_{o}^{\infty} b^{-1} F(p,b)\, db \underset{m',k'}{\overset{m.k}{\rightleftarrows}} \int_{o}^{\infty} b^{-1} f(x,b)\, db$$

and

$$\int_{o}^{\infty}\int_{o}^{\infty} (ab)^{-1} F(a,b)\, da\, db \underset{m',k'}{\overset{m,k}{\rightleftarrows}} \int_{o}^{\infty}\int_{o}^{\infty} (ab)^{-1} f(a,b)\, da\, db$$

provided that the integrals involved exist.

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STUDY OF MEIJER G-FUNCTION OF TWO VARIABLES (PART II)

by

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ABSTRACT

The sum of infinite series involving Meijer's G-function of two variables defined by Agarwal (1) have been given in Meijer's G-function of two variables.

INTRODUCTION

In the first part we had given a few finite series (2). Meijer's G-function of two variables defined by Agarwal (1) is

$$G^{n_1,n_2,n_3,m_1,m_2}_{p_1,q_1,r_2,q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\left\{(a_{p_1});(d_{p_2})\right\}:\left\{(b_{q_1});(c_{q_1})\right\}:\left\{(e_{q_2});(f_{q_2})\right\}\right]$$

$$=\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\phi(s+t)\,\psi(s,t)\,x^s y^t\,ds\,dt \qquad \cdots(1.0)$$

where
$$\phi(s+t)=\prod_{j=1}^{n_1}\Gamma(1-a_j+s+t)\left[\prod_{j=1+n_1}^{p_1}\Gamma(a_j-s-t)\prod_{j=1}^{p_2}\Gamma(d_j+s+t)\right]^{-1} \qquad \cdots(1.1)$$

$$\psi(s,t)=\frac{\prod_{j=1}^{m_1}\Gamma(e_j-s)\prod_{j=1}^{n_2}\Gamma(b_j+s)\prod_{j=1}^{m_2}\Gamma(f_j-t)}{\prod_{j=1+m_1}^{q_1}\Gamma(1-e_j+s)\prod_{j=1+n_2}^{q_1}\Gamma(1-b_j-s)\prod_{j=1+m_2}^{q_2}\Gamma(1-f_j+t)}\cdot\frac{\prod_{j=1}^{n_3}\Gamma(c_j+t)}{\prod_{j=1+n_3}^{q_1}\Gamma(1-c_j-t)} \qquad \cdots(1.2)$$

$o \leqslant m_1 \leqslant q_2,\ o \leqslant m_2 \leqslant q_2,\ o \leqslant n_2 \leqslant q_1,\ o \leqslant n_3 \leqslant q_1,\ o \leqslant n_1 \leqslant p_1$ .. (1.3)

$$\left.\begin{array}{l}(\alpha_p) = \alpha_1, \alpha_2, \alpha_3, \ \ , \alpha_p\ ;\ (\alpha_{m,p}) = \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_p \\ (\alpha_{p_1})+\gamma+\gamma_1=(a_1+\gamma+\gamma_1),(a_2+\gamma+\gamma_1),(a_3+\gamma+\gamma_1), \ldots ,(a_{p_1}+\gamma+\gamma_1)\end{array}\right\} \quad ..(1.4)$$

$$\omega < 2\bar{\omega}\ |\arg x| < \pi\,(\bar{\omega}-\tfrac{1}{2}\omega);\ \omega < 2\bar{\omega}_1,\ |\arg y| < \pi\,(\bar{\omega}_1-\tfrac{1}{2}\omega) \quad ..(A)$$

$$\omega = p_1+q_1+p_2+q_2,\ \bar{\omega} = m_1+n_2+n_1,\ \bar{\omega}_1 = m_2+n_3+n_1 \quad ...(1.6)$$

Also the sequence of parameters (e_{m_1}), (f_{m_2}), (b_{n_2}), (c_{n_3}) and (a_{n_1}) are such that none of the poles of the integrand coincide. The paths of integration are indented, if necessary, in such a way that all the poles of $\Gamma(e_j-s)$, $j=1,2,..,m_1$, and $\Gamma(f_k-t)$, $k=1,2,\ldots,m_2$ lie to the right and those of $\Gamma(b_j+s)$, $j=1,2,\ldots,n_2$, $\Gamma\, c_k+t)$, $k = 1, 2, \ldots, n_3$ and $\Gamma(1-a_j+s+t)\ j=1,2, \ \ ,n_1$ lie to the left of the imaginary axis. Also if the parameters are given as in the l.h.s. of (1.0), we shall abbreviate it as $G(x,y)$ or $G\left[\begin{array}{c}x\\y\end{array}\right]$ and in case any one of them, is different, only the different one will be mentioned e.g.

$$G^{n_1,n_2,n_3,m_1,m_2}_{p_1,q_1,p_2,q_2}\left[\begin{array}{c}x\\y\end{array}\ \{(a_{p_1});(d_{p_2})\}:\{b_{q_1});(c_{q_1})\}:\{e_1+\gamma,(e_{2},_{q_2});f_1+\gamma_1,(f_{2},_{q_2})\}\right]$$

will be written as

$$G\left[\begin{array}{c}x\\y\end{array}\middle|\ \{\ \}:\{\ \}:\{e_1+\gamma,(e_{2},_{q_2});f_1+\gamma_1,(f_{2},_{q_2})\}\right]$$

here after

Further the following known results will be required in the proof of the sequel

$${}_2F_1\left(\begin{array}{c}a,b\\c\end{array}:1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ R\,(c-a-b) > o \quad (1.7)$$

$${}_5F_4\left(\begin{array}{l}\alpha, \frac{\alpha}{2}+1, \beta, \gamma, \delta\\ \frac{\alpha}{2}, \alpha-\beta+1, \alpha-\gamma+1, \alpha-\delta+1\end{array}:1\right)$$
$$= \frac{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)\Gamma(\alpha-\delta+1)\Gamma(\alpha-\beta-\gamma-\delta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)\Gamma(\alpha-\gamma-\delta+1)\Gamma(\alpha-\beta-\delta+1)},$$
$$R\,(\alpha-\beta-\gamma-\delta) > -1 \quad (1.8)$$

$$ {}_4F_3\left(\begin{matrix}\alpha, \frac{\alpha}{2}+1, \beta, \gamma \\ \frac{\alpha}{2}, \alpha-\beta+1, \alpha-\gamma+1\end{matrix} : 1\right) = \frac{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)}, $$

$$ R(\alpha-2\beta-2\gamma) > -2 \quad ...(1.9) $$

$$ x^{-2b_1-2s} {}_1F_0(b_1+s ; - ; t/x) = (x^2-xt)^{-(b_1+s)} \quad .. (2.0) $$

$$ \frac{(s)_\gamma}{(1-s)_\gamma} = \frac{\Gamma(s+\gamma)\Gamma(1-s)}{\Gamma(s)\Gamma(1-s+\gamma)} \quad ...(2.1) $$

$$ (\operatorname{Sin}\frac{\theta}{2})^{-2s} = \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(\frac{1}{2})\Gamma(1-s)}\left[1+2\sum_{\gamma=1}^{\infty}\frac{(s)_\gamma}{(1-s)_\gamma}\operatorname{Cos}\gamma\theta\right], $$

$$ o \leqslant \theta \leqslant \pi, \ R(1-2s) > o \quad ...(2.2) $$

$$ (\operatorname{Sin}\theta)^{1-2s} = \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{3}{2})\Gamma(2-s)}\sum_{\gamma=o}^{\infty}\frac{(s)_\gamma}{(2-s)_\gamma}\operatorname{Sin}(2\gamma+1)\theta, \ o \leqslant \theta \leqslant \pi, $$

$$ R(1-2s) \geqslant o \quad ...(2.3) $$

Theorem I : If conditions (A) and $R(c)>0$, $R(b)>0$, then

$$ \sum_{\gamma,\gamma_1=o}^{\infty} \frac{(b-b_{q_1})_\gamma (c-c_{q_1})_{\gamma_1}}{\gamma!\gamma_1!\Gamma(1+b_1-b_{q_1}+\gamma)\Gamma(1+c_1-c_{q_1}+\gamma_1)} \times $$

$$ G\left[\begin{matrix}x\\y\end{matrix}\middle| \{\ \} : \{b_1+\gamma, (b2,q_1-1), b ; c_1+\gamma_1, (c2,q_1-1), c\} : \{\ \}\right] $$

$$ = \frac{1}{\Gamma(b_1-b+1)\Gamma(c_1-c+1)} G\left[\begin{matrix}x\\y\end{matrix}\right] \quad ...(2.4) $$

Proof :—In the l.h.s. of (2.4), write the value of $G(x,y)$ in the contour form by (1.0), change the order of integration and summation which is justified under the conditions $R(c)>0$, $R(b)>0$, simplify, use (1.7) and interpret again with the help of (1.0), the r.h.s. follows.

