LATER WITTGENSTEIN ON LANGUAGE AND MATHEMATICS

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A Non-foundational Narration

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Preface

Ludwig Wittgenstein (1889–1951) in his later writings threw down an interesting and intriguing challenge to the notion of theory, or rather to the art of theory-building itself. All theorisations across all disciplines (comprising all that traditionally goes by the names of 'humanities' and 'science') labour under an impossible ideal of forging their foundations in such a way that they stop one step short of the founded, and yet spurt forth the latter from their hidden reserves. In other words, all theories or foundations claim to have a dignified status, aloof from their applications, and yet entail the latter through a magnetic or magical power. Philosophers have proposed several alternative foundations of language—universals of Platonic or Aristotelian character, verbal rules, physical ostensions, mental images, brain patterns, neural firings, etc. The later Wittgenstein demonstrated in graphic detail how none of these foundations had the required extra-linguistic or self-interpretive character, how each of them called for an interminable series of interpretations of interpretations of interpretations..., until we are forced to merge the foundation with the founded, the theory with its applications. This amounts to meshing language with its meaning and the meant realities in a single and seamless complex.

Obviously, such an anti-foundationalist endeavour cannot be complete or convincing unless it extends to the sciences and mathematics as well, unless it opens up a path to show how the experimentally consolidated grounds of the sciences and the rigorously defined notions of space and time in mathematics too are vulnerable to endless indeterminacies and opacities. It is from this sphere of anxiety and motivation that this work takes off—with the clear intention of weaving a single anti-foundationalist narrative that will connect later Wittgenstein's views on 'ordinary' language with those on the 'specialist' regime of mathematical language. To put it the other way round, if we learn how later Wittgenstein breaks through the absolute quantitative identity of mathematical units, or the unique implicative power of mathematical rules, these fissures will automatically be seen to cut through the entire field of ordinary, non-technical language as well.

Though the logistic force of such programmes should absorb Wittgenstein's treatment of the sciences as well, we are obliged to confine ourselves only to his view on mathematics (and that too with a primary focus on arithmetic), both on account of availability of primary literature and the feasibility of our project.

The present book consists of eight chapters (including the Conclusion), followed up by an Appendix. As I will provide a brief synopsis of their content in the Introduction, here I merely touch upon the issues dealt with by each of them, without going into a description of the arguments.

The first chapter sets the basic theme of anti-foundationalism through a general account of how, according to Wittgenstein, concepts are formed in a flow of resemblances. It takes special note of the fact that neither these resemblances nor the representational tools of definition, ostension or measurement have a non-relational core of identity that sets a pre-established harmony between the two sets. Rather language, meaning and reality blend into an indeterminate flow of practices.

The second chapter is in two parts. The first part presents an account of how the major philosophies of mathematics treat mathematical propositions as descriptive of, and thus parasitic on, some domain of reality—whether abstract or concrete, inner or outer. The second part of the chapter shows how, for Wittgenstein, mathematical propositions turn out to be not descriptions, but *paradigms* of description, whereby experience is frozen into a physiognomic cycle of practices. There is no pre-interpretive content of external reality, inner experience, or a priori forms of mind that lends itself to mathematical description or abstraction.

The logicists like Frege and Russell attempt to circumvent the patent problems of mathematical foundations—those pertaining to the indeterminacy and contingency of physical proofs, or the temporal and synthetic character of mathematical operations—by reducing mathematics into a purely logical system based on definitions and rules of inference. The third chapter is an elaborate critique of logic itself, and virtually turns back the logico-mathematical proofs into spatial constructions of the rules of sign-geometry.

The fourth and the fifth chapters attempt to construe mathematical cognition in terms of a special kind of perception that Wittgenstein calls 'aspect-perception'. A probe into the nature of this perception shows mathematical cognition—not to be based on pre-given bits of sensations, brain patterns (Gestalts) or on nervous excitements—but as a special technique or a mode of employment or practices. So while the second and the third chapters cleanse mathematics of the demands of Platonism, intuitionism and logicism, the fourth and the fifth chapters prevent it from lapsing into some form of psychological or physiological foundations.

The sixth chapter takes up a special issue in Wittgenstein's critique of logicism, viz., his charge of circularity against the Fregean definition of number in terms of one–one correlation. The chapter clears certain possible misconceptions about Wittgenstein's family resemblance notion of numbers, one–one correlation and the phenomenon of subitisation, thus placing the exact significance of the charge within the broader purview of aspect-perception and actions.

The final chapter highlights the non-revisionary game of mathematics—that of freezing experience and experiments into physiognomic pictures—as ensconced in a particular form of living. It also explores the exact nuances of deviant mathematical practices described by Wittgenstein, with an attempt to resist such accounts from lapsing into some form of deviantism or relativism, which I have shown to be nothing but another version of foundationalism.

I conclude this work with the suggestion that any philosophical discourse on forms of life will itself be a form of life, and each philosophy goes on expanding and enriching this notion, rather than pushing it back to the dubious domain of silence.

A word of explanation as to why I put the chapter on formalism at the end of the book—as an Appendix. While I should ideally have written it after the chapter on Wittgenstein's critique of logicism (chapter III), as a matter of fact I wrote it months after the completion of this entire work. It was the insightful reviewers' comments (which I received long after I completed the book) that motivated me to extend the scope of this work into the new area of Wittgenstein's critique of formalism. This further exercise of pitting Wittgenstein against the anti-foundational stance of formalism helped me reframe all the issues that I had discussed previously—language-games, family resemblance, rule-following, aspect-perception, semantic opacity, forms of life—in a space that became further expanded, magnified and thickened out in a new dimension. Here I provide a brief outline of the contents of the Appendix. Formalism professes a deflationary ontology of mathematics, declaring its propositions to be merely about signs or blind rules of manipulating these signs. This anti-foundational stance, however, breaks down in many directions. First, the formalist, in order to make sense of his or her own position, has to go beyond sign-tokens to sign-types, and further beyond sign-types to an abstract structure embodied in the sign-reality itself. Thus ironically formalism comes to substitute a new kind of realism in lieu of Platonism or intuitionism—an ethereal sign-essence hovering on the concrete specificity of the signs themselves. On the other hand, to shirk off this unwanted intrusion of sign-essentialism, the formalist has to ground his rule-formalistic version on the logical power of entailment, thus lapsing into the logical fictions of classes, sets, numbers—a position that formalism had sought to avoid at the very outset. I have argued that Hilbert's theory—the most sophisticated version of formalism—labours under this mould of fallacies, oscillating between two equally destructive options: that of logicism on the one hand, and the naïve commitment to a pre-conceptual transparency of sign-intuitions on the other. Applying Wittgenstein's tools anew to track these pitfalls in Hilbert's system effectively gives a cutting edge to these old tools themselves—already used in the previous chapters. This final chapter (in the shape of an Appendix), by the very demands of its title, creates a new space for dealing with some technical issues, viz., irrational numbers, imaginary numbers, Cantor's proof of denumerability of the set of irrational numbers, Gödel's programme, etc. In fact, it creates a space for redirecting these technical issues to a non-technical solution.

Within the vast arena of literature on later Wittgenstein's philosophy of mathematics, this book carves out a niche of its own. As already specified, it attempts to synthesise later Wittgenstein's general account of language and his philosophy of mathematics in a single continuum. Some popular notions about his later views—those that confuse his idea of indeterminacy with mere ambiguity, read him as substituting sociological or physiological foundations for the classical ones, or brand his anti-foundationalism as a form of relativism—are all addressed within a single framework, and resisted with the same anti-foundationalist tools. It is the same insights of Wittgenstein that are found again and again to merge ostension with the ostended, concepts with their extensions, rules with applications, or the insular data of perception with open and public practices—across all forms of language, whether mathematical or non-mathematical. Again within the realm of mathematics, it is the same tools that are activated against the dominant doctrines like Platonism, intuitionism, logicism and formalism or any possible theory that seeks to pose psychological, physiological foundations for mathematics. In other words, this book attempts to steer the technical treatments of logicist issues (like number theories and proofs) towards epistemological enquiries into the nature of mathematical cognition, creating overall a style of narrative where philosophy of mathematics will be merged with philosophies of logic, psychology and action in an immaculate whole.

Later Wittgenstein's views, particularly his critique of the foundations of mathematics, are usually found to be counter-intuitive and technically wrong, while the various foundationalist versions make a smooth entry into our schemes of thought. Dismissing his later insights, or creating an apparently clinching demonstration as to how his views give way under internal inconsistencies, is a much easier task than a thoroughgoing defence. It is the latter project with which this work has entasked itself, not in a spirit of a sympathetic exposition, but in a spirit of matching my intellectual obstinacy with that of Wittgenstein; often striving to provide better arguments than he himself has done, often stretching beyond the limits of his explicit articulations to a non-standard and polemical mode of interpretation. Overall this work is a dogged attempt to take anti-foundationalism to the point of its saturation. In the long run, it is for readers to decide whether this study yields to the pressures of foundationalism or whether it teaches us to assimilate an unavoidable intellectual vertigo into our lives and academics.

I have tried to keep this work free of the enormous load of commentaries and have sought to glean the necessary insights from Wittgenstein's texts themselves. Many of his texts, like *Philosophical Grammar*, *Philosophical Remarks* and also certain portions of *Remarks on the Foundations of Mathematics*, do contain intensely detailed analyses of technical issues and intricate symbolic notations. I have avoided them as much as possible in a conscious attempt to prioritise a general conceptual approach over specialised and discipline-centred exegesis. However, what I have attempted to achieve against the foundations of mathematics, in an extremely general and non-technical manner, may hopefully be extended to highly intricate and technical issues of mathematics by experts on the subject. Overall, this book can be used by *all* readers across *all* disciplines, provided s/he has a flair for problematising the 'obvious', and an incisive interest in what constitute the conditions of possibility of language and mathematics.

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Abbreviations

BB	*The Blue and Brown Books*
NB	*Notebooks*
PG	*Philosophical Grammar*
PI	*Philosophical Investigations*
PR	*Philosophical Remarks*
RFM	*Remarks on the Foundations of Mathematics*
ROC	*Remarks on Colour*
TLP	*Tractatus Logico-Philosophicus*
RPP	*Remarks on the Philosophy of Psychology*
LPP	*Last Writings on the Philosophy of Psychology*

Introduction

We have an incurable craving for a foundation of language, something other and more primordial than language itself. It is often inspired by the ceremony of laying a foundation stone for a building, and hardens into the myth of an isolated structure, external to and yet lying underneath our usage of language. Such a foundation is supposed to have a definite origin and boundary that marks it off from language, and yet is presumed to have the magical power of drawing the entire corpus of language to come to rest on it. This myth has to be dissipated by a continual toil, by a relentless process of spreading out the proposed foundations through the unending flow of language itself. One has to lay the foundation throughout the building; we cannot stop with a single, solitary act of laying a unique and isolated base to take care of the entire superstructure. To understand this peculiar 'foundational' and 'non-foundational' nature of language, one has to start with the traditional foundations proposed by classical philosophy, spread them out, dissolve them, integrate them into language itself. In his early works, Wittgenstein had been satisfied with laying a foundation stone, or rather with laying a host of foundation stones (the 'objects', 'pictorial form', 'logical form' of the *Tractatus* [*TLP*]), believing that the foundation would magically pour the entire edifice of language out of itself. These foundation stones however fizzled out as he confronted newer and newer language-games, lived through newer and newer forms of life. This programme of dissipating foundations was most fully carried out in *Philosophical Investigations* [*PI*], while other texts, like *Remarks on the Foundations of Mathematics* [*RFM*], *Zettel*, *On*

Certainty and *Remarks on Colour* [*ROC*] represented his new and more mature philosophy.

In this work, I have tried to catch some glimpses of this enormous philosophical labour carried out by the later Wittgenstein—the labour of flattening out the hidden depths of language (proposed by classical philosophers) into an open expanse. It is an expanse where nothing is hidden, everything is open to view, and yet is unimaginably rich and complex, where the 'foundation' and the 'founded' are woven into an indissoluble whole, a whole which is never complete, never determinate and never predictable. To be more precise, the principal objective of this work is to show that for the later Wittgenstein, language cannot be founded upon anything other than language. None of the usually proposed foundations—universals, physical ostension, mental images, verbal rules, nervous excitements, brain patterns or even forms of life—can be claimed to have a pre-linguistic or extra-linguistic character that can serve as the desired origin and justification of language.

This work consists of eight chapters and an appendix. In the first chapter, I present Wittgenstein's view of concept-formation and attempt to tune myself as well my readers to the general anti-foundationalist theme of his philosophy. In the remaining chapters, I concentrate chiefly on mathematical language, rather than addressing the non-technical and ordinary language of our daily discourse. In other words, I have attempted to appreciate the general theme of anti-foundationalism particularly with respect to mathematical language. General non-mathematical language like colour predicates or pain expressions do come in occasionally. While discussing the nature of mathematical cognition as a kind of aspect-seeing (in the fourth and fifth chapters), I resort mostly to non-mathematical examples to explain the point. Some significant concepts—particularly psychological ones like 'intention', 'wish', 'will', 'desire'—are dealt with peripherally in so far as they relate to mathematical cognition. The preoccupation with mathematical language may be justified on the count that Wittgenstein himself treated mathematics as a privileged entry point into his philosophy of non-foundationalism. Speaking of the nature of concepts being formed through a continual process of adding and shedding fibres, Wittgenstein cites 'number' as one of the prime examples to explain the point. While emphasising the inherent incompleteness and indeterminacy of language, Wittgenstein asserts that we can get a rough picture of this from the changes in mathematical language, i.e., from the adding of new notations, new number systems, new calculus, and the dropping of old ones. Though this book does not address such technical issues as the changing history of mathematical language, I have attempted, on the whole, to follow Wittgenstein's strategy of attacking the invincible identity of mathematical units and the unquestionable rigour of mathematical rules as an effective method of rejecting foundational metaphysics. If we learn to break through the absolute quantitative identity of mathematical units, or the unique implicative power of mathematical rules, these fissures will automatically cut through the entire field of ordinary, non-technical language as well. In other words, the non-foundational character of mathematical language may easily be extended to the entire flow of language itself.

However, as already mentioned, except for addressing some of the technical issues in mathematics in broad, sweeping strokes, I have chosen to keep many of these issues (discussed extensively by Wittgenstein in *RFM*, *PR* and *PG*) out of the scope of the present work. I have only attempted to grasp the *spirit* of Wittgenstein's resistance with reference to Russell's theory of class and numbers, certain elementary proofs of progression of natural numbers and numerical equations. Avoiding the intra-systemic intricacies of Russell's logicism like type theory, the axiom of reducibility and the axiom of infinity, I have sought to build up pressure against the basic ideal of a system itself, and to destabilise the underlying logic of necessitation. I try to justify this omission with some reflections of the philosopher himself.

It is my task, not to attack Russell's logic from within, but from without.

That is to say, not to attack it mathematically—otherwise I should be doing mathematics—but its position, its office.

My task is, not to talk about (e.g.) Gödel's proof, but to pass it by. (*RFM* V:16)

In this work, I have not adopted any of the standard mode of commentaries written on Wittgenstein. Rather, I have tried to draw my insights as much as possible from the texts themselves, particularly *PI* and *RFM*, occasionally using supportive commentary in favour of my arguments. Here I present an outline of the work through a brief summary of each of its chapters.

The first chapter, as already mentioned, narrates Wittgenstein's view on how concepts are formed through family resemblances. I show how a cursory reading of Wittgenstein's remarks on 'twisting fibre on fibre', or on 'overlapping and criss-crossing', tends to retain a somewhat dilute theory of foundations—as a host of common features, though temporary and short-ranged, residing out there in reality. People are said to pick and choose from this vast repertoire of reality, to add new fibres and shed old ones, make different permutations and combinations according to their specific needs, interests, or wider community factors. This reading also comes to be loaded with a more cumbrous version of the Augustinian model, a model totally alien to Wittgenstein's philosophy. In this book, I seek to work my way out of these misinterpretations stage by stage, to get into the exact nuances of his writings. I show that phenomena like physical ostension, mental image or semantic rules are themselves linguistic practices having no pre-linguistic status that could explain and justify our language. Nor can an extra-linguistic reality serve as the required foundation, for the only way in which language can work is not through clamping labels passively on to a host of external essences, but rather like the tools or levers of a mechanism that incorporates reality into its own body, into a complex, functional and organic whole. Neither the 'normal' nor the 'deviant' modes of concept-formation can be grounded on a single, solitary base—say, a characteristic mode of ostension, or a specific semantic rule (normal or deviant as the case may be). Rather, normal and deviant modes of concept-formation are two styles of activity, different wholes that integrate reality and language in different ways.

The second chapter is divided into two parts. The first part attempts to show how all the classical theories seek to base mathematics on extra-linguistic foundations, assumed to be given unproblematically, be it Platonic numbers, set of sets (Frege and Russell), rules or conventions (analyticists), or a priori forms of intuition (Kant and the intuitionists). The task of mathematical language is to *describe* these foundations, to *discover* the necessary relations they entail through an act of 'ostension' (taken as either the crude physical gesture or in a reified sense of 'logical' experience' or 'immediate flash of intuition', as the case requires). Thus, all the classical theories of mathematics, though belonging to mutually opposed brands, actually adhere to the Augustinian model—a pervasive malady shared by all forms of foundationalism.

The second part of this chapter sketches out Wittgenstein's own views of mathematics. For Wittgenstein, the ideal units and sequence of arithmetic, or the ideal space of geometry, or the necessary relations between two numerical or spatial concepts are not given out there to be captured and explored by transparent acts of ostension. A mathematician does not *describe* or *discover* spatial and numerical relations but creates *paradigms* for describing or discovering, or rather judging such properties or relations. A mathematician freezes the intractable flow of experience—say the experience of putting 2 and 2 marbles or 2 and 2 blobs of paint side by side—into a flat isomorphic pattern (here, 2 + 2 = 4); and then uses it as a paradigm for judging isomorphic correlations between similar experiences and similar patterns. The mechanism of creating patterns consists not in a unique and transparent flash of intuition, but in an indeterminate cluster of repeated rituals. This cluster channelises the experience of 2 + 2 into that of 4 by interlocking both of them in a single circular motion, rather than extracting one from the inner essence of another as is generally assumed in the classical theories. It is also suggested that mathematical paradigms do not entail their subsequent applications through an inner implicative power, but redirect experience into newer and newer channels, through forming new criteria for re-identifying old experience with the new. Mathematics would turn out to be not a rigorous system issuing from a single set of foundations, but a flow of newer and newer techniques related by family resemblances.

The third chapter shifts its focus from picture paradigms to verbal paradigms or rules, or rather to a critique of the logicistic reduction of mathematics. For Wittgenstein, the verbal proofs of mathematics do not overcome either the opacity and indeterminacy of physical pictures, or their lack of a unique implicative power. Verbal or logical proofs of mathematics (with its vast resources of quantifiers, variables, constants, rules and definitions) may claim to reign over an infinite and unsurveyable mass of applications. But for Wittgenstein, in spite of all its supposedly non-sensual, non-specific and abstract content, it is at bottom nothing more than a physical picture. A logical proof like any other proof is a measure, and as a measure it must be surveyable. But when the logical proofs of mathematics claim to demonstrate a one–one correlation between a surveyable and an unsurveyable spread of numerals, or to range over an infinite space before reaching a unique conclusion, these proofs themselves fail the criterion of surveyability. To say that logical proofs are *logically* surveyable (though not *actually*) is to reduplicate the meaning of logic into a hopeless sophistry. The only way a logical proof can prove its desired conclusion is not by splitting it from the premises, magically traversing an infinite space in between, and then arriving at the conclusion with a great flourish; on the contrary, a logical proof can only prove a conclusion in the way physical proof-pictures do, i.e., by enclosing the premises and the conclusion in a single circle. However, since neither the picture paradigms nor the rule paradigms have an inner pre-applicational meaning, there are innumerably deviant ways of carving out these circles, i.e., innumerably deviant ways of applying the pictures or rules of mathematics. In fine, neither pictures nor rules can serve as the foundations of mathematics.

The fourth chapter turns to the nature of mathematical cognition, which for Wittgenstein is a new kind of 'seeing' or experience, what he calls 'aspect-seeing'. The second chapter clarified how mathematical propositions differ from empirical propositions by channelising experience into a flat pattern, thereby putting the result of experience into the process itself. In this chapter, I attempt to show how this creates a necessary transition between one experience and another, and helps us to see the old experience of 2 + 2 in a new aspect, i.e., as different from and yet identical with the new experience of 4. Now the empiricist notion of experience as consisting in primordial, irreversible and discrete bits of sensation precludes the possibility of *experiencing* either a necessary relation of identity or an aspectual *transition* of one experience into another. Wittgenstein's critique of the empiricist theory of sensation and aspect representation is discussed in this connection. For Wittgenstein, sensations are not pre-lingual or pre-aspectual grounds for inferring aspects, as the empiricists hold. Rather, sensations, including those of colour and pain (held to be typical examples of sensations by the empiricists), are aspectual in character. Two sections in this chapter devoted to the aspectual nature of colour and pain carry the suggestion that they are not as remote from mathematical aspect-seeing as is generally thought to be the case. On the whole, aspect-seeing for Wittgenstein cannot be grounded on given fragments of sensations; rather, sensations, description, inference and behaviour all go towards forming its rich and complex body. In the course of this chapter, we learn to appreciate that the empiricist notion of experience as compounded of given bits of sensation is as much contrived as the patterns created by mathematicians. The distinction between a mathematical and an empirical proposition, or for that matter the distinction between a necessary and a contingent proposition, is not a distinction between the given and the constructed, nor between their respective ontological or epistemological status. The distinction lies in the respective modes of their acknowledgement, the respective roles they play in relation to the surrounding propositions.

The fifth chapter is a conceptual investigation into the nature of aspect-seeing. It aims to show that, like the difference between mathematical and empirical propositions, the difference between aspectual and non-aspectual cognition is not ontical or psychological, but rather conceptual or grammatical. There is not a specific domain of pure non-relational objects reserved for object-perception, nor a special range of aspectual relations allotted to aspect representations. There is no new flash of intuition or a distinct state of inference, no new image, impression or a new pattern in the brain (as the Gestalt theorists say) that underlies a change of aspect. An enquiry into the Gestalt theory of perception undertaken at this juncture attempts to show that the representation of a mathematical aspect and the transition to new ones cannot be explained by a similar pattern in the cerebral cortex, or a characteristic figure–ground reorganisation. The concept of seeing, like all other concepts in Wittgenstein's philosophy, is fluid: there is no scope for an exclusive boundary between seeing and not-seeing, seeing and interpreting, seeing and understanding what it is supposed to be. The significance of aspect-seeing, too, consists in its grammatical contrasts with other related notions of seeing—seeing an object, knowing what it is supposed to be, taking it as a blueprint or a working drawing, and so on. The difference among all these cases of seeing is, to repeat, not ontical or epistemological, but a difference among different clusters of behaviours, or rather, among 'fine shades of behaviour'. This remodelled notion of seeing, viz., aspect-seeing, which is principally a technique or a cluster of activities, also preserves a meaning of mathematical novelty and necessity without being loaded with the ontical and epistemological foundations of classical philosophy.

We can say while the second and the third chapters critique logicism or conceptualism, the fourth and fifth chapters are directed against psychological and physiological foundationalism. The sixth chapter seeks to combine all these strands in a more comprehensive framework. The second and the third chapters had already exhibited the futility of the Frege-Russell strategy of transition to the second level (i.e., the level of concepts) in their definition of number, and offered a resourceful resistance against their persistent presupposition of the notion of 'one' or a single unit. The sixth chapter takes up this issue, viz., the charge of circularity levelled by Wittgenstein against the Fregean definition of number in terms of one–one correlation, to give it a different orientation. It visits this charge in the light of de Bruin's well-known commentary on the subject and a more direct engagement with Wittgenstein's observations in *Philosophical Grammar* (*PG*) and *Philosophical Remarks* (*PR*), which it eventually links up with the notion of aspect-perception. De Bruin summarises Wittgenstein's charge of circularity in the following form. As the *de dicto* knowledge of possible one–one correlation already presupposes *de re* knowledge, i.e., knowledge of actual cardinality, Frege's definition is circular. On the other hand, if *de dicto* knowledge of one–one correlation is independent of *de re* knowledge of number, as is actually the case with subitisation (where one has an immediate perception of number without performing the procedure of one–one correlation), Frege cannot define number in terms of one–one correlation. While agreeing summarily with de Bruin's construal of the charge, I have attempted to note some marked anti-foundational strands in Wittgenstein's notion of number, one–one correlation and the phenomenon of subitisation, possibly missed by de Bruin. Altogether, I have attempted to situate the charge of circularity in a wider purview to connect it with Wittgenstein's family resemblance notion of number, aspect-perception and action.

To appreciate that mathematical and empirical propositions play different roles, or that aspectual and non-aspectual perceptions embody different styles of behaviour, is to appreciate them as different language-games or different forms of life. The last chapter sees mathematics as a language-game or a form of life, related with and emerging from the empirical games, or empirical form of living, both nested in the broader stream of life. As Wittgenstein points out, the same array of signs, say 81 + 81 = 162, can be played out in at least two different ways—either as the fallible game of experiment and prediction, or by freezing it and making it non-revisable in the face of experiences. Freezing experience into non-reversible patterns is as much a game as leaving it open to experience. Further, it is a *new* game, a game of channelising experience into newer directions, effecting newer and newer aspectual transitions. Failure to understand this *new* language-game character or form of life character of mathematics, coupled with Wittgenstein's rejection of all ontical and psychological foundation, often lapses into a conventionalist reading of Wittgenstein, as one who demands that new conventions be stipulated at every stage of mathematical operation. Such misinterpretations can easily be avoided by a very simple and very important insight of Wittgenstein—that setting rules or conventions and obeying them is itself a language-game, or a form of life, which cannot be taught through further conventions. Neither can mathematics be founded on factual regularities, for such regularities being dubitable can at most form the background of mathematics, giving it its *sense*, but they cannot explain mathematical *truth*. We have also given a detailed account of deviant mathematical practices embedded in different forms of life. The purpose is to show that deviant mathematics does not depend on a characteristic deviant foundation (a set of deviant beliefs, deviant rules, deviant mental images or brain patterns). The difference between 'normal' and 'deviant' mathematics, as well as that between normal and deviant language-games in general, consists in different styles of painting, different whirls of organism.

I conclude this work with the tentative suggestion that forms of life themselves are a form of language that provide the background for the emergence of more and more complex language-games. Any philosophical discourse on forms of life will itself be a form of life, and each philosophy goes on expanding and enriching this notion, rather than pushing it back to the dubious domain of silence.

My reasons for putting the discussion of Wittgenstein's critique of formalism in an appendix have already been explained in the Preface. The crux of this chapter is to exhibit how, from Wittgenstein's perspective, formalism—reading mathematics as either about signs or about blind techniques to operate with signs—lapses either into a kind of logicism, or into sign intuitionism or sign essentialism. This basic line of the argument is unfolded through the three major sections of this chapter. The first section is an exposition of Hilbert's theory that improves on the naïve versions of term formalism and game formalism and consolidates them as a kind of logicism. In his later work, David Hilbert came to look upon mathematics as being about pre-conceptual sign intuitions, which are to be extended to the formal systems of ideal mathematics. The second section is an account of Wittgenstein's treatment of specific issues—like irrational numbers, infinity, Cantor's proof of denumerability, contradiction, etc.—which is designed to show that while it is claimed that the notion of irregular numbers speaks in favour of formalism, the latter ends up with a treatment that virtually goes against their non-realistic and non-foundational stance, lapsing ultimately into essentialism in a new disguise. Wittgenstein's approach to the exact significance of GÖdel's programme is also addressed in this section. The third section winds up with a more pointed articulation of the major discords between Hilbert and Wittgenstein, beyond their apparent proximity. On the whole, as already mentioned, the appendix will hopefully thicken out all the crucial issues in Wittgenstein's philosophy—family resemblance, language-games, aspect-perception and forms of living—not by discovering new depths, nor even by showing an absence of depth, but by recasting the so-called hidden depths as a newly expanded surface with newly enriched details.

CHAPTER I

Spinning Concepts

Laying Fibre on Fibre

Wittgenstein's view of concept-formation is set against the classical philosophical scenario, where concepts were grounded on universal essences (of the Platonic or Aristotelian variety), and definitions were made to operate with necessary and sufficient conditions, supposedly shared by all the defined items. Wittgenstein often describes the process of concept-formation in terms of fibres 'overlapping and criss-crossing', 'common features' that 'appear' and 'drop out', features that he characterises as 'family resemblances' (*PI* 66, 67). The account is often prone to certain misinterpretations, its deeper implications not being always effectively worked out. I shall, however, consciously start with a minimalist interpretation of the notion of 'family resemblance' and the 'fibre on fibre' account of concepts. Ironically, this leaves us with a multiplicity of temporary and short-ranged features (in lieu of classical 'universals'), thus retaining the overworn dichotomy between particulars and properties, and perhaps also a cumbrous version of the Augustinian model of concept-formation, the model that Wittgenstein rejected both in detail and in principle.

I shall attempt to find a way out of this impasse through an extensive critique of the Augustinian model. I shall focus particularly on the false cleavage between language and reality, and the dubious transparency of ostensive definition, claiming to bridge the two in an isomorphic relation. A total rejection of this model brought Wittgenstein to a fresh analogy between language and tools, an analogy which he effectively used to overcome the cleavage. This chapter ends by offering a rough idea of Wittgenstein's vision of language, as to how language, concepts and reality are woven together in a single and seamless complex.

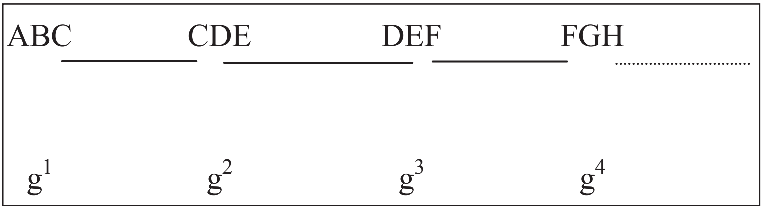
1. The Minimalist Interpretation of Family Resemblances

It is both customary and convenient to start with the concept of games, an ingenious choice with which to dissipate the notion of a fixed and unitary essence. Wittgenstein cites the examples of board games, card games, ball games and Olympic games. The features we consider important in the board games—like throwing dice, moving counters on the board—manifestly drop out in the card games and others appear. These again drop out in ball games. Obviously we have to look for certain other commonalities of apparently a broader range—like amusement, competition, winning and losing, skill and luck. Bullfighting and boxing, often involving bloodshed and casualties, do not satisfy the amusement condition. Moreover, the kind of amusement we find in chess drops out from noughts and crosses; another fibre—let it be called 'amusement' again—reappears, which will again drop out from the next kind of game we encounter. Winning and losing—the element of competition (an apparently invariable feature in all games)—do not feature in patience. Considering the fact that skill in chess is so different from skill in tennis, we cannot posit skill as a recurring feature of all games. Moreover, skill in a very general sense drops out altogether from games like Ring a Ring o Roses. '[W]e see a complicated network of similarities overlapping and criss-crossing: sometimes overall similarities, sometimes similarities of detail' (*PI* 66). In *PI* (67), Wittgenstein further observes:

> And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the fibre does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres.

The talk of overlapping fibre on fibre naturally leads to the following picture (Figure 1.1) most commonly used by Wittgenstein's commentators.

Figure 1.1

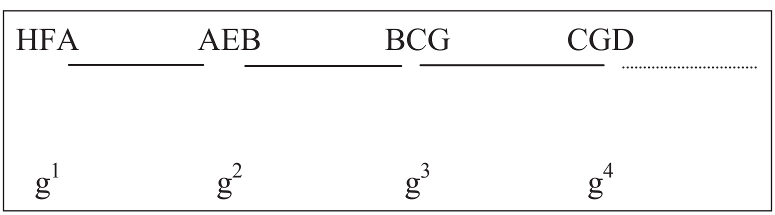


Note: Lower-case 'g' represents games, and capital letters are used for overlapping features like 'amusement', 'winning and losing', 'skill in chess', 'skill in tennis', etc.

The particulars that we call 'game' do not even share a common necessary condition, not to speak of a common sufficient condition. Nor can we construct a subset from the given set of overlapping features and claim it to be the necessary and sufficient conditions of any game whatsoever. The fibres go on overlapping in an ever-expanding horizontal line, never converging to a single point.