Theorem II : If conditions (A) and $|h| < 1$, $|h_1| < 1$, then

$$ \sum_{\gamma,\gamma_1=0}^{\infty} \frac{(-1)^{\gamma+\gamma_1} h^\gamma h_1^{\gamma_1}}{\gamma!\gamma_1!} G\left[\begin{matrix}x\\y\end{matrix}\middle| \{\ \} : \{(b_{q_1-1}), b_{q_1}+\gamma ; (c_{q_1-1}), c_{q_1}+\gamma\} : \{\ \}\right] $$

$$ = (1-h)^{-b_{q_1}}(1-h_1)^{-c_{q_1}} G\left[\frac{x}{1-h}, \frac{y}{1-h_1}\right] \quad ...(2.5) $$

Proof :—Identical as for theorem I.

Theorem III : If conditions (A) and $R(e_{q_1-1}+k)<1$, $R(f_{q_1-1}+k_1)<1$, then

$$\sum_{\gamma\cdot\gamma_1=0}^{\infty} \frac{\gamma(\frac{\lambda}{2}+1)_{\gamma}\,\Gamma(\lambda+\gamma)\,(e_{q_1-1})_{\gamma}\,(K)_{\gamma}\,\lambda_1(\frac{\lambda_1}{2}+1)_{\gamma_1}}{(\frac{\lambda}{2})_{\gamma}\,\Gamma(\lambda-e_{q_2-1}+1+\gamma)\,\Gamma(\lambda-k+1+\gamma)\,\Gamma(\lambda_1-f_{q_2-1}+1+\gamma_1)}$$

$$\frac{\Gamma(\lambda_1+\gamma_1)\,(f_{q_2-1})_{\gamma_1}\,(k_1)_{\gamma_1}}{\Gamma(\lambda_1-k_1+1+\gamma_1)\,\gamma\,!\,\gamma_1\,!\,(\frac{\lambda_1}{2})_{\gamma_1}}\times$$

$$G\left[\begin{matrix}x\\y\end{matrix}\middle|\{\}:\{(b_{q_1}-1),\lambda\,;(c_{q_1}-1),\lambda_1\}:\{\lambda+\gamma,(e_{2},_{q_9-2}),e_{q_2-1}+k,\right.$$

$$\left.-\gamma\,;\lambda_1+\gamma_1,(f_2,_{q_2-2}),f_{q_2-1}+k_1,-\gamma_1\}\right]$$

$$=\left[\Gamma(\lambda-e_{q_2-1}-k+1)\,\Gamma(\lambda_1-f_{q_2-1}-k_1+1)\right]^{-1}\times$$

$$G\left[\begin{matrix}x\\y\end{matrix}\middle|\{\}:\{(b_{q_1-1}),\lambda\,;(c_{q_1-1}),\lambda_1\}:\{\lambda,(e_2,_{q_2-1}),k\,;\lambda_1,\right.$$

$$\left.(f_2,_{q_2-1}),k_1\}\right] \quad \cdots (2.6)$$

Proof :—Identical with theorem I. Use result (1.8) to sum the inner series.

Theorem IV : If conditions (A) and $R(\lambda+2e_{q_2})<2$, $R(\lambda_1+2f_{q_2})<2$, then

$$\sum_{\gamma,\gamma_1=0}^{\infty} \frac{\lambda.\lambda_1(\frac{\lambda}{2}+1)_{\gamma}\,(\frac{\lambda_1}{2}+1)_{\gamma_1}\,(e_{q_2}-\lambda)_{\gamma}\,(f_{q_2}-\lambda_1)\,(-1)^{\gamma+\gamma_1}}{(\frac{\lambda}{2})_{\gamma}\,(\frac{\lambda_1}{2})_{\gamma_1}\,\gamma\,!\,\gamma_1\,!\,\Gamma(2\lambda-e_{q_2}+1+\gamma)}$$

$$\frac{\Gamma(\lambda+\gamma)\,\Gamma(\lambda_1+\gamma_1)}{\Gamma(2\lambda_1\quad f_{q_2}+1+\gamma_1)}\times$$

$$G\left[\begin{matrix}x\\y\end{matrix}\middle|\{\}:\{\}:\{2\lambda+\gamma,(e_2,_{q_2-1}),\lambda-\gamma\,;2\lambda_1+\gamma_1,(f_2,_{q_2-1}),\lambda_1-\gamma_1\}\right]$$

$$=G\left[\begin{matrix}x\\y\end{matrix}\middle|\{\}:\{\}:\{2\lambda,(e_2,_{q_2})\,;2\lambda_1,(f_2,_{q_2})\}\right] \quad \cdots (2,7)$$

Proof :—Identical with theorem I. Use (1.9) to sum the inner series.

Theorem V : If conditions (A), $|\arg(x-t)| < \pi(\bar{\omega}-\frac{1}{2}\omega)$, $|\arg(y-t_1)| < \pi(\omega_1-\frac{\omega}{2})$, $|t/x| < 1$, $|t_1/y| < 1$

then

$$\sum_{\gamma,\gamma_1=0}^{\infty} \frac{x^{2a_1-2+\gamma}\, t^{\gamma}}{\gamma!} \frac{y^{2a_2-2+\gamma_1}\, t_1^{\gamma_1}}{\gamma_1!} \times$$

$$G\left[\begin{matrix} x^{-2} \\ y^{-2} \end{matrix} \middle| \{(a_{p_1})+\gamma+\gamma_1 ; (\alpha_{q_1})-\gamma-\gamma_1\} : \{b_1,(b_{2_{q_1}})-\gamma ; c_1,(c_{2,q_1})-\gamma_1\} : \right.$$

$$\left. : \{e_{q_2})+\gamma ; (f_{q_2})+\gamma_1\}\right]$$

$$=(x^2-xt)^{-b_1} (y^2-yt_1)^{-c_1}\, G\left[(x^2-xt)^{-1}, (y^2-yt_1)^{-1}\right] \quad ..(2.8)$$

Proof :—Almost identical to theorem I. Use (2.0), replace s by $s+r$ and t by $t+r_1$.

***Theorem VI** : If conditiens (A) and $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi\leqslant\pi$ $R(1-2b_1)>0$, $R(1-2c_1)>0$,

$$\text{then } G\left[\begin{matrix} x \\ y \end{matrix} \middle| \{\ \} : \{(b_{q_1}), b_1 ; (c_{q_1})\} : \{\tfrac{1}{2}-b_1, (e_{q_2}) ; (f_{q_2})\}\right]$$

$$+2 \sum_{\gamma=1}^{\infty} \frac{\cos\gamma\theta}{x^{\gamma}} G\left[\begin{matrix} x \\ y \end{matrix} \middle| \{(a_{p_1})+\gamma ; (d_{p_2})-\gamma\} : \{b_1,(b_{2,q_1})-\gamma, b_1-2\gamma ; \right.$$

$$\left. (c_{q_1})\} : \{\tfrac{1}{2}-b_1+\gamma, (e_{q_2})+\gamma ; (f_{q_2})\}\right] + G\left[\begin{matrix} x \\ y \end{matrix} \middle| \{\ \} : \{ b_{q_1}) ; (c_{q_1}), c_1\} : \right.$$

$$\left. \{(e_{q_2}), e_1-\tfrac{1}{2} ; (f_{q_2})\}\right]$$

$$+2 \sum_{\gamma_1=1}^{\infty} \frac{\cos\gamma_1\phi}{y^{\gamma_1}} G\left[\begin{matrix} x \\ y \end{matrix} \middle| \{(a_{p_1})+\gamma_1 ; (d_{p_2})-\gamma_1\} : \{(b_{q_1}) ; c_1, (c_{2_{q_1}}), \right.$$

$$\left. c_1-2\gamma_1\} : \{(e_{q_2}), e_1-\tfrac{1}{2}+\gamma_1 ; (f_{q_2})+\gamma_1\}\right]$$

$$=\sqrt{\pi}\,(\sin\tfrac{\theta}{2})^{-2b_1}\, G\left[\begin{matrix} x/\sin^2\theta/2 \\ y \end{matrix}\right] + \sqrt{\pi}\,(\sin\tfrac{\phi}{2})^{-2c_1}$$

$$G\left[\begin{matrix} x \\ y/\sin^2\phi/2 \end{matrix}\right] \quad ...(2.9)$$

Proof :—Identical with theorem V. Use (2.1) and (2.2) to obtain the inner sum.

***Theorem VII** : If conditions (A) and $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi\leqslant\pi$, $R(2-2b_1)\geqslant 0$, $R(1-2c_1)\geqslant 0$

then

$$\sum_{\gamma=0}^{\infty} x^{-\gamma}\ G\left[{x \atop y}\middle| \{(a_{p_1})+\gamma\,;(d_{p_2})-\gamma\}:\{b_1,(b_{2,q_1})-\gamma,\,b_1-1-2\gamma\,;\ (c_{q_1})\}\right]$$

$$+\sum_{\gamma_1=0}^{\infty} y^{-\gamma_1}\ G\left[{x \atop y}\middle| \{(a_{p_1})+\gamma_1\,;(d_{p_2})-\gamma_1\}:\{(b_{q_1})\,;\,c_1,(c_{2,q_1})-\gamma_1,\ c_1-1-2\gamma_1\}:\{(e_{q_2})\,;\,-c_1+\frac{3}{2}+\gamma_1,\,(f_{q_2})+\gamma_1\}\right]\sin(2\gamma_1+1)\,\phi$$

$$=(\sin\theta)^{1-2b_1}\frac{\sqrt{\pi}}{2}\ G\left[{x/\sin^2\theta \atop y}\right]+(\sin\phi)^{1-2c_1}\ \frac{\sqrt{\pi}}{2}\ G\left[{x \atop y/\sin^2\phi}\right] \quad \ldots(3.0)$$

Proof :—Identical with theorem VI. Use (2.1) and (2.3) to obtain the inner sum.

Particular cases :

We give only for theorem I in the sums, for other theorems by following the same scheme a large number of particular cases can be easily arrived at.

(1) If $n_1=p_1=p_2=0$, then we have

$$\sum_{\gamma,\gamma_1=0}^{\infty}\frac{(b-b_{q_1})_{\gamma}\,(c-c_{q_1})_{\gamma_1}}{\gamma\,!\,\gamma_1\,!\,\Gamma(1+\gamma+b_1-b_{q_1})\,\Gamma(1+\gamma_1+c_1-c_{q_1})}\times$$

$$G^{m_1,n_2}_{q_1,q_2}\left[x\middle|{1-b_1-\gamma,1-(b_{2,q_1-1}),(1-b) \atop (e_{q_2})}\right]\times$$

$$G^{m_2,n_3}_{q_1,q_2}\left[y\middle|{1-c_1-\gamma_1,\ 1-(c_{2},q_1-1),\ 1-c \atop (f_{q_2})}\right]$$

$$= \left[\Gamma(b_1-b+1)\ \Gamma(c_1-c+1) \right]^{-1}$$

$$G^{m_1,n_2}_{q_1,q_2}\left[x \middle| \begin{matrix}(b_{q_1})\\(e_{q_2})\end{matrix}\right] G^{m_2,n_3}_{q_1,q_2}\left[y \middle| \begin{matrix}(c_{q_1})\\(f_{q_2})\end{matrix}\right] \quad ...(3.1)$$

(2) Result (3.1) will yield a very large number of very interest in particular cases by giving various values to the parameters in G and using H.T.F. Vol. I.

(3) If $n_1=n_2=n_3=m_1=m_2=p_1=p_2=q_1=q_2=1$ then

$$\sum_{\gamma,\gamma_1=0}^{\infty} \frac{(b-b_1)_\gamma\,(c-c_1)_{\gamma_1}\,(\mu)_\gamma\,(\nu)_{\gamma_1}}{\gamma!\,\gamma_1!\,\Gamma(1+\gamma)\,\Gamma(1+\gamma_1)} F_1(\lambda;\mu+\gamma,\nu+\gamma_1;\theta;-x,-y)$$

$$= [\Gamma(b_1-b+1)\,\Gamma(c_1-c+1)]^{-1} F_1(\lambda;\mu,\nu;\theta;-x,-y) \quad ...(3.2)$$

where $\lambda=1-a_1+e_1+f_1$; $\mu=b_1+e_1$, $\nu=c_1+f_1$, $\theta=d_1+e_1+f_1$

(iv) If $n_1=n_2=m_1=m_2=p_1=q_1=n_3=1$, $p_2=0$, $q_2=2$ then

$$\sum_{\gamma,\gamma_1=0}^{\infty} \frac{(b-b_1)_\gamma\,(c-c_1)_{\gamma_1}\,(\mu)_\gamma\,(\nu)_{\gamma_1}}{\gamma!\,\gamma_1!\,\Gamma(1+\gamma)\,\Gamma(1+\gamma_1)} F_2(\lambda;\mu+\gamma,\nu+\gamma_1;A,B;-x,-y)$$

$$= [\Gamma(b_1-b+1)\,\Gamma(c_1-c+1)]^{-1} F_2(\lambda;\mu,\nu;A,B;-x,-y)$$

where $A=1-e_2+e_1$, $B=1-f_2+f_1$

(v) If $n_1=p_1=0$, $n_2=n_3=q_1=2$, $m_1=m_2=p_2=q_2=1$ then

$$\sum_{\gamma,\gamma_1=0}^{\infty} \frac{(b_1+e_1)_\gamma\,(c_1+f_1)_{\gamma_1}\,(b-b_2)_\gamma\,(c-c_2)_{\gamma_1}}{\gamma!\,\gamma_1!\,\Gamma(1+\gamma+b_1-b_2)\,\Gamma(1+\gamma_1+c_1-c_2)}$$

$$F_3(\mu+\gamma,\nu+\gamma_1,b_2+e_1,c_2+f_1;\theta;-x,-y)$$

$$= [\Gamma(b_1-b+1)\,\Gamma(c_1-c+1)]^{-1}$$

$$F_3(\mu,\nu,b_2+e_1,c_2+f_1;\theta;-x,-y) \quad ..(3.4)$$

(vi) If $n_2=n_3=q_1=p_2=0$, $m_2=1$, $e_1=b_1$, $b_{q_1}=1-a_{p_1}$ and if we take the limit as y tends to 0, we get results of Bhise (3) and Jain (4) by the property Agarwal (1).

$$\underset{y\to 0}{dt}\; G^{n_1,0,0,m_1,1}_{p_1,0,0,q_2}\left[\begin{matrix}x\\y\end{matrix} \middle| \begin{matrix}(a_{p_1})\\(e_{q_2}),0\end{matrix}\right] = G^{m_1,n_1}_{p_1,q_2}\left[x \middle| \begin{matrix}(a_{p_1})\\(e_{q_2})\end{matrix}\right]$$

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**Note* :—In theorems VI and VII parameters inside G should be Carefully Connected with $n_1, n_2, n_3, m_1, m_2, p_1, q_1, p_2, q_2$ before writing the contours.

GENERALISED STUDY OF THE FORM-FUNCTION OF A PARTIALLY MODIFIED MULTITUBULAR CHARGE WITH HOLES DISTRIBUTED IN A 2-DIMENSIONAL SPACE

By

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Summary :

This paper consists of two sections; in section A the general theory of symmetrical distribution of points about various axes of symmetry in a 2-dimensional space has been discussed. In section B the formfunction of a cylindrical charge having its holes distributed along n rings in a 3-fold axis of symmetry has been studied. The cases of 4-fold and 6-fold axes of symmetry have been discussed in subsequent papers.