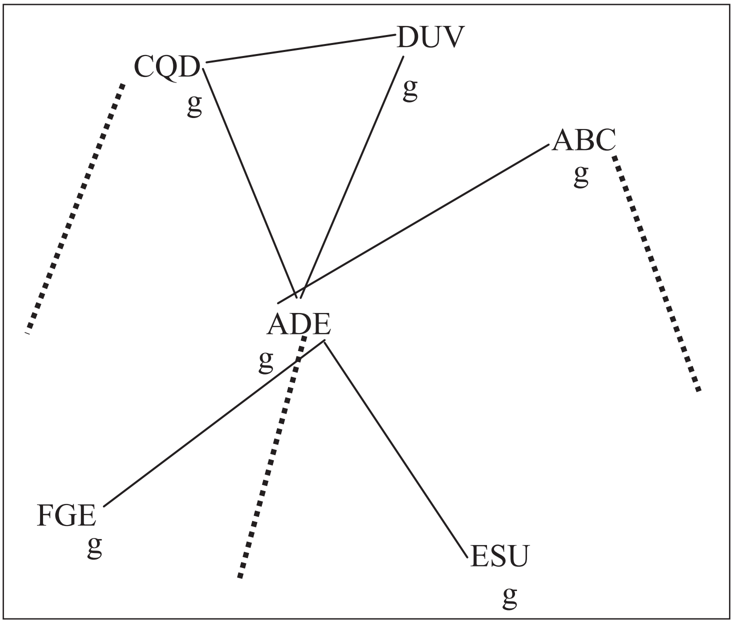
There is also no reason to suppose that all persons start with the same set of fibres, with exactly the same sets mediating between in the same order. Different language users would spin concepts in different lines like in Figure 1.2, and also in many other conceivable alternative tracks.

Figure 1.2



It is necessary to appreciate that the fibres do not only move through a horizontal track of time, jumping from game$_1$ to game$_2$, from preceding moments to successive ones. There is, as already stated, a complicated network of fibres that both overlap and criss-cross, a network that has no point of origin, where games cannot be numbered in an ordinal series of 1, 2, 3, . . ., and each individual game at any moment is a cross-section of many fibres simultaneously crossing each other. Figure 1.3 may be taken as a rough indication of what this network is like and how it expands.

Figure 1.3



Note: Here again we take ‘g’ for individual games (this time without being numbered); A, B, C as features; and the dotted lines as some of the possible modes of expansion.

Wittgenstein challenged not only the notion of a *unitary* essence but also that of a *fixed* essence. The process of old fibres disappearing and new fibres cropping up is one of continuous expansion, and not a permutation and combination of a pre-given, finite set.

Wittgenstein describes these overlapping features or fibres as ‘family resemblances’ (*PI* 67). Large families, where we can survey a number of siblings and cousins, their parents, grandparents and their offspring together, clearly exhibit how features like build, shape of the eyes and nose, structure of the jaws, curve of the lips, colour of the eyes, gait, temperament, etc., overlap and criss-cross in the same way. None of the above features at any point can be attributed to all the members in common. Thus, though starting with the instance of game, Wittgenstein privileges the case of family as well, as an exemplary case to understand how other concepts, i.e., concepts other than game too, are spun through overlapping and criss-crossing fibres, and not on the basis of a putative set of necessary and sufficient conditions. And a family expands forever, its new members continually being born, and old members passing away, generating new features to be added to the network and old features dying out.

In point of fact, though the notion of family resemblance (and therewith that of language-game) is spaced throughout the entire body of Wittgenstein's works, explicit analysis of phrases like 'overlapping and criss-crossing', 'twisting fibre on fibre', are somewhat confined to the regions around *PI* 66, 67 and 68. Apart from game and family, Wittgenstein makes important observations on the concept of number, which will be explained on a later occasion. Scattered throughout his writings are many other examples—'language' (*PI* 108, 203), 'proposition' (*PI* 134–36), 'reading' (*PI* 156–71), 'pain' (*PI* 448–49), 'rules' (*PI* 82–85), 'exact' (*PI* 69, 70), 'chair' (*PI* 80). Wittgenstein breaks through the fossilised lump of these concepts and spreads them out in a network of fibres. While some of these examples are not directly relevant to our present purpose, others will be clarified in the natural course of this investigation. But since Wittgenstein warns us not to think, but to 'look and see', we cannot now just stop with two examples—we have to examine some simple and familiar concepts, especially those which, unlike 'game' and 'family', *do* seem to have an essential property in common.

To take the example of 'gold'—a neat, scientific concept, dressed up in a complete set of necessary and sufficient conditions.[1] A definite spectral line, a certain atomic number (79), a certain atomic weight, a characteristic odour, a certain degree of malleability, a certain melting point, entering into certain chemical combinations and not others. Suppose something occurred with the same atomic number but was not yellow but purple, not malleable, had a different melting point, and produced a different series of spectral lines. Many chemists who take the atomic number itself to be the sole defining characteristic would still call it gold. Others who consider each of the above conditions to be necessary cannot call it by the same name—a position rather dubious in view of the fact that an isotope has a different weight from that normally characterising the element, yet chemists call it 'X' (X, but an isotope of X), as long as it has other characteristics of X. And we can stretch our imagination a little further to the emergence of different metals, each with a different set of 'goldish' fibres, overlapping and criss-crossing, but not a single fibre commonly running through all of them. Conceived in this way, one cannot rule out the possibility of newer and newer samples of gold with newer and newer fibres, hitherto unrecorded. This is one reason why one cannot posit a 'disjunctive property' shared in common by all particulars of the same name—whatever fibres you may have incorporated in that disjunctive set, you cannot ever put a last member. On the other hand, speaking of such common properties—a disjunctive set with an indefinite number of elements—is only 'playing with words'. 'One might just as well say: "Something runs through the whole thread—namely the continuous overlapping of those fibres."' (*PI* 67). These are the kinds of philosophical sophistries that we find parodied in nonsense prose like *Alice in Wonderland*, where the King, hearing that Alice knew nothing whatever about a theft, noted down 'nothing whatever' as very important evidence.[2]

Concepts may be seen to spread out in a network of overlapping fibres, but the myth of a common starting point—a minimal necessary condition—is not so easy to dissipate. To take the word 'cat', a rather indefinite number of characteristics are usually associated with the word—being four-legged, bearing fur, having whiskers, stalking its prey and eating it, purring, meowing, and so on. Two-faced or three-legged cats, though rare, do get reported in the newspapers. One cat may not purr or meow, another might be a vegetarian. While each of these has some of the other characteristics, no two of them share exactly the same set. But how about a common necessary condition? Surely we cannot call anything a cat unless it maintains a definite size at a given time, unless it is a mammal, unless it is available to a stable and continued perception. But a creature might emerge that purrs, meows, has some other feline features, and yet lays eggs, and suckles its babies like the platypus. Even the mammalian fibres may vary in different cases—depending on the nature of the mother's teat, the nature of the womb, the composition of the milk it suckles—very much in the way the skill in chess differs from the skill in tennis. We may think of certain extreme examples where a particular cat suddenly expands to a hundred times its normal size, or can be revived from death. We might call it a cat, or we might not; we do not have rules ready at hand for all imaginable possibilities.

Wittgenstein cites an example of a chair defying the minimal condition of continued perceivability. Seeing a chair, I go up to it meaning to fetch it, and it suddenly disappears from sight.

> 'So it wasn't chair, but some kind of illusion.'—But in a few moments we see it again, and are able to touch it and so on.—'So the chair was there after all and its disappearance was some kind of illusion'.—But suppose that after a time it disappears again—or seems to disappear. (*PI* 80)

Interestingly, Friedrich Waisemann traces exactly the same anomalies in the concept of a human. He describes a friend disappearing and reappearing, alternately becoming tangible and intangible, alternately characterising his own perceptions, i.e., the perception of his friend and that of his disappearance as valid and hallucinatory.[3]

This phenomenon of spinning concepts through dropping familiar fibres and adding unfamiliar ones has indeed taken a dramatic turn. We now seem to have dropped our necessary presuppositions like steadiness of size, continuous perceivability, and are adding fibres that are their exact opposites. These examples are designed not merely to evoke a sense of amusement or excitement, but to break through a certain fetishised notion of conception, understanding and communication. To have a concept, or to understand the meaning of the relevant term, or to communicate that meaning to others, we need not and cannot have a precise set of defining characteristics ready at hand, that sets the mind at rest once for all. Of course, it seems we can circumscribe a concept if we want, put a rigid outline around it (*PI* 68). We can indeed refuse to call anything gold unless it bears the prescribed atomic number, whichever other characteristics it may possess; or refuse to call anything a dog unless it is a mammal with the usual kind of teats that other dogs have. We may define a game by a prescribed number of characteristics that will leave out patience, Ring a Ring o Roses, or a child's play of throwing a ball and catching it again. But this approach is far from practicable; it encumbers us with the task of inventing heaps of new terms, new concept-words for both the new fibres and the particulars that bear those fibres.

As Wittgenstein observes at *PI* 68:

> I *can* give the concept 'number' rigid limits in this way . . . but I can also use it so that the extension of the concept is *not* closed by a frontier. . . . [H]ow is the concept of a game bounded? What still counts as a game and what no longer does? Can you give the boundary? No. You can *draw* one, for none has so far been drawn. (But that never troubled you before when you used the word 'game'.)

One might have played lots of games, taught how to play tennis or basketball, written various articles on them, been on the editorial board of a sports weekly. Or, he might be a scientist who had done elaborate research on gold, conducted successful experiments, and led research teams on the subject. On the classical theory of essences, the emergence of a new game or a new sample of gold with a completely new fibre, would have rather unpalatable consequences—viz., that the person concerned never understood the meaning of the word 'game' or 'gold', never entertained a proper concept about it, had never been able to communicate anything to anybody on the subject.

Wittgenstein says that one can draw a boundary around a concept 'for a special purpose' (*PI* 69). We may give a strategic definition of games in favour of choosing certain games (karate, boxing) and excluding others like carom, chess, tennis and badminton from our college curriculum. These definitions, if followed outside the particular context, would virtually put the word 'game' out of circulation, i.e., out of use. Suppose we want to redefine the length of the corridor in our university department (which we know to be X metres) in terms of how many paces it takes to walk through. For this 'special purpose' we define one pace as 75 centimetres and match up the two definitions as X metres = Y paces (*PI* 69). But apart from serving this very special purpose, this definition cannot be made to put an absurd demand on everybody's pace to measure up exactly to 75 centimetres every time they walk, thus making the very concepts of 'pace' and 'walk' unusable.

The entire process of concept-formation, the ever-expanding network of overlapping and criss-crossing similarities, will appear to be ever more complex once we realise that people differ in their mode of concept-formation, i.e., on which particulars should be subsumed under which concept. Firstly, a person or a particular community, under the influence of specific needs, interests, or of a particular history, culture or physiology, may assimilate the *same* object (i.e., what other people call the 'same' object) under a *different* concept. Secondly, she/they can assimilate 'different' objects (i.e., what other people call 'different' objects) under the 'same' concepts.

A very interesting example given by R. Bambrough may profitably be used to clarify these points. Bambrough asks us to imagine a tribe—the 'South Sea Islanders'—whose island is thickly clad with a rich variety of trees, and for whom trees are of the greatest importance in their life and work.[4] Their ways of classifying trees do not conform to the botanists' principle of classification. They do not classify trees as orange trees, date-palms or cedars, but as 'house-building trees', 'boat-building trees', or in terms of their height, thickness or maturity—features that are specially relevant to the necessities of their life. Here of course, as in all other cases, the botanist's conceptualisation of, say, a 'mango tree' and the islander's classification of 'boat-building trees' work, not with a unitary essence, but with overlapping fibres. But while the botanists' fibres of classification either go undetected, or are deemed irrelevant by the islanders, similar charges will apply to us or the botanist. The South Sea Islander may assimilate the same trees (say mango) under different concepts; say he calls one mango tree a boat-building tree, he classifies another mango tree as a house-building tree, and so on. On the other hand, he may also assimilate different trees (mango, pine and oak) under the same concept of a boat-building tree. At any point of time, an existing network of concepts is already invaded, or rather, made intricate by more and more tracks and features.

We do not always have to imagine a remote island with a remote way of life to appreciate the diverse modes of concept-formation. Modern society with its widely ramified professions, technologies and industries offers ample examples of the issue. Animals are divided in one way by the zoologist, another way by the fur industry, still in another way by the leather industry. Houses are classified in one way by the architect, in another way by the gas inspector, and in still another way by the fire department.

We may now concentrate on some of the examples cited by Wittgenstein himself on different modes of concept-formation. In *RFM* (V:42), he imagines a person (or a group of persons) who observes a surface only as coloured red, white and blue, and does not observe that it is also red. A kind of colour adjective is used for things that are partially red, partially blue and partially white—they are said to be 'bu'. And someone can be trained to observe that it is 'bu', and not to observe whether it is also red, blue or white. Such a person could only report 'bu' and 'non-bu'. Here, Wittgenstein invites us to imagine that the 'observation happens by means of a psychological sieve, which for example only lets through the fact that the surface is blue-white-red (the French tricolour) or that it is not.' Here the person obviously misses out the distinction between separate fibres; he assimilates the three distinct colours, red, white and blue, under one colour concept, 'bu'; he obfuscates the distinction between the other colours, and calls each of them 'non-bu'. The situation is somewhat like the South Sea Islanders who consider three different kinds of trees—mango, pine and cedar—to be the same. They assimilate separate fibres like the shape of the trunk, or the quality of the wood, under the same fibre, say 'maturity', and subsume the three different trees under the same class name.

At *PI* 47, Wittgenstein cites some other examples—where the same thing can be looked at from different points of view and be subsumed under different concepts. These examples are specially directed against the Tractarian theory of absolute simples. These examples seem to convey quite a simple message—whether you call a thing 'simple' or 'composite' would depend very much on what you mean by these words in different contexts, which again would be determined by your taste, temperament or focus of interest. There will be no sense in speaking of the absolutely simple elements of a chair. A carpenter would count the seat, the backrest, the legs and the arms as its elements, somebody with a physicist's approach might look upon it as atoms and molecules, and an artist as a design, and finally in a situation where we are desperately looking for some fuel to make up a fire, a chair will seem to be composed of nothing but pieces of wood. In one case, a chair may be assimilated with a car, a bag, or a goat, for all of them are composed of atoms and molecules; the artist will put the chair on par with a wood carving, a marble statue, or a picture—all that he considers being similar in artistic nuances, and so on. To consider two other examples—our visual image of a tree may be looked upon as a complex of different colours, or a complex of trunks and branches, or a broken outline composed of straight bits. A chessboard is not an absolute composite of 32 white squares and 32 black squares. It may be said to be composed of the colours black and white and the schema of squares (*PI* 47). In this way, the same individual object can be put on different tracks of family resemblances.

It is time to pause and reflect a bit on the foregoing account with its excess of examples. Is Wittgenstein merely providing an alternative theory of language where the usage of general words is founded on features or fibres residing out there in reality, from which we select and reject, make various permutations and combinations according to our specific ways of life? Are these fibres 'common features' of a different status—temporary and of a smaller range, unlike the eternal and ubiquitous universals of Platonic and Aristotelian philosophy? Are they identically shared by individuals, be they of a small group? Unfortunately, on a few occasions, Wittgenstein's phrases do provide some fuel for this kind of interpretation. 'Now pass to card-games . . . many *common features* drop out, and others appear. When we pass next to ball-games, much that is *common* is retained, but much is lost' (*PI* 66; italics mine). The metaphor of physical overlapping of one fibre on another, transferred to the context of concept-formation (*PI* 67), may also entail some misleading suggestions. On a fragmentary reading of Wittgenstein's texts, the notorious ontology of common features, over and above individuals and identically shared by them, remains unscathed. No doubt Wittgenstein draws attention to the all-important role played by the subject's attitude, history and locality in concept-formation—something that went totally unrecognised in classical philosophy. But does Wittgenstein merely succeed in breaking up the lump of eternal and all-pervading universals into temporary, smaller and local lumps? What we have to see now is how his philosophy goes *beyond* that.

2. The Augustinian Model

This myth of detachable common features, whether eternal or temporary, one or many, all-pervasive or restricted, is appended with another myth—the Augustinian model within which all language, all signs are supposed to work. According to this model or theory, each sign reaches out to its corresponding object in reality, which is its meaning, and stamps a label on it. Baker and Hacker give a neat summary of this theory in a few sentences:[5]

a) Every word has a meaning, and it is the object for which the word stands.
b) Ostensive definition is the fundamental form of explaining the meaning of a word.
c) Every sentence is a description of something; description is a combination of names.
d) Naming and describing are two essential functions of language

Baker and Hacker claim that the Augustinian theory of language is a model which all philosophers, like Plato, Aristotle, Frege, Russell, the early Wittgenstein and the logical positivists, accepted in its rudiments, improving upon it and branching out in various directions.[6] Without going into the technical details as to how exactly this was done, we may still appreciate the common strain which all these philosophies shared. In the classical scheme of Plato, Aristotle and Russell, the word 'table', for instance, would name a single entity, an Ideal Tablehood for Plato, the immanent tablehood for Aristotle, and the meaning or concept for Russell. For the British empiricists too, the sign 'table' would either be the name of an abstract mental image (Locke) or a logically abstractable essence, or, in the most non-committal nominalistic version, it would still name, not a single individual, but a single group of individuals, arbitrarily selected, which professedly do not share any common characteristic. For Frege and early Wittgenstein too, for whom sentences dissolved into a unique set of ultimate simples, the scheme of names catching their unique referents worked in full swing, with the variation that now naming occurred in the context of a sentence.

The theory of multiple, overlapping fibres too may be shown to follow the Augustinian model in principle, though with some interesting differences. In the classical scheme of universals, each name was correlated with its meaning in a single act of correlation, and each time the word recurred in different sentences it would automatically reach out to its unique meaning with which it had already been correlated once for all. The same holds for nominalists, for whom names name a particular group of individuals; and so also for Frege and early Wittgenstein. But here we have two kinds of general words—one that we may call 'concept-words', like 'game', 'family', 'gold', 'tree'; and others we may call 'feature-words'—like 'skill', 'luck', 'amusement', 'colour', 'shape', 'gait', 'malleability', 'rationality', etc. Whenever we use a concept-word, we use a different cluster of names, a different cluster of feature-words, so to speak—sometimes ABC, sometimes CDE, sometimes DEF, in the manner already explained. Now it is each of these feature-words that counts as a name; each names a uniquely corresponding feature, that is its meaning. So while in the essentialist or the nominalist scheme, there was one act of naming associated with each concept-word, now we have several features to be named, and several acts of naming. With the emergence of new features, new names and new acts of naming have to be introduced. In place of the old model of *one* concept, *one* word, *one* named entity, and *one* act of naming, now we have *one* concept, *many* named features, *many* names, and *many* acts of naming in a continuous process.

The different modes of concept-formation, including the ones Wittgenstein himself describes at *PI* 47, can, on a superficial reading, be lumped under the Augustinian model. Let us recall the instances where a chair was seen either as a composition of atoms, or of separate parts, or as an unanalysable artistic design. A chessboard can also be looked on not only as 32 white squares and 32 black squares, but also as a complex pattern which cannot be analysed into simpler elements. Firstly, according to the Augustinian model, the purported simplicity of the chair or the chessboard is only apparent, for a chair even when seen as a design can be described, and if it can be described at all, that must be in terms of its simpler elements. Secondly, whenever we put the same individual (say the same chessboard) under different concepts, the concept will be declared as ambiguous between definite alternatives. The mechanism of naming and describing would still function, only the same word would pick out different meanings in different situations.

3. Failure of Verbal Definitions

Now, can the name reach out to a unique fibre or feature, and sever it from the other features, and from the object to which it belongs? Suppose a dog is defined in terms of the features 'being carnivorous', 'barking', 'having four legs'. Will this last word, for instance, be able to hook on unfailingly to a single, detachable feature of four-leggedness commonly shared by all dogs? Each species of dog would show characteristic features of its legs, demarcating it from other species. We have to detail our definitions, introduce sharper rules in terms of the specific shape of the legs, the structure of the bones, the texture of the hair on the legs, in order to demarcate the four-leggedness, say of a Dalmatian from that of a Doberman. Instead of the single term 'four-leggedness', we will now have a host of terms, one term for a particular shape of legs, another term for the particular structure of bones, another for a specific layout of muscles. Each of these words should be able to name its meaning or its corresponding object unfailingly. But even then, the bone structure of Dalmatian_1 and Dalmatian_2 may differ considerably in other respects. Their respective bones may have different kinds of dents or undulations, the hair in their legs may have different degrees of smoothness of texture or varying shades of colour. It seems we have to break down the fibres or features further, to brush off the unwanted variations, the anomalous eruptions of boundaries, and get to the common core, quite in the same way as we go on breaking the hard cover of the coconut, tearing away the fibre, breaking the seed finally to get to the smooth neat and round copra.

It seems that to some extent the disseminations can be checked by rules. The bone structures of Dalmatian_1 and Dalmatian_2 have to be further analysed and specified as being similar in respect of another feature, say a common angle of bent at the mid-joints, which again, when shown to exhibit further individual variations, has to be analysed and specified to be similar in respect of another feature, say 'Y'. In point of fact, whatever rules we may specify, however we may detail the features of similarity, words will lead to words and to further words. This often gives the impression that while reality itself is neat, round and smoothly bounded, it is language that is inadequate to capture reality. Language is full of holes, cracks and crevices, and whatever words we might use to plug these holes and cracks themselves have fresh cracks, and so on. This way of looking at things has naturally led philosophers to rely on ostensive definitions as the last resort. In point of fact, ostensive definition was recommended by Augustine, and widely applied by other philosophers, as the only way in which language can work, i.e., the only way the sign can capture its unique meaning, cutting it out neatly from its unwanted spill-outs.

4. Failure of Ostensive Definitions

It is with very simple examples that ostensive definitions start losing their sanctity. Pointing to a pencil, I may say 'This is tove' (*BB* p. 2) (Wittgenstein deliberately chooses a fictitious word which does not have a lexical meaning). This ostensive definition can be variously interpreted to mean:

This is pencil.
This is round.
This is wood.
This is one.
This is hard, etc.

I can try to define the number 'two' by pointing to two nuts. If the hearer does not know what I want to call 'two', he will suppose that 'two' is the name given to *this* group of nuts. He might also make the opposite mistake: when I want to assign a name to this group of nuts, he might understand it as a numeral. And he may equally well take the name of a person of whom I want to give an ostensive definition as that of a colour, race or even of a point of the compass. This means that the ostensive definition has to be supplemented by words, or rather phrases like 'This *number* is called two,' 'This *colour* is called so-and-so,' 'This *length* is called so-and so' (*PI* 28).

To go back to our example of Dalmatians, how can I point to their common coat apart from the individual spot patterns that each Dalmatian has? How can I point to the common texture of their hair apart from the varying degrees of softness or roughness? Suppose there are two or more Dalmatians sitting in a sunroom in different positions and postures, the sun falling at different angles and producing different filigrees of light and shade on the body of each. How will an ostensive procedure be able to cut out their common Dalmatian coat, except perhaps by being backed up by such phrases as 'Do not look at the size, shape, number or configuration of black spots, just note that the dogs are all white with black spots.' 'Do not look at the light and shade effect on their body, just *feel* the texture of their hair.' Now, is there only one way of taking the words ‘colour’, ‘length’ or ‘texture’, ‘black and white spots’, ‘coat’, or ‘hair’ (*PI* 29)? To take ‘colour’, for instance, I point to a transparent green glass on the table and then to the same glass painted in a picture on the wall, and say ‘This colour is green.’ What do I mean by ‘colour’ in this case? Do I mean the colour in the transparency, or the opaque green as painted on a wooden door or as a pigment on the palette? In the first alternative, the colour of the green glass and that of the painted glass will not be the name, for it is the complex of colour patches that depicts the glass in the picture that is its colour. The second alternative too has no greater prospect of presenting a pure opaque green colour as a single object of ostension (*ROC*, I:18). Colour takes different dimensions, depths and hues depending on the thing that has the colour and depending on its environment; one cannot find a self-identical, saturated sample of green or white that can be captured by ostension. As Wittgenstein observes in *ROC* (I:61), ‘We are inclined to believe the analysis of our colour concepts would lead ultimately to the *colours of places* in our visual field, which are independent of any spatial or physical interpretation; for here there is neither light nor shadow, nor highlight, etc., etc.’ Of the two Dalmatians, I may see one as being white with black spots, and the other as black with white spots, putting black and white alternatively in the background and foreground. Light falling on their bodies at different angles and with different intensities will produce tonal variations of white and grey on the different parts of the body. There will be intractable variations if the light happens to filter through curtains of different colours. Differences in the sitting postures and the movement of muscles too may cause a subtle redistribution of shades. A painter who depicts each of these dogs in its characteristic posture and position with the individual light-and-shade pattern of its body has to use a different combination of colours on his palette for each of them. The ostensive definition along with the explanatory phrase, ‘Look at the common white and black coat,’ will be of little help to him.

Similar remarks would apply to the alternative modes of concept-formation with even stronger force. How would the islander pick out the characteristic feature of the boat-building trees, say the maturity of wood or the girth of the trunk, in isolation from the colour of the wood, its thickness or texture? To take Wittgenstein's own example at *PI* 47, how can one alternately point to two exclusive features of the tree—first to its broken outline composed of straight bits and then to the complexity of its colours? Any ostensive technique that might be adopted would lead to words, and words to further ostension, and neither can be privileged as the originary foundation.

5. Opacity of Acts of Ostension

Reality is gradually turning out to be too complex—too irreducibly complex to be structured even into multiple, overlapping, short-ranged, temporary features to be selected/rejected, permuted/combined, to be captured by rules or ostensive procedures. And along with this realisation, the invincible sanctity of ostensive definitions too can be seen to lose its ground in several spheres. The myth of self-identical, detachable features out there in reality, waiting to be captured by names, needs another myth of there being uniform acts of putting labels on to each of these features. All ostensive definitions are looked upon as uniform in structure and content, as uniform as the repetitive acts of a factory worker pasting identical labels on identical bottles, one after another. Taking a close look at what actually happens when we point to the shape, or point to the colour, we see that on each occasion we perform different kinds of activities that exhibit no commonly identifiable essence. How does one first point to the colour of a piece of paper, then to its shape, and then to its number? She may reply that she *meant* a different thing each time she pointed, by concentrating her attention on each of them in turn. Wittgenstein asks '[D]o you always do the *same* thing when you direct your attention to the colour?' He asks us to imagine a few cases like:

'Is this blue the same as the blue over there? Do you see any difference?'—
You are mixing paint and you say 'It's hard to get the blue of this sky.'
'It's turning fine, you can already see blue sky again.'
'Look what different effects two blues have.'
'Do you see the blue book over there? Bring it here.'

'This blue light-signal means . . . .'
'What's this blue called?—Is it "indigo"?' (*PI* 33)

Pointing to the common green colour of the painted glass and the real one, whether as transparent or opaque, is not encapsulated in a single act of correlation, but spread out in all these different kinds of activities. We sometimes attend to the colour by putting our hand up to keep the outline from view, or by not looking at the outline of the thing; sometimes by staring at the object and trying to remember where we saw the colour before. We attend to the shape sometimes by screwing up our eyes so as not to see the colour clearly, and in many other ways. And even if there were a characteristic process of attending to the shape—say, following the outline with one's finger or eyes, this by itself would not constitute what we call pointing to the shape and not to the colour (*PI* 33). It is weirder to talk of a single act of pointing to the common black and white coat of a Dalmatian—an act which brushes away the variant effects of light and shade, variant sizes and shapes and configurations of their spots. Can it possibly be done by screwing up our eyes to have a blurred image of black and white, which will, so to speak abstract from individual variations in colour and spot patterns? Such a blurred image, which has a rather stronger potential to throw out similarity relations in numerous directions, has still less chance of catching a single detachable correlate.

Wittgenstein further argues at *PI* 85:

Does the sign-post leave no doubt open about the way I have to go? Does it shew which direction I am to take when I have passed it; whether along the road or the footpath or cross-country? But where is it said which way I am to follow it; whether in the direction of its finger or (e.g.) in the opposite one?—And if there were not a single sign-post, but a chain of adjacent ones or of chalk-marks on the ground—is there only *one* way of interpreting them?

There is not a single way of interpreting a single act of pointing with the finger. I can read not only in the direction of the wrist to the finger, or from the finger to the wrist, but also in the direction in which the knuckles move (i.e., upwards), the direction in which a sliver of sunlight falls on the palms, or even the direction in which the hair stands on the arms. And whatever corrective techniques may be adopted—rubbing the knuckles, flattening out the bristles of the hair, patting my back every time I do it in the 'right' way, putting a cross in the 'wrong' direction—all these pictures are again available to innumerable ways of reading. All ostensive procedures are pictures that are ruptured from within; they disseminate into an unending flow of more and more words, and more and more pictures.

6. Failure of Inner Ostension

For the Augustinians, the fact that verbal language and gesture language fail to capture a unique meaning does not show that there is no such meaning to be captured. It only shows that we need something stronger, something 'deeper' or 'inner', something different, to effect the correlation between the word and its meaning. And the Augustinian finds it in the mental act of 'meaning' or 'understanding'. To mean something by a word is to *intend* that it be understood in a particular way, that a particular object is associated with it. When I intend that a particular name should mean a particular object, and the hearer understands that intended meaning—these are both mental occurrences, or internal acts of correlation between the word and its meaning. For the Augustinian, while a physical act of ostension, or a physical image or icon may miss its target, a mental ostension gets unfailingly hooked on its unique meaning.

The mentalists, however, cannot provide us any satisfactory answer to the question as to what this mental or spiritual act consists in. It is sometimes suggested that apart from a physical procedure or a physical picture, a person uses a mental picture, a picture that is essentially different from a physical picture in that it can represent only one meaning. In this way, one might come to regard a mental picture as a 'super-likeness' or a super-picture which makes it an image of *this* and of nothing else (*PI* 389). The Augustinians do not appreciate the fact that like the physical picture, a mental picture too, say of a white dog-skin with black spots, can be read in many different ways; it cannot by itself get hooked on to its unique meaning-entity, the unique Dalmatian coat, so to speak. We need further explanations, further acts of meaning to glue a particular mental image to a particular meaning-entity.

Usually, one is prone to having a picture of a conscious mental process running concurrently with the physical process of speaking. When someone says something and means it, he says to himself, while or just before he speaks, the same sentence or its equivalent, whereas if someone says something and does *not* mean it, he says nothing to himself inwardly. The person who means his words to stand for the coat of the Dalmatian, must say something like this to himself: 'It is the common white and black-spotted skin they all share despite their differences.' But apart from the fact that such an inner speech may not occur in many cases, it does not have any explanatory value even in the cases where it does occur. An array of unspoken words, mouthed silently, cannot have any magical quality that makes it perform a feat that a physically uttered sentence cannot. We cannot hold up a single act or occurrence, whether mental or physical—a characteristic 'feeling' of meaning, a sincere tone of voice, or an earnest facial expression—as a plausible agent to do the trick. Meaning or understanding what we say, meaning or understanding one thing rather than another, one aspect of a thing, say colour, as distinct from its shape, is not encapsulated in a single word, a single sentence, a single ostensive procedure, a single mental image, or a single inward utterance. It is a plethora of linguistic and non-linguistic activities, 'a variety of actions and experiences of different kinds before and after' (*BB* p. 145 and also *PI* 35).

We cannot fully appreciate the above statement unless we understand how ostensive definitions (including mental ostensions too) presuppose a vast background of explanations and illustrations, training and experience, practices and uses. We have already noted that an ostensive definition can explain the meaning of a word only when 'the overall role of the word in language is clear' (*PI* 30). We have seen that the definition 'This is tove' actually applied to a pencil, leaves a wide latitude for interpreting the word as colour, number, point of a compass, etc. Supplementary phrases like 'It is the number, not the shape, it is the colour, not the length, it is the texture, not the reflection of light' have to be understood before the ostensive definition can get off the ground. These words like 'colour', 'number' and 'texture' indeed show the 'grammar', 'the post at which we station the word' (*PI* 29). And to know this 'grammar' or 'post' is not to know any definitions or rules of usage, but to have a rough idea of the contexts and associations in which it is appropriate or inappropriate to use the word in question. To understand the ostensive definition, viz., 'This colour is called tove,' I must already have used sentences like 'Rose is red,' 'Sky is blue,' 'Blue is soothing or sad,' 'This colour lies between red and orange'; and not have used sentences like 'Red is walking fast,' 'Blue and green are going to the horse race' (except in metaphors or humorous non-sense).