Introduction :

To obtain as progressive a charge as possible, the formfunctions of different shapes of multitubular powders have been studied by various authors, e.g. bi-tubular powders by Kothari (1963), tri-tubular powders by Jain (1962), quatretubular powders by Patni and Jain (1963), pentatubular powders by Jain (1964), heptatubular powders by Tavernier (1956 a, b), Gupta (1959 a, b), Kapur and Jain (1961) and nineteentubular by Kothari (1964). Jain (1964, 65) has studied the formfunction of more general types of multitubular powders having $2n$, $2n+1$, $3n$ and $3n+1$ holes, In all these cases the centres of the holes are symmetrically situated on the circumference of a circle or on two circles termed as rings, and all holes having their centres on a ring touch their respective adjacent holes at the end of the first phase of combustion. There is, however, no contact between the holes of the first ring and those of the second ring at the end of the first phase. Thereafter the unburnt space is consumed in various stages.

The aim of the present paper is to study the possibilities of distributing the holes along any number of rings in such a manner that their centres are symmetrically distributed over the entire area enclosed by the outermost ring. This would result in every hole touching all the holes adjacent to it at the end of the first phase of burning. If the space now lying outside the outermost ring be inhibited, the slivers between the holes, which are exactly identical, will burn up in one stage so that the number of stages of combustion will be reduced to two. As will be clear from the following discussion, such an arrangement of holes is possible only along the sides of similar and similarly situated regular polygons and not along concentric circles. Further, the choice of such polygons too is severely restricted. We shall first discuss the theory of symmetrical distribution of points.

SECTION A

Theory of symmetrical distribution of points in 2-dimensional space

As already stated, this section deals with the symmetrical distribution of points with particular reference to a 2-dimensional space. For clarity, it is desirable first to define certain basic terms used in the following discussion.

1. *Translation* : A figure is said to be translated when every point of it is moved through the same distance in the same direction. Such a motion is called *translation* and can be represented in magnitude and direction (but not in position) by a *directed line segment*.

2. *Rotation* : A figure is said to be rotated about a straight line (called the axis of rotation) when every point of it moves as if rigidly connected with the straight line (which is completely fixed in space). The angle between the initial and final positions of any plane parallel to the axis and rigidly connected with the figure is called the *angle of rotation*.

3. *Congruence* : Two figures are said to be congruent to one another when each can be brought to coincide with the other by the movement of its parts without altering their relative positions. A figure can be brought into concidence with any congruent figure by a translation followed by rotation or vice versa.

4. *Operation* : A translation, rotation or combination of these is called an operation. An operation may be represented by means of symbols. Thus, a rotation about an axis through an angle α may be represented by $A(\alpha)$ or A, and a translation through a distance t by T.

If two or more operations are written one after another, it will mean that they are applied to the figure from left to right. Thus $A\ B$ means that first the operation A acts and then the operation B.

5. *Repetition and Symmetry* :

(*a*) *Repetition* : In a collection of like figures, the entire set can be produced from any one member by repetition, i.e. by the successive applications of the same operation. Any kind of operation—translation, rotation or a combination of both—may be used to generate a periodic repetition.

In such a case one figure is regarded as the progeniter of the others. The rest of the neighbours are produced by the application of the operations (say) A, A^2 ; A^3, ... etc. The operation which produces the first figure from itself is called the identity operation and is denoted by 1. Thus the complete set of operation A, A^2,... produces the complete set of figures and has one to one correspondence with them. Such a set is called a group.

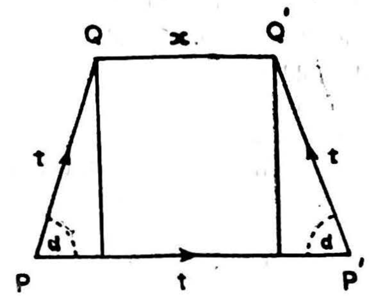
(*b*) *Symmetry operations and symmetry elements* : Any operation which produces symmetrically arranged objects in space seperated by equal intervals is called a *symmetry operation.* Thus translation, rotation or a combination of both are all symmetry operations.

The geometrical locus about which a group of symmetry operations acts is called a *symmetry element.* Thus a point about which an operation of translation T, T^2, ... acts or the line normal to the page about which the rotation A,A^2, .. all act, are symmetry elements.

It is clear that for rotational symmetry the permissible angular repetition interval α is $\frac{2\pi}{n}$, where n is the number of repetitions of the angular interval to complete a circle. The symmetry element corresponding to this α is called n-fold axis, because it is associated with an n-fold duplication of any figure.

Limitations of α set by translation periodicity

In order that a 2-dimensional space may be filled up symmetrically with points, as required in the present case, there will be simultaneous occurence of repetition by rotation and translation. This imposes severe restrictions on the value of α.



Let the starting point P undergo the symmetry operation $T.\ A.$ consisting of a translation t and rotation α resulting in the points P' and Q such that $PP'=t$, $\angle P'PQ=\alpha$ and $PQ=t$. Similarly the symmetry operation about P' produces Q' such that $P'Q'=t$ and $\angle PP'Q'=\alpha$. Now QQ' has the same direction as PP' and in order that it may be consistent with translational symmetry, it should be an integral multiple of the original translation PP'. Hence if $QQ'=x$, we have

$$x=mt, \text{ where } m \text{ is a positive integer.}$$

but $$x=t-2t\cos\alpha, \text{ from the figure}$$

$$2\cos\alpha=1-m.$$

Since m is an integer, $1-m$ is also an integer, say N.

$$2\cos\alpha=N$$

or $$\cos\alpha=\frac{N}{2}.$$

Thus the value of $\cos\alpha$ is restricted to one-half of an integer. But, as shown in the following table, there are only a few such values as give possible values of $\cos\alpha$.

N	$\cos\alpha$	α	$n=\frac{2\pi}{\alpha}$	$x=t-2t\cos\alpha$
-2	-1	π	2	$3t$
-1	$-\frac{1}{2}$	$\frac{2\pi}{3}$	3	$2t$
0	0	$\frac{\pi}{2}$	4	t
1	$\frac{1}{2}$	$\frac{\pi}{3}$	6	0
2	1	2π	1	$-t$

It is therefore clear that symmetries of rotations combined with those of translation which can be applied to points to fill a two-dimensional space are about 1-fold, 2-fold, 3-fold, 4-fold and 6-fold axes. The symmetries about 1 and 2-fold axes are trivial, and thus those about 3, 4 and 6-fold axes are the only symmetries which would meet our requirements. Consequently, the centres of the holes will be distributed about these axes only. In the following pages, the case of 3-fold axis of symmetry has been discussed. The cases of 4-fold and 6-fold axes have been studied in subsequent papers.

SECTION B

Formfunction of a Partially Inhibited Cylindrical Change with the Distribution of Holes Over *n* Rings in 3-Fold Axis of Symmetry

Notations :

D = The diameter of the charge grain.

d = The diameter of holes in the charge.

e_r = The distance between any two holes or between any hole and the curved surface of the grain.

L = The length of the charge.

m = The ratio of the diameter of the charge grain to the diameter of any hole.

ρ = The ratio of the length of the charge grain to the diameter of the charge grain.

S_o = The initial surface of the charge.

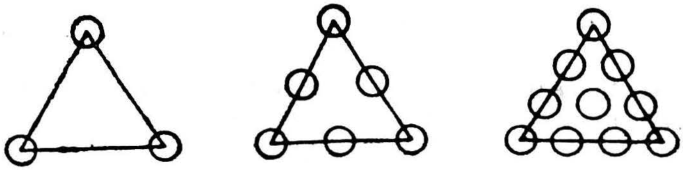
S = The surface of the charge at any instant t.