And how does one learn to use a word in certain contexts and not to use it in certain others? Simply by using them. I may use a definition to teach a person the meaning of the word 'game', but that definition will not *entail* any usage whatsoever. What is most important is that we should describe games to the person, give examples of various kinds of indoor and outdoor games like ludo, snakes and ladders, backgammon, basketball and badminton, and add, 'This and other *similar* things are called games' (*PI* 69). We also explain how other sorts of games like chess, othello, cricket, football can be constructed on the analogy of these (*PI* 75). What is important here is that the person should already be using the word in certain contexts, travelled through some of the routes of similarity that the word has led him through, and already been debarred from using the word in certain associations. Knowing the grammar of 'game' or having the place ready for the word consists in going through a cluster of uses, and it is only then that we can try some ostensive methods of teaching the person some details of the nomenclature—like 'indoor games', 'outdoor games', 'board games', 'card games', 'good move' and 'bad move', 'masculine games', or a new game like 'squash' with which he had not been acquainted before. One can learn the game of chess without ever learning or formulating rules. One might simply have learnt simple board games at first by watching, and then progressed to more and more complicated games. A person might be given an ostensive definition, 'This is the king,' if he were shown a chessman of a shape he was not used to. The ostensive definition tells him the use of the piece because the place for it was already prepared—he has already played games, used game pieces and used these words on various occasions. Only in those conditions will he be able to ask relevantly, 'What do you call this?' about a piece of game (*PI* 31).

7. The False Bridge between Language and Reality

Both common sense and classical philosophy have been staggering under the burden of a false imagery—a false picture. There is 'reality' with its neatly detachable, self-identical features on one side, and a vast storehouse of names or labels on the other, each connected with its corresponding meaning-entity. And along with them are the human acts of correlation (definition or ostension), each act joining each label with its unique meaning. And in so far as ostension and verbal definitions are looked upon as bridges striving to join a unique label with a unique meaning, their failure to do so is put down to an essential privation in the nature of physical ostension and the nature of language itself, just as a defective bridge built with defective materials and a flawed method of construction may break, leaving a yawning gap between the two sides. Language and physical ostension are looked upon as nebulous radiations or fulgurations which fail to serve as a bridge to a unique point of reality; they radiate simultaneously in various directions. One cannot construct a bridge with a radiation, one needs a well-bounded linear structure. The Augustinians put all their trust in a mental bridge with the hope that it would provide the solid and definite structure that a physical bridge cannot.

Traditionally, language has also been looked upon as a vertical structure, a building founded upon reality. Given the three-tier scheme laid down by the essentialists—'names, definitions/ostension, reality'—we are constrained to look upon the second tier as some kind of a strange adhesive that joins the foundation with the building in some localised areas. And it is a peculiar weakness of the adhesive that it fails to join the two—the building with its foundation, the names with their unique referents. The house wobbles, the adhesive cannot hold it in its proper place, on its proper foundation.

To get out of this metaphor is to get out of the literal. The building and foundation are not to be looked upon as separate architectural features but as a continuous course of plinth, shaft, fillets, columns, walls, cornice, floor and roof. So also for the bridge. Language, reality and the preposterous bridge in between have to be dissolved together into a single complex whole. Successful ostension and definition are not encapsulated in a single act or occurrence, with a self-interpretive and self-identical content, neatly detachable from and yet hooked to an equally neat, equally detachable piece of reality. Successful communication is a cluster of uses and behaviours with no beginning or end, indeterminate and incomplete, overwhelmingly complex and overwhelmingly trivial, with all the nitty-gritty and painstaking details of humdrum daily existence. Wittgenstein's philosophy is perhaps the only one of which these details form an integral part; his texts are perhaps the only philosophical works where they have been worked out with relentless energy. Till the course we have followed, we have tried not to ignore them entirely. Once we seriously attempt to handle this indeterminate cluster, sooner or later we come to realise that we cannot by any means demarcate it from the 'meant reality'; we cannot make it stop one step short of reality as if to enable it to get hooked on to reality. We cannot ask the ocean wave to freeze just the moment before it breaks, and then attempt to extract the crushed expanse of its foams from the frozen reservoir. We cannot beckon the object (through ostension) to make its appearance in the room, in the limelight, the 'object' which had so far been standing in the dark, on the threshold of reality (*Zettel* 59).

8. Language as Tools or Levers

Wittgenstein asks us to look upon language as a toolbox and its concepts as tools (*PI* 11). A tool like a hammer, a pair of pliers or a screwdriver does not name the object it produces, nor can the process of working with tools, say driving a nail with a hammer or cutting wood with a saw, be regarded as a bridge joining the tool with its product. It makes even less sense to maintain that a verbal definition can entail its application than to speak of a tool as entailing its product. A tool can be said to produce an object only in so far as it is an integral part of a complex process—of being used by the worker along with other tools and material. In this sense we can say that a tool produces its object only in so far as the object is incorporated into it. More accurately, all of these—the tools, the materials, the process of working—are incorporated into a functional and organic complex. Language and ostension can relate to reality only in so far as reality is woven into them —in a filigree of words, descriptions, gestures, mental images, pictures, affirmations, denials, behaviours, actions.

When we speak of a tool as incorporating its effect or product into itself, we also mean that each tool performs its function in a different way, in a different complex, where none of these complexes can be said to share any common structure or content. A hammer, a pair of pliers and a screwdriver may be said to share a similarity of appearance, just as words looked upon as mere printed marks or uttered sounds may all be said to look alike. Wittgenstein invites us to look inside the cabin of a locomotive (*PI* 12), where all the different handles, looked upon as inert projections placed in different parts of the cabin, look more or less the same. It is only when we actually *use* these handles that we come to see how each works in a different mechanism, in a characteristically different style. One is the handle of the crank, i.e., the bent part of the axle, keyed to it in a right angle, which is moved continually to impart circulatory motion to the axle regulating the position of the valve. The handle of the switch is connected to the wires, batteries and the light bulb, having only two effective positions, 'off' and 'on'. Another is the handle of the brake: the harder it pulls, the harder it brakes, i.e., the more effectively does it stop the motion of the vehicle. Nothing is gained by such assimilative phrases as 'all tools and levers serve to modify something' (*PI* 14); the crank lever modifies the flow of the fluid, the switch handle modifies the light bulb, the brake modifies the motion of the vehicle. It is as vacuous to say that all language is composed of names, and all names name a unique entity out there. The same word is passed over from a description to a question, to commands, requests, exclamations, singing, passing jokes, guessing riddles—and even from one description to another—with no self-identical, repeatable content to follow the trail (*PI* 23). What is identically repeated when we describe the colour of a thing as 'blue' in different contexts? (see *PI* 33, quoted earlier). Every time it is the concrete details of description, illustration, behaviour and action that constitute what we call meaning, understanding and communication.

In point of fact, the entire proposition of giving specific names to specific objects is something one has to learn through a laborious process of drill and exercise. Had language been only a composition of names which, given the effective technique of correlation, picked out specific meaning-entities without further ado, this would not have been necessary. Rather, it is like children being constantly drilled to raise their left foot in a particular angle to a specific beat of the drum. It is the entire game of question and answer that they have to be programmed into, since the questions do not by themselves entail unique answers. It is the triadic scheme of 'name–ostension–named' that one has to be tuned into, through a constant ritual of chanting and repetition.

Naming by itself says nothing. When we say, 'This is called tove,' we have said nothing whatever, apart from the fact that we might have wanted to distinguish this word from words without meanings such as occur in Lewis Carroll's poems, or words like 'Lilliburlero' in songs (*PI* 13). In this way it can be said to be preparatory to the use of a word. But what we do after naming, i.e., the subsequent uses and behaviours, cannot in any sense be given in a single act of naming (*PI* 27). A rod by itself is nothing. It may be preparatory to being connected with the lever and serving as a brake. But the entire mechanism in which it works as a brake is not given in the rod. The variety of language-games that are played with the same word in a multitude of contexts is not given in the word—it is not given at all. It is generally thought that names take us to reality through ostension and all these three determine what we are to do with the word, how we are going to use it. But as we have seen, ostension or definition is not a unique act but itself spreads out in a motley of uses that are assumed to follow from it.

Just as a rod without being connected to the lever and without the support of the entire mechanism is nothing, it might also be anything (*PI* 6). Being just a preparation, it says nothing, but at the same time it can be a preparation for anything. The same word 'tove' may be geared to different procedures of ostensive training, different styles of drill, to yield different results, different meanings—'This is hard,' 'This is blue,' 'This is a pencil,' 'This is one.' We should rather say that each of the different training procedures incorporates a different result, just as each different drill consists in moving our limbs in a different way. Drills do not entail but include a particular style of movement. So also the different ostensive trainings—none of them by themselves yields a particular use, or takes us to a unique object; they absorb the uses and the 'objects' into themselves.

One has to see similarity or resemblance in a new light—not as relation *in respect of* a self-identical common feature shared by both the relata. Relations of family resemblances are not grounded upon non-relational features that foreshadow different routes of similarity relations. For we have seen how these putative features themselves dissipate into an incomplete and unpredictable cluster of language-games. Thus, concepts themselves amount to such clusters which do not have a beginning or end, which do not start in a vacuum, but are already integrated into other clusters, equally shapeless and incomplete. To ask what a particular object is called (i.e., which concept it falls under) or to learn its meaning through ostension or definition, one has already gone through a vast, complicated and indefinite network of relations—similarity relations without a non-relational *respect*.[7] Shorn of these '*respects*', i.e., ostensible common features, and also of unique and unfailing acts of ostension, the concepts used *in* our language, as well as the concept of language *itself*, turn out to be a motley of language-games, behaviours and practices, without any common structure or content. 'Instead of producing something common to all that we call language, I am saying . . . that they are *related* to one another in many different ways. It is because of this relationship, or these relationships, that we call them all "language"' (*PI* 65). The emphasis on 'related' is indeed designed to wean us away from the non-relational or non-foundational core of relations.

9. The 'Normal' and the 'Deviant'

What is it to say that different people of different communities under the influence of their specific histories or cultures form concepts in different ways? They do not operate with alternative sets of common features, selecting some and rejecting others from the vast repertoire of reality. This is a kind of story that is usually constructed by the pluralists. Nor is it a case of setting alternative conventions or constructing alternative rules, like 'The word "P" should be applied to all objects having x, y, and z feature,' or alternatively, 'to those having s, r, and t'. Rules or conventions cannot foreshadow their applications, for what constitutes the required features is itself a cluster of uses and behaviours which cannot be deduced from rules. To say two persons engage in two different modes of concept-formation, one that we call 'normal', and the other appearing as 'strange' or 'deviant', is to say that they participate in different clusters of uses and behaviours. Neither of these clusters can be explained in foundational terms, i.e., neither the normal uses nor the deviant ones can be said to be deduced from their respective foundations, one normal and the other deviant. It is no use trying to track down a deviant foundation—a unique rule or a unique act of ostension—unique but deviant in nature, which will explain why the person in question talks and behaves in what seems to us a deviant manner. Such a foundation, even if postulated, cannot predict all possible deviant language-games that the person would come to practise. Here again, for the deviant practitioner, the putative ground of his behaviour has to be spread out in a continuous flow of uses, images, pictures, gestures—a flow as unpredictable and indeterminate as the normal ones.

Take the example of the person who sees a chessboard not as 32 white squares and 32 black squares, but as the colours black and white and a schema of squares (see *PI* 47, also referred to earlier). What does this different way of seeing consist in? Not in picking out a detachable feature of the chessboard by a unique gesture, a unique mental image, a unique inward speech. It consists in the uses and behaviours she might indulge in—she might frame it and hang it on the wall, put it on the table and place dinner plates on it, saying 'I would like to buy other designs of black and white combination to go with it.' One can go on contriving more and more examples with each of the alternative modes of conception mentioned before—the islanders' way of assimilating different trees under the same concept, looking at a table either as a design or as pieces of wood, or as made up of parts, and many such others. In all these cases, these are language-games of inexhaustible number and variety, and in no case can they be deduced from a single and isolated ground.

Three things may be noted about what we call a 'deviant' cluster of behaviours (and this applies fully to the 'normal' uses as well): (*a*) The cluster does not follow a predictable pattern. The person who places a dinner plate on the chessboard may also place chess pieces on the board, move the pieces around, and play a game as we do. 'The same savage, who stabs the picture of his enemy apparently in order to kill him really builds his hut out of wood and carves his arrows skillfully and not in effigy.'[8] (*b*) It is for this reason that a deviant cluster cannot be marked off from a normal one, for neither of them has a well-bounded content, and both clusters exhibit an interesting admixture of both kinds of behaviours. (*c*) Since reality always comes as dissolved into the cluster of language-games, Wittgenstein cannot be construed as a conceptual relativist—as propounding a theory of a unique extra-linguistic reality being moulded by different conceptual schemes. For one thing, we do not have an extra-conceptual reality that allows itself to be moulded in different ways; moreover, we do not have conceptual *schemes* or conceptual *systems*, but unsystemic, unpredictable, endless and beginningless clusters of uses and behaviours.

The fossilised dichotomy of concept and object, one founding the other, cannot be displaced through some cryptic pronouncements. Instead of professing that concepts consist in a cluster of uses, Wittgenstein has actually sought to disperse the concepts into an inexhaustible variety of language-games. He has handled concepts in various fields—colour concepts, psychological concepts like belief, desire, pain and intention in their relation to actions, concepts of culture and value, and principally concepts of number and space in mathematics. So far we have only tried to attune ourselves to some anti-foundational strains in Wittgenstein's philosophy, to see whether and in what sense they can be carried over to his philosophy of mathematics. This will be our primary concern in the following chapters.

NOTES

1. I have been obliged to borrow many examples from John Hospers, *An Introduction to Philosophical Analysis* (New Delhi: Allied Publishers, 1986), for their wonderful variety and simplicity.
2. Lewis Carroll, *Alice's Adventures in Wonderland* (Wordsworth Classics, Hertfordshire, 2001), p. 136.
3. Hospers, *An Introduction to Philosophical Analysis*, p. 73.
4. See R. Bambrough, 'Universals and Family Resemblance', in George Pitcher (ed.), *Wittgenstein: The Philosophical Investigations* (London: Macmillan, 1966), pp. 202–3.
5. G. P. Baker and P. M. S. Hacker, *Wittgenstein*: *Understanding and Meaning* (Oxford: Basil Blackwell, 1980), vol. 1, p. 33.
6. Ibid., pp. 45–59.
7. The proposed *respects* of similarity relations always take us to an indefinite regress, as already explained earlier.
8. Ludwig Wittgenstein, 'Remarks on Frazer's *Golden Bough*', in C. G. Luckhardt (ed.), *Wittgenstein*: *Sources and Perspectives* (Hassocks, Sussex: Harvester Press, 1979).

CHAPTER II

Forming Concepts in Mathematics

1. The Classical Backdrop

As suggested earlier, the Augustinian picture of language might be represented as a prototype towards which many philosophical theories gravitate. The crude picture of pointing with the index finger or other physical gestures together with uttering names naturally developed into many sophisticated philosophies—philosophies that professedly contradict each other. To be more precise, the Augustinian model grew up into a pervasive malady that infected the entire philosophical scenario. Before we move on to Wittgenstein's philosophy of mathematics, we might try to see, as a prelude, how this malady crept into most of the prevalent theories of mathematics—Platonism, logicism, empiricism, and to some extent perhaps even in intuitionism or constructionism. Such an exercise may be expected to provide a privileged entry point into Wittgenstein's philosophy of mathematics.

1.1 *Stretching Out the Augustinian Model*

Let us in the first place, remind ourselves of the rudiments of the Augustinian model.

(a) Every word has a meaning, which is the object for which it stands, or with which it is correlated.
(b) Combinations of names make a sentence. All sentences are descriptions of something, some actual domain of reality.

(c) The correlation between a word and its meaning, a name and the named, is usually effected by an act of pointing, and an utterance of the name.

This model has its obvious limitations, and philosophers, striving to make it more comprehensive, have stretched it out or thinned it into a set of vacuous and rarefied pronouncements:[1]

(d) Physical ostension need only have a proximate role in explaining the meaning of a word. Since all words are names and all sentences are combinations of names, each word has to have a unique correlate. Language, in order to be language, must have a reality waiting out there, and a reality worth its name must be a vast repertoire of nameable objects, ready to be picked up by names. Consequently, ostensive definition is now taken to mean something like a 'bare fact of correlation', over and above the specific methods of correlation employed, whether it be the act of pointing with the index finger, moving one's eyeballs, or calling up a mental image. Such descriptions fall outside the philosophy of meaning and are of merely historical or psychological interest. The fact that the methods of correlation are mostly unsuccessful or inadequate is immaterial to the pre-established metaphysical harmony between reality and language.

(e) Ostensive definition taken in this rarefied sense forges a timeless link between a word and a thing.

(f) It is only when both language and reality are analysed down to their simplest elements that the metaphysical harmony between the two comes to light. Ordinary techniques of ostension which never go down to that level remain vague, ambiguous and incomplete, calling forth further explanations. Ostensive definition of an unanalysable word is a complete explanation of its meaning, needing no verbal supplementation.

(g) If the final ostensive definition of an unanalysable name is self-complete, it should by itself capture the essential nature of the object, and its essential or necessary relations with other objects as well. Such an ostensive definition should reflect all the combinatorial possibilities of the word, governing all its correct uses, precluding the incorrect ones. The necessary connections between simple objects are those that the ostensive definition explores. These necessities are 'synthetic': they are really out there in the object and not forged by verbal stipulations. On this view, words like 'red' and 'green' stand for simple colours and are available to complete ostensive definitions. Such definitions would explore synthetic compatibilities and incompatibilities, like 'Nothing can be red and green all over'—a truth which is necessary at the same time. Ostension in this sense has clearly taken a rationalist mould, which does not make much sense for the empiricists and positivists. For them, ostension is simply a case of perceptual demonstration, and sense perception cannot take us beyond the discrete items of reality to any real necessary relations. The necessary statement cited above would be declared analytic in the empiricist framework.

(h) The notion of ostensive definition brings with it the notion of understanding—an act of correlation, proximate or immediate, usually taken to be a mental activity apart from the physical act of pointing.

(i) Direct acquaintance, which is the foundation of understanding, is always indefinable (unanalysable). If acquaintance requires a mental representative of the object of acquaintance, then every simple object presupposes a corresponding experience or impression. Every distinct word would require a corresponding experience—e.g., perception for perceptual properties, and logical experience for the logical indefinables.

(j) Like ostensive definition, full understanding also governs the logical grammar, and thus all the possible uses of the word. And full understanding is achieved, if at all, in an instantaneous flash.

(k) Two extreme views—psychologism and anti-psychologism—can both be shown to fall under the Augustinian paradigm. At one extreme, i.e., for the anti-psychologists, the object named by the word is never in the mind, although understanding the word consists in a mental correlation with the object. On this view, mental images, emotive tones, feelings, are never relevant to understanding words. At the other extreme, only an object in the mind can be named by the word, since a mental correlation can hold exclusively between mental objects. On this view, all words should stand for mental images, and all must be directly given in experience.

1.2 *The Rationalistic Allegiance*

Given such a vacuously comprehensive model, it is not surprising that a great chunk of the prevalent philosophy of mathematics, with its wide assortment of opposites—realism, logicism, nominalism, conventionalism, and perhaps even intuitionism—can be lumped under it. It is convenient to start with mathematical realism (coupled with logicism), with Plato, Frege and Russell as its representatives.

Plato

For Plato, mathematics is about abstract entities that exist in their own right, independently of human thought, language or actions. Number theory is to be regarded as the description of objective, self-subsistent mathematical objects that are timeless, non-spatial, non-mental and non-linguistic. Facts concerning them do not involve any relation to language or thought—concrete, or in principle. A mathematician's task is to explore or discover the eternal realm of being, through the faculty of intuition—a direct and instantaneous revelation. Thus, Plato's theory of mathematics can be regarded as quite a transparent instantiation of the Augustinian model.

Frege

With Frege, mathematics was sought to be reduced to logic through a formal system—a system which was essentially a system of set theory. This logistic theory of mathematics too was Platonic in spirit—in the sense that both numbers, and sets or sequences of numbers, were treated as existing in themselves. Gödel too was a Platonist in so far as he wrote:

> the objects of the transfinite set theory,... clearly do not belong to the physical world . . . . But, despite their remoteness from sense-experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.[2]

Rather, the growth of mathematical logic was in effect an attempt at 'Platonising' logic into some kind of 'ultra-physics', which explores the quintessential reality of mathematics that cannot be captured either by ordinary experience or by the empirical sciences.[3]

In Frege's philosophy, what looks like a steadfast resistance to the Augustinian model actually involves techniques to improve on the model itself.[4] The old dictum that every word must name something is now redefined as: every word, if it occurs non-vacuously in a sentence, must play a definite role in determining its truth value. And although Frege classified words into different logical types, this differentiation was rather superimposed on a fundamental uniformity of function shared by all words—viz., each of them necessarily having a reference in any sentence bearing a truth value. His novel theory of sense as distinct from reference does not outgrow the name–designation model of meaning, but only reinstates it with a new rigour. Defined as the 'mode of presentation of the reference', the sense of a word is actually designed to provide a criterion for deciding whether or not a given object is the reference of the name in question. Indeed, Frege presents us with an overpopulated world of reference—the reference of a proper name is an individual object, that of a first-level predicate is a first-level concept, first-level relational predicates stand for first-level relations, and even sentences are names of their respective truth values. And reference is conceived as a timeless correlation between a word and an entity of the appropriate type. It is this mode of inquiry—as to which particular word stands for which kinds of things—that led Frege to his revolutionary number theory. Numerals, for Frege, do not stand for concrete individual things, nor for properties of things (such as 'being red'), nor do they name a heap of things. He defines the number of Fs as the extension of a second-level concept, i.e., the extension of the concept of being equinumerous with the concept F. That is, the number 'two' is the extension of the concept of being equinumerous with the concept of a couple.[5] To a pile of boots no number as such is attributed, but to a set of boots in a pile, or to the concept 'boot in this pile', a number does belong, as the extension of the concept of being equinumerous with the set of boots in this pile. Number 'two' named by the numeral '2' is a set containing all and only those concepts under each of which falls something and another distinct from the first. The noteworthy point here is that for Frege, this extension that a number is is an abstract object, it is not a property of individuals or of concepts.[6] The sense of the numeral '2' would be the means of determining a second-level concept, or a set of concepts as its referent, and thereby of determining whether the first-level concept belonging to the set really has two instances. Since numerals are names, mathematical sentences too are combinations of names. And the fact that Frege never mentions anything like an ostensive definition only shows his allegiance to a more refined and rarefied mode of correlation—a timeless correlation purified of all physical gestures or procedures, mental images or emotive tones. In fine, Frege's theory of 'meaning' in general and his theory of mathematics in particular, instead of challenging the Augustinian paradigm, is rather 'an uncommonly luxuriant specimen of the weed Wittgenstein is eager to root out'.[7]

Russell

Russell identified number not with the set of concepts but with the set of sets.[8] Two classes are said to have the same number when there is a one-to-one correspondence between them.

For Russell, the number '2' would be the set of all couples. The number of a given class is the class of all classes similar (having one–one correspondence) to the given class. One can say that when Platonism combines with logicism, these sets have to be looked upon as real entities, and the definitions involved have to be regarded not as stipulations, but as descriptions of a pre-existing reality. Mathematical set theory discovers and describes the realm of an abstract reality, and thus is quite in keeping with the Augustinian model of naming and description.

Of course, there is a tension between Russell's earlier and later works—i.e., between his *Principles of Mathematics* on the one hand, and *Principia Mathematica* (*PM*) and *Introduction to Mathematical Philosophy* on the other. In his later works, he recognised incomplete symbols—an admission that not every expression that occurs in a sentence is a name. Nor need every expression stand for a constituent of the 'proposition' or fact making it true. Among incomplete symbols are definite descriptions, class concepts, class names, ordinary proper names, demonstratives and logical constants. But once we appreciate Russell's theory of logical atomism, and the distinction between surface grammar and philosophical grammar, this departure from the Augustinian model turns out to be superficial. Sentences expressed in ordinary grammatical form do not conform to the Augustinian model. Apparent components (the parts of speech of ordinary language) are not real components, but have to be discovered by logical analysis—dissolving complex terms by definition to ultimate simples. Incomplete symbols should be successively removed until real components are revealed. It is only when the sentence is fully analysed that it fits the Augustinian picture. And Russell's theory also demands that one must be *acquainted* with each of the 'real' components of a fully analysed sentence.

Russell had argued that general words can have meaning only by being names of universals. Relational terms like 'in', 'of' and 'on' should accordingly name relational universals. The proposition 'The cat is on the mat' is about cat, mat and the relation between them—a relational universal truly subsisting in reality. Parallely, in *PM*, Russell tries hard to resist this line of argument extending to the propositions of mathematics, so that the proposition 3 + 2 = 5 does not turn out to be *about* 3, 2, 5 and a relation between them. However, as we shall argue in chapter III, there are strong reasons to believe that Russell could not prevent mathematical propositions from being descriptions of a particular domain of reality. He did not only accept the Augustinian model for his general philosophy, but actually turned it into a full-fledged theory of mathematical realism.

1.3 *Empiricist Theory of Mathematics*

For the empiricists, knowledge consists of discrete simple ideas having no necessary connection among themselves, and which can only be compounded into complex ideas. These simple ideas are completely independent of thought and language. Need for communication leads us to tag each simple idea with a public mark by arbitrary convention. Language too being an external representation of reality dissolves into simple symbols—each of them in their immediate signification stands for a simple idea. When empiricism is pushed to nominalism, the model of inner ostension is bound to change its character. All ideas are particulars having no common content; there is nothing like a general image, or a characteristic idea, say of red, for the name 'red' to pick out. Naming in the nominalist model is like putting an arbitrary label to a group of ideas that are at best similar, but do not share any universal property.

As for the question what mathematical propositions are about, all empiricists do not have the same answer. For Mill, mathematical propositions were inductive generalisations on the behaviour of empirical objects. Mill probably took numerals to be the names of accidental properties of particulars or groups of particulars, just as the black colour is an accidental property of ravens. Based on the repeated observation of two things and two things coming up to four things, one arrives at the general proposition 2 + 2 = 4. Mathematical nominalism, on the other hand, takes numbers to be concrete items in the world occurring at particular times. Two unsophisticated forms of this theory have gained popularity.[9]

According to one, numbers are concrete marks written on the blackboard, or sounds uttered; according to another, numbers are particulars ideas occurring in people's minds. Frege and other realists have strongly criticised this view on several counts—one of which is that while (for the realists) each number-word should denote a specific or unique entity or a specific set, the nominalists cannot provide for such uniqueness. And on this view too (like that of Mill), mathematical propositions would be contingently true descriptions of groups of marks or ideas, always running the risk of falsification by a negative instance.

Most empiricists, however (except Mill and perhaps a few others), find it obligatory to preserve the necessity of mathematical propositions. And in their theory, ostension is always about empirical reality, whether this reality is tables and chairs, written marks, uttered sounds or ideas in the mind; and ostension cannot unearth any necessary connection among all these particular items of reality. Hence, they declare mathematical propositions to be not about any particular item of reality, but about verbal symbols and the meanings we confer on them. The proposition '1 + 1 = 2' would be true by virtue of the meaning we have put into the symbols '1', '+', '=' and '2'. The question arises whether one has to go on stipulating conventions at every successive stage (i.e., 1 + 2 = 3, 2 + 2 = 4, 2 + 3 = 5 and so on), or whether conventions once stipulated can entail further conventions as their conclusions through logical rules which again are nothing but conventions. The first alternative is plainly unpalatable, and empiricists and logical positivists opt for the second. Let us try to understand how this alternative works with a very simple example. We start with following set of conventions:

(a) Df i 2 = 1 + 1 (immediate successor of 1)
(b) Df ii 3 = 2 + 1 (immediate successor of 2)
(c) '+ 1' for being 'immediate successor of'
(d) Law of Association: (x + y) + z = x + (y + z)

Now we may try to deduce 3 from 1 + 2:

1.	$1 + 2 / \therefore 3$	
2.	$1 + (1 + 1)$	1 Df i
3.	$(1 + 1) + 1$	2 Assoc.
4.	$2 + 1$	3 Df i
5.	3	4 Df ii

Even such a simple derivation would presuppose a lot—the meanings of '∴', 'immediate successor' and 'all'. The law of Association is clearly preceded by a universal quantifier 'for all values of x, y and z', which involves a knowledge of the range of their values, a knowledge of the identity criterion as to what may be considered a legitimate substitution instance of the law of Association. One cannot introduce a further convention to settle the issue. There will always be a gap between a convention and its supposed consequence, and any attempt to bridge that gap with a further convention will create a fresh gap. To account for mathematical necessity, conventionalists would be obliged to revert to the Augustinian model. They would have to regard mathematical propositions as being about objective meaning-entities and objective necessary connections, thus taking us back to the shackles of Platonism they sought to overthrow. One might say that the conventionalist theory of necessity was in a way already embedded in the Augustinian model before it attempted to break away from it.

1.4 *Kant and Intuitionism*

Before we move on to Kant's philosophy of mathematics, let us remind ourselves of the basic tune of the traditional rationalistic approach. For Plato, Frege, Russell and many others, mathematics consists in a supersensible realm of sets, numbers, triangles, circles, etc. Mathematics having been made completely independent of sense-experience, the mind had to be invested with a special faculty of rational intuition to discover these entities and their necessary connections. As we have already noted, the programme of reducing mathematics to logic (as initiated by Frege) was not to deontologise mathematics, but to ontologise logic itself, into an ultra-physics, fit to explore the special domain of mathematics—one beyond the reach of ordinary experience and empirical sciences. Axiomatising mathematics is a gradual process of opening up this mathematical realm, where one starts with self-evident axioms, moving progressively to more complicated theorems, with definitions and inference rules figuring as keys to open up one door after another, leading to larger and still larger rooms. When mathematical propositions are declared analytic in this realistic framework (as by Frege and Russell), negatively, it is to distinguish them from tautologies or verbal stipulation on the one hand, and contingent propositions on the other. Positively speaking, in this rationalist-analyticist approach, mathematical space is looked upon as a special space where larger spaces can be encapsulated into smaller ones. Axioms, definitions and inference rules are looked upon as instruments to explode the condensed spaces gradually into larger ones, thus opening up the mathematical reality into a gigantic and rigorous superstructure. The pre-established harmony that language and knowledge bore with reality is now shifted on to logic. Among all this, however, mathematical reality stands by its own right: while it has to be discovered and displayed through logical axiomatisation, its ontological status does not depend on this display. Mathematics for the realists depends on neither sense-experience nor rational intuition for its existence.

Talking generally, the Augustinian model of mathematics is one in which the mind can enter into a transparent relationship with an independent reality, be it the supersensible entities of the rationalists or the particular objects and images of the empiricists. The mind can fully capture the object of mathematical knowledge either through rational intuition or through sense-experience. Kant[10] problematised this realistic model both for knowledge in general and mathematics in particular. What we know, whether in mathematics or in any other sphere, is not reality, not things-in-themselves, but appearances or phenomena. More precisely, mathematics is about forms of intuition which, along with other things, construct full-fledged phenomena. According to Kant, to know is to condition; the unconditioned can never be known.

The conditioning mechanisms are two distinct faculties of the mind—viz., sensibility and understanding—neither of which can do this job of creating or conditioning by itself. Sensibility receives the manifold of intuitions under the a priori forms of space and time. While imagination combines the manifold into a complex intuition, being unconscious about the rule of synthesis, it ends up in merely an indefinite picture or image. Understanding judges it in terms of concepts, through a reflective awareness of the rule or plan of imaginative synthesis. The distinction between intuition and conception is not physical or temporal, but logical or conceptual. They are opposed to and dependent on one another at the same time. Intuitions are blind without concepts, and concepts are empty without intuitions.

The status of mathematical knowledge and mathematical propositions fits into this general framework. How are arithmetical propositions, say 1 + 1 + 1 + 1 + 1 = 5, established? It is through the successive addition of homogeneous units, say fingers, dots on paper, etc., that we arrive at specific number-concepts and numerical relations. And in this process of successive advance from unit to unit, from moment to moment, time itself is generated. The concept of number of course involves the empirical concepts of dots, fingertips, which in their turn would presuppose a received manifold in space and time, an imaginative synthesis of this manifold into a complex image, and a conceptual representation of the general plan or rule of synthesis. But though the simplest arithmetical proposition (say 1 + 1 = 2) involves all these, it abstracts its subject-matter i.e., the pure form of time, from this complex whole. So does geometry—the science of the pure intuition of space—which is based on the successive synthesis of imagination from part to part involved in the generation of figures (A 163, B 204, pp. 198–99).