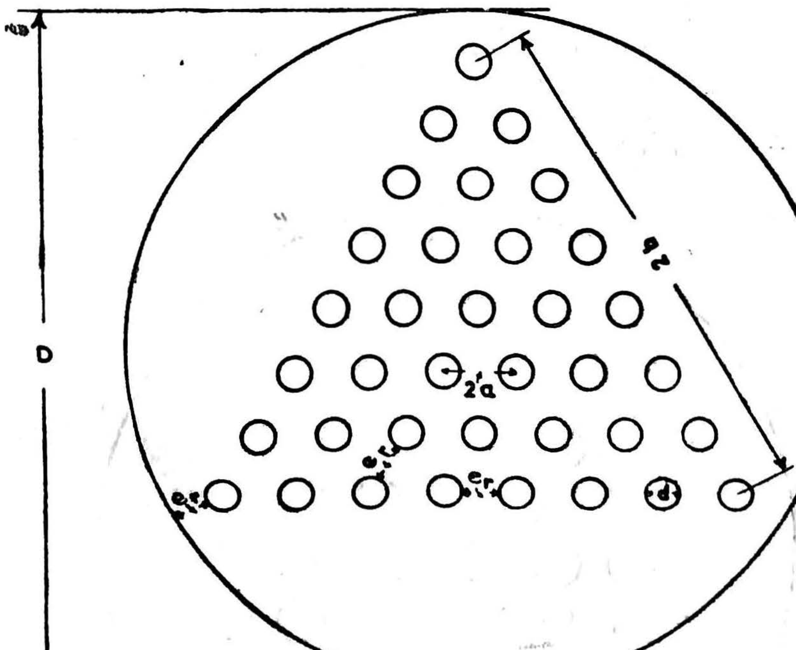
δ = The density of the charge grain.

n = The number of rings.

N = Number of holes.

It is clear that in the case of three-fold axis of symmetry, the centres of the holes will be situated on the vertices of exactly alike equilateral triangles. Moreover, these holes wlll be distributed along the sides of similar and similarly situated equilateral triangles. We shall call these triangles as rings. The number of holes on a side of the innermost triangle may be 2, 3, or 4 as shown in the figures





We shall designate these arrangements as category I, II and III respectively. In the case of category III, there will be an extra hole at the centre of the innermost triangle.

Now the number of holes on a side of the nth ring

$$= 3n-2+r = \mu_r+1, \qquad r = 1,2,3. \qquad \dots(1)$$

and the number of holes in the charge is given by

$$N = \tfrac{1}{2}(3n+r-2)\ (3n+r-1)$$
$$= \tfrac{1}{2}(\mu_r+1)\ (\mu_r+2), \qquad r = 1,2,3 \qquad \ldots(2)$$

where $\mu_r = 3n+r-3$ ...(3)

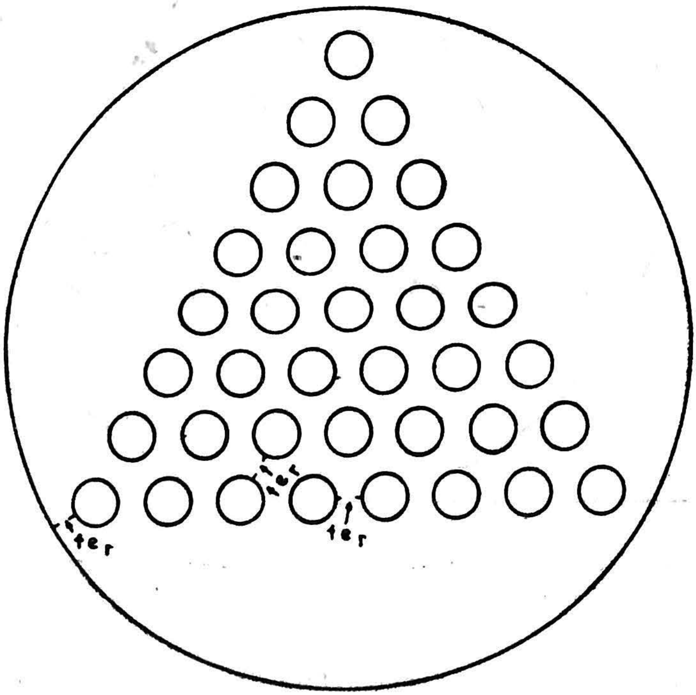
Let $2a$ be the side of the innermost triangle (i.e. the 1st ring) and $2b$ that of the outermost triangle (i.e. the nth ring).
Then

$$b = \frac{\mu_r}{r}\,a \qquad \ldots(4)$$

Now,

$$D = \frac{4a\,[\,3\,(n-1) + r+\sqrt{3}\,]}{r\,\sqrt{3}} - d = md$$

$$\therefore\ a = \frac{r\,\sqrt{3}\,(m+1)\,d}{4\,(\mu_r+\sqrt{3})} \qquad \ldots(5)$$



Charge at any time t

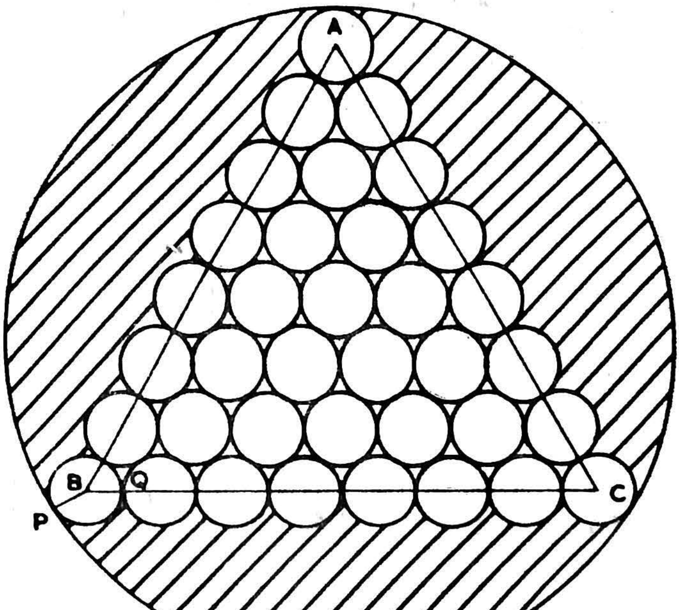
Also

$$e_r = \frac{1}{r}(2a - rd), \qquad r = 1,2,3$$

$$= \frac{[\sqrt{3}\,(m-1) - 2\mu_r]\,d}{2\,(\mu_r + \sqrt{3})} \qquad \text{...(6)}$$

Clearly we should have

$$m \geqslant \frac{2\mu_r + \sqrt{3}}{\sqrt{3}}$$



Charge at the end of first phase of combustion

At the end of the first phase of combution, when all the circles in the rings touch each other, the charge grain consists of

(i) μ_r^2 curvilinear triangular prisms having the same cross-section;

(ii) three curvilinear prisms in the outermost region of the charge having equal and similar cross-sections.

The diameter of the charge at this instant

$$= 2\left(b \sec \frac{\pi}{6} + \frac{a}{r}\right)$$

$$= \frac{2a}{\sqrt{3}}\left(2\mu_r + \sqrt{3}\right) = D' \text{ (say)} \qquad \ldots(7)$$

and the diameter of any hole at this instant

$$= \frac{2a}{r} = d' \text{ (say)} \qquad \ldots(8)$$

The area of the bases of the three outer curvilinear prisms

$$= \frac{\pi D'^2}{4} - \triangle ABC - 6 \text{ sector } BPQ - 3(\mu_r - 1)\frac{\pi d'^2}{8}$$

$$= \frac{\mu_r (m+1)^2 \Delta_r}{32\left(\mu_r + \sqrt{3}\right)^2} d^2 \qquad \ldots(9)$$

where $\Delta_r = (8\pi - 5\sqrt{3})\,\mu_r + (8\sqrt{3} - 9)\,\pi$ (10)

Now when these outer curvilinear prisms are inhibited from burning,

$$V_o = \left[\frac{\pi D^2}{4} - N_r \frac{\pi d^2}{4} - \frac{\mu_r (m+1)^2 \Delta_r}{32\left(\mu_r + \sqrt{3}\right)^2} d^2\right] \times L$$

$$= \frac{\rho m d^3}{32\left(\mu_r + \sqrt{3}\right)^2}\left[8\pi\left(m^2 - N_r\right)\left(\mu_r + \sqrt{3}\right)^2 - (m+1)^2 \mu_r \Delta_r\right] \quad (11)$$

and

$$V = \left[\pi\left\{\frac{D}{2} - \frac{e_r(1-f)}{2}\right\}^2 - \pi N_r\left\{\frac{d}{2} + \frac{e_r(1-f)}{2}\right\}^2 - \frac{\mu_r (m+1)^2 \Delta_r}{32\left(\mu_r + \sqrt{3}\right)^2} d^2\right] \times \left[L - e_r(1-f)\right]$$

$$= \frac{d^3}{32\left(\mu_r + \sqrt{3}\right)^2}\Big[8\pi\left(\mu_r + \sqrt{3}\right)^2\left(m^2 - N_r\right) - (m+1)^2 \mu_r \Delta_r$$
$$- 8\pi\left(\mu_r + \sqrt{3}\right)\left\{\sqrt{3}(m-1) - 2\mu_r\right\} \times \left(m + N_r\right)(1-f)$$
$$- 2\pi\left\{\sqrt{3}(m-1) - 2\mu_r\right\}^2\left(N_r - 1\right)(1-f)^2\Big] \times$$
$$\left[\frac{\rho m - \left\{\sqrt{3}(m-1) - 2\mu_r\right\}(1-f)}{2\left(\mu_r + \sqrt{3}\right)}\right] \qquad (12)$$

$$\therefore z = \frac{V_o\delta - V\delta}{V_o\delta} = 1 - \frac{V}{V_o} = (1-f)\left(A - Bf - Cf^2\right) \qquad (13)$$

where

$$A = \{\sqrt{3}(m-1)-2\mu_r\}\,[4\pi(\mu_r+\sqrt{3})\,\{4\mu_r+\sqrt{3}\,(N_r+3)\}\,m^2\rho$$

$$+4\pi(\mu_r+\sqrt{3})\,\{N_r(2\mu_r+3\sqrt{3})+(2\mu_r+\sqrt{3})\}m\rho$$

$$+\{((2\mu_r+\sqrt{3})^2-3N_r)2\pi-\mu_r\,\Delta_r\}\,(m+1)^2]\div$$

$$[\,2m\rho(\mu_r+\sqrt{3})\,[\,8\pi(m^2-N_r)\,(\mu_r+\sqrt{3})^2-(m+1)^2\,\mu_r\,\Delta_r\,] \quad ...(14)$$

$$B = \{\sqrt{3}\,(m-1)-2\mu_r\,\}^2\,[\,4\pi(\mu_r+\sqrt{3})\,(N_r-1)\,m\,\rho$$

$$-\,4\pi(2\mu_r+\sqrt{3}\,N_r+\sqrt{3})\,(m+1)\,]\div$$

$$[\,2m\rho(\mu_r+\sqrt{3})\,\{8\pi(m^2-N_r)\,(\,\mu_r+\sqrt{3})^2-(m+1)^2\,\mu_r\,\Delta_r\,\}\,] \quad ...(15)$$

$$C = \frac{2\pi\,\{\sqrt{3}(m-1)-2\mu_r\,\}^3\,(N_r-1)}{2m\rho(\mu_r+\sqrt{3})\,[8\pi\,(m^2-N_r)\,(\mu_r+\sqrt{3})^2-(m+1)^2\,\mu_r\,\Delta_r]} \quad ...(16)$$

Now

$$L = \rho\,D = \rho\,md$$

$$= \frac{2(\,\mu_r+\sqrt{3}\,)m\rho}{[\sqrt{3}(m-1)-2\mu_r]}\,e_r \quad ..(17)$$

In order that all burnt position of the charge may not occur before or coincide with the rupture of the grain, we must have

$$L > e_r$$

or $$\rho > \frac{\sqrt{3}(m-1)-2\mu_r}{2(\mu_r+\sqrt{3})m} = \rho\ \text{min} \quad ...(18)$$

where $$\rho\ \text{min} = \frac{\sqrt{3}(m-1)-2\,\mu_r}{2(\mu_r+\sqrt{3})m} \quad ...(19)$$

(14) can be written as

$$A = A_0\,(m,\,N_r) + \frac{A_1\,(\,m,\,N_r\,)}{\rho} \quad ...(20)$$

where

$$A_o(m, N_r) = \frac{\{\sqrt{3}(m-1)-2\mu_r\}}{2(\mu_r+3)\{8\pi(m^2-N_r)\}(\mu_r+\sqrt{3})^2-\mu_r\Delta_r(m+1)^2\}}$$

$$\Big[4\pi(\mu_r+\sqrt{3})\{4\mu_r+\sqrt{3}(N_r+3)\}m+4\pi(\mu_r+\sqrt{3})$$

$$\{N_r(2\mu_r+3\sqrt{3})+(2\mu_r+\sqrt{3})\}\Big] \quad ...(21)$$

and

$$A_1(m,N_r) = \frac{\{\sqrt{3}(m-1)-2\mu_r\}[\{(2\mu_r+\sqrt{3})^2-3N_r\}2\pi-\mu_r\Delta_r](m+1)^2}{2(\mu_r+\sqrt{3})[8\pi(m^2-N_r)(\mu_r+\sqrt{3})^2-\mu_r\Delta_r(m+1)^2]m} \quad (22)$$

Hence for a given value of N_r, we have

$$A\max = A_o(m, N_r) + \frac{A_1(m, N_r)}{\rho\min}$$

$$= 1, \text{ a value independent of } m \text{ and } N_r \quad .. (23)$$

Also for given values of m and N_r, A is minimum when $\rho=+\infty$, so that from (20)

$$A\min = A_o(m, N_r) \quad ...(24)$$

Further if $m = \frac{\sqrt{3}+2\mu_r}{\sqrt{3}}$, $A_o(m, N_r)=0$

and if $m=\infty$, $A_o(m, N_r) = \frac{2\pi[4\sqrt{3}\mu_r+3N_r+9]}{8\pi(\mu_r+\sqrt{3})^2-\mu_r\Delta_r}$

Taking the minimum values or N_r for $r=1, 2, 3$ and $n=1$, we have

$$\mu_1=1, \quad N=3, \quad \Delta_1=30.02$$
$$\mu_2=2, \quad N_2=6, \quad \Delta_2=44.76$$
$$\mu_3=3, \quad N_3=10, \quad \Delta_3=59.51$$

so that $A_o(m, N_1) = \frac{2\pi(4\sqrt{3}+18)}{8\pi(4+2\sqrt{3})-30.02} = .99$

$$A_o(m, N_2) = \frac{2\pi(8\sqrt{3}+27)}{8\pi(7+4\sqrt{3})-89.52} = .98$$

$$A_o(m, N_3) = \frac{2\pi(12\sqrt{3}+39)}{8\pi(12+6\sqrt{3})-178.53} = .97$$

Also for $N_r=\infty$, $\mu_r=\infty$ and $A_o(m, N_r)=0$.