The necessary and synthetic character of mathematical propositions hereby comes to light. For Kant, we cannot have any knowledge, any concept, without the aid of intuition, and without the pure forms of space and time in which all intuitions are determined a priori. Nothing can be apprehended, nothing can be taken into our empirical consciousness, save through the synthesis of the manifold, that is, through the combination of the homogeneous manifold and consciousness of its synthetic unity. And mathematical knowledge, in so far as it confines itself only to these universal and necessary forms of appearances, i.e., to space and time, is necessary.

On the other hand, since appearances involve a synthesis of the manifold, and since space and time are generated in this synthesis, mathematical propositions, being about space and time, are also synthetic. For Kant, the talk of mathematical propositions as being 'analytic' in the sense of opening up the supersensible realm into a rigorous superstructure is fallacious—it carries concepts beyond the range of sensible intuitions, i.e., beyond the legitimate range of their application. Such fallacies as we know might form the subject-matter of Transcendental Dialectics. In the Transcendental Aesthetics, however, Kant's scheme of ascribing syntheticity to mathematical propositions is specially designed against the tautologous and stipulative character of mathematics popularised by the empiricists. For them, mathematical propositions are about concepts, and for Kant concepts, unlike intuitions, are mediate representations that operate through certain common marks, affording no further knowledge of individuals beyond these marks. An analytic proposition like 'All bodies are extended' by its very nature blocks any direct confrontation with an individual (an individual body in this instance), thus precluding the possibility of any information about the individual body (it's being heavy or being coloured, etc.). On the other hand, representations of space and time are direct and immediate; their properties are determined synthetically in intuition, and not blocked by mere conceptual analysis. Moreover, intuitions always represent a single individual, and space and time, each being unique in its kind, have no instances. They contain many parts within themselves, but they do not subsume many instances under themselves. And since we cannot speak of common marks of spatiality and temporality, supposedly shared by all instances of space and time, we cannot also speak of a conceptual analysis of these pure forms of intuition. Simultaneity and succession are a relation, holding among different parts of time; it is not a property identically shared among them.

Thus in adding 5 to 7 we have to call in the aid of intuition (five fingers, or five dots), move successively from one unit to the next, and see what number the process comes to. Since the concept of number and numerical relations is to be derived from this successive movement from unit to unit, one cannot speak of the number 5 as assorted of and reducible back to five discrete units. In other words, one cannot represent 1 + 1 + 1 + 1 + 1 = 5 as an analytic statement.

The concept of 12 is by no means already thought in merely thinking this union of 7 and 5; and I may analyse my concept of such possible sum as long as I please, still I shall never find 12 in it . . . . [H]owever we may turn and twist our concepts, we could never, by mere analysis of them, and without the aid of intuition, discover what … is the sum. (B15, B16, p. 53)

Similar remarks would apply to geometry:

Take, for instance, the proposition, 'Two straight lines cannot enclose a space, and with them alone no figure is possible,' and try to derive it from the concept of straight lines and of the number 2. Or take the proposition, 'Given three straight lines, a figure is possible.' (B65, p. 86)

We have to resort to intuition, which is an actual progressive movement from part to part—a process that generates time and space itself.

Apparently, there are quite good reasons to regard Kant's philosophy of mathematics as anti-Augustinian. The straightforward correlation between language or knowledge on the one hand, and an independent reality on the other, is put out of court; creation of objects through a complex of many faculties, many levels of synthesis, takes its place. One can neither talk of an instantaneous correlation between the mind and supersensible 'sets' or particular objects or images. One cannot teach or learn the concept of number 'five' solely through a physical ostension to a group of five nuts, or a suitable mental image. Conceptual identification of such gestures, images or physical groups itself presupposes their a priori representation in time. Ostensive techniques to teach mathematical concepts may at best be regarded as occasions when the pure intuitions of space and time come into play.

In spite of all this, Kant could not supersede the model; he merely shifted it from reality to appearances, from noumena to phenomena. To repeat the major pitfalls of the Augustinian model: it creates a false dichotomy between knowledge or language on the one hand and reality on the other; characterises each with a well-bounded and saturated content; proclaims a pre-established harmony between the two; and strives to bridge the gap with various mechanisms—physical or inner ostension, rational intuition, logical experience. Such false dichotomies multiply in Kant's philosophy; they appear in various forms in various spheres, posing false foundations, demanding false bridges. Since Kant was committed to an independent and immaculate reality, he put it beyond all relations and characterisations—as 'things-in-themselves'. He met our need for a reality constraint through the given matter of intuitions, preserved the necessity and novelty of knowledge through a priori forms of mind—space, time and categories—forms which the human mind is determined to create and operate in the same way, in the same situation. To match with this dichotomy between pure matter and pure form, a fresh dichotomy between sensibility and understanding is invoked. Let us note some problems that all this leads to, particularly for his theory of mathematics.

(*a*) The conceptual distinction between matter and form of intuition cannot be maintained. Whatever analogies one may avail of to explain a pure form organising a pure matter—like plasticine cast into moulds, wood sawn into specific shapes—are all cases of formalised matter and materialised forms. There is nothing like an unspatialised and untemporalised mass on the one hand, and pure space and time on the other. The 'metaphysical exposition' of space and time demands that sense-experience presupposes space and time, and hence cannot be derived from from sense-experience itself, and that we have immediate and unique representation of pure and empty space and time. All such arguments themselves beg the issue and do not prove the proclaimed purity and a priority Kant desires.

(*b*) The absurdity of a purely passive manifold, independent of any spatio-temporal or conceptual characterisation, is betrayed in Kant's own arguments. The very notion of manifold involves complex levels of spontaneous synthesis. A manifold is a manifold in so far as the mind already distinguishes its elements and represents them in time, as occurring one after another. And as one successively advances to the latter parts, one must also reproduce the earlier parts in imagination, so that all the parts together form a whole. Reproduction in imagination further involves conceptual recognition of the reproduced parts to be the same as what we have apprehended before. And yet, among all this, Kant doggedly persists with a conceptual distinction between all these three levels of synthesis, and entrusts each to a respective faculty of mind—viz., sense, imagination and understanding, in that order.

(*c*) It is a strangely delicate bridge that Kant invokes to show how forms of judgement, i.e., concepts, apply to matter, i.e., objects or intuitions, both being mutually exclusive and yet correlative. 'Concepts are altogether impossible and can have no meaning if no objects are given for them . . . [T]he only manner in which objects can be given to us is by modification of our sensibility.' The notion of understanding thus must contain *a priori* formal conditions of sensibility, viz., that of inner sense, and to this the employment of concepts is restricted. This a priori formal condition of inner sense is called the schema of a concept (B 179 = A 140). This schema may well be described as the representation of the procedure of the understanding by which the sensibility is so determined to give us a content suitable for the application of concepts. The schema of a concept enables us to have a sensible image of a concept, but it (the schema itself) is more general than the image. If we put down five points one after another, we can have an image of number 5, but the method by which this image is produced, the successive combination of units into a whole, is the same whether the number is five or hundred. No image is adequate to a concept; we can never have an image of a triangle which will possibly represent triangles of different kinds, but we can have a schema of a triangle which represents the self-same procedure of the understanding in drawing in imagination triangular figures in space. 'This representation of a universal procedure of the imagination in producing an image for a concept, I entitle the schema of this concept' (B 180).

It is on this impossible task that Kant's philosophy of mathematics thrives. He attempts to split the whole into its impossible morsels—into pure space, pure time, pure matter, pure forms or concepts, and pure schema to bridge the gap in between. In this scheme of things, mathematics (unlike in the Platonic model) depends on the human activity of synthesis, synthesising concrete objects in concrete situations. Yet this entire model of constructing mathematics—receiving intuitions, forming a complex image through blind imagination, conceptualising this image into a series of homogeneous units, generating pure space and time in the process, representing the schema in thought—is loaded with false abstractions. This model is in fact more Augustinian than the other usual versions. The Augustinians argued that since there is a (pre-established) harmony, a foundational correspondence between language and reality, there must be a bridge to cover up any proposed gap or anomaly. If no physical procedure is adequate to this task, there must be an inner mental bridge—a mental image, an inward speech, a characteristic feeling, an instantaneous flash of rational intuition. Kant takes the argument still further: he claims that since none of these is adequate, it must be hidden in the depths of our mind. This is the stance that Kant takes with regard to his schema, which is in pure thought, which is the universal rule by which our imagination delineates images, in a general manner,

> without limitation to a single determinate figure, such as experience, or any possible image that I can represent in concreto, actually presents. This schematism of our understanding, in its application to appearances and their mere form, is an art concealed in the depths of the human soul, whose real modes of activity nature is hardly likely ever to allow us to discover, and to have open to our gaze. (B 181)

This Kantian theory of mathematics, its synthetic necessity, is professedly a model of description and discovery, discovering not supersensible sets or concrete particulars, but our mind itself—its putative pure forms, mechanisms and operations.

L. E. J. Brouwer's theory too aims at a complete rejection of Platonism.[11] Like Kant, he held that numbers, points, triangles exist in so far as they are created by the activity of human thought. His theory conforms to the Augustinian model since, for him, numbers really belong among the ultimate furniture of the universe, and the laws of number theory literally hold to be true. Like Kant, Brouwer too maintained that a pure intuition of temporal counting serves as a point of departure for the mathematics of number. Hence, the names 'intuitionism' and 'constructivism' are given to the philosophy of this group.

Believing that numbers are creatures of the mind, the intuitionists follow Kant in supposing that whatever the mind creates it must be able to know through and through. This leads to the belief that nothing is true about numbers except what is verifiable by counting and constructive procedures, and nothing is false about numbers except what is constructively refutable.

Brouwer would, however, take construction in a relaxed sense, as possibilities of construction, which refer to the idealised possibility of construction. Since for Brouwer mathematical constructions are mental, they derive from perceptions, i.e., physical perception of outer objects and mental images, but the idealised possibility of construction is of much wider scope than actuality—actual construction, actual perception and mental phenomena. It is based on the intention of mind, in which we abstract from: (*a*) concrete qualities and existence (i.e., from concrete marks and sounds, actually drawn and uttered in specific situations); and (*b*) from the limitation of generating sequences, for 'construction', by its very meaning is a process in time which is never complete. The 'infinite' in constructivism must be potential rather than actual. However, the rules by which infinite sequences can be generated are not merely tools in our knowledge but part of the reality (mental reality) that mathematics is about.

For the intuitionists, every natural number can be constructed, like 8 + 2 = 10 can be constructed as:

I I I I I I I I + I I = I I I I I I I I I I

But there is no construction which contains within itself the whole series of natural numbers.

Like Kant, the intuitionists genuinely wished to work away from

Platonism and yet preserve mathematical necessity. They wanted to be 'critical' and concrete, and attempted to bring mathematics down to actual constructions. Yet, scared of sounding too 'mundane' or 'plebeian', they had to make windows, so to speak, through which concrete specific constructions can escape into 'thin air', into idealised possibilities. While they do not carry the entire transcendental package of Kant, they do commit themselves to pure intuitions of space and time, and also to a theory of transparent mental intention—another theory to add to other mechanisms of inner ostension.

1.5 *Summing Up the Anti-Wittgensteinian Positions on Mathematics: Is the Notion of 'One' Presupposed?*

We have attempted a rough outline of the prevalent philosophy of mathematics, taking into account as many thinkers as Plato, Frege, Russell, a few empiricists, Kant and Brouwer. What we need to appreciate is that in spite of the variations in their respective positions, they all fit into a general characterisation. Each holds that mathematics is a rigorous superstructure, set upon paradigmatic foundations—i.e., ideal unit, ideal sequence, ideal space—building itself step by step, where in each step the same premises lead to the unique and unfailing conclusion. Arithmetic is claimed to be founded on the paradigmatic unit, the paradigmatic sequence, and all arithmetical operations are ultimately one–one correlations with such a sequence—a sequence where each unit maintains its definite position and identity. Geometry professes to start with ideal points, ideal length, ideal straightness and the like, and strives to build definitions and theorems upon them. Now, mathematicians in a way choose to ignore any further questions about these idealities—about what the criteria of their identity might be. In the Augustinian model, especially in its Platonic cast, such criteria are totally unnecessary, for the ideal units, ideal sequence and ideal space do not in any way depend on methods of identification. As for Frege, since all words have sense and reference, numerals and geometrical terms too should have their mode of presentation or route to identifying their reference. But the way Frege defines ‘number’ presupposes the very concept of number itself, or at least the notion of a single unit. So also for Russell, who defined ‘number one’ as a set of all sets which have a single member, ‘number two’ as a set of all couples, and so on.

It will naturally be insisted that the notion of one–one correlation between concepts (or classes in Russell’s case) is defined in terms of second-order logic with identity *without presupposing* the notion of *one* or a single unit. According to Frege, the fully symbolised form of this relation of one–one correlation or equinumerousity, viz., R, between two concepts F and G runs thus:

$$\exists R\ ((x)\ (Fx \rightarrow ((\exists y)\ ((z)\ (xRz.Gz) \leftrightarrow y = z))\ .\ ((x)\ (Gx \rightarrow (\exists y)\ ((z)\ ((xRz.Fz) \leftrightarrow y = z)))$$ [12]

To explain the first conjunct: This relation R between F and G is such that for any F, there exists one or more y such that for all z-s, F is related by this relation R with any z that is G if and only if y is identical with z. Now to break down the *if* clause and the *only-if* clause separately:

What the *if* clause says is: The relation R between F and G is such that if any F and any G are related by this R then for every F there has to be at least one (one or more) G. In other words, R cannot be such that any F is related with a non-existent G, that is, there is an F without a G.

What the *only-if* clause says is this: It cannot be that for any F there exists at least one (one or more) individual y such that whatever z this y is identical with, that z is not G, or that z is not related by this relation R with each F.

The second conjunct holds exactly the same statement from the side of G.

The first conjunct leaves open the possibility that for every F there might be more than one G, in which case there will be at least one G corresponding to which there is no F. To rule out this possibility, the second conjunct states: For every G that is related to every F by this relation R, there has to be at least one y identical with each F.

Let us explore the flaws of this definition.[13]

Firstly, the definition making claims about 'every x that is F' and 'every z that is G' is evidently imbibing the notions of what it is to be just *one* or an *individual* instance of F and G. It *is* using the notions of individual variables and individual quantifiers that take in or range over individual constants respectively.

Secondly, according to the notion of the existential quantifier, '∃y' is to be taken as one or more y-s, which renders the bit 'y = z' (along with their governing quantifiers) synonymous with 'whatever z-s the one or more y is identical with'. But it has already been presumed that the universal quantifiers (x) and (z) take single or just one constant as their values, R is already stated to be a relation that holds between each individual constant in (x) (Fx Ɔ (z) (xRz.Gz)), etc., etc. When identity with the variable governed by an existential quantifier is brought in the scenario, there arises a discrepancy between two kinds of identities—s-identity and j-identity (i.e., those which take several or just one individual(s) as their respective argument. Examples of s-identity might be: 'The five wealthiest people in USA are the five with most political clout.' But we cannot possibly read the identity in the above definition as s-identity, for the x and z have already been presumed to be individual variables taking singular terms as their values. Such a reading as 'There exists one or more y that is identical with each of the individual values of Gz' will not be grammatically well formed. Frege can of course explicitly choose to reinterpret the '=' sign here as j-identity, in which case he has to redefine the '∃y' as shorn of the second disjunct, viz., 'more than one'. One has to reread the relevant bit of the definition as 'for every z . . . there is just one y such that y is identical with z.' This generates two further problems: first, the definition loses all promise of a genuinely non-circular advancement to the notion of *one* that seemed to be procured by the optional and flexible character of '∃y'. Second, the second conjunct of the above definition of equinumerousity becomes redundant because the first conjunct, if interpreted in terms of j-identity, will negate the possibility of there being any Gs without a corresponding F.

For Frege and Russell, none of the conjuncts presupposed the notion of *one*, and the two together were supposed to build it up along with the much-desired notion of equinumerousity. However, what emerges is that none of the conjuncts makes any partial contribution to the required non-circular move, creating the space for the other conjunct to add up to it and complete the definition.

Thus, the notion of one–one correlation cannot be derived from that of the supposedly primitive notion of identity (unless we define identity as the relation which takes only singular terms). If we take the notion of identity with a fresh start without the baggage of *one*, we can never demand that there are individual immaculate identities given as pre-given objects, adding upon which we manufacture the compound notions of plural identities. The identity of friendship, of successful teaching, are relevant examples in favour of this point.

The greatest stake in favour of the non-circular definition of *one*, of numbers in general and the notion of equinumerousity, is placed on a shift to the second level—to the level of concepts. Whenever philosophers face trouble in defining certain notions, they attempt to shift to the *concept* of that notion, thereby seeking to capture the original indefinable term as an *instance* of the concept. To give some examples: existence is not presupposed when it is predicated not to the objects but to the concept of objects; truth is not presupposed when it is predicated not to objects of the first realm like pictures, models, photographs or events, or to ideas, feelings, images in the second realm, but to thoughts (in the third realm) *qua representing* objects in the first and the second realms.[14] When Russell's logical atoms are defined as unanalysable and irreducible terms of relations, it is the *concept* of the simple that is addressed, not the simples themselves.[15] Similarly, the notion of *one* is claimed not to be presupposed when it is predicated of a second-level concept or relation.

Wittgenstein's response to this objection, which can be best worked out with reference to specific examples of proof constructions (to be provided shortly), shall have at least to be hinted at this juncture. While the problem of identifying certain notions, say the notion of existence or the invalidity of inference itself, can be passed on to the concept *of* object or second-level inference *about* inference,[16] it is doubtful whether the crucial notion of numerical identity itself can be solved by the simple strategy of passing the buck to the second level. Enigmas like relativity of space and velocity cannot be dissolved by invoking a second level of space, itself absolute and stationary, accurately keeping track of *which* object in the first-level space is relative to *which* other. Any such attempt will invoke higher and higher levels of space *ad infinitum*. Similarly, any attempt to pass the question of a numerical unit or *one* to the level of concepts claiming to have 'one–one' correspondence with other concepts will invoke the second-level concept about what it is for the first-level concepts to have matching instantiations; this second level concept will have to invite a third level inquiry —trapping one into the patent problem of infinite regress.[17] Passing over from Frege and Russell to the empiricists, let us recall that they discarded the supersensible realm of numbers and ideal space to claim that sense-experience should produce a characteristic mental image for each mathematical term as its respective meaning. They too never seriously took up the question as to whether and in what sense particular mental images, say of one nut or a particular line drawn on the blackboard, could be available for universal conceptual demonstration. Kant on the other hand claimed that the mind, if it is to have any knowledge at all, must create such pure paradigmatic identities in the shape of space and time, and pure schema of thought, which is then up to the mathematicians to explore and describe. Postulating them as pure and a priori foundations of mathematics, and of knowledge in general, Kant blocked any further question as to how they themselves can be known or identified in any ordinary sense of the term. Intuitionists too stopped their query at equally vacuous pronouncements—that these pure identities must be there as ideal possibilities of construction, which our mind can abstract through a transparent flash of intention, from concrete qualities and existence. It was Wittgenstein who for the first time worked out a different way to deal with certain fundamental queries in mathematics.

2. Wittgenstein's Philosophy of Mathematics

To start with, we must appreciate that there is nothing like pure foundational identities mentioned above, which can be picked out through a single, instantaneous act of ostension. Let us first consider the prospect of numerals like 'one' or 'two' hooking on to ideal oneness and twoness through a unique and instantaneous revelation. We shall see that this presumption leads to the same absurdities we noted in Wittgenstein's general critique of Augustine. We may start with the initial and obvious difficulties of ostension.

2.1 Failure of Ostension in Mathematics

(*a*) Suppose we try to teach the meaning of the numeral '2' by showing two nuts and uttering the word 'two'. The hearer may think 'two' is the name given to this group of nuts (*PI* 28). We shall have to show different groups of unlike objects—two nuts, two books, two tables, two drops of water, two different gases filled in two containers. Whatever possible variations we may try out, we shall never exhaust the possibility of the hearer taking something else, what he thinks to be common to all the groups shown—say the shadow cast by them, or perhaps the fact that all these groups of objects face towards the north, or the angle of light falling upon them—to be the meaning of 'two'. The same problems recur with the ostensive teaching of 'one': the hearer may take it to be the act of pointing itself, or even the act of uttering sounds.

(*b*) If the hearer already does not know what it is that is defined, we have to use supplementary phrase like, 'It is the number that is called "one", "two", and not its shape, colour, or the ray of light you see falling upon them.' Ostensive definitions of 'one', 'two', 'three', etc., can explain the meaning of numerals only when the overall use of the word in language is clear (*PI* 29, 30). To understand the ostensive definition of numerals, one must already have gone through the ritual of counting, and this implies that she has already learnt the concept of number. However, one of the main purposes of the set-theoretic view of number is to show that one can understand the essence of twoness and threeness without knowing how to count—by simply noting the same cardinality (i.e., the exact one–one correlation without remainder) among specific sets. But this again would require us to know what 'one' is, and oneness cannot be captured in a well-bounded essence, through a single act of ostension. To identify the act of ostension, one must be able to know, among other things, what one finger is, as singled out from other fingers, what one word is, as opposed to other words and letters. Even to learn the meaning of 'one', a child needs to have gone through some elementary counting with apples, sticks, or marks on paper, played with beads of abacus, chanted tables, rehearsed nursery rhymes with numbers. One does not learn the meaning of 'one' through a sudden instantaneous flash, nor can the question of the essential identity of 'one' being neatly repeated in progressively larger numbers as 1 + 1, 1 + 1 + 1, etc., meaningfully arise. Like all words, the meaning of 'one', 'two', 'three', etc., spread out little by little, through a shapeless cluster of uses, having no end, nor any originary foundation.

(*c*) The concept of paradigmatic unithood being itself vacuous cannot be posed as a foundation of arithmetical operations. The problem may be fleshed out with a specific example discussed in *RFM* (I:25). Suppose we draw two patterns and name them 'Hands' (H) and 'Pentacles' (P) respectively (Figures 2.1.A and 2.1.B). Then each hand may be correlated with each angle as in Figure 2.1.C.

Figure 2.1.A

Figure 2.1.B

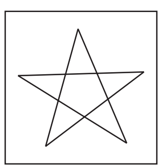
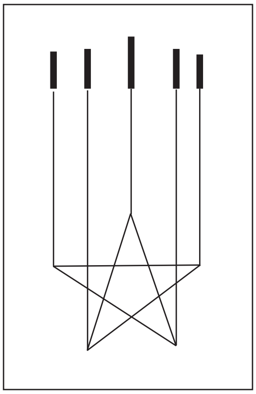


Figure 2.1.C



Now the picture may be conceived as the following mathematical proof: Any pair of shapes or structures similar to H and P must be equinumeral. Traditionally, the mathematicians are presumed to discover the ideal essence of H and P and their isomorphic connection which necessarily follows therefrom. This ideal identity is either a real essence in a subsistent realm or one that the mind necessarily creates and discovers through an immediate apprehension.

We know that for Wittgenstein the mind cannot, in any sense, be said to create or discover anything like a saturated, well-bounded identity of two patterns like the Hands and Pentacles, or their ideal isomorphic connection, through a single act of ostension. To take Figure 2.1.A, a simple sequence of linear strokes, there may be innumerable ways of looking at it. One may see it horizontally as rungs of a ladder, suggesting a continuous movement upward, where the separate identity of each rung recedes to the background. One may also see it, vertically, as the structure of a front gate. All corrective pictures and directions that we may employ, like outlining the strokes from top to bottom, pointing an arrow downward, will only invite further pictures and further directions. This becomes more evident with the star (Figure 2.1.B), with any attempt to cut out a unique paradigmatic structure. Attempts to pin down each of the outermost angles, through further gestures, pointers and diagrams, will only disseminate into intractable directions. Any verbal instruction employed will have to be loaded with words, and more words, endlessly. Let us consider the following: 'Look at each of the triangles jutting out in different directions—each forming the outermost angle, each forming one corner of the star. It is to each of these outermost angles that you should correlate the lowest point of each stroke.' These instructions have to be phrased in terms that the hearer already understands, but not in isolation, not through a single act of ostension. One needs to have gone through a considerable spread of uses—drawing lines and schematic shapes of various kinds, seeing and drawing stars or star-shaped patterns, being trained to draw lines of correlation between different shapes, and similar other practices—before he can be said to avail of such instructions.

(*d*) And no amount of such prior uses and practices will ensure that the hearer correlates the patterns in one way and no other. He might draw it in a way 'as if' he takes inner angles and not the apex as points of correlation, like in Figure 2.1.D. Or he might do it in a way still stranger—for him, all angles jut out, or none of them does, in which case he draws it as in Figure 2.1.E.

Figure 2.1.D

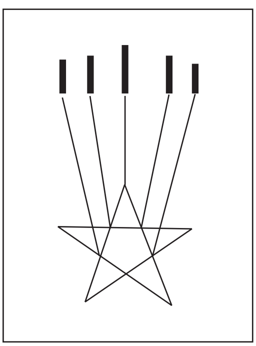
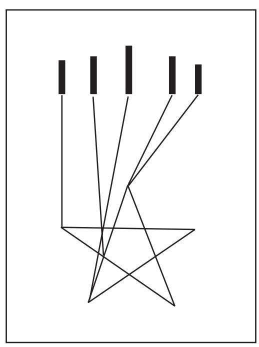


Figure 2.1.E



Drawn in this way, these two patterns, H and P, by their very essence may be said to entail that an isomorphic connection is impossible. In no way can we specify the meaning of expressions like 'jutting out', 'outermost angle', 'apex' or 'base' that would entail a unique mode of one–one correlation.

Wittgenstein has clearly argued (though not exactly in the above fashion) against the supposed transparency of self-identical units, the unique isomorphism it is claimed to entail, showing the endless regress or the inevitable circularity it falls into. He anticipates interesting objections and provides interesting rejoinders. Suppose his opponents claim that H and P have an essence whereby they entail a unique one–one correlation, in a way it wouldn't have worked if the top figure had been | | | | | | instead of | | | | |. Wittgenstein would reply that it still works; it still entails a conclusion, e.g., Figure 2.1.F.

Figure 2.1.F

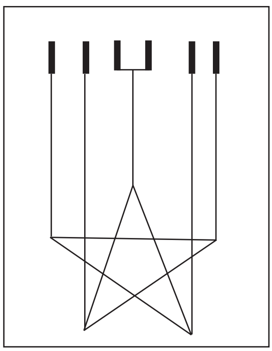
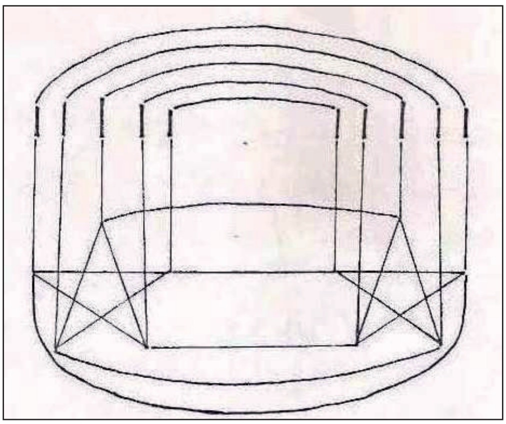


Figure 2.1.F can be used to prove that groups of these forms cannot be given a one–one correlation. The opponent would surely reply that 'I didn't mean it like that,' but whatever he 'meant' has to be cashed out in further pictures, diagrams, instructions—which would again have to go on endlessly. What the opponent cannot say is that the number of the strokes of H has to be 'five' and not 'six'. Since he claims to prove an equinumeral connection between H and P, solely from the ideal essence of their respective units, without reference to whatever the number of the units might be, he cannot smuggle the number of units itself in the premises (*RFM* I:40).

Again at *RFM* (I:41), Wittgenstein asks, 'Now isn't it possible for me to get into difficulties when I want to correlate the shapes H and P—say by there being an angle too many at the bottom or a stroke too many at the top?' The customary reply would be, 'But surely not if you have really drawn H and P again.' The essentialists might go on to claim that a sure and transparent way to check whether one has drawn the same figures H and P is to draw another pair of figures by the side of the original ones and match each apex and each stroke of the first pair with those of the second pair respectively (Figure 2.1.G).

Figure 2.1.G



Since for all traditional theories of mathematics, the ideal essence of two figures—here the Hands and Pentacles—are already given as well-defined entities, either as a real essence or as a necessary creation of the mind, such essence would not stay undetected too long. If one picture fails to capture the essential isomorphism, another picture (like Figure 2.1.G) is sure to dig it out. '[B]ut can't I still get into difficulties when I want to use this model as a guide?' (*RFM* I:41). Fresh lines of correlation may only spread out into fresh images, fresh constructions, fresh modes of one–one correlation or its denial. That we do not normally get into such difficulties is not because we stumble on some irreducible essential identity.

The persistent objection against Wittgenstein's approach, already made in defence of the non-circular definition of number supposedly achieved by a simple switch of orders, may reappear here with renewed force. One can urge: all these contrived anxieties about the inherent opacity of the pictures, the 'different ways of correlating Hands with Pentacles'—depend crucially on treating one–one correlation as a first-level relation between objects (diagrams and pictures). Don't they just go away if one follows Frege in explaining one–one correlation as a second-level relation between suitable concepts (e.g., strokes in the array and vertex of the star)?

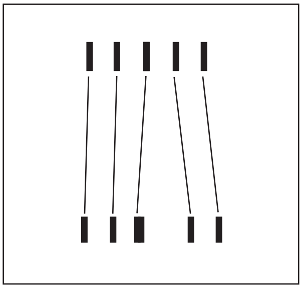
The point of this objection can be rephrased in the following manner. Let it be granted that there are different ways of interpreting language, that the lines of connection between language and reality are not transparent, and the bridges themselves stand in need of further and further interpretations. But one has to make the indeterminacy intelligible; one has to articulate what is becoming indeterminate, in what manner or along what routes. Unless one had got hold of the concepts of strokes in an array and vertex of the star, one would not have been able to articulate these indeterminacies. These concepts have always been there in the background along with other alternative concepts, and only then can one play around in the Wittgensteinian fashion with the several options of their instantiation. So why not hold on to a particular concept at the very outset, whereby one can fix a unique mode of one–one correlation as the unique instantiation of the particular concept, thus brushing away all unwanted lines of fulgurations?

If we have been able to situate ourselves in Wittgenstein's current of thought, his mode of response is not hard to imagine. He would say that even if one starts with the second level, i.e., with the concepts of objects, to brush out every other mode of conceptualisation, this concept would require a third level, i.e., a concept of the concept that is supposed to determine how this concept instantiates. That third level would require a fourth level, and so on. This is precisely the point of Wittgenstein's repeated insistence of every route of interpretation being opaque and taking you on to further and further interpretations. His assertion that grammar shows the post at which a word is stationed (*PI* 29) does not put him on the same footing with his essentialist opponents—it does not betray a switch to the second level as a sure strategy of dissipating all indeterminacies. For as we have already noted, he immediately goes on to say: 'But is there only *one* way of taking the word "colour" or "length"?' I can use the phrase 'the colour of the pencil' to refer to the colour impression, the play of light and shade on its surface, the pigments that a painter has to use on his canvas to capture the colour impression, the detachable flat coat of paint stuck on the pencil. A length can be seen as two bits: 1 centimetre each, while it may also be looked upon as consisting of two bits—one bit 1 centimetre and another bit 3 centimetres measured in the opposite direction (*PI* 47).