Using (13) in the relation

$$\frac{S}{S_o} = \frac{dz/df}{(dz/df)_{f=1}} \quad \text{.. (25)}$$

we get

$$\frac{S}{S_o} = \alpha - \beta f - \gamma f^2 \quad \text{...(26)}$$

where

$$\alpha = \frac{A+B}{A-B-C} = \Big[(m+1)\,[8\pi(\mu_r+\sqrt{3})\,\{2\mu_r+\sqrt{3}\,(N_r+1)\}\,m\rho + 2\pi\,\{4\mu_r^2-9\,\mu_r-3\}m+2\pi(12\mu_r^2+12\sqrt{3}\mu_r+4\sqrt{3}\mu_r N_r+3N_r+9)-\mu_r\Delta_r\,(m+1)\,]\Big] \div \Big[\,8\pi\,(\mu_r+\sqrt{3}\,)^2\,(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\,\Delta_r\,(m+1)^2\Big] \quad \text{...(27)}$$

$$\beta = \frac{2(B-C)}{A-B-C} = 2[\sqrt{3}(m-1)-2\mu_r]\,[4\pi(\mu_r+\sqrt{3})\,(N_r-1)\,\rho m - 2\pi\,\{4\mu_r+\sqrt{3}\,(3N_r+1)\}\,m-2\pi\,\{\sqrt{3}\,(N_r+3)-2\mu_r(N_r-3)\}\,]\div[8\pi(\mu_r+\sqrt{3})^2\,(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\Delta_r\,(m+1)^2\,] \quad \text{(28)}$$

$$\gamma = \frac{3C}{A-B-C} = \frac{6\pi(N_r-1)\,[\sqrt{3}(m-1)-2\mu_r]^2}{[8\pi(\mu_r+\sqrt{3})^2\,(2m^2\rho+2m\rho N_r+m^2-N_r)-\mu_r\Delta_r\,(m+1)^2\,]} \quad \text{.. (29)}$$

From (26), we get

$$\frac{d}{df}\left(\frac{S}{S_o}\right) = -\beta - 2\gamma f \quad \text{...(30)}$$

and

$$\frac{d^2}{df^2}\left(\frac{S}{S_o}\right) = -2\gamma \quad \text{.. (31)}$$

It is clear from (29) that γ is always positive, hence $\frac{d^2}{df^2}\left(\frac{S}{S_o}\right)$ is always negative. Thus $\frac{S}{S_o}$ can have only a maximum value for some value of f. Now for a maximum,

$$\frac{d}{df}\left(\frac{S}{S_o}\right) = -\beta - 2\gamma f = 0, \text{ so that}$$

$$f = -\frac{\beta}{2\gamma}$$

Since $1 \geqslant f \geqslant 0$, we have

$$-\frac{\beta}{2\gamma} \leqslant 1 \qquad \ldots(32)$$

and

$$-\frac{\beta}{2\gamma} \geqslant 0 \qquad \ldots(33)$$

From (28) and (29), we get

$$-\frac{\beta}{2\gamma} = [\{\sqrt{3}(N_r+3) - 2\mu_r(N_r-3)\} + \{4\mu_r + \sqrt{3}(3N_r+1)\}m - 2(\mu_r+\sqrt{3})(N_r-1)\rho m] \div 3(N_r-1)[\sqrt{3}(m-1) - 2\mu_r] \qquad \ldots(34)$$

$\therefore$ (32) gives

$$[\{\sqrt{3}(N_r+3) - 2\mu_r(N_r-3)\} + \{4\mu_r + \sqrt{3}(3N_r+1)\}m - 2(\mu_r+\sqrt{3})(N_r-1)\rho m] \div 3(N_r-1)\{\sqrt{3}(m-1) - 2\mu_r\} \leqslant 1$$

or

$$\rho \geqslant \frac{2(m+N_r)}{(N_r-1)m} = \rho_1 \text{ (say)} \qquad \ldots(35)$$

Again (33) gives

$$[\{\sqrt{3}(N_r+3) - 2\mu_r(N_r-3)\} + \{4\mu_r + \sqrt{3}(3N_r+1)\}m - 2(\mu_r+\sqrt{3})(N_r-1)\rho m] \div 3(N_r-1)\{\sqrt{3}(m-1) - 2\mu_r\} \geqslant 0$$

or

$$\rho \leqslant \frac{(\sqrt{3}+3\sqrt{3}N_r+4\mu_r)m + \sqrt{3}(N_r+3) - 2\mu_r(N_r-3)}{2(\sqrt{3}+\mu_r)(N_r-1)m} = \rho_2 \text{ (say)} \qquad \ldots(36)$$

(35) and (36) give the two values of ρ between which the value of ρ should lie for the maximum value of $\frac{S}{S_0}$ to occur between the beginning of the combustion and the rupture of the grains. If $\rho=\rho_1$, the maximum of $\frac{S}{S_0}$ occurs at the beginning of the combustion and if $\rho=\rho_2$ this maximum occurs at the rupture of the grains.

Using (22), (34), (35) and (36) in

$$f = -\frac{\beta}{2\gamma}$$

$$f = 1 - \frac{\rho - \rho_1}{\rho\ \text{min}} = \frac{\rho_2 - \rho}{\rho\ \text{min}} \qquad \ldots(37)$$

which gives the value of f for a maximum of $\frac{S}{S_o}$.

From (30), (28), (29) and (35), we have

$$\left[\frac{d}{df}\left(\frac{S}{S_o}\right)\right]_{f=1} = -\beta - 2\gamma$$

$$= \frac{8\pi(\mu_r + \sqrt{3})\{\sqrt{3}(m-1) - 2\mu_r\}\, m\, (N_r - 1)(\rho_1 - \rho)}{[8\pi(\mu_r + \sqrt{3})^2 (2m^2 \rho + 2m\rho N_r + m^2 - N_r) - \mu_r \Delta_r (m+1)^2]} \qquad \ldots(38)$$

and from (30), (28) and (36),

$$\left[\frac{d}{df}\left(\frac{S}{S_o}\right)\right]_{f=0} = -\beta$$

$$= \frac{8\pi(\mu_r + \sqrt{3})\{\sqrt{3}(m-1) - 2\mu_r\}\, m\, (N_r - 1)(\rho_2 - \rho)}{[8\pi(\mu_r + \sqrt{3})^2 (2m^2 \rho + 2m\rho N_r + m^2 - N_r) - \mu_r \Delta_r (m+1)^2]} \qquad \ldots(39)$$

Hence, in general, for any given value of N_r

(i) if $\rho\ \text{min} \leqslant \rho \leqslant \rho_1$, $\frac{d}{df}\left(\frac{S}{S_o}\right)$ is always positive right from the beginning and the charge is throughout degressive;

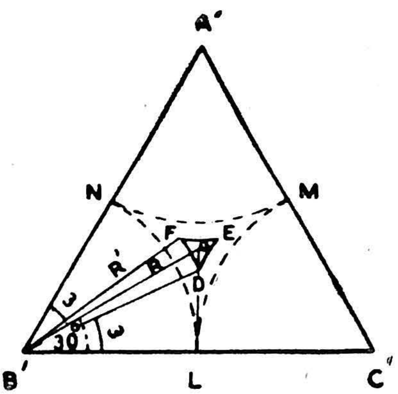
(ii) if $\rho_1 < \rho \leqslant \rho_2$, $\frac{d}{df}\left(\frac{S}{S_o}\right)$ is negative in the beginning and then positive, so that the charge is first progressive and then degressive;

(iii) if $\rho > \rho_2$, $\frac{d}{df}\left(\frac{S}{S_o}\right)$ is always negative and the charge is throughout progressive.

The above results can be summarised as below :

	$\rho_{min} \leqslant \rho \leqslant \rho_1$	$\rho_1 < \rho \leqslant \rho_2$	$\rho > \rho_2$
$\frac{S}{S_o}$	Decreasing function of z throughout and the charge is degressive.	Increasing function of z in the beginning and then decreasing function. The charge is progressive at first and then degressive.	Increasing function of z throughout. The charge is progressive throughout.

Formfunction for the second phase of combustion.



At the beginning of the second phase of combustion, the number of curvilinear triangular prisms

$$= [\ 3\ (n-1)+r\ ]^2 = \mu_r^{\ 2} \qquad \ldots(40)$$

The length of the charge at this instant

$$= L - e_r = \frac{2m\ \rho\ [\mu_r + \sqrt{3}] - \sqrt{3}(m-1) + 2\mu_r}{2\ (\ \mu_r + \sqrt{3}\ )}\ d \qquad \ldots(41)$$

$$\text{The radius of an arc like NL} = \frac{a}{r} \qquad \ldots(42)$$

$$\text{The side of a triangle like } ABC = \frac{2a}{r} \qquad \ldots(43)$$

Let R be the circum radius of a triangle like ABC and R' be the radius of an arc like DE of a curvilinear triangle at any instant during the second phase of combustion.