Now, with the present concepts at hand, one has to further articulate whether it is the collection of strokes and the collection of vertices, or each of the individual strokes and a vertex that is meant. 'Stroke in an array' may need to be defined as vertical lines drawn one after another in a chain; 'vertical' may be rephrased as 'standing upright'; 'line' as 'length with the minimum breadth'. Relevant articulations will be needed for the phrase 'vertices of the star' as well. But will these clarificatory exercises with these two phrases secure a unique way of one–one correlation between these two pictures? Someone sensitive to the breadth of the lines, however minimal and suppressed it may be to the normal naked eye, will take the first phrase to mean all the horizontal breadths from top to bottom with which each stroke is constituted; he takes the vertex of each star too as each position with the minimal magnitude. For him, a proposal of one–one correlation between these two pictures or concepts will be a proposal of correlating the respective breadths. Now as the number of breadths in each stroke far exceeds the breadth(s) in each of the vertices, the person will rather come up with a proof that these two pictures do not admit of a one–one correlation. The real agenda of these arguments is not to be deliberately perverse or stupid, but to make the point that however we may try to chisel out the concepts, their instantiation into a unique mode of isomorphic correlation is not an automatic and transparent phenomenon. Let us try to reread the Fregean definition of the equinumerousity between F and G in terms of these two specific concepts at hand. The entire symbolisation (with its tools of quantifiers, individual and predicate variables, logical operators and sign of identity) leaves the crucial question of what it is to be the value of the individual variables x, y and z (i.e., what it is to be a stroke in an array and the vertex of a star) unanswered.

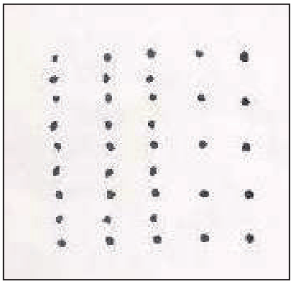
(*e*) Once we learn to problematise the acclaimed transparency of the units in figures like Hands and Pentacles, we should be able to extend this strain into simpler ones as well. All arithmetical equations can be constructed into an array of linear strokes. When a simple arithmetical equation like 5 = 3 + 2 is constructed as | | | | | = | | | | | and is correlated as in Figure 2.1.H, the putative identity of each unit and the isomorphic connection claimed to follow therefrom may similarly be questioned.

Figure 2.1.H



Suppose the lines are drawn in dots as in Figure 2.1.I, or drawn on an unusual kind of paper where the lines drawn diffuse into irregular patterns, or where the paper itself is marked with vertical lines. Wittgenstein himself had spoken of geometrical constructions carried out in flowing colours, or partially in black on a white background and partially in white on a black background (*RFM* III:50). Now one may try to specify the conditions of paradigmatic identity which are evidently not satisfied in these cases. One must specify that one should draw full lines and not dots, that the material on which the figures are drawn must have a uniform colour and texture, and the material should not interact with pencil or ink marks in an unpredictable fashion, etc. There will be lots of other conditions which, apart from being necessarily incomplete, are themselves too complex, too loaded and too opaque.

Figure 2.1.I



2.2 Failure of Measurement

It is natural to propose quantitative measurement, specifying the required dimensions of the units, the exact distance among them, their exact position in terms of grid and coordinates, or brushing away all unwanted qualitative fulguration. It is expected to present us with purely quantitative or paradigmatic identity of mathematical units. A little reflection will however show that a measuring scale fares no better than physical or inner ostension. Any attempt to pin down a fixed originary moment of complete identification—be it with ostension, or rational intuition, or measurement—will produce an endless regress of origins. In the first place, let us recall that to identify an object, say as 'blue', through ostension, we must already have identified it as having some feature, coloured, shaped or hard, etc. Similarly to put the measuring scale against the object (here the arithmetical units—say marks on the paper, or objects like tables or chairs), one needs to identify the two points within which the object lies, i.e., to have already determined its quantity. Secondly, we also need to identify the beginning and end-point of the measuring scale, which cannot be further decided by another scale without repeating the problem. Similarly, we also need to conceptualise the ostensive procedure itself—as an act of pointing with the finger, or a movement of the eyeball, or a mental image. Thirdly, the comparison between the measuring device and the measured object can no more be decided by measurement than the comparison between the ostender and the ostended can be decided by ostension. Whether the act of pointing is matched up with the table lying in the direction of the finger, or with the bed lying in the direction of the wrist, or whether the mental image of the ashtray is matched up with the purple colour of the actual ashtray lying in front, or with its oval shape, cannot be passed over to further ostensions. Measurement too would involve at least two more identifications: (*a*) coinciding the left end of the object with that point of the scale from which the markers begin; and (*b*) determining the two marks of the scale between which the right end of the object lies. Thus the limits of an object, the coincidence of points, their relative position are all presupposed and not decided by measurement.

Lastly, counting the marks of the scale is the heart of measuring. And counting as we know is mathematically defined as one–one correspondence with a paradigmatic sequence—a sequence which consists of mutually discrete and non-overlapping elements, in short, paradigmatic units. Thus, ironically, we are back to the vexed problem of deciding the paradigmatic identity of units.[18]

To say that measurement presupposes identity is not to push it further back into an extra-linguistic and ineffable realm, too subtle and rarefied for measurement to get hold of. This is quite clear from Wittgenstein's scattered comments on the issue. In *PI* (p. 225), Wittgenstein addresses a possible objection, which runs somewhat like this: what length is, is independent of the uses and practices of the method of determining length. That is to say, the absolute spatial identity of an object is very much out there, and the 'more accurate' the measurement is, the 'nearer and nearer approach' we have of an object. Wittgenstein replies that in certain cases it is, and in certain cases it is not clear what 'approaching nearer to the length of the object means'. And to make it clear is to specify certain uses and practices. A thin plastic ruler might be more effective than a thick wooden one with jagged edges, steel measuring rods may be more accurate than those made of canvas and having curves and folds. Setting a ruler against a chalk mark is not accurate since a chalk has breadth and irregular edges. A colour edge might be more exact, but this might still demand a definition of what is to count as overstepping the boundary, how and with what instruments it is to be established (*PI* 88). This will lead to an endless series of further measures, further specifications, definitions and rectifications, for as we have seen, a measure, even though posed as 'more accurate', cannot by itself determine the absolute length or quantity of the object measured; it cannot hook on to the precise quantity of the object left out or crossed over by the less accurate one, previously applied. We can talk of 'accurate and inaccurate measurements', we can go on posing measures 'more sophisticated and accurate' than the preceding ones, but at no point can one reach an absolute measure capturing an absolute quantity. All talk of 'boundaries', 'edges', 'surface', 'accuracies', 'definite', 'indefinite' are to be cashed out through a cluster of uses, practices, specifications and rectifications. It does not point to an uncashable, untouchable, pure identity, eluding all techniques of measurement. What 'determining the length' (or any other quantitative dimension) means is not learnt by learning what 'length', 'quantity', 'determining' are, once for all, through an instantaneous revelation. The meanings of these words are learnt, by learning among other things what it is to determine quantity (*PI* p. 225). To insist on a pure ideal identity, an absolute exactness, independent of all language-games, is to put language out of usage, out of circulation, to make the engine idle (*PI* **88)**.

The notion of absolute spatial discreteness, or that of paradigmatic mathematical unithood, is obviously bound up with the 'absolute simples'—an extremely popular demand of the essentialist thinkers, and thus of the Augustinian model. These simples would be unproblematically given as the ultimate primitives of both mathematical language and reality; they are claimed to dictate a unique way of composition and a unique reversal back into themselves—thus shaping up the definite quantitative identity for each mathematical unit and sequence. Now Wittgenstein always comes forth with suitable examples against such unbreakable simple identities, such unbreakable mental blocks.

We use the words 'composite' and 'simple' in an enormous number of different and differently related ways. . . . Is this length 2 cm simple, or does it consist of two parts, each 1 cm long? But why not of one bit 3 cm long and one bit 1 cm long measured in the opposite direction? (*PI* 47)

And as we have seen, no theory of measurement or counting can teach us a unique identity of the object to be measured. '[I]s there only one way of taking the word . . . "length"?' (*PI* 29, quoted earlier). One can apply similar remarks to the word 'quantity'. Wittgenstein points out that expressions like 'division of a line by a point outside it', and 'composition of forces', clearly show that sometimes we tend to look upon a greater area as composed by a division of the smaller and a smaller area as composed of greater area (*PI* 48). The second example brings an interesting analogy between matter and meaning into play. Neither matter nor meaning should be looked upon as a composite, tightly packed up with hard little balls or absolute, simple elements. Matter is to be conceived as a swarm of electrical particles, widely separated from each other and rushing about at great speed—thus creating a network or field of forces. The particles are not inert little balls, resting smugly in an equally inert, external and empty space. They are forces which can be said to occupy space only by buffeting away anything that tries to enter. Thus, they are not in space; they create space, they are space. And in this sense they create a 'composition of forces', where the smaller area can be said to be composed out of greater areas. One cannot look upon matter (or meaning) as assorted out of smaller elements inertly adding up to progressively larger ones, for the smaller can only be understood as exploding into or creating bigger space.

2.3 The Talk of Definite or Absolute Identity

In spite of all this, we are prone to talking of space as an external void, where objects with 'definite boundaries' lie inertly in 'definite positions'. There are 'impenetrable', 'solid' objects which do not allow any other object to occupy the same place at the same time. In the Augustinian model, the usage of such words and expressions, like all others, has to be founded on corresponding realities on the one hand, and a transparent flash of understanding on the other; where I do not only know the meant object but also know that I know it. Now Wittgenstein asks, 'Must I know whether I understand a word?' (*PI* p. 53, footnote). And he adds in a parenthesis, 'I thought I knew what "relative" and "absolute" motion meant, but I see that I don't know.' If one sets aside for a moment the rotation of the earth and its orbit around the sun, one could say that the earth was at rest and the train on it was travelling north at 90 miles per hour, or that the train was at rest and the earth was moving south at 90 miles per hour. Any attempt to discover the real position and real velocity of the earth would be subject to a similar pattern of relativisation, with respect to other planets in the system.[19] Suppose one tries to talk meaningfully of the distinction between 'absolute motion' and 'relative motion', with reference to a special planet in an absolute static position in space which can keep track of all planets and bodies in the universe—as to who is at rest and who is moving, and at what speed. The inhabitants of this special planet would know the absolute motion and velocity of all other planets around, while we and all these other planets caught in the relational network can only perceive relative motions. To talk of such a planet beyond relations, beyond gravitational field, without motion and mass, i.e., beyond space, is to put language 'on holiday', where words are no longer played out in language-games, in actual association with other words in concrete context. What we can at most say is that the phrases 'absolute motion' and 'relative motion' can be put to some use. We do not feel the jerks while on a station platform or on a road, while we do so in a bus or train, we see the driver starting a vehicle, while we do not see anyone starting or moving a planet. We can take an aerial motion picture of a busy city showing buses and trains moving and the land as stationary. One can of course design a film in animation graphics—a film of the universe, with different planets and stars moving in different orbits, where each planet can be shown to calculate the velocity of others from its own relative standpoint and velocity—while away from all these, at the limit of the universe so to speak, is this supreme planet looking over all of them. If we always saw ourselves and everything else jerking and moving about, we could not contrive such a picture, and the phrases 'relative motion' and 'absolute motion' would be unusable. But that we can play out these phrases in certain uses, practices and pictures does not point to a real foundation, an absolute space, or to any self-revealing flash of understanding reflecting such a space. One can always extend this imaginary picture of the universe, go on adding one planet after another, demoting the preceding ones to the relational network, and privileging the last. Such an exercise does not take us nearer to the limit of space, just as more accurate measurements do not take us nearer and nearer to the object—to an absolute boundary thereof (*PI* p. 225, mentioned in the previous section). Phrases like 'absolute space', 'absolute motion' or 'absolute boundary' are idle lumps that we eternally push beyond and further beyond all language-games, behaviours, practices or engagements.

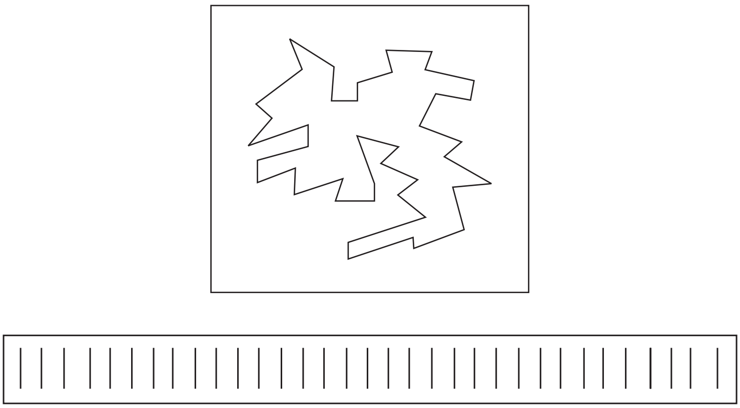
What does the talk of 'spatially discrete objects' with 'definite boundaries' consist in? A motley of behaviour and customary practices—like passing our hands along what we call the 'edge' of the table, putting things on them, carrying them around from one place to another, putting a measuring scale along its side. We cannot push our finger into tables, chairs or stones as we can into a pot of jelly; a chair or a stone does not suddenly expand or dwindle, while certain things like a gas balloon, camphor or mercury do. If everything that we term as 'solid bodies' and to which we ascribe 'definite shapes and sizes' always expanded, contracted, melted or floated like gases, the tripartite scheme of solid, liquid and gas would have been unusable. We can do certain exercises with things like apples, beans, sticks and stones, put them on the table side by side, count them, reshuffle them; and provided nothing shakes the tables, nobody comes near it, the result of such counting and reshuffling would always be the same. This is how children learn sums. If all objects that we call 'solids' like apples, beans, sticks and stones always disappeared, split or merged, mathematical propositions like '2 + 2 = 4' would be unusable (*RFM* I:37). But that such propositions can be cashed out in terms of certain uses and practices, does not point to any paradigmatic unithood—namely, any pure quantitative identity founding all uses. Just like we can talk about 'absolute' and 'relative motions', the fact that we can construct a picture of moving planets with one supreme body at the limit does not point to any absolute limit of space.

To talk meaningfully of 'absolute' and 'relative' motion, of 'absolute identity' of arithmetical units, of 'necessary connection' between '2 + 2' and '4', one has to produce certain suggestive and memorable pictures. These pictures are usually produced in the background of repeated rituals and laborious training, which can impress a procedure, a memorable pattern. Certain uses, practices, images and experiences are consciously associated through a constant drill, while others are shoved to the background. Children are constantly exposed to repetitive acts of pointing, and utterances like 'This is a table,' 'This is a chair,' 'Those are sticks,' 'This is me,' 'That is he,' and 'Those are others,' 'These are my two eyes,' 'These are my two ears,' 'This is my mouth,' 'This is inside,' 'That is outside,' 'This is indoor,' and 'That is outdoor.' They are trained not to 'bump' into others when working together, or standing in a line, and thus to respect others' 'space'. It is through the same kind of exercise that we rehearse raising our right foot (and not the left) to a particular drumbeat, train a dog to run and catch the ball in the direction in which it is thrown and not the other way. Right from our childhood we have been programmed compulsively to draw and paint in definite chromes and not to smudge, forced to put a black outline around every object we draw, assigned to fit specific shapes into specific moulds and not into others. That the table once it catches fire readily reduces to ashes, that a karate player breaks through hard bricks, that we take in and give out, that we are constantly shaped through others' thoughts and ideas, that the earth is what it is due to its atmosphere, its relation with other planets—these are all looked upon as causal relations bridging the gap between entities already separated, differentiated and distanced from one another. Causation in this model is usually supposed to work over empty space—both at the macroscopic and at the microscopic level.[20] And this separation, identification, differentiation, distancing is a matter of recurrent practice, not grounded upon any ontological difference or identity.

2.4 Mathematics Forms Pictures or Paradigms

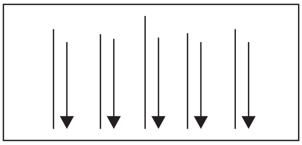
It is through constant rituals and reinforcements that the talk of discrete and homogeneous units gets its meaning. Let us see how the talk of necessary or conceptual connection between two number-concepts, or two spatial concepts, turns out to be significant. Let us take a slightly more complicated example than the ones we have used in the previous sections. We may draw an arbitrary polygon and then some arbitrary series of strokes like in Figure 2.4.A.

Figure 2.4.A



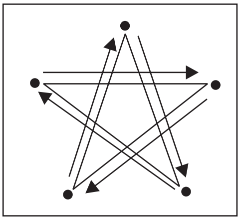
Now, we can find out by correlating them whether we have as many angles in the top figure as strokes in the bottom figure. We do not know how it will turn out (*RFM* I:27). Similarly, if we consider Figure 2.1.C again, we can also say that by drawing projection lines, we have also ascertained that there are as many strokes at the top of Figure 2.1.C as the star beneath has points. And we have drawn and ascertained the projection lines at a specific time. However, looked at in this fashion, the figure is not like a mathematical proof, just as it is not a mathematical proof when we divide a bag of apples among a group of people and find that each can have just one apple. Now suppose the whole process of drawing the stars, the strokes and the projection lines is filmed. On the screen we see a pencil moving from top to bottom like in Figure 2.4.B, and an accompanying voice chanting 'one stroke', 'two strokes', etc.

Figure 2.4.B



Then we see the star being chalked out as in Figure 2.4.C, each of its outermost points being highlighted along with the vocal commentary. The two figures (2.1.A and 2.1.B) are captioned as 'Hands' and 'Pentacles' respectively, and finally the lines of correlation are seen to move from each stroke to each point of the star below. Thus the picture is made non-temporal, it is turned into a paradigm, in the light of which we can judge all patterns similar to H and P to have an isomorphic connection. As long as the above picture is seen just as an ornament or a wallpaper design, where one line is seen to turn at this curve, at this angle, in that direction, only as a matter of fact, it is not a proof. It is by repeating and reinforcing the design in a motion picture that it is made into a paradigm (*RFM* I:28).

Figure 2.4.C

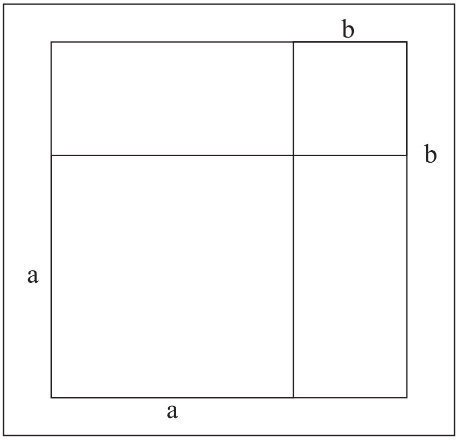


Let us consider a straightforward example of an arithmetical proof at *RFM* (I:36), where Wittgenstein gives a full description of the entire process. We are asked to imagine that we have a row of marbles and we number them with Arabic numerals which run from 1 to 100. Then we make a big gap after every 10, and in each 10 a rather smaller gap in the middle with 5 on either side: this makes the 10 stand clearly as 10. Now we take the sets of 10 and put them one below another, and in the middle of the column we make a bigger gap so that we have 5 rows above and 5 below. Finally, we number the rows as 1 to 10. 'We have, so to speak, done drill with the marbles' (*RFM* I:36). But now suppose the entire process of experimenting with the marbles were filmed in the way described in the Hand–Pentacle example. What we now see on the screen is surely not an experiment but a picture of an experiment. There first appear a hundred spots on screen, then they are arranged in tens, then fives, and so on. We claim to 'see the "mathematically essential" thing about the process in the projection too' (I:36). We do not describe this process but make it a paradigm for describing all other collections similar to that of the marbles seen on the screen.

A proof is a picture of an activity performed, an experiment conducted at a particular point of time. It is not a still picture, but as we have seen, a motion picture, or a cinematographic picture (*RFM* I:36, II:22). A still shot of a collection of marbles on screen already arranged into rows, separated at the middle as described above, will not serve as the required proof, viz., $100 = \{(5 + 5) + (5 + 5) + (5 + 5) + (5 + 5) + (5 + 5)\} + \{(5 + 5) + (5 + 5) + (5 + 5) + (5 + 5) + (5 + 5)\}$. It is only by filming this ritualistic activity of putting one marble after another, the process of assigning a number to each, the act of moving 10 marbles into a row, motioning others into rows of 10, in the successive movement of putting one row after another—that a procedure is impressed on us, is made into a memorable pattern. While a still picture can be read in many ways, it is only by mobilising the picture into a particular pattern of activity that it inclines to one direction, one line of interpretation among the rest. It is as if the motion picture guided our experience into a particular channel barring all others, so one experience is now seen together with another one in a new way (*RFM* III:32).

One may at this juncture come forward with some alleged counter-examples of proofs that work typically with still shots. Let us consider the still, diagrammatic picture of $(a + b)^2 = (a^2 + 2ab + b^2)$ (Figure 2.4.D). Some supplementary explanations of this diagram perhaps need to be provided, and the dynamic acts of drawing sequences performed by the learner have to be there in the background before he can react to this proof. But does this amount to showing that the proof itself involves an act of drawing, of movements of lines up and down, putting captions—all of which again need to be frozen in a ceremonial climax? The diagram *qua* a still shot is adequate to prove the theorem, and the learner does not need to draw these lines of movement even in his mind's eye.

Figure 2.4.D



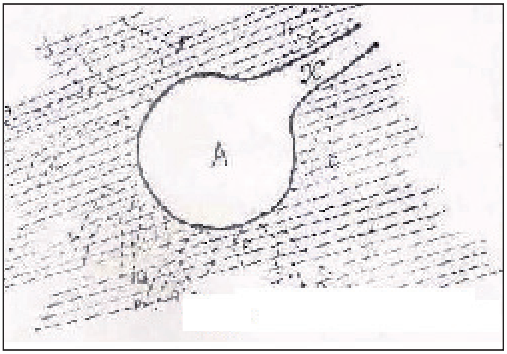
Here again we need to unlearn the obvious—we again need to impress upon ourselves that the above mass of lines has to be fixated into the required pattern through a ritualistic exercise. The lengths a and b have to be conjoined into a single line; they need to be squared up through exact repetition into a closed figure, and the line *a* has to be forced to multiply itself *b* times and vice versa to forge the two rectangles answering to 2*ab*. Without such an animated imposition, the other possible exercises—say of conjuring two layers of space where the square of *a* with the rectangle *ab* on top forms a door covering and cutting up the more expansive space beyond, a space whose structure is hidden from our eyes—cannot be blotted out. With a little imagination we can come up with many other options, none of which would be conducive to the proof of our good old algebraic formula.

We may try to clarify the matter further with a simple non-mathematical illustration. Suppose we have a site, with a roundish space that runs off in several paths, somewhat like in Figure 2.4.E.

Figure 2.4.E

Now suppose one undertakes a renovation of this site—the roundish space is paved out in a rounder shape, along with a particular track in the north-east (viz., x) while the rest of the space, and the rest of the paths remain unkempt, where dust accumulates, and are gradually covered up by weeds and undergrowth. Now this whole process is filmed—filmed in rapid motion, suitably edited and scissored, deliberately obscuring the slowness, the strenuous labour of the process of renovation, the difficulties and resistances encountered at every step. The renovated site is projected on screen and is frozen. It will look somewhat like Figure 2.4.F.

Figure 2.4.F

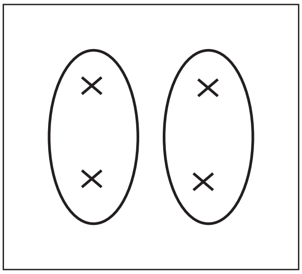


In this way our experience is guided into a definite channel, our experience of that path running north-east is now seen together with the round space itself in a new way, as if that space is essentially extended to that path, while all the other tracks that previously surrounded it are now seen as external to it. Wittgenstein has given certain other examples: he compares the mechanism of a mathematical proof with an optical instrument where light coming from various sources is made to fall into a particular pattern. Or, more interestingly, he talks of a proof as a 'work of fiction, stage-play' (*RFM* III:33). The proof is like a persuasive narration where the readers or the viewers take up a particular track as the only one in which the characters can possibly develop, the only turn the incidents can possibly take. And this is how a proof-picture can serve as a paradigm. Figure 2.4.F serves as a paradigm as to how all round spaces like A in Figure 2.4.E essentially move into paths like x and no other. The pattern made by the optical instrument serves as a paradigm as to how light sources coming in this way necessarily fall into this particular pattern and no other. The stage play serves as a paradigm of how the characters necessarily behave and develop into this climax and no other. It is in the same way that the motion picture of the process of drilling marbles into a pattern of rows and columns, or that of 'Hands' and 'Pentacles' joining up into a one–one correlation, serves as a paradigm—paradigms determining how all objects similar to the given cluster of marbles, or to the given pictures of H and P, can be drilled into similar rows and columns, or in a unique mode of isomorphic correlation.

2.5 *Circle of Reason and Experience*

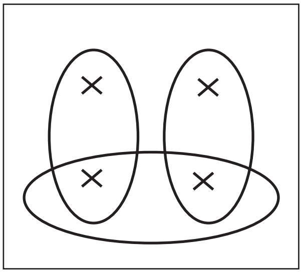
It often seems that the same stage play can be written in a different way, the same round shape, A, can move into any or all of the other tracks, rays of light coming from the same sources can be made to fall into a different pattern. But could a group of marbles, if they are really similar to the given set in the picture, i.e., if they are really a group of 100 marbles, be arranged into say (10 + 10) + (10 + 10) + (10 + 10)? Or to take simpler examples, do we not just need to look at the Figure 2.5.A to know that 2 + 2 = 4?

Figure 2.5.A



Wittgenstein says we need only to look at Figure 2.5.B to convince ourselves that 2 + 2 + 2 = 4 (*RFM* I:38).

Figure 2.5.B



One cannot argue that 2 + 2 + 2 is not a standard sequence, since the third unit merges with the fifth, and the fourth unit with the sixth, and hence cannot be available for arithmetical formulation. To argue in this way is to blatantly close oneself into a circle, to smuggle the conclusion into the premises. And this is exactly what we do with our mathematical paradigms: we persuade experience to flow in a definite channel, closing all others; we release the paradigm from the face of experience. We close our experience in the sense that we decide not to call it the experience of a particular object or a particular process unless it unfolds into these properties or leads to these results. We shall not call it an experience of the process of adding 2 and 2 unless it leads to 4, or of the same play or of the same round shape A, unless it leads to this climax or moves to this track. We incorporate the result into the process: '[i]

n mathematics, the process and result are equivalent' (*RFM* I:82). We usually think, '"It is a property of this number that this process leads to it"—But, mathematically speaking, a process does not lead to it; it is the end of a process (is itself part of the process)' (*RFM* I:84). But why do we feel that the property of a row (say of marbles) is unfolded, is shown when I assign each with a number, and then for instance split it into (5 + 5) + (5 + 5) . . . ? 'Because I alternately look at what is shewn as essential and as non-essential to the row. Or again: because I think of these properties alternately as external and as internal' (*RFM* I:85). We see 4 as internal to 2 + 2, and yet look upon it as something that 2 + 2 comes up to, or reaches at the end. One might still object, 'You surely unfold the properties of the hundred marbles when you shew what can be made of them.' Now, 'that it can be made of them no one has doubted, so the point must be the kind of way it is produced from them. But look at that, and see whether it does not perhaps itself presuppose the result' (*RFM* I:86). Suppose in that way we got one time this result and another time a different result. But it would not be considered valid since we cannot accept different results arrived at through the same process. We would say that we have made a mistake—the same units, operated in the same way, will always have to produce the same result. This shows that we are 'incorporating the result of the transformation into the kind of way the transforming is done' (*RFM* I:86). When we say 'This face turns into that through this alternation,' we cannot define 'this face' and 'this alteration' in a way that remains external to, and yet necessarily ends up in, 'that face' (*RFM* I:87). Similar remarks would apply to light coming from various sources resulting in this pattern after going through this process in the optical instrument; or 2 marbles and 2 marbles leading to 4 marbles after going through this process of addition.

This is why a proof cannot be a still shot, it has to be a motion picture. An isolated freeze shot in a cinema (without a repeated moving sequence in the background) can suggest or predict nothing. Or rather, it can suggest or predict anything. It is only when we have seen the hero ritualistically fighting against the villain, jumping from high above into a group of gangsters, that a particular shot (of the hero) frozen in space can help us predict the immediate next shot or sequence. One cannot freeze halfway and pretend to extract the other half therefrom, just as one cannot ask the ocean waves to freeze just before they break and then pretend to extract the crushed expanse of foams from its frozen reservoir. (This is a remark we have already made against the general Augustinian model of language in chapter I.) A still picture of a cluster of marbles, or Hands and Pentacles, so far as they are not put to any use, is like an inert projection sticking out from the sides of a locomotive, or like a severed rod of a vehicle, which by itself is nothing, or anything. A picture, if it is to serve as a paradigm—particularly a mathematical paradigm—has to be activated into a circle of experience, one that the mathematicians themselves carve out and in which they infuse the result into the process.

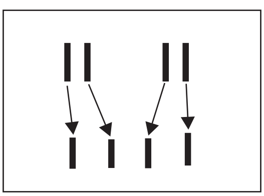
The rift that the mathematicians seek to make between the process and the result, the premises and the conclusion, the foundation and the founded, falls flat on the ground. So does the supposed transparency, the unique implicative power of the proof paradigm. An interesting analogy drawn between mathematics and kinematics may further accentuate the point. At *RFM* (I:119), Wittgenstein asks us to imagine a machine made of a material harder than any other and completely rigid to all the forces to which we subject it. Now suppose the machine is contrived to move in one particular way—like when the rod is brought out of the horizontal into the vertical, it shrinks, or it bends when set upright. Or suppose the rod bends when a certain mass is brought near it; for instance, the guide rails of the crosshead bend and straighten again as the crank approaches and retreats. Now if we are unable to relate the crosshead and the crank under a single mechanism, if we look upon each of them as a severed projection, we might think that the rails were something alive, moving by a hidden internal essence. This leads one to look upon kinematics as a theory that deduces the movements of the machine from the absolute rigidity of its parts. Given the absolute rigidity or identity of its parts, given the fact that the parts do not react to external influences, the theory of kinematics predicts their unique movement. This manner of speaking obscures the simple truth that the 'absolute rigidity' or 'self-identity' of the machine parts consists in precisely those movements which cannot be externalised to and extracted from it. The paradigmatic identity of 2 + 2 is made to consist in 4, and hence cannot be deduced from it. Kinematics does not specify any method of measurement when it speaks of the sameness of the machine parts, or of the constancy of their length (*RFM* I:120). Nor does mathematics specify a criterion of the ideal sameness of units, or of geometrical lines and figures. To put all anomalies in mathematical conclusions or in mechanical movements down to a violation of this 'ideal identity' of units or of absolute rigidity of machine parts, and to pose this 'rigidity' or 'identity' as independent of and yet entailing a unique conclusion, is to reify these phrases into an unbreakable and unusable lump, putting language 'on holiday' (*PI* 38) 'like an engine idling' (*PI* 132).

2.6 *Some Reflections against Mathematical Realism*

The exact manner in which the paradigmatic character of mathematics is made to work against Platonism is adequately clear.

Mathematics creates paradigms, rather, paradigmatic motion pictures, in which it carves out a circle of experiences in a way that it decides not to call something a picture of 'x' unless it is also a picture of 'y'. The picture II II is not to be called a picture of 2 + 2 unless it is also called a picture of 4. Diagrams like Figures 2.1.A and 2.1.B are not to be called pictures of strokes and pentacles unless they join up to a unique one–one correlation. By creating II II as a paradigm of 2 + 2, we also consider it as a paradigm of 4.

Figure 2.6.A



The proposition 2 + 2 = 4 does not express something as the essence or internal constitution of these two numbers. The mathematical proof or paradigm '2 + 2 = 4', as we have seen, is a usable picture, it provides a criterion of meaning. Figure 2.6.A suggests that it makes sense to say 2 + 2 = 4 and not 2 + 2 = 3. Let us return to our old picture of the closed universe, with one static planet placed at a vantage point to keep track of the real position and motion of the other planets moving in the relational network. Such a model of absolute space makes the phases 'relative' and 'absolute' motion usable. Let us also remind ourselves of the old example of two measuring tapes—the first a canvas one with irregular edges, curves and folds, the second made of metal with relatively smooth edge and texture. When these two tapes are successively placed against an object, one can use this to point to the part of the object left out or overcrossed by the first measure. Now we can use a picture of this phenomenon as a model to give meaning to the phrases 'definite and indefinite boundaries', 'accurate and inaccurate measurements'. But none of these pictures points to a real foundation, an absolutely static space, or an absolute boundary of objects, impenetrable to all interactions. Models or paradigms always work with levels of comparison, of varying degrees placed serially, just as grammarians place adjectives in ascending levels of 'positive', 'comparative' and 'superlative'. The fact that it is grammatically correct to use adjectives in three levels—say sweet, sweeter and sweetest—does not correlate to an ontically highest order of sweetness. No more do mathematical paradigms like 2 + 2 = 4 point to an absolute identity of each number on each side of the equation, or to a real, necessary connection between the two. Mathematical paradigms too have this play of differences in the background. Certain things break, bend, dwindle, disappear in varying degrees which enable us to use some objects as units, speak of them as rigid, hard and stable. It is the use of adjectives in different grammatical levels like 'soft', 'softer', 'softest', 'hard', 'harder', 'hardest', that goes a long way in giving mathematical paradigms their required use or significance.

Wittgenstein has many interesting observations to make against mathematical realism. In *RFM* (III:11), he remarks that our entire thinking is penetrated with the idea that arithmetic is the natural history or mineralogy of numbers. And to think in such a way is also to think numbers (not numerals) as shapes, one shape containing the other, and it is the task of arithmetic to explore the properties of these shapes, much in the same fashion as mineralogy explores the chemical composition of ores, classifies minerals and so on. But then to describe the properties of shapes in this way would imply the description of facts: whether this square paper can be folded and pushed into this hole, in what way does this shape react to heat and cold. We know that arithmetic does not concern itself with such issues; the properties of shape that arithmetic may undertake to describe are possibilities, not properties with respect to things having this shape (*RFM* III:11). And these possibilities emerge as physical or psychological possibilities of separation and arrangement. And as arithmetic is restricted to possibilities (shoving off actualities like the irregular behaviour of units), the role of these shapes (numbers) is merely that of pictures—pictures which are by definition robbed of reality, understood as the vast and complex continuum where it does not make sense to speak of an isolated identity, or idealised possibility of separation and arrangement. What these pictures would give are not real properties of shapes, of shaped things, but transformation of shapes, set up as paradigms of some kind or other (*RFM* III:11).

We are often inclined to think of number and shapes as fine drawings or fine frames on which a concrete group of objects, or a thing of a concrete shape, is stretched. Such a picture Wittgenstein thinks stands in comparison with Plato's conception of properties as ingredients of a thing (*RFM* I:71). Geometrical shapes and arithmetical numbers are conceived as ethereal entities, ethereal frames that are created once for all (by God perhaps) along with all the concrete empirical objects fitted on them (*RFM* I:72).

In the Platonic scheme, the Ideas—including numbers, triangles, circles, etc.—'subsist' in the third realm. The word 'being' is used for a 'sublimed ethereal kind of existence' (*RFM* I:72). One needs to remind oneself that sentences like 'Number 5 is,' or 'The triangle exists,' or 'The proof has shown that such and such is the case' have meaning only through a transition of similarities or a flow of differences. The phrase 'This name "p" designates this object Q' becomes meaningful not by picking out a neat chunk of reality through a singular act of ostension, but through a play of contrasts—contrasts with nonsense words like 'Lilliburlero' in a nonsense poem (*PI* 13), or with words used in singing, or in play-acting (*PI* 23). The phrase 'Red is' can be uttered meaningfully when a red colour sample is placed along with many close shades of deep orange, magenta, crimson, peach, etc. One is tempted to pronounce a sentence like 'Red is' when one is looking attentively at the colour; that is, in the same situation as that in which one observes the existence of a thing (of a leaf-like insect for example) (*RFM* I:72). One can invent similar uses for existential statements about numbers and geometrical figures, or for the description of their properties.

2.7 Mathematical Paradigms as a Stoppage of Experience

To form a picture is to form a representation of reality, to rob reality of its multidimensional continuity and reduce it into a truncated fragment. The house we see on stage is a fragmentary front which does not have a back. The staircase that seems to go up from the stage floor is cut off after a few steps, once it goes out of sight. The woman who weeps for her son on stage, cuts off her grief as soon as she disappears behind the wings. More shadowy and more fragmentary is the celluloid movie, where shadow men and women, shadow tables and chairs, are completely bereft of any dimension; we cannot get into the screen to interact with the shadow characters, or sit on that shadow chair. Pictures of reality merely provide us with a severed surface, and so do the paradigmatic motion pictures that mathematicians create. The picture of a dying hero is not death, the picture of the weeping woman is not grief, the picture of the palace is not the real palace. 'The picture of an arrangement is not an arrangement; the picture of a separation is not a separation; the picture of something's fitting is not a case of fitting' (*RFM* III:12). Arithmetical units and geometrical figures are not real objects that are really arranged, separated or fitted. They are thinned out, truncated surfaces, engaged in equally unsubstantial mathematical operations.

Mathematical representation of reality is, in a way, stoppage of thoughts, practices and experiences.[21] It is a stoppage of the ongoing flow of words in the chain of similarity relations where reality and language are woven together. To close experience in a circle amounts to severing reality into a one-dimensional fragment and thereby to stop further investigations. We do not investigate why Figure 2.7.A and Figure 2.7.B must join up in a one–one correlation as Figure 2.7.C and not as Figure 2.7.D.

Figure 2.7.A

Figure 2.7.B

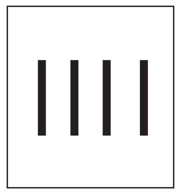


Figure 2.7.C

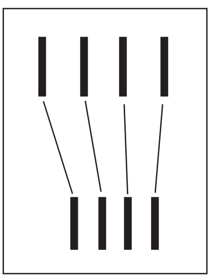
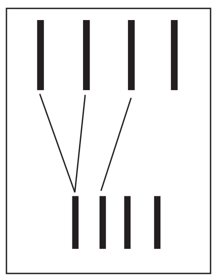


Figure 2.7.D



Once Figures 2.7.A, 2.7.B and 2.7.C are locked together in a motion picture, we do not investigate why Figure 2.7.A and Figure 2.7.B must close up as a paradigm for 2 + 2 = 4, and not as a paradigm for 2 + 2 ≠ 4.[22] By closing experience into a circular representation of reality, we have deliberately closed the scope of investigation. And one can close one's investigation only with respect to a picture, not with reality, not with the vast continuum extending indefinitely in multiple directions. There is nothing to investigate behind the flat celluloid, behind the fragmentary affront of the theatre set, behind the arithmetical numerals and geometrical figures drawn on paper, which are, ex hypothesis, unreal, truncated, fragmentary surfaces. Mathematicians can create these unreal fragments; they can claim to show certain connections as rigid, paradigmatic and necessary, only in so far as we do not investigate these connections; 'we look away from [them] and at something else. We turn our back upon them, so to speak. Or: we rest, or lean on them. . . . We acknowledge it by turning our back on it' (*RFM* III:35).

We close our investigations not only with the paradigms, but also with the actualities judged by these paradigms. We do not actually perform the task of correlating two groups of apples to see whether they can be isomorphically connected. We simply compare the actual units with the picture in which they are already correlated, in which they are already locked in a circle and shorn of their reality dimensions. And in comparing reality with the picture, in explaining all the real processes of transformation and interaction of units as anomalous exceptions to their ideal identity, we are virtually reducing reality into pictures, the paradigmed into the paradigm, flattening it out into a fragment, to match it up with another fragment.

The above account is by no means a complete or detailed analysis of Wittgenstein's philosophy of mathematics. What we have chiefly attempted is an extension of the non-foundational strains of his thought to his view of mathematical language as well. More specifically, the key issues addressed in this chapter show the various ways in which his philosophy of mathematics breaks away from the all-pervading Augustinian model. We close this chapter with a brief resume of its salient points:

1. Mathematics does not describe a supersensible domain of number, sequence or spatial properties. All talk of ideal unit or sequence, or of ideal space, gets its meaning through contrastive uses and relations.
2. One cannot pick out any such ideal unit, or ideal space (entities that are traditionally posed as the foundations of mathematics), through an act of ostension, or verbal rules, or measurement.
3. There is no question of pinning down such ideal sequence

or space (i.e., the putative foundations of mathematics) in a manner that will entail another such sequence or spatial property. To show any such necessary relation we have to close up both the sequences, the premises and the conclusion, the foundation and the founded, in a single circular motion.

4. The complex (of the foundation and the founded) is not a reality to be picked out by ostension. It is a picture or representation of reality, where reality is reduced to a fragmentary surface, severed from the real continuum.
5. To repeat, it is a picture of reality, not an item of reality itself, be it the supersensible realm, or a mental entity like a transparent image, an *apriori* form of intuition, or an instantaneous flash of understanding. The mathematical pictures of reality are created not in a single saturated act of ostension, either in its crude or the most sophisticated senses. Mathematical pictures spread out in an incomplete cluster of uses and practices.
6. When this picture is made to serve as a paradigm for determining how similar pictures are to be interlocked in a similar one–one correlation, it is not through a recurring common essence. As we shall see in the course of this work, one proof-picture leads to another proof-picture, or to a concrete calculation, through a transition of similarity relations, in a motley of techniques that mathematics consists in.

Notes

1. My exercise of thinning out the Augustinian model is mainly a summary of what Baker and Hacker have shown in *Wittgenstein: Understanding and Meaning*, vol. I, pp. 37–41.
2. K. Gödel, 'What Is Cantor's Continuum Problem?', in P. Benacerraf and H. Putnam (eds), *Philosophy of Mathematics*: *Selected Readings* (Cambridge: Cambridge University Press, 1984), pp 470-85. The quoted sentences figure in pp 483-4.
3. See *RFM* (I:8): 'But still I must only infer what really *follows*! . . . Here what is before our minds in a vague way is that this reality is something very abstract, very general, and very rigid. Logic is a kind of ultra-physics, the description of the 'logical structure' of the world, which we perceive through a kind of ultra-experience (with

the understanding e.g.).'

4. At least, this is what Baker and Hacker think in *Wittgenstein: Understanding and Meaning*, vol. I, pp. 47–52.
5. The revolutionary idea that number statements are statements about concepts is brought out in G. Frege, *The Foundations of Arithmetic*, trans. J. L. Austin (Oxford: Basil Blackwell, 1986), section 46. Section 68 develops this idea to numbers being extensions of a concept.
6. Ibid., section 68
7. Baker and Hacker, *Wittgenstein: Understanding and Meaning*, vol. I, p. 52.
8. B. Russell, *Introduction to Mathematical Philosophy* (New York: Dover Publications, 1993), chapters II and XVII. I have also used the exposition provided by Baker and Hacker, *Wittgenstein: Understanding and Meaning*, vol. I, pp. 52–57 .
9. Paul Edwards (ed.), *The Encyclopaedia of Philosophy* (New York: Macmillan and Free Press, 1967), vol. 5, p. 528.
10. All references hitherto are from Kant's *Critique of Pure Reason*, trans. N. K. Smith (London: Macmillan, 1992). I have also substantially drawn from R. Das, *A Handbook to Kant's Critique of Pure Reason* (Calcutta: Progressive Publishers, 1992), pp. 52–56, also pp. 80–83.
11. My account of Brouwer's theory is drawn from Edwards, *Encyclopedia of Philosophy*, vol. 5, pp. 204–5.
12. Frege expounds this idea of equinumerousity as a second-level relation between the two concepts F and G in *The Foundations of Arithmetic*, sections 68–73. A full symbolic version is presented in 'Frege's 'Logic, Theorem and Foundations for Arithmetic', section 3.1, *Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu/entries/frege-theorem/ Accessed on 3.8.16.
13. I have borrowed the essential points of this critique from Laurence Goldstein, 'The Indefinability of "One"', *Journal of Philosophical Logic*, vol. 31, no. 1 (February 2002), pp. 29–42. While Goldstein's critique was chiefly directed against Russell's formulation of definite description, I have attempted to glean the points that can be effectively utilised against both the formulations of definite descriptions and equinumerousity.
14. G. Frege, 'The Thought: A Logical Enquiry', in P. F. Strawson (ed.), *Philosophical Logic* (London: Oxford University Press, 1967).
15. B. Russell, 'The Philosophy of Logical Atomism', in R. C. Marsh (ed.), *Logic and Knowledge*, Lecture II (New York: Routledge, 1994).
16. This is how the Cārvākas, popularly known as the Indian materialists,

can seek to defend their inference claiming to establish the invalidity of inference itself on the ground of invalidity of *vyāpti* (universal concomitance).

17. Wittgenstein launches a pointed charge of circularity against Frege's definition of number in *PG* and *PR*. We shall take it up at the end of chapter V and VI, where we shall be able to situate it in a wider canvas along with other issues like aspect-seeing and action.
18. This analysis of measurement is derived largely from R.S. Jones, *Physics as Metaphor* (London: Wildwood House, 1982), pp. 18–30.
19. A background reading of issues like Fitzgerald contraction and relativity of space in *The Nature of the Physical World* by Sir Arthur Eddington (London: Comet Books, 1982), pp. 17–25, has been extremely helpful in framing my arguments in this section.
20. A refreshingly new conception of space showing the inadequacies of the old ones has been worked out by R.S. Jones in *Physics as Metaphor*, pp. 55–63. This has immensely helped me in steering my arguments in the desired direction.
21. The qualifying phrase is important, for looked at from a different viewpoint, mathematics opens up experience through newer and newer routes of aspectual transition. This has been fully discussed in chapters IV and V.
22. It is possible to see the first stroke of Figure 2.7 B as a fusion of two strokes, in which case we would count it twice, and correlate it twice with two strokes of the other group as shown in Figure 2.7.D. Wittgenstein in *RFM* (I:168) gives a similar example.

CHAPTER III

Critique of Rules and Logic

1. Logicism: Setting the Prelude

In spite of Wittgenstein's resourceful contentions against the transparency of concepts or against the sanctity of a switch over from the first level to the second level, there is a persistent pressure to transform non-verbal proofs or proof-pictures to verbal proofs. Let us browse through this logicist programme of reducing mathematics to logic, chiefly conceived and executed by Frege and Russell.

The core idea and the claimed advantages of reducing mathematics to logic may be enlisted as follows:

(a) It will get rid of the ambiguity and opacity of the proof-pictures that patently pertain to its physical and sensuous content. While physical pictures waver in different routes of interpretation, logic with its tools of primitives, definitions, specified categories of variables and constants, precise rules of derivation, quantification and substitution will close all possible tracks of dissipation.

(b) The sensuous character of the physical pictures goes against the ideal of necessity. Whatever pictures one may use for mathematical demonstrations are indeterminate and imperspicuous. (We have already noted such examples—numerals may be written in flowing colours, Euclidean diagrams may be drawn partly in white on a black background and vice versa. Units disappear and coalesce erratically.) A pictorial proof involved in Euclid's theorems may be inexact in the sense that the straight lines are not straight, the circles are not exactly circular. Now if mathematics cannot provide a criterion for its ideal unithood, or ideal sequences, surely logic can, with the powerful mechanism of logical identity, quantification and set theory.[1] A mathematical proof must be surveyable, i.e., it must have a precise criterion of its quantitative identity. A proof which is one mile long, or that which operates with a thousand signs, is no proof. The definitions or rules of equivalence claim to abbreviate the unsurveyable and indefinite series of numbers into a condensed sequence. A large number written with a few thousand signs as {{{{{(1 + 1) + 1} + 1} + 1} + 1} + 1} . . ., when converted by decimal notation into a shortened sequence, is claimed to be an operation of a purely logical inference.

(c) The mathematical proof must also be perspicuous, invested with a clear-cut criterion of its qualitative identity. It must be possible to decide with certainty whether we really have the same proof twice over, notwithstanding the difference in colour, handwriting or any other conceivable difference in the physical medium. In all cases one needs logic to detect what is essential to the proof apart from the variant, peripheral and contingent accompaniments.

(d) Mathematics seems characteristically to deal with enigmatic entities, numbers of various kinds, ideal arithmetical and geometrical entities, infinity, etc., that are not found in the world of common experience. If the core primitives of mathematics can be transformed to some logical terms and axioms from which one can deduce all the truths of mathematics as theorems with the help of the standard rules of transformation used in logical systems, this will establish mathematics on secure ground, purging it of its dubious load of ontology.

(e) Even had the physical proof-pictures been determinate, perspicuous and surveyable, mathematics cannot possibly be based on experience (of the sensible pictures). This is for the patent reason that experience only shows what it *is*, not what *must be*. The physical pictures do not present any pure and minimal identity that can be extracted by physical or inner ostension, and be consolidated as the essential and inviolable connection between two numerical or spatial concepts. The Kantian theory of a priori intuition of space and time along with its entire transcendental package of phenomena and noumena did not attract many philosophers. Under the circumstances, logic comes up with the promise of reducing all mathematical statements to purely logical statements, or purely logical consequences of such statements—statements that are true solely by virtue of logical constants ('some', 'not', 'all', 'if–then', 'identical with')—delightfully general, content-neutral, freed from any specific theory of knowledge or ontology. Logic creates the impression of being the most innocuous and yet the unfailing saviour of mathematics.

(f) In *PM*, Russell and Whitehead's proposal to logicise mathematics roughly followed the track of reducing geometry to arithmetic, i.e., space to time, and then abstracting from time itself by detemporalising numbers. Statements about cardinal numbers were then reduced to second-order statements about propositional functions, shorn of all referential stance of being about numbers and spaces *qua* objects with describable properties. Mathematics, previously taken to be a first-level enquiry into the ontology of space and numbers in a subsistent world, will now be about propositional functions which may be true or false, without any referential commitments.

Wittgenstein's attack against the logicisation of mathematics may be seen to unfold in the following steps:

a. The first step is to undertake a moderate attack that, while acknowledging the validity of the proposal itself, questions certain specific applications *within* the programme. That includes questioning Russell's type theory, the axiom of reducibility, the axiom of infinity, the notion of implication, etc. It is not only Wittgenstein but other philosophers too (not necessarily sharing the anti-foundationalist commitments of the later Wittgenstein) who have launched various attacks against one or another of these foundational props in Russell's programme of logicisation of mathematics. (We shall not, however, be engaging in this area of the critique, whether with Wittgenstein or with other philosophers.)

b. The more radical attack takes the shape of questioning the sanctity of logic itself. This chiefly consists in the insight that logic with its seemingly invincible armour of quantifiers, variables and precisely defined logical constants cannot bridge the gap between the 'all' and 'some', it cannot encapsulate each of the individual applications within the reserves of 'all'. In other words, logic cannot determine the significant range of values for its variables (whether they are of the 'propositional', 'individual' or 'predicate' category), it cannot determine what is to constitute the valid substitution instance of its variables of the appropriate sort. Let us take the argument: 'All figure–ground reorganisations are cases of alternative Gestalt formations. This chessboard can be seen alternatively as 32 black squares and 32 white squares, or as the colours black and white and the schema of squares. Therefore, seeing this chessboard consists in alternative Gestalt formations.' That the second premise has to be rephrased as a particular case of figure–ground reorganisation is not contained in the first premise, which logic has simply to pull out as its conclusion. This was the fundamental insight of Wittgenstein's attack against rule-following: A semantic rule formulated as: 'The word "P" has to be applied to all cases having a, b, c features,' does not determine what these features are, or what being relevantly similar to these features is. In other words, the 'all' does not encapsulate the individual cases; rather the 'all' fleshes out bit by bit with every individual application. (This point has already been discussed at length in chapter

I in the section titled 'Failure of Verbal Definitions'.)

c. All this is to say that the typical stance of logic, which is to abstract itself from ontological enigmas and turn itself into an unproblematic second-order enquiry, fails. Such a stance rests on a loaded commitment to a special realm of facts, semantic facts or logical facts, whereby a semantic rule once formulated magically encapsulates all actual and possible applications in its infinite reserve, so to speak.
d. To put the point more specifically, Russell's strategy with mathematical terms as an operational move from the priority of reference to that of propositional function did not achieve the required deontologisation of mathematics. It falls back on the predicative functions determining a unique range of values, thus committing itself to a different ontology, that of the occult power of semantic rules and definitions.
e. Further, if for Russell all second-order functions reduce to first-order functions *about* individuals, the nagging question as to what the particulars are that mathematical propositions are actually about, cannot be avoided.
f. It is a customary defence to say that the task of logic is confined to constructing uninterpreted formal systems, where the task of proving or checking validity pertains merely to spatial operations with signs, a task that can be performed by machines as well. But a machine in order to identify the syntactic categories—i.e., to identify which ones are propositions (atomic or compound), which are the variables as contrasted with the constants, and of which genre—needs to understand the semantics embedded into them. A machine will always require a non-mechanical intervention to make the data suitable for it to work on; it will at each stage need an extra-mechanistic aptitude of redefining the 'all' in the light of each individual application. Overall, the task of appreciating a contrivance as 'mechanical' requires compensatory interactions and adjustments at various levels—a conceptually loaded operation that is far from mechanical.

g. This amounts to saying that a logical proof can never bridge the gap between the 'all' and the 'some'. If it has to be surveyable, it cannot go beyond what 'can be taken in', it cannot go beyond what is given as a finite content, it cannot pretend to reach out to a possible range of unsurveyed 'all'. This means that logic cannot go beyond the finite and sensuous content of the signs, the physical character of the proof. It turns out to be nothing but a physical picture, and thus liable to all the typical limitations pertaining to its finiteness, indeterminacy and opacity.
h. Thus, while Russell and Whitehead in their *Principia Mathematica* wanted to reduce geometry into arithmetic, space to time and time to mere non-temporal cardinality, Wittgenstein steers the way in a reverse direction, to show that all logical proofs and the logicised proofs of mathematics turn back to spatial configurations of signs—to proofs of sign-geometry.

While Wittgenstein's attack on logicism is immensely complex in its purview and internal details, we shall be confining ourselves to the following tasks:

- Working out Wittgenstein's response to Russell's treatment of classes and numbers as to whether it achieves the required deontologisation.
- Arguing against the logicist attempt to generate the series of natural numbers with set-theoretic tools.
- Describing Wittgenstein's attack on Russell's programme of turning the non-decimalic notation of numbers into decimalic notation.
- Arguing against the infallible prediction of achieving a unique number at each stage in continuing a series.
- Arguing against deriving specific numerical equations like 2 + 2 = 4 from a set of axioms and rules of inference. We have attempted to reconstruct a Wittgensteinian refutation of such derivations as worked out in the Q system of Robinson Arithmetic.

- Reducing the logico-arithmetical proofs into proofs of sign-geometry.

2. Russell's Treatment of Classes and Numbers[2]

There are some patent problems regarding the ontological status of a class. While extensionally a class is a collection or aggregate of elements, with both an empty class and a unit class, it is the class concept or the intensional interpretation that gets priority over the extensional one for obvious reasons. Again, two classes may be intensionally different but extensionally the same. Further, the irreconcilable elements of both oneness and manyness imbued in the notion of a class disqualify it from any entitative status.

It is such considerations that motivate Russell to deontologise classes and numbers by a particular treatment of the class symbols and numerals. To start with class symbols first, Russell in *PM* I[3] says that symbols for classes are incomplete symbols in the sense that their *uses* are defined in the context of a full sentence; they themselves do not mean anything in isolation. Just as the meaning of a definite description is unpacked in the context of a conjunction of sentences (stating unique existence), similarly, class symbols too have to be pulled apart in a sentence stating an equivalence relation between two predicative functions.

We shall try to present a simple and less technical account of the matter before finishing it off with its more jargonised and symbolic formulation.

The interesting feature about class symbols is, Russell thinks, their extensional character—the fact that we are oriented towards the reality of the members as such, independent of the mode in which we conceive these members. The typical notion of a class makes it impossible to restrict its identity to just one propositional function. It is not difficult to follow up the validity of this claim. As classes are completely determined by their members, if the members are distinct, the classes too have to be distinct. So if there are two propositional functions true or false of the *same* range of objects, they have to determine the *same* class. So we can roughly say that every propositional function determines a class consisting of those objects of which the function is true. There has to be one class, no more and no less, for a group of formally equivalent propositional functions: the class of men has to be the same for featherless bipeds or rational animals. So what one states when one uses a class symbol, say, 'the class of men', or 'the tigers' (ψ standing for the noun) is that there is a relation of equivalence between ψx (x is tiger) and φx (x is mammalian cat with stripes).

What emerges as the interesting feature of a class is that in the function φx, the predicate-variable φ may not be cashed out in terms of a predicate-constant co-extensive with φ. When I use the symbol 'the class of tigers', I might not be cognisant of a predicate co-extensive with that of tigers. All that is required is that I state that there is a predicative function φx formally equivalent to ψx (where ψx determines the class.)

To put the matter a bit differently: when there is an intensional operator attached to the class symbol in a proposition, it is possible to derive an extensional operator from the intensional one, in the sense that the belief operator will now operate not only on the predicative function ψx, but also on φx that is formally equivalent to it. In other words, if I believe in any proposition in which the class is determined by the propositional function ψx, I must also believe that there is a propositional function φx formally equivalent to it; though, to repeat, I may not know which specific predicate(s) answer to φx. This in a nutshell is the dominant message of the class symbol—its extensional force overpowering/going beyond the mode of presentation or intension. On the other hand, a predicative function has no characteristic force of going beyond the intensional operator to the extensional one. From the statement 'I believe the quality of being virtuous is becoming rarer day by day,' I cannot derive a belief-statement about there being a predicative function co-extensive to 'x is virtuous,' so that whatever function operates on 'x is virtuous' will also operate on the other function as well.

Once we appreciate the characteristically extensional character of the class symbol, we can smoothly pass over to the slightly roundabout way in which Russell puts it, by way of invoking second-level functions, i.e., functions of functions, and then making the extensional/intensional dichotomy within these second-level functions. Consider the two functions: (1) For all x, if x is human, then x is mortal; and (2) S believes that for any x, if x is human, then x is mortal. The intensional character of (2) as opposed to the extensional character of (1) is explained by Russell with the special example that I may believe the Phoenix to be a rational animal which is immortal, in which case the substitution (of 'rational animal' for 'human') will not preserve the truth value of (2). The crucial point to be noted here is that when the second-level function operates on the class determined by the propositional function, then it (i.e., the second-level function) attains the status of an extensional function. This is because, as already explained, my belief with respect to a class will pertain to the actual members and not to any special feature of the members by which I represent them in my belief. In modern terminology, the *de dicto* belief about a class can be smoothly transformed to a *de re* belief about the same. Thus, to put it simply and pointedly: With respect to a second-level function on a class, one can derive an extensional function from an intensional one. The derived extensional function is: 'There is a function formally equivalent to "x is human," such that I believe that whatever satisfies it is mortal. This remains true when we substitute "x is a rational animal" for "x is human" even if I falsely believe that the Phoenix is rational and immortal.'[4]

Thus, one can define the derived extensional function as having the class determined by the propositional function ψx as its argument and as asserting *f* of this class. Russell goes on to define a proposition about a class more explicitly as follows:

> To assert that 'the class determined by ψx has the property *f*' is to assert that ψx satisfies the extensional function derived from *f*.[5]

Once we understand the motivation behind this definition, we can also handle the slightly technical definition that he offers in *PM*:

$$f\{z \mid \psi z\} = \exists\varphi\ [(x)\ (\varphi!x \equiv \varphi\ x)\ .\ f\varphi!z]\ \mathrm{Df}\ (20.01)^6$$

In the above, $f\{z \mid \psi z\}$ is in reality a function of the propositional function ψz, which is defined whenever ψz is formally equivalent to the predicative function φ!z such that *f*{φ!z} is significant.[7]

Let us remind ourselves that this task of deriving an extensional function from any function on a class symbol by overriding the subjective screen posed between the believer and the objects (in the shape of an intensional operator) is achieved by articulating the distinction between levels or types of propositional functions, between those that operate on objects and those that operate on the functions themselves. With a conflation of different levels of propositional functions, this derivation would not have been feasible. It is this presupposition that Russell articulates in the formulation of the axiom of reducibility:

> There is a type τ such that if φ is a function that can take a given object *a* as its argument, then there is a function ψ of the type τ that is formally equivalent to φ.

To this he adds the further definition:

> If φ is a function which can take given object *a* as its argument, and τ the given type mentioned in the above axiom, then to say that the class determined by φ has the property *f* is to say that there is a function of the type τ formally equivalent to φ and having the property *f*.[8]

Russell concedes that the axiom of reducibility rests on an extra-logical intuition which he equates with Leibniz's identity of indiscernibles, viz., if two ranges of objects have the same predicates, they are identical. While Leibniz gives it a self-evident status, Russell ranks it as pragmatic, not having a world-independent necessity.[9]

3. Russell's Definition of Numbers

Russell defines numbers as classes of classes.[10] Two classes have the same number when they have one–one correspondence, and this relation of one–one correspondence is defined as that which is reflexive, transitive and symmetric. For Russell, the number of a class is the class of all classes similar to the given class. This class of classes is the unique extension of all the properties applicable to only those classes that are similar to the given class. Thus, every class belongs to the class of classes similar to it, which is its unique number. Thus, corresponding to the number 5 is the class of all pentads or five-membered classes. To use the numeral 5 is to use the symbol for the class of all pentads, and such propositions are to be paraphrased as stating an equivalence relation between all predicative functions like 'x is a vertex of a star,' 'x is a finger of the human hand,' 'x is a claw in the paw of mammals,' 'x is a petal of a flower.' And to use the numeral '5', for instance, is to derive an extensional function from a second-level function about these propositional functions with the aid of the axiom of reducibility.

4. Working Out Wittgenstein's Response to Russell's Theory of Classes and Numbers

Let us start by addressing a major disanalogy between Russell's treatment of definite descriptions and symbols for classes and numerals. We know that with respect to the former, the universal and existential quantifications (in the paraphrase statements) are to be analysed down to a conjunction or disjunction of atomic propositions, and are even supposed to keep a residue of a general fact or an existence fact respectively. Thus, the conjunction of the three propositions is only a secondary stage of analysis, awaiting the final stage. Naturally Russell's logicist programme with the class symbols is distinct from that with the definite descriptions, in so far as the former has no commitment to individual constants, or to any further decomposition of the recommended analysis. In other words, a statement about number 5 has no commitment to actual realities of human fingers and flower petals, or to the logical atoms to which they are further supposed to boil down. Besides, Russell's derogation of numbers as being 'fictions of fictions' also clearly shows that he does not admit his prescribed analysandum of the class-propositions as representing any fact of generality or existence with a patently problematic ontology. However, we shall see that though seeming to start off from a different note, the tussle between Wittgenstein and Russell on the status of class-propositions and numerical propositions ultimately rests on the issue of their being absolute simples as the ultimate referents.

In the predominant current of analytic philosophy taking off from Frege, the distinction between extensional and intensional operator has been woven with the distinction between referential transparency and referential opacity. This in its turn falls back on a direct encounter with a pre-linguistic and pre-conceptual reality that lends itself to different modes of presentation, description, or the subject's attitudes like beliefs, desires, etc. The subject's mediation is by no means private or incommunicable for Frege; it is an intersubjectively shareable entity that may well be referred in indirect contexts.[11] For Wittgenstein, on the other hand, one plays the game of referential transparency or referential opacity in two different modes of activities or clusters of behaviours. The former consists in simple preparatory moves like putting pieces on the board (before actually starting to play) (*PI* 49), participation of teachers and learners in the ostensive teaching and learning of words (*PI* 6), playing memory-games (*PI* 47), the builder's and the assistant's engagement in the command 'Slab', 'Blocks', 'Beams', 'Pillars' (*PI* 2). To these we will have to add the language-games with class-expressions and numerals: when we utter expressions like 'the class of tigers' or 'the number 5', we simply place ourselves at the rudiments of the game. The more complex descriptive moves like tracing the mutual configuration of the pieces in relation to their placements on the board, constructing sentences with words learnt in ostensive definitions, understanding how the building blocks are integrated in the actual construction—all these will constitute the games of description. Similarly, breaking through the 'outer' boundaries of a class into its members, ascribing properties to each of them, will figure as the appropriate descriptive moves with class-expressions for Wittgenstein. As for the referential stance with the number-words, Wittgenstein invests them with a more interesting character; they act rather like the measuring scale in putting up a means of representation or frame of reference—the frame in which all quantitative and qualitative operations are to take shape (*PI* 50). At the same time, we need constantly to remind ourselves that the means of representation or the rudiments of reference, for instance the classes and numbers in particular, are both externally and internally ruptured; they are not only replaceable by other modes of reference or other modes of identification, but, within its own frame of reference, its putatively cohesive body will undergo constant metamorphosis in and through each move of description. Hence Russell's analyticist strategy of displacing the referential commitment of class-expressions and numerals to the semantic priority of predication and truth, his move of deriving an extensional operator from the second-level function on these expressions, the ceremonial declaration of the principle of Interchangeability Salva Veritate and *de re* beliefs, boil down simply to certain referring games or a cluster of activities—activities like putting pieces on the board. There is neither the reality of the irreversible referents, nor the semantic priority of the propositional functions being determinately true or false of a particular range of objects in determining a class or achieving equivalence with another propositional function.

How would the entire world fall into neat sets of pentads answering to number 5, ensuring each has a one–one correspondence with the others? How to claim that to speak of number 5 is to derive an objective and extensional one–one correlation between all pentads? As we have seen in chapter II, such global claims only flesh themselves out through each act of charting out certain lines of correlation between two pictures, rather, between two cinematographic pictures, through each act of freezing these pictures into definitions. Russell's slogan that numbers are fictions of fictions,[12] or his intellectual honesty regarding the extra-logical character of the axiom of reducibility, ironically rests on a tacit admission of an occult semantic fact—the fact of rules or predicates magically foreshadowing an infinite world of applications.

This becomes interestingly obvious when Russell, despite his project of deontologising mathematics, refuses to reduce the nominal definition of numbers to arbitrary and trivial stipulations. He claims that definitions, in a way, have a life of their own, independent of their use in specific propositions. They do not simply achieve a non-informative paraphrase of the original symbol; rather, they make a notable advance by clarifying and sharpening the common ideas that are usually vague and blurred.[13]

What Russell claims here is that definitions in logic carve out a circle, or forge a reservoir whereby each of them can encapsulate an infinite number of applications, in a way which was never appreciated by our untutored common sense. Thus the traditional charge of circularity against formal logic turns out to be its credit, the credit of contriving a circle to anticipate all applications; while our common ideas in their unskilled attempts to draw the circle fail pathetically to close up the figure, and end up leaving ugly yawning gaps. Here again we need to highlight the insight of the later Wittgenstein, irreducibly original in its force:

> The tool of definitions of formal logic never succeeds in carving out an all-encompassing enclosure for good, it only carves out newer and newer circles in each step of its applications.

To put it a bit otherwise, Russell's take on mathematical existence can at best be described as 'agnostic', who commits himself neither to their existence nor to their denial. Now to be agnostic is also to commit oneself to an epistemological gap between our cognitive powers and the extra- linguistic and extra-conceptual reality, while for Wittgenstein language and reality boil down to an indeterminate flow of uses.

As we hinted in the beginning of this section, there seems to be an internal tension persisting in Russell's logicism. On the one hand, there is the attractive project of purging mathematics of its ontological load, and thus the urge to reduce all problematic entities to propositional functions. On the other hand, there is the opposite pull of eliminating the alleged references to universals or properties by turning predicate variables to individual variables. All higher-order functions are reducible to elementary propositions loosely concatenated by the truth-functional 'stroke' operation. Terms which occur in these propositions cannot occur in any position other than the subject. On the other hand, it is the nature of universals that the appropriate symbol for them would have a configurational or structural character, which renders them incapable of being symbolised in isolation. Defining particulars and universals in this syntactic way, Russell claims to evade the perennial perplexities regarding the ontological dichotomy between particulars and universals. To put it in simpler terms: all statements seemingly about universals can be reduced to statements about individuals or particulars. There cannot be a statement where a term for a property or universal occupies the subject position. Here we see that Russell's operations in either of the two directions—whether it is to turn apparently referring expressions into non-referring propositional functions, or the reverse exercise of dissolving the ontological dichotomy between particulars and universals in terms of a difference of syntax—equally bypass the two nagging queries: how predicates govern a specified range of their values, and what constitutes the individual values of a variable.

5. Against the Logicist Definition of Specific Numbers

This section deals with the general logicist claim of deriving numbers progressively from 0 to the larger ones, through a method professedly non-circular and encased in the *atemporal* vocabulary of set theory and other definitions. We shall argue that such a derivation, independent of any picture or a dynamic constructive procedure, cannot possibly be upheld in the Wittgensteinian perspective. This will also make it clear why the specific arithmetical facts $2 + 2 = 4$ cannot be derived in the model of logical proofs. We shall address the latter issue with reference to the derivations laid out in the Q system of Robinson arithmetic.

We know that Russell's project of reducing arithmetic to logic needs him to show how the proposed axioms of arithmetic (principally borrowed from Peano) can be deduced on the basis of logical axioms. To put the matter simply, the axioms of arithmetic say things about numbers, relations between numbers, and the relation of a number with its successor; while the axioms of logic do not speak of such things. Thus, Russell needed to show how the seemingly exclusive and primitive vocabulary of arithmetic can be translated into the vocabulary of logic, i.e., the vocabulary of set theory.

We shall look into two methods of deriving the series of natural numbers in the vocabulary of logic. Of these, it is the second technique that was generally adopted by Russell,[14] while sometimes the first method is also wrongly ascribed to him.[15] We shall, however, be interested in tracking down a common strain of presumption underlying both the methods and being equally antithetical to Wittgenstein's approach.

5.1 *The First Method*

Definition of zero: 0 = {Ø}
Zero is the set whose only member is the empty set.
Definition of successor: The successor of a set *a* is that set which contains every set which contains a member x, such that if x is eliminated from that set, what remains is a set which is a member of *a*.

So the successor of zero is the set which contains a member such that, when that member is eliminated from the set, the remaining set is a member of zero. But there is only one member of zero: the empty set. So the successor of zero is just the set of all one-membered sets, which have a member which, when eliminated, yields the empty set. So the successor of zero—i.e., the number 1—is the set of all one-membered sets. The number 2 is defined as the successor of 1, and so on.

One may object that here we are defining a successor in terms of *having one item less than its predecessor*. Such definitions will naturally invite the question as to the meaning of 'predecessor', which cannot be defined in terms of that to which the addition of one term yields its 'successor'. The upholders of this technique however claim that the notion of predecessor is not packed into the definition of 'successor of set *a*'. The empty set from which we start has already been defined and identified independent of the notion of its preceding any number. When the successor of any set is sought to be defined, it does not presuppose the notion of successor, but only those of set, member and taking away one member from a set. Thus, the notion of taking away a member is claimed not to imbibe the notion of subtraction, or receding back in time to the predecessor.

To prevent any occasion of such charges of circularity, this technique uses the additional tools of set theory—the operations of intersection and complement—and seeks to define the notion of a successor in terms of these. But before we take a look at these definitions, let us note what seems to be a legitimate worry. When S or the successor of a particular set is yet to be defined, how can one perform the operation of taking away a member from that set? One can perhaps perform the operation of advancing from a defined set, but can one recede from a set whose content is yet to be determined?

To see that the same kind of presumption infects the purportedly stronger definition of successor in terms of complement and intersection, we need to consider the definition at some length. On this definition, the successor of 0 is the set of all sets S which should meet the following condition. Let us consider the three sets:

a. a one-membered set, viz., {x} whose sole member x belongs to S
b. the set S
c. the complement of the one-membered set {x}

Now, the successor of any set A is the set S such that there exists at least one x that is a member of S, and the common members of S and the complement of {x} are members of A. Symbolically, this condition of S being the successor of A is:

$$(\exists x)\ [x \in S \ \&\ (S \cap \text{Comp}\{x\} \in A)]$$

Starting from 0, we can say the successor of 0, i.e., 0', is the set S such that

$$(\exists x)[x \in S \ \&\ (S \cap \text{Comp}\{x\} \in 0)\ ]$$

Given our definition of zero as the set whose only member is the empty set, viz., {Ø}, 0' = the set of all sets S which meet the following condition:

$$(\exists x)\ [x \in S \ \&\ (S \cap \text{Comp}\{x\} \in \{\emptyset\})]$$

Similarly, 0'' = the set of all sets S which meet the following condition:

$$(\exists x)\ [x \in S\ \&\ (S \cap \text{Comp}\ \{x\} \in \{0'\})]$$

In this way, all the numbers 2, 3, etc., are claimed to be generated as successors.

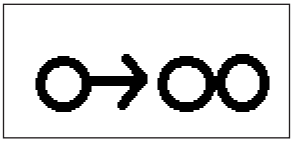
The simple trick behind this process of generating all the natural numbers in progression is quite obvious. In each case, the successor of A, viz., S, is defined by identifying a set that is obtained by removing a member x from the set S, and to remove a member is the same as to procure an intersection of set S with the complement of {x}.

By way of evaluating this procedure, let us first note that the complement of a one-membered class is introduced as a new notion, and this set is indeed an infinite set in which the set A, the set whose successor is sought, falls, and within which the operation of intersection of the two sets is conducted.

Now the crucial question is whether this way of generating the natural numbers rests purely on definitions and axioms without the help of constructing any cinematographic picture, and without the subsequent operation of freezing it in a flat configuration—the way Wittgenstein conceived the nature of all mathematical operation to be. The question can be rephrased in this way: does the logic of set theory itself achieve a generation of progressively larger and larger sets, where the new set formed at each stage will be increasing in its content in the required manner? The crucial notion of the complement of the set {x} which is infinite, i.e., where you cannot put the last member—is a notion that is inherently problematic; and on top of that one has to perform the further complex operation of intersecting it with the successor set itself. The obvious problem is, when neither the successor set nor the complement of {x} has been defined or identified, how can one go on to perform an operation of intersection between them? That these sets and their intersection seem intelligible is because of a rich plethora of activities going on in the background, a background where we have been constructing motion pictures, somewhat in the manner of Figure 3.1.

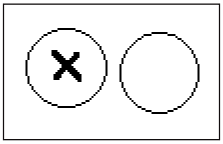
Figure 3.1

1st Phase



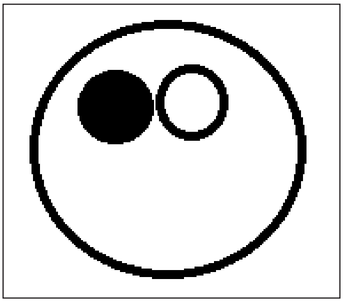
One set or unit moving on to its successor
(The first circle is the set of one and the second duo of circles is the proposed successor of the set of one, named S)

2nd Phase



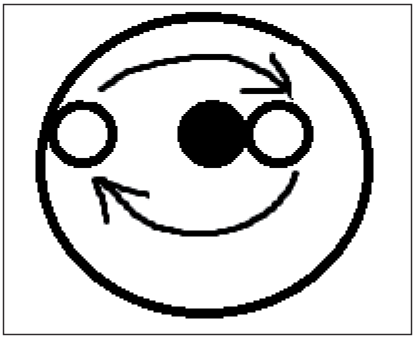
$\exists x\ (x \in S)$
Putting a member in S

3rd Phase



Making a complement of {x}
One circle of S is blotted out and the whole region with the unblotted circle and the rest of the bigger circle is to be laid out as the required class-complement

4th Phase



To construct the intersection of the complement of {x} and S
This comes back to the first circle in the first picture

So, to move from **O** to **OO,** one has to construct a complement set of **O**, intersect it with **OO** where the set **O** is re-identified.. Similarly, to go from **OO** to **OOO,** one has to find the complement of **OO** and intersect it with **OOO,** where again the set **OO** is re–identified. . In all cases, the intersection of the complement of {x} and the successor set re-identifies itself with the predecessor set with which we started.

5.2 *Russell's Method*

While Russell's definition of successor and the procedure for generation of natural numbers is more consistently tuned to his dismissal of classes as real entities, yet it seems to betray a kind of circular navigation at each stage in a subtle and subdued fashion. In this mechanism, successor is defined as follows:

The successor of the number of terms in a class *a* is the number of terms in the class consisting of *a* together with *x* where *x* is any term not belonging to the class *a*.

The definitions of the numbers are laid down as:

$$
\begin{aligned}
0 &= \{\wedge\} \\
1 &= \{\wedge \cup \{x\} : x \notin \wedge\} \\
&= \{\{x\}\} \\
2 &= \{\{x\} \cup \{x'\} : x' \notin \{x\}\} \\
&= \{\{x, x'\}\}
\end{aligned}
$$

and so on.

Here we take a set of one-membered sets and then again couples from which we subtract one member as not belonging to any of the one-membered sets in 1. The content of the couple is prefabricated so as to include a member which precisely is excluded from its predecessor, thus to forge a union between the two. It is not that the predecessor reaches out to the successor through a real movement, but in each case a new circle is carved out by mutual compensation and conspiracy.

We opened this chapter by way of addressing an opposition to Wittgenstein's approach to mathematics, the opposition that insists that once the mathematical proof-pictures are cast into non-pictorial logical proofs, all the indeterminacies vanish into nothing. But in the long run, the only sense we can make of this generation of natural numbers is not by definitions, but by constructing a dynamic series of pictures. Each of these pictures through its specific direction and movement reinforces a particular content, blotting out other modes of interpretation, much in the same way that lights coming from different sources are, by a repeated drill, made to fall into a pattern. When we say that *this* number through *this* procedure leads to its unique successor, we freeze the picture in such a way as to make the process and the result equivalent.

6. Against the Q System of Robinson Arithmetic

Let us try to activate Wittgenstein's insights against the Q system of Robinson Arithmetic, with respect to one specific proof of a simple numerical equation like 2 + 2 = 4. We shall confine ourselves only to those details of his system as are required to run the current of Wittgenstein's thoughts against this kind of logicist endeavour. Robinson uses '′' for the successor function, the name 'o' for the number 0 and two-place symbol '+' for addition.[16]

We shall refer only to the axioms of addition, used in the proof of 2 + 2 = 4. The domain of Q is the set of positive integers 0, 0′, 0″, etc.

To lay down the axioms:

$$Q1.\ (x)\,(y)\,(x \neq y \rightarrow x' \neq y')$$

$$Q2.\ (x)\ o \neq x'$$

$$Q3.\ (x)\,(x \neq o \rightarrow (\exists y)x = y')$$

$$Q4.\ (x)\,(x + o) = x$$

$$Q5.\ (x)\,(y)\,(x + y') = (x + y)'$$

Let us note that Q4 and Q5 have already been formulated by Russell as the axioms of addition. Let us also note at the outset that the very *look* of Q5 betrays the typical logicist strategy of generation of natural numbers against which we had worked out Wittgenstein's objections. To enhance a number by one unit is, as we have seen, to take away one number from the successor and identify it with the predecessor. Here also we track down the same idea with a slightly higher degree of complexity. Q5 can be read as 1 + (1 + 1) = (1 + 1) + 1 which is nothing but the rule of association. In other words, to take the sum of the first number and the successor of the second number is to recede to the sum of the first and second numbers (without the successor added to the second), and then project the successor as external to the entire sum. And it is the same restructuring strategy that we see throughout the tenor of the following proof: Proof of 2 + 2 = 4 or $o'' + o'' = o''''$

Proof

1.	$(x)\,(x + o) = x$	Q4
2.	$(x)\,(y)\,(x + y') = (x + y)'$	Q5
3.	$o'' + o'' \neq o''''$	(negation of conclusion)
4.	$o'' + o = o''$	(from 1, substituting o″ for x)
5.	$(y)\,(o'' + y') = (o'' + y)'$	(from 2, substituting o″ for x)

6.	$(o'' + o') = (o'' + o)'$	(from 5, substituting o for y)
7.	$(o'' + o') = (o'')'$	(substituting on the right hand side of equation 6 on the strength of 4)
8.	$(o'' + o') = o'''$	(from 7)
9.	$(o'' + o'') = (o'' + o')'$	(from 5, substituting o′ for y)
10.	$(o'' + o'') = (o''')'$	(from 9 and 8)
11.	$(o'' + o'') = o''''$	(from 10)

Proved

Here again we see that in step 7, 2 + 1 is reformatted as (2) + 1, and 2 + 2 = 4 is reformatted as (2 + 1) + 1, i.e., we can say about the latter case that 2 + 2 is projected as a successor of a preceding number, and the preceding number is seen as one less than the successor number.

This is an occasion to appreciate how this so-called non-pictorial abstract proof of logic actually feeds on some cinematographic visions—constructing two pictures with an equation in between, laying out two configurations with a cyclic transition between the two. The first transits to the second only to revert back to itself, one reorganises, reassociates itself in terms of the other, which is, to repeat, comparable to rays of light coming from different sources falling into a particular pattern, blotting out all other modes of radiation.

It may be noted that this is precisely the game that the so-called analytic propositions play. They claim to be definitional rules that set a paradigm of meaning, assessing whether a particular individual falls under that paradigm. But as we have seen time and again, the rule does not foreshadow its applications; rather, it goes on identifying and re-identifying itself through each and every application. Statements like 'Bachelors are unmarried men,' 'Every rod has a length,' or 'Everything is identical with itself,' if claimed to be exact repetitions of their identities, are not valid language-games. The imaginative exercises associated with these propositions are, as we have seen, the vision of a thing coming out from its own skin, leaving a socket behind and again settling back into it. Wittgenstein observes that these propositions have the same force as 'Every coloured patch fits into its own surrounding,' for they create a surrounding for themselves, define their own boundary in that surrounding, move out of that boundary and resettle into the same position (*PI* 216, further amplified in the next section). This exercise of self identification and re-identification is a matter of uses: it is not the same conceptual scheme of formatting the same barrage of sensory stimulations as Quine would have it. One can always play different games to project a rod under the sway of atmospheric factors, never preserving its elongated shape but regularly flattening out; two and two not becoming four; or, seeing a lamp shimmering in the daylight, we say 'The lamp is different from itself,' and also find it quite sensible to characterise a bachelor as married.

For the last case, let us recall the story 'The Japanese wife', where the hero named 'Snehamoy' has not registered officially at a marriage bureau, yet he and his Japanese penfriend Miyage both declare themselves as 'married' to each other through their letters, exchanging rings and ritualistic gifts as befitting the ceremony. They carry on this 'married life' for 15 years, sharing their joys and sorrows through letters, without ever seeing each other in person. Here we can say that Snehamoy is a bachelor who is not unmarried. To insist that once you have termed him a 'bachelor' you must term him as 'unmarried' is just to assume a stance of moving from bachelor to 'unmarried man', only to force it back to itself—just as 2 was made to move towards its successor only to take an element out of the latter and come back to itself. This is characteristically the art of theory building, to put up a stance of leading the explanandum to the explanans, making it stop one step short of the explanans, and claiming that the former has traversed the gap and reached out to the latter through a genuine movement. But that movement is a fake, for the explanandum is defined as that whose content is the same as the explanans except for the step to cover the gap, and the explanans is defined as the same as the explanandum except for having that extra element that is left out by the gap. To repeat, there is no genuine movement from 2 + 1 to 3, or in the generation of natural numbers. The movement of reorganising or reassociating the content cannot qualify as what the logicists claim to be the movement of making the implicit explicit.

The more important point goes deeper than the imputation of circularity, that with which the logicists are more or less comfortable. Once we appreciate that definitions merely turn out to be cyclic operations, we must also digest that one cyclic closure between two items does not encapsulate other subsequent closures the logicist would like to achieve. For each step one has to carve out new circles afresh. If we have carved out circles till *n* numbers, we have only demonstrated our point till *n*. We cannot assume automatic generation of cycles for n + 1, n + 1 + 1, and so on. Once we have the cycle between bachelors and unmarried man for a few cases, this cannot be assumed to take care of future instances as well. Thus, Q5 fleshes out its content through each of its substitution instances; it cannot possibly carry an infinite number of circular constructions under the garb of its universal quantifier and individual variables. This crucial point shall be further elaborated in the final section of this chapter.

7. Continuing a Series: Following a Rule

That the rules do not cast a long shadow ahead of the actual applications carried out becomes clearer particularly with the rules of continuing a series. The rule for placing squares of consecutive numbers does not determine ahead what one is going to place in the 25th position. The rule for continuing the + 2 series does not preclude the possibility of someone calculating in the 'normal' way, as 2, 4, 6, 8, . . . up to 1,000, and then adding as 1,000 + 2 = 1,004; 1,004 + 2 = 1,008; 1,012, etc. (*PI* 185; *RFM* I:3). This is what Wittgenstein regards as doing 'the same'; this is what he regards as being in 'accord with the rule' (*RFM* I:113). 'If from one day to the next you promise: "Tomorrow I will come and see you"—are you saying the same thing every day, or every day something different?' (*PI* 226). It will be of no avail to hypostatise numbers as timeless entities beyond all relations, variations and interactions. For in what does this non-temporal and non-spatial identity of the number 2, or of any number, consist? If it is a distinctive shape or sound, as per our earlier discussion, it cannot be fixed by a rule. Nor is it a distinctive feeling or a mental image that a person carries every time she uses the word '2' in all + 2 operations. Neither the 'normal' nor the 'deviant' calculator bases her calculations on a pre-applicational saturated content of '2'. As we shall see, neither the normal nor the deviant practices are founded respectively upon a pre-applicational 'normal' rule and a 'deviant' one.

Even rules like 'Continue the same number as 2, 2, 2, 2, 2 . . .' do not entail that one has to write '2' in the 500th position (*RFM* I:3) 'But isn't *the same* at least the same? We seem to have an infallible paradigm of identity in the identity of a thing with itself. I feel like saying: "Here at least there can't be a variety of interpretations"' (*PI* 215). But Wittgenstein remarks, 'There is no finer example of a useless proposition' than this statement of identity (*PI* 216). And he further adds that such statements as 'A thing is identical with itself' or 'p = p' is yet 'connected with a certain play of the imagination. It is as if in imagination we put a thing into its own shape and saw that it fitted.' Thus the identity statement may also be expressed as 'Everything fits into itself,' or 'Everything fits into its own shape,' where 'at the same time we look at a thing and imagine that there was a blank left for it, and now it fits into it exactly.' We also use it in such situations where there was already a socket and a thing of that shape is fitted into it. We may further use it in a still different situation like 'Every coloured patch fits exactly into its surroundings' (*PI* 216). All these show that the law of identity, i.e., p = p, is either an unusable lump, or it lives through its various uses through a transitional flow of similarity relations. It does not have an uncashable reserve that repeats itself fully, or even partially (as is frequently mistakenly assumed after the model of physical overlapping), in each of its occurrences. Similarly, when we demand that 2 must identically repeat itself in each step, it is such exercises of imagination like 2 coming out of its socket and getting back to itself again, or dissipating and reverting back to its own shape, that give meaning to it. And such imageries do not have a self-interpretive content. There are different ways in which 2 in its different associations can indulge in this process of coming out and getting back to its own shape; there are different ways in which it fashions its own identity. There is nothing in 2 = 2 that fixates it to one mode of identical repetition.

As we have repeated too often, a rule does not have a pre-applicational content. We do not learn the application from the rule; rather, we learn the rule from the applications. Here again we smuggle the application into the rule, and close them in a circle. And this is why Wittgenstein says, 'Can't anything be derived from anything by means of *some* rule—or even according to any rule, with a suitable interpretation?' (*RFM* I:7). The student who operates the '+ 2' rule as 1,000 + 2 = 1,004, 1,004 + 2 = 1,008, 2,000 + 2 = 2,006, 2,006 + 2 = 2,012, . . . , acts as if he hears the '+ 2' rule in a different way, as 'Add 2 up to 1,000, 4 up to 2,000, 6 up to 3,000, and so on.' In other words, he takes the word 'plus' to mean 'quus' whereby the second addendum '2' will increase by 2 after every thousand.

For the classical theorists, this does not threaten the status of rules, the unique implicative power of logic. If one comes up with a weird application of a rule, it is only because he has attached a different meaning to one or more of its constituent words. Under that different interpretation, the rule will be deviant, but will imply a uniquely deviant set of applications. The rule will still have the inner power to produce a unique conclusion. Now here we think that an interpretation is a fixture to be attached to a rule to produce its application, and thus look upon a different (or deviant) application as produced by a different fixture attached to the rule. 'It strikes us as if something else, something over and above the *use* of the word "all" must have changed if "fa" is no longer to follow from "(x) fx", something attaching to the word itself' (*RFM* I:13). And something over and above the use of the word '2', '+', '1,000', etc., must have changed if 1,002 does not follow from the '+ 2' rule.

One only needs a little insight to see that even if we attempt to attach a different interpretation to '+ 2', whereby 2 would be larger by 2 after every thousand, this will not entail a unique set of conclusions. The deviant user might write 1,000 + 2 = 2,004, 2,004

+ 2 = 4,012 . . . and he might try to justify that if 2 or '+ 2' increases by 2 on and after every thousand, i.e., if every occurrence of 2 *within* 1,000 + 2 increases by 2, 1,000 + 2 will be 2,004 (because 2 occurs 500 times in 1,000 and once in + 2, which increased by 2 will yield 4 times 500 plus 4); and once we get to 2,000, now 2,004 + 2 will be 4,012 (for now there are 1,002 occurrences of 2 in 2,004 and one occurrence in + 2; each occurrence will now have to be increased by 2 + 2, which will yield 4,012).

To ruminate once more on the exact significance of Wittgenstein's slogan: 'Anything can be derived from anything by some rule or *some* interpretation of any rule.' The second 'anything' does not have a separate identity prior to the derivation of the first 'anything'; none of these deviant interpretations of the '+ 2' series can uphold a semantic status of its own that is distinct from, and yet inviolably entails, each of the uniquely deviant applications in each step. That we have to pose the two 'anythings' as separate, that we have to pose two separate linguistic bits before we claim to merge them into identity, is a linguistic accident. The rules live in and through their interpretation and application. It makes no sense to split up mathematics into isolated splinters—here the numerical formula, here the logical rules and definitions, here its interpretation, and here the application. Two people cannot share the same formula, the same rules and definitions, and yet produce different interpretations and applications.

8. Russell's Demonstration of Contradictions Involved in the Concept of a Class

Let R be a class determined by the propositional function φx, i.e., let R be the class of all and only those x-s of which φx is true.

So R is the class of x-s, and membership of each x is defined in such a way as to exclude the possibility of any of them being a member of itself. That is, for any x, x belongs to R if and only if x does not belong to itself.

But the question arises with regard to R, due to the strange one–many character of the class, so far as it poses a unitary character over and above the plurality of its members. Hence, the possibility of substituting R itself as a value of x comes up. And the question arises whether R is a member of itself or not.[17]

Now we have seen that x belongs to R only if x does not belong to itself. That means: R belongs to R → R does not belong to R.

We have also seen that R (as a value of x) belongs to R if it does not belong to R.

That is, R does not belong to R → R belongs to R.[18]

In the absence of the level difference, such contradictions are a common incidence. To say that inference as source of valid cognition is invalid because *vyāpti* (universal concomitance between the probans and the probandum) cannot be established, is itself an inference and so itself is invalid. So *if all inferences are invalid*, then this inference is also invalid, in which case *it is not the case that all inferences are invalid* (because this inference failed to establish the invalidity of all inferences). Again, if this inference is valid (i.e., if it validly establishes the invalidity of all inferences), then, itself being an inference, it will be invalid. The same applies to the liar paradox: I am lying.

However, let us see how Russell tackles the problem. In *PM* I, Russell proposes to solve the problem of contradiction by introducing a hierarchy of logical types among the propositional functions as well among the corresponding classes formed by the propositional functions of various orders. That is, R being a higher order than the x-s, the question of its being a substitution instance of x will not arise.[19] Secondly, he also seeks to avoid the contradiction by avoiding the very assumption of there being classes. And this, we have seen, is achieved by paraphrasing the proposition having class symbols as equivalence relations between predicative functions.

9. Working Out Wittgenstein's Response to Russell's Type Theory

Interestingly, these kinds of charges of contradictions may be levelled against Wittgenstein's view of language in general, and with his approach to mathematical language in particular. His objectors may insist that Wittgenstein's attack on essences (whether with respect to ordinary discourses or to mathematical ones) simply amounts to putting deviant meanings or interpretations to words—a deviance determined by deviant ways of living. To validate this point, Wittgenstein has to allow his schema of deviance to be universally intelligible, i.e., let the modes of relativisation be absolute themselves. If the schemes of relativisation were themselves relative, Wittgenstein would not have established his point. Let us try to chart out how this entire mode of attack on Wittgenstein can be recast in the same style as the charges of contradiction that were framed against Frege's definition of number.

If relativism is true then it is not true (because it itself will be relative.)

If relativism is not true (i.e., is not absolute) then relativism is true (as a converse of absolutism).

Now, one may insist further that to get out of such contradictions, Wittgenstein has to follow the track of Russell. He has to set up a hierarchy of types, saying that while his insistence on relativities and deviations pertains to the usual formation of concepts and usage in the first order, the discourse about how such relativities are shaped belongs to a higher order; this discourse is itself absolute. And once Wittgenstein is in this mode of defence, his opponents can further urge that any anti-essentialist philosophy that seeks to base the force of its incisions on a schema of hierarchy that is itself universalisable, will be pushed into further options, none of which will be conducive to a professed refutation of essences. According to the first option, all deviances or deconstruction of essences will have to fall back on a basic agreement on some basic essences. According to the second option, his attack on essences is just a trivial operation of framing weird semantic rules instead of the standard ones—an operation which falls back on a tacit commitment to a special unsurpassable essence of rules—an essence that makes even a deviant rule encapsulate a uniquely deviant range of applications. Looked at in this way, Wittgenstein, in spite of his marked crusade against Russell, will fall back into a position not radically different from his opponent's. It is precisely this path of slippage that we have been trying to resist all through, and had explicitly highlighted in the previous section on the critique of rule-following. Wittgenstein never seeks to back up his anti-essentialist insights on a hierarchy of order or types, for any such theory would invoke specific essences for specific types—i.e., investing each order of variables with a specific and predetermined range of substitution instances. This virtually amounts to investing each range of concepts with a reservoir containing all its actual and possible applications—a typically essentialist commitment that Wittgenstein had been struggling against. In fine, for Wittgenstein, a transition to the type theory will not serve to secure mathematics on the much-coveted foundations.

10. Logical Proofs Lapsing Into Physical Pictures

Logic was assigned the task of emancipating mathematical proof-pictures from their limitation: first, the limitation of indeterminacy; second, the limitation of finitude. The second limitation might require a little bit of explanatory recapitulation. The physicality of a proof binds it to that picture only. To say that this picture stands for or represents all other pictures of the same sort, we have to come out of the picture and resort to verbal definitions, verbal tools of logic.

10.1 *Unsurveyability of Logical Proofs: Against the Decimalic Reduction of Numbers*

The logical rules of equivalence do not put the scattered granules into a capsule that can open and spread out when required. When we imagine the cardinal numbers explained as 1, 1 + 1, (1 + 1) + 1, [(1 + 1) + 1] + 1 . . . , we assume that the 'logical' definitions converting them into the decimal notation merely serve to abbreviate the expression for the convenience of the calculator (i.e., the person who calculates), just as we find it convenient to put all the small granules in a closed capsule (*RFM* II:2). If so, the logical rule of equivalence should work both ways—one must be able to construct the calculation 703,000 × 4,000,101 in that 'wearisome notation' too, and to lay out the vast sprawling mass of signs in isomorphic correspondence with the other calculation in the decimal notation (*RFM* II:3). But Wittgenstein points out that one can prove a one–one correspondence only between two surveyable pictures, two surveyable series of marks. The entire proof of one–one correspondence has to be surveyable, perspicuous, one that can be 'taken in', and the lines of correlation can be drawn only between two surveyable series. What happens when one attempts to draw such lines between two groups of calculation—say, '103,000 × 4,000,101' (arrived at in the decimal notation), and another carried out in the first notation, viz., 1, (1 + 1), (1 + 1) + 1 . . . ? After carrying on for say half an hour, can one be sure that the former part of the proof, one that is out of our sight now, has not changed? Or even after presenting two such calculations—one performed in the decimal notation, another in an unwieldy spread of numerals and signs (say one mile long)—can we be sure that it has not changed after half an hour when we look at it again? 'For one cannot command a clear view of it' (*RFM* II:3). It will be of no avail to press on 'logical' unithood that precludes all anomalies—like breaking, bending, changing or coalescing. We cannot invent a 'logical' possibility of carrying out an unsurveyable correlation—a possibility over and above empirical or practical possibility. The very *meaning* of 'logical' equivalence is exhausted with the *actual* survey of one–one correlation, the extent to which it can be 'taken in'; one cannot reduplicate 'logic' to bring in a vast unsurveyed correspondence hidden behind the surveyed one.

A logical proof fails to stand apart from a pictorial one in any *purely* 'logical' sense of the term. It too is a picture—a closed picture, a closed circle of experience. Logic cannot perform the extraordinary feat in which ordinary experience and ordinary pictures fail; it cannot magically jump across a vast or infinite expanse, magically transport itself from one point to another, without actually taking the intermediate steps in between. Wittgenstein categorically denies the existence of an unsurveyable proof, say one of 7,034,174 + 6,594,321 = 13,628,495 (II:3); for him, a proof has to be a closed picture, a closed series of numerals, which cannot be splintered into two independent fragments, each compelling the other to join up in a unique one–one correlation. Rather, *we* join them up—we join up the antecedent (7,034,174 + 6,594,321) with the consequent (13,628,495) in a single circle. And the mechanism of closing two pictures into one can work only if they are surveyable.

It is interesting to see how one picture or imagery leads to the other. If we suppose units as given prior to language and practice, we are also prone to seeing them as automatically joining up to form bigger and bigger numbers, even when we do not actually care to join them up. If space is pictured as a vast empty container assorted out of simple, hard little balls, then looked at the other way round, these balls are presumed to assemble inertly into a bigger and bigger space, whether or not one actually traverses that space. The extension from a pile of building blocks to a larger pile is carried over to our extension from smaller numbers to bigger numbers, and finally to our extension of smaller measurements to unimaginably bigger ones, from measuring the table with a foot rule to measuring the distance between the earth and the sun. Numbers are envisaged to move on unit by unit, along a single, one-dimensional line in space—all devoid of dimension, depth, relation or interaction, that is, devoid of reality. So also are the unit of measurement the foot rule, for instance is a thinned-out entity assumed to repeat itself identically, jumping from point to point on the single one-dimensional line in space, right up to the sun. We extend our ideas from calculations with small numbers to ones with large numbers in the same kind of way as we imagine that it is 'logically' possible to measure the distance from here to the sun with a foot rule, and in that case we would get the very result that we get in a quite different way (*RFM* II:4).[20]

The idea of transforming non-decimalic notation to the decimalic one is principally one of making bundles, each with 10 units. For one set of 10 units we make a bundle, for 10 sets of 10 units we make another bundle with two 0s, i.e., represent it with 00. But actually we are not making bundles, for the items that are supposed to be gathered in the bundles named as 00, 000, etc., are gradually becoming unsurveyable. Similarly, in our process of forming concepts, we do not tie bundles, for the things which are to be tied into bundles are simply not there. It is just a starting point, an entrance door suggesting a journey beyond, giving us the illusion of a real bundle. To recall the argument with the chessboard and alternative Gestalt formations in chapter II, we do not pull out the specific instance of the chessboard from the previously tied bundle; rather, with each derivation of specific instances, they gradually accumulate to form a bundle, which progressively increases in its content. The role of definitions, say that of 'man' in terms of 'rational animal', is not to provide us with a tool for procuring as many rational animals for as many men that are there; it does not enable us to effect an isomorphic connection between the items of two pre-existent bundles, for the simple reason that the bundles did *not* exist beforehand. We should rather be saying that we are in the process of procuring for every man a rational animal—a process that is ever indeterminate, ever incomplete.

Put in a slightly different way, the talk of condensing a sprawling mass of units into a small space makes sense only in terms of reorganising the relationship among the units themselves. . Scooping or bundling up a scattered spread of granules into a small capsule amounts to lessening the distance among the granules and widening s the distance between the granules and the other objects to which they were lying closer. But the logicist programme of turning the non-decimalic notation into a decimalic one treats number as a spatial container of all objects—an ethereal or incorporeal wall surrounding all of them—so that one can squeeze the larger container into a smaller one, or turn the large bundle into several smaller ones, keeping their inner content intact.

In writing numbers in the long notation as 1, 1 + 1, (1 + 1) + 1, or as | | | | | | | | | |, we are obliged to write 'and so on', or put the customary triple dots (. . .) at some point or other; we do not seem to realise that such dots or phrases are not tokens of laziness, they are not abbreviated notations. We do not appreciate that the meaning of 'and so on' or of such dots has to be cashed anew with each new element that we may care to add; their meaning does not stretch beyond a single mark that we have actually written or used in some way. When we draw a picture with a blurred shadowy edge at one side of the canvas, or when we put dots or write 'and so on' at the end of a number series or a process of calculation, we do not know more than we say (*PI* 208).

> It is not logic . . . that compels me to accept a proposition of the form (∃) (∃) ⊃ (∃) when there are a million variables in the first two pairs of brackets and two million in the third. . . . Something *else* compels me to accept such a proposition as in accord with logic. (*RFM* II:16)

Suppose we took 100 steps of the logical calculation at a time and got trustworthy results, while we do not get them when we take the steps singly. It is not logic but something other than logic that persuades us to claim that the calculation is based on prescribed unit steps. We can define 10 as 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1, and 100 × 2 as 2 + 2 + 2 + 2 + . . . , but we cannot necessarily define 100 10 as 10 + 10 + 10 + . . . or as 1 + 1 + 1 + 1 + . . . (*RFM* II:17, 18). It is not a forward movement from 10 to 100, but a regressive navigation from the latter to the former. 100 = 10 + 10 . . . guides us as to what to expect from 100 = 1 + 1 + 1 + . . . in the unshortened notation, not the other way round (*RFM* II:18). Extension of the decimal notation from one sequence to another is like extending general words, say, 'courageous', beautiful', 'games', or colour words like 'red', 'blue', 'green', etc., from an agreed cluster of individuals to new and different ones. '*How do I know* that in working out the series "+ 2" I must write 20,004, 20,006 and not 20,004, and 20,008?' And we may add: 'How do I know that if 10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 then 100 is also 1 + 1 + 1 + 1 + . . . 100th n?' 'The question: "How do I know this colour is 'red'?" is similar' (*RFM* I:3). 'The rule can . . . produce all its consequences in advance only if I draw them as a *matter of course*' (*PI* 238). An attempt to ground the use of 'blue' on a detachable image of blue will endlessly call for another image to identify the previous images. An attempt to ground mathematics on logic, seeking to ground each occurrence of decimal notation on a self-identical base of 10, or seeking to make 'the shortened procedure as a pale shadow of the unshortened one' (*RFM* II:19), falls into the same kind of impasse.

When logic is looked upon as magically condensing a vast expanse into a short space, it feeds itself on another picture—that of discrete units unproblematically given prior to any language or practice, one from each series magnetically pulling its unique correlate from the other. But suppose we proved by Russell's method that (∃a . . . g) (∃a . . .l) ⊃ (∃a . . . s), can we reduce our result to g + l's being s (*RFM* II:4)? Russell's proof, or any logical proof for that matter, will not show that one can take the three bits of the alphabet as representatives of the proof. One could obviously have carried out logical proofs with a group of signs in the brackets whose sequence makes no characteristic impression on him, so that he could not represent the group of signs between brackets by its last term. Wittgenstein further insists that even assuming that Russell's proofs were carried out with a notation x_1, x_2 . . . x_{10}, x_{11} . . . x_{100} as in the decimal notation, and there being 100 members in the first pair of brackets, 300 in the second and 400 in the third, the proof itself would not show 100 + 300 = 400. The proof might lead at one time to this result and at another time to a different one, say 100 + 300 = 420. Logic cannot teach how to identify each member and thus the last member of the series (*RFM* II:4).

In fine, logic does not enable us to condense a massive stretch of symbols or numerals into a short, surveyable notation. For a proof is a measure, and as a measure, it must be surveyable, and the correspondence between two such notations (one surveyable and the other unsurveyable) cannot be displayed in a proof. Any proof in which it is sought to be displayed will be a long, sprawling mass of notations. A doubt might always creep in whether we have mistakenly counted a unit twice, or whether the units changed in the long course of the proof. 'Where a doubt can make its appearance whether *this* is really the pattern of *this* proof, where we are prepared to doubt the identity of the proof, the derivation has lost its proving power' (*RFM* II:21).

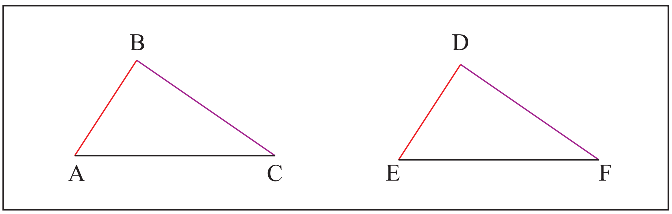
10.2 *Logico-arithmetical Proof: Its Geometrical Character*

We have seen that a proof consists in actual one–one correlations, leaving no room for *possible* correlations between a surveyable and an unsurveyable notation, the latter being either absent under the guise of an infinite series, or having a finite but unwieldy content, say a mile long. In both cases, the foundation of the proof is alienated from the founded, the explanandum is severed from the explanans, throwing up several doubts and options as to the particular method of collation adopted, or to the real structure of the proof beyond irregular interactions between units and temporal vicissitudes. This is the point to be appreciated even with respect to the surveyable proofs with a perspicuous content. And as we have already noted, here Wittgenstein is obliged to put his point in a fashion which is refreshingly paradoxical. While proving 400 + 20 = 420 logically, i.e., while proving a statement of the form '$(^1)\ (^2) \supset (^3)$' is a tautology, Russell does not succeed in externalising the units of the first two brackets and generating the same number of units in the third. He does not teach us to write as many variables in the third pair of brackets as there were in the first two together. What he really teaches us is to write a variable in $(^3)$ for every variable in $(^1)$ and $(^2)$ (*RFM* II:7). That is, what he teaches us is to draw the premise and the conclusion blended in a single stroke of the brush, in a single circular motion.

To put the matter a little differently, rules or definitions of logic cannot make the signs go beyond their own physical content, either to something outside the signs, or even to *similar* signs. One circle drawn between two sets of signs cannot ensure *similar* circles will be drawn for *similar* sets of signs. It cannot represent anything specifically, still less generically. All this narrows down the content of logical proofs to specific spatial configurations—to the manner in which one sign is made to transform to another sign, i.e., how 'successor of 0' is led to '00', of which the element '0' is again made to revert back to the '0' in the previous sign, or how 'o′ + o″' is led to 'o‴', and 'o″ + o‴' is made to recast itself as '(o‴)″'. While Russell wanted to reduce geometry to arithmetic, arithmetical proofs for Wittgenstein ironically turn out to be proofs of sign-geometry. In both a logical proof and an Euclidean demonstration, it is by construction of certain signs that we tend to compel the acceptance of a sign. Accepting a proof, whether a logical or a pictorial one, is like accepting a paradigm that *this* pattern or formation of signs must arise when *these* rules are correctly applied to *these* patterns or signs initially given. And if this is what a proof proves first and foremost, then, Wittgenstein thinks, the proposition proved (by a professedly 'logical' or 'verbal' proof) is a geometrical proposition, which can also be alternatively characterised as a proposition of grammar concerning the transformation of signs (*RFM* II:38).

Here we need to present an illustrative clarification of the issue. To take a particular theorem of Euclid (I.4):

> If two triangles have two sides equal to two sides respectively, and if the angles contained by those sides are also equal, then the triangles will be equal in all respects.[21]



Euclid's proofs consist in pictorial constructions, i.e., in the present case, constructions of what it is to be two triangles of which the two sides of the first triangle, viz., AB and AC, are respectively equal to DE and DF of the second, and what it is to have the two contained angles ABC and DEF equal to one another. In drawing the premises of the proof, we also thereby draw the conclusion. Any attempt at non-pictorial or verbal demonstration of the proof will spurt forth indeterminacies that we have noted with respect to rule-following. One can imagine two three-dimensional (though liquid) angles created with water jets, satisfying the criteria of equality between the two pairs of sides and that between their contained angles; but any attempt to turn the angles into finished triangles inevitably tilts the water jet of their baselines, making them unequal. One can draw two equal triangles (following the instructions of the premises) on a paper which disproportionately absorbs the pigments of the bases of the two triangles, making them inequal. Such imageries may be taken to prove a theorem contrary to I.4—i.e., two triangles having two sides equal, etc., etc., are *not* equal. Our natural rejoinder would be the same as we noted in the cases of two apples and two apples coalescing into one or spawning into a thousand small apples, viz., in these imagined cases, we never had two finished triangles with the required equality of sides and angles! Such responses betray the fact that Euclidean demonstrations are already cast in the required mould—they are cinematographic pictures that freeze experiments into definitions. (Of course, the further Euclidean claim that this construction stands for any other similar triangles whatever is to be taken with the same reservations as Wittgenstein exhorted.) As for arithmetical proofs like 2 + 2 = 4, the only way to pre-empt the possibility of 2 + 2 acting in intractable ways is to *draw*, not write, the proof in the Euclidean manner; laying out the body of 'o'' + o''' as '(((o'')')')', as incorporating layers of '1's within itself, thus embodying the unique conclusion. None of the Euclidean and the arithmetical proofs makes advances from the first picture to the second; rather, they prefabricate the premises and the conclusion in a single picture of mutual compensations and adjustments.

As we have seen, Wittgenstein on several occasions characterised mathematical propositions as grammatical ones (*RFM* II:26, 31, 38, 39, pp. 161, 164, to mention a few instances), and more specifically claimed that the logical proofs were grammatical rules of sign-geometry (*RFM* II:38, for instance). Both the terms 'grammar' and 'geometry' have interesting shades of meaning that may relevantly be considered with respect to this claim. A grammatical statement virtually states that it makes sense to change *these* signs to *these* according to *these* rules of gender, number and syntax. In logical proofs as well, 'it is proved that it makes *sense* to say that someone has got a sign . . . according to these rules from . . . and . . . ; but no sense etc., etc.' (*RFM* II:38). More specifically, one can say that grammar formulates rules as to how one sign-pattern, considered mainly in terms of its physical features or physiognomy, gives rise to another physiognomy through changes of gender, number, tense, person, inflection. The phrase 'to inflect' also extends to other contexts, to mean a change in the pitch of voice, to turn from the direct line or course, a change in curvature from convex to concave (or conversely)—all adding a geometrical or spatial dimension to grammar. On the other hand, the term 'geometry' is often extended to art based on simple curvilinear or rectilinear motifs. The fact that the English alphabet and numerals have simple geometrical and symmetrical shapes lends cogency to this 'geometric' conception of logical proofs. The term 'geometric mean' is applied to the nth root of the product of n numbers, and a sequence is called a 'geometric progression' when the ratio of a term to its predecessor is always the same (e.g., 1, ½, ¼). These uses suggest a reduction of numbers into space or rather into spatial symmetry.

Now, one can always raise a question at this juncture: geometrical constructions always involve measurement—with rulers, protractors and compass. They also involve counting the units, which in its turn is supposed to rest upon a one–one correlation with a standard arithmetical sequence. This naturally is claimed to speak for an inalienable status of arithmetic, its pure units and pure sequence, which cannot be reduced to space, to geometry. Without entering into any theory of space or geometry at this juncture, we can at least remind ourselves what Wittgenstein had to say about measurement (see chapter II, section 2.2 titled 'Failure of Measurement'). The identity of units and sequences do not found but actually *consist in* the measuring activities. Putting the ruler against the line segment, ticking off the units on the scale, placing the protractor with its horizontal line along the hypotenuse of the angle, identifying its central point, identifying the 0^0 mark, moving the eyes along the 'rim', matching the mark with the other arm of the angle—all constitute the units and sequences, are all part of the process of sign transformation, and not external foundations to it.

What exactly does Wittgenstein gain by characterising the logical proofs (of mathematical propositions) as proofs of sign-geometry? Or saying that 'cogency of logical proofs stands or falls with its geometrical cogency' (*RFM* II:43)? As we have already seen, the term 'geometry' is sometimes used to mean not so much a branch of mathematics, but surface shapes, configurations of objects (here signs), and some of their properties like linearity, symmetry, angularity, circularity, etc. Even when taken as a proper branch of mathematics, there are interesting differences between arithmetic and geometry in a stance that arithmetic puts up and geometry does not, at least not in the same way. Reducing the cogency of logical proofs of arithmetic to their geometrical cogency is to purge arithmetic, or rather logicised arithmetic, of that stance.

Notwithstanding the geometer's claim of sweeping generalisations, it must be noted that the 'geometrical' cogency of geometrical proofs itself goes only up to the extent of actual construction; it does not extend to the indefinite extension of the 'line segment', i.e., to the 'ray', or the breadthlessness of a line. (Admittedly, points, lines, angles in their geometrical definitions, cannot be constructed.) The logicistic theory of arithmetic on the other hand (along with the notion of 'set') is perhaps more ambitious, aspiring for abstractions at various levels, thriving on reified *possibilities* of construction. The notion of a standard sequence and its units is supposed to range over numerals, or to strokes like I, II, III, or to little stars like *, **, ***, or to sounds, and to what not—and is thus left completely undetermined, unconstructed. So is the criterion of reproducing the logical proof. If, for instance, the sign I IIIIIIII occurs in a proof, it is not clear whether the same number of strokes, or little crosses, or even some other number is to count as a reproduction of it (*RFM* II:44), for the very notion of a unit and that of the same number of units is pushed beyond actual usage or construction. Whatever be the specific character of units or of the sequence, it is supposed to go beyond itself to an abstract structure—a structure of generating successive units. Every number defined as a set of sets becomes vacuous—the number three, for instance, is conceived as an abstract property shared by collections, or sets of collections, however dissimilar, however wide apart. Not only would 'the number three' have to have the colours of the Indian national flag, the ideals of the French Revolution, the antigen injections given to babies, as its extension—a range appreciably difficult. Consider a more difficult situation: on the instruction of a psychiatrist, I utter a list of three unconnected things at random, say {a beam of light, the third letter of the English alphabet, a drop of water}; or the teacher of Vaiśeṣika metaphysics gives examples of three divergent things as all coming under the notion of sayable and existent objects (*padārtha*s), e.g. {a star, a dream and an act of jumping}. These two sets too have to be included under the extension of 'number three'. The extensional enterprise of deontologising numbers has a more obscurantist impact than setting things right.

Moreover, as we have seen, arithmetic always puts up a stance of leaping over an indeterminate region, a blank space, unused, unconstructed, whether it is the conversion of large numbers into decimal notations, or continuing a series (like '+ 2', or inserting the squares of consecutive numbers), indefinite expansion of π, or the fundamental principle of recursive induction. Geometry, on the other hand, does not usually feed upon such an unusable lump of space or reified possibilities of construction.

On the whole, the logical tools of quantifiers, variables, constants, cannot take this magical leap from the surveyable to the unsurveyable, from the present to the absent, any more than geometrical demonstrations do. It is in this sense that logical proofs stand on a par with geometrical ones. What makes a proof so unshakeably certain is its geometrical and grammatical character, where we ourselves lock up the premises and the conclusion in a closed construction with a 'grammatical trick' (*RFM* V:3). And this geometrical and grammatical closure, as we have seen, consists in its peculiar depthlessness. 'It must not be necessary to make a physical investigation of the proof configuration', an investigation beyond the flat, two-dimensional figure to show what has been proved (*RFM* II:39). When we are shown the picture of two men, we do not first say that one man appears smaller than the other, and then that he seems further. It is perfectly possible that one man's being shorter than the other does not strike us at all, but only his being behind (*RFM* II:40). It is not real space behind the picture, but the picture itself, the two-dimensional geometrical lines of perspective (invented by the Renaissance artists), that constitutes the third dimension, the spatial distance between the two men. A logical proof too, like a picture, 'must be a procedure plain to view' and 'a procedure that is *plain to view*' (*RFM* II:42). It is not something *behind* the proof, the primitives, quantifiers, variables, rules of inference and substitution—all that is claimed to be the logical foundation *behind* the proof—that does the work of proving. It is only by cutting it off from all depth, all back and beyond, that I have made it indubitable, that I can declare the same substitution instance will yield the same result (*RFM* II:42, 43). This is what surveyability is; it is not what logic achieves by condensing a long expanse of marks into a short one, or by converting the third dimension into a two-dimensional surface. When the surveyability of a proof is destroyed, it is not for some silly and unimportant reason that logic can easily bypass (*RFM* II:43). Units smudge, split, disappear, a proof engraved on rock may alter in appearance over the years, and above all, all the techniques of physical and mental ostension, rituals and reinforcements may fail to convince one, may not persuade her into the desired closure. Under such circumstances, logic cannot salvage the proof. Rules, language, logical proofs cannot achieve what pictures cannot, they do not move a single step beyond the sensible quality of pictures. The kernel point is that if mathematics fails in terms of representing any real entity, or in terms of semantic transparency, then floating up logical rules and concepts cannot lay out an infinite and transparent space beyond surveyability. '[L]ogic as the foundation of all mathematics does not work, and to show this, it is enough that the cogency of logical proof stands or falls with its geometrical cogency. . . . The logical certainty of proofs . . . does not extend beyond their geometrical certainty' (*RFM* II:43).

* * *

We cannot leave this discussion without at least touching upon a crucial and provocative point that I have already mentioned. In his operation of reducing logico-arithmetical proofs to geometrical ones, Wittgenstein is not attempting to reduce number to space, but rather both number (time) and space to a motley of uses and activities. Wittgenstein's view of activities or actions can be best be understood in stark contrast with that of the Nyāya Vaiśeşika, for whom action is a cause of conjunction or disjunction with pre-given objects in time and space—the two infinite and eternal containers. For Wittgenstein, on the other hand, it is our actions that forge the identity of separate bodies and units, and not the other way round. There is no minimal identity of our actions as events located in specific space and time, wherefrom it is ready to receive alternative descriptions. Any attempt to locate such a pre-descriptional identity is already to have enmeshed it in a holistic pattern, from which it cannot emerge with a self-identical core to refigure in another pattern. What looks like a plain and simple act of counting, making one–one correlations with pre-given objects like the protractor, the marks inscribed on it, the tips of my fingers touching them, the distinct identity of my unitary body—all these could be radically reconstituted beyond any recognition in a different network. They could all be fragments in a concerted effort of several bodies, performing a larger action totally different from counting marks on the protractor. There is no pre-behavioural, pre-applicational content of arithmetical units and sequence that persistently spills over the activities that we call counting or calculation. And we must add: there is no pure identity of ideal space, ideal points, angles, lines, which persists over and above the process of construction.

NOTES

1. The previous chapter has already exposed the inherent lacuna in logic in so far as it attempted to show how logic presumes the notion of *one* under the guise of the individual variable and individual quantifier. For Wittgenstein, there are still more myths of logic waiting to be dispelled, which I take up more systematically in this chapter.
2. I am chiefly indebted to the remarkably lucid and comprehensive exposition of such a difficult topic by Sanjukta Basu, 'Significance of Russell's Incomplete Symbols in His Philosophy of Mathematics', in A. K. Mukhopadhyay et al. (eds), *Revisiting Principia Mathematica after 100 Years* (Kolkata: Gangchil, 2011).
3. A. N. Whitehead and B. Russell, *Principia Mathematica* (London: Cambridge University Press, 1967), vol. I.
4. Russell, *Introduction to Mathematical Philosophy*, chapter XVII.
5. Ibid.

6. Whitehead and Russell, *Principia Mathematica*, part I, section C, p. 188.
7. One can profitably refer to Basu, 'Significance of Russell's Incomplete Symbols', p. 243, for a clarification of the above definition.
8. Russell, *Introduction to Mathematical Philosophy*, chapter XVII.
9. Ibid.
10. Russell, *Introduction to Mathematical Philosophy*, chapter II, and also Russell, *Principles of Mathematics* (London: Routledge, 2012), chapter 11. Basu's article contains a short and pointed account of Russell's definition of number.
11. G. Frege, 'Über Sinn und Bedeutung', in Michael Beaney (ed.), *The Frege Reader*, Oxford: Blackwell, 1997.
12. Russell, 'The Philosophy of Logical Atomism'.
13. This point is also clarified in Basu, 'Significance of Russell's Incomplete Symbols'.
14. Russell, *Introduction to Mathematical Philosophy*, chapters VIII and XVII. A simple exposition of this progression will also be found in A. K. Mukhopadhyay, 'Reading *Principia Mathematica* after 100 Years', in A. K. Mukhopadhyay et al. (eds), *Revisiting Principia Mathematica after 100 Years* (Kolkata: Gangchil, 2011).
15. 'Russell's Logicism', 26 September 2007, available at: http://www3.nd.edu/~jspeaks/courses/2007-8/43904/ (accessed on 13 July 2016).
16 I have borrowed the following proof from R. Jeffrey, *Formal Logic*: *Its Scope and Limits* (New York: McGraw Hill, 1981), 2nd edn, chapter 5.
17. This possibility of duplicating R—as generating another layer above it, and itself being included as a member of this ever-proliferating layer—may be put in a different, or what I believe is a less artificial way. If you project a universal property as covering all particulars—but as itself being an identity—this will throw up the question as to how and under which property this universal is to be identified. Is it to be identified under a second-level property?
18. Whitehead and Russell, *Principia Mathematica*, vol. I; and Russell, *Principles of Mathematics*, Appendix A, section 492; Appendix B, section 500. I have also relied heavily on Mukhopadhyay, 'Reading *Principia Mathematica* after 100 Years'.
19. For Frege, the extension or value-range of a concept is itself an object that can figure as an argument of a concept. That is to say, a first-level concept can take both individuals as well as extensions of concepts or classes as their arguments. Thus the problematic concept

being not a member of itself can take as argument its own extension, i.e., the propositional function $x \notin x'$ is satisfied by the extension of the concept being a spoon, being a table, etc. As within Frege's system there is no hierarchy of objects and class, i.e., the extension of a concept is itself an object, it inevitably falls within the trap of a contradiction.

20. The comparison between small numbers and the foot rule, on the one hand, and that between bigger numbers and interplanetary space is quite provocative in different ways. The absurdity of measuring the interplanetary distance with a foot rule is quite in keeping with the revolution in physics when all matter, including the measuring rod, were shown to be a delicate balance of opposing forces, changing volume with the velocity of the respective planet. These theories as we know formed the backdrop of the final emergence of Einstein's theory of relativity (see Sir Arthur Eddington, *The Nature of the Physical World*, pp. 17–34). But while space, time and matter were revolutionised and relativised in the Einsteinian theory, number was still considered to be absolute. Wittgenstein, on the other hand, is clearly putting the idea of absolute space and time, as well as absolute numerical units, on the same level of absurdity.
21. 'Some Theorems of Plane Geometry', The Math Page, available at: http://www.themathpage.com/atrig/theorems-of-geometry.htm (accessed on 16 July 2016).

CHAPTER IV

Aspect-Perception, Sensation and Mathematics

We have seen that both visual demonstrations and verbal proofs of mathematics are pictures, ones that have locked the result into the process, the conclusion into the premises, and thus foreclosed experience into definite channels. Mathematics thus stands apart from empirical propositions, not in the way it did in the rationalist or the Kantian scheme, i.e., by being about a transcendent reality, or a priori forms of mind. It stands apart from experience in a special Wittgensteinian way which can now be spelt out in clearer terms. The talk of blocking or channelising experience, or turning it into flat paradigms, on closer analysis shows mathematical cognition as a new kind of perception, what Wittgenstein calls perception of *aspects* or *seeing* aspects. To understand a proof-picture is to see it in a new aspect, where the old picture is re-identified with the new. This phenomenon of seeing aspects displaces the traditional empiricist theory, where pre-lingual and discrete bits of sensation are given irrevocably, and aspects are inferred from these bits through a separate state of inductive inference, admittedly fallible. We have sought to construct from Wittgenstein's writings a consistent critique of the empiricist theory of sensation and aspect-perception. While taking into account various kinds of sensations, we have focused primarily on *colour* and *pain*, in view of the primordial, irrevocable and non-aspectual character patently ascribed to them in the empiricist tradition. We shall attempt to see how Wittgenstein works out a remodelled notion of seeing, i.e., aspect-seeing, not inferred from pre-lingual bits of sensations, but as a complex process where sensations, images, language and behaviour are all blended together. Mathematical cognition, viewed as a kind of aspect-seeing, will be relieved of all classical demands of a foundation, and yet preserve its required novelty and necessity at the same time.

1. The Distinction between Mathematical and Empirical Proposition

How do mathematical propositions differ from the empirical ones? We may start with a preliminary distinction—that between 'causal connections' found in experience and 'connections of a pattern' exhibited in mathematical proofs and calculations (*RFM* V:15, p. 170; V:40, p. 190). Unlike causal connections, patterns are what we create, where *we* choose to start and end, *we* carve out a trajectory of our own. And patterns are usually flat, plain configurations designed on a two-dimensional surface. Even when constructed in three-dimensional models, the third dimension or depth is what *we* create, a depth robbed of reality, of a real causal continuum. On the other hand, when two substances undergo a chemical change in the beaker, show frothing and finally red crystals, it is a real *causal* process that we *experience* but do not create (*RFM* III:33).[1]

Numerical or spatial relations, in so far as they are known in experience, are causal, not configurational or mathematical. Experience can in some sense be said to inform us as to how many objects of a specific kind are present in a specific situation, or how many objects a combination of two groups would practically yield in a particular causal process. Experience tells us that we can lay marbles side by side on a table, that they do not get stuck to the table or to our hands, they make a clicking sound as they bump into one another. It also tells us that they wobble, roll and tend to fall off the table unless we put up the necessary supports; and provided with that support, two groups of marbles, each containing five, and laid out with proper care, will actually add up to ten marbles. What is it to experience a proof written out on paper? It is to experience the properties of the paper and the ink marks, that we can move our hands and fingers on the paper, that we cannot do so on a jagged rock, that what has been written out can be preserved in a certain way. Counting, in so far as it is known through experience, is a real causal process which a mathematician transforms into a pattern. And in doing so, he has to shove out those interactions that may come to disturb the pattern.

Of course, nature abounds in objects with marvellous intricacy and symmetry, all formed by causal connections. Patterns in mathematics are apparently of a simpler nature, presenting a change of configuration or alteration of physiognomy. And the unpredictable causal interactions always come to disrupt the predictable and monolithic reshuffles that a mathematician seeks to contrive. To take a very simple and unambitious example of a pattern formed by a natural causal process—say a blob of paint trickling over a rocky surface full of cracks and pores. At no point can one predict its direction, or the angle the next curve will take, or the exact change of shade. Similarly, we can have the impression of two marbles and two marbles placed on the table and yet not coming to four marbles when we count them all together. It is possible to have the experience or impression of a reversal of the sequence 321, yet with 123 not arising thereby (*RFM* III:50). Wittgenstein characterises mathematical relations as 'formal' relations which cannot be learnt from experience, for if we did so we would have two impressions—one of five marbles and five marbles placed on the table, and another of ten marbles arising out of it. In mathematics, '...the result (is) put as equivalent to the oiperation'. (*RFM* III:40) 'The reason why one really cannot say that one learns that formal proposition from experience is—that one only calls it this experience when this process leads to this result' (*RFM* III:50). And when an experience consists of this process with this result, it is no longer an experience but rather a stoppage or closure of experience.

The mechanism of closing experience, we may remember, has been compared with an optical instrument persuading light rays coming from various sources to fall into a definite pattern. Or it is like a mass of marbles wobbling, rolling, colliding in different directions, caught and frozen into a typical configuration. To create a mathematical paradigm is like attaining a special shade of colour—e.g., a special reddish blue—through mixing a variety of pigments; this special mixture of the resultant colour is then 'fixed' by a ritualistic procedure and a name is given to it (*RFM* II:31). When it comes to colours, one can permit oneself to stretch the analogy a bit further. Let us say that creating mathematical paradigms is like freezing the swirling colours of marvellously complex shades (that mingle and flow in intractable directions) at *one* particular point and shade; and then branding it and putting it into the archives for good. Wittgenstein has, as we have seen, emphasised the role of constant drills and rehearsals in freezing experience into mathematical patterns. 'Pay attention to the patter by means of which we convince someone of the truth of a mathematical proposition.' It is that by which 'the machine of a calculating technique is set in motion' (*RFM* III:27). It is through this patter that the real flow of experiences is robbed of its depth and reduced to a flat configuration. And since it itself has no depth to investigate, we can lean on it, use it as a *model* for our investigations, as a criterion for deciding which experience to regard as valid and which false. 'The mathematical proposition has, as it were officially, been given the stamp of incontestability. I.e.: "Dispute about other things; *this* is immovable—it is a hinge on which your dispute can turn"' (*On Certainty*, 655).

Mathematical calculation does not uncover facts, for one cannot show what fact is meant by pointing with one's finger or any other of the acclaimed methods of ostension. Rather, it takes mathematics to define the character of the fact. Mathematics does not merely teach us how many vibrations this note has, not just the answer to the question, but rather the whole language-game with questions and answers. One can even say that mathematics initiates us into the notion of discrete and homogeneous units, and in this sense it teaches us to count (*RFM* V:15). It is not the outcome of experience, but gives a new direction to experience. It does not describe, but provides a framework for description (*RFM* V:2).

Now, one might object to ever being conscious of two processes—of there being a many-sided flow of experiences on the one hand, and another directing this flow into definite channels.

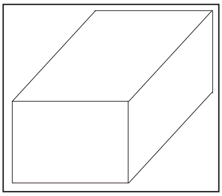