From Δ^s BDL and OBL,

$$R \cos 30° = R' \cos \omega = \frac{a}{r} \qquad \ldots(44)$$

$$\therefore\ R = \frac{2a}{\sqrt{3}r} \text{ and } R' = \frac{a}{r} \sec \omega \qquad \ldots(45)$$

For the complete combustion of the curvilinear triangular prisms, $R' = R$ and (45) gives

$$\cos \omega = \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6}$$

$$\therefore\ \omega = \frac{\pi}{6} \qquad \ldots(46)$$

which is independent of r. Hence for all the three categories, the complete combustion of the charge always takes place when $\omega = \frac{\pi}{6}$. At any instant during the second phase of combustion, when the base of the curvilinear triangular prism, i.e. LMN becomes DEF, the common length is given by

$$L' = L - e_r - 2\left(R' - \frac{a}{r} \right)$$

$$= \left[(\rho m + 1) - \frac{\sqrt{3}(m+1)}{2(\mu_r + \sqrt{3})} \sec \omega \right] d \qquad \ldots(47)$$

For L' to remain positive up to the complete combustion of thegrain, i.e. when $\omega = \frac{\pi}{6}$, we should have

$$\rho m + 1 - \frac{\sqrt{3}(m+1)}{2(\mu_r + \sqrt{3})} \cdot \frac{2}{\sqrt{3}} \geqslant 0$$

or

$$\rho \geqslant \rho_3$$

where

$$\rho_3 = \left[\frac{(m+1)}{\mu_r + \sqrt{3}} - 1 \right] \frac{1}{m} \qquad \ldots(48)$$

Now area DEF $= \Delta ABC - 6\,\Delta\,BDL - 3$ sector BDF

$$= \frac{a^2}{r^2} \left[\sqrt{3} - 3 \tan \omega - \frac{3}{2} \left(\frac{\pi}{3} - 2\omega \right) \sec^2 \omega \right]$$

Area of the bases of all the curvilinear triangular prisms at any instant during the second phase of combustion

$$= \frac{3\mu_r^2 (m+1)^2 d^2}{16(\mu_r + \sqrt{3})^2} \left[\sqrt{3} - 3 \tan \omega - \frac{3}{2} \left(\frac{\pi}{3} - 2\omega \right) \sec^2 \omega \right]$$

$$= \frac{3\mu_r^{\,2}(m+1)^2 d^2}{16(\mu_r+\sqrt{3})^2} F(\omega) \qquad \ldots (49)$$

where $F(\omega) = \sqrt{3} - 3\tan\omega - \frac{3}{2}\left(\frac{\pi}{3} - 2\omega\right)\sec^2\omega \qquad \ldots (50)$

Denoting the volume of a prism having a base DEF at any instant during the second phase of combustion by V (DEF), the volume of the charge at this instant

$$= \mu_r^{\,2} V(DEF)$$

$$= \frac{3\mu_r^{\,2}(m+1)^2 d^3}{16(\mu_r+\sqrt{3})^2}\left[m\rho+1-\frac{\sqrt{3}(m+1)}{2(\mu_r+\sqrt{3})}\sec\omega\right]F(\omega) \qquad \ldots (51)$$

$$\therefore z = 1 - \frac{\mu_r^{\,2} V(DEF)\,\delta}{V_0\delta} \qquad \ldots (52)$$

which on using (51) reduces to

$$z = 1 - \frac{3\mu_r^{\,2}(m+1)^2\left[2(m\rho+1) - \frac{\sqrt{3}(m+1)}{\mu_r+\sqrt{3}}\sec\omega\right]F(\omega)}{m\rho\,[8\pi(m^2-N_r)(\mu_r+\sqrt{3})^2-(m+1)^2\mu_r\Delta_r]} \qquad (53)$$

$$= 1 - \frac{3(m+1)^2\mu_r^{\,2}G(\omega)}{m\rho\,[8\pi(m^2-N_r)(\mu_r+\sqrt{3})^2-(m+1)^2\mu_r\Delta_r](\mu_r+\sqrt{3})}$$

$$\ldots (54)$$

where $G(\omega) = [2(\mu_r+\sqrt{3})(m\rho+1) - \sqrt{3}(m+1)\sec\omega]\,F(\omega)$ (55)

Initially $\omega = 0$, $F(\omega) = \sqrt{3} - \frac{\pi}{2}$ ... (56)

and $G(\omega) = \left[2(m\rho+1)(\mu_r+\sqrt{3}) - \sqrt{3}(m+1)\right]\left[\sqrt{3}-\frac{\pi}{2}\right]$

$$\ldots (57)$$

and $z = [\sqrt{3}(m-1) - 2\mu_r]\,[4\pi(\mu_r+\sqrt{3})^2\{4\mu_r+\sqrt{3}(N_r+3)\}$

$m^2\rho + 4\pi(\mu_r+\sqrt{3})\{N_r(2\mu_r+3\sqrt{3})+(2\mu_r+\sqrt{3})\}$

$m\rho + \{(2\mu_r+\sqrt{3})^2 - 3N_r)\,2\pi - \mu_r\Delta_r\}(m+1)^2]$

$\div\ 2m\rho(\mu_r+\sqrt{3})[8\pi(m^2-N_r)(\mu_r+\sqrt{3})^2-(m+1)^2\mu_r\Delta_r]$

$= A,$ ... (58)

which is the value of z at the end of the first phase of combustion. Defining f now as the ratio of the distance receded (from the beginning of the second phase of combustion up to the instant considered) to the initial thickness e_r, we get

$$f = \frac{2\left(\frac{a}{r} - R'\right)}{e_r} = \frac{\sqrt{3}\,(m+1)\,(1 - \sec\omega)}{\sqrt{3}\,(m-1, -2\,\mu_r} \qquad \ldots (59)$$

and

$$\frac{dz}{df} = \frac{dz/d\omega}{df/d\omega}$$

$$= \sqrt{3}\,\mu_r^{\,2}\,(m+1)\,\{\sqrt{3}(m-1) - 2\,\mu_r\}\;[\;\{\;2\,(m\rho+1)\;\;(\mu_r+\sqrt{3})$$
$$-\sqrt{3}\,(m+1)\sec\omega\;\}\;(6\,\omega - \pi)\sec\omega - \sqrt{3}\;(m+1)\;F\,(\omega)\;]$$
$$\div\left[(\,\mu_r+\sqrt{3})\,m\,\rho\,[\,8\pi\,(m^2 - N_r)\;(\mu_r + \sqrt{3}\,)^2 - (m+1)^2\,\mu_r\,\Delta_r]\right] \qquad \ldots (60)$$

$$\therefore \frac{S}{S_o} = \frac{dz/df}{(dz/df)\;f=1}$$

$$= -2\,\sqrt{3}\,\mu_r^{\,2}\,(m+1)\,[2\,(m\rho+1)\;(\mu_r+\sqrt{3})\;(\,6\omega - \pi\,)\;\sec\,\omega\;-$$
$$\sqrt{3}\,(m+1)\;H\,(\omega)\;] \div [\,8\pi\,(\mu_r + \sqrt{3})^2\,(2\,m^2\,\rho+2m\rho\,N_r+m^2 - N_r)$$
$$-\,\mu_r\,\Delta_r\,(m+1)^2\,] \qquad \ldots (61)$$

where $\qquad H(\omega) = (6\omega - \pi)\sec{}^2\omega + F\,(\omega) \qquad \ldots (62)$

Initially $\qquad \omega = 0,\; H(\omega) = \frac{\sqrt{3}}{2}\,(2 - \sqrt{3}\pi) \qquad \ldots (63)$

and

$$\frac{S}{S_o} = \frac{\sqrt{3}\mu_r^{\,2}(m+1)\;\{\;4\,(m\rho+1)\;(\mu_r + \sqrt{3})\;\pi - 3\;(\sqrt{3}\pi - 2)\;(\;m+1\;)\;\}}{[\;8\pi\;(\mu_r+\sqrt{3})^2\;(\;2m^2\;\rho+2m\;\rho\;N_r+m^2 - N_r - (\;m+1\;)^2\;\mu_r\;\Delta_r\;]} \qquad \ldots (64)$$

At the end of complete combustion of the charge grain

$$\omega = \frac{\pi}{6}\,, \qquad G\,(\omega) = 0, \qquad H\,(\omega) = 0,$$

so that $\qquad z = 1 \quad$ and $\quad \frac{S}{S_o} = 0.$

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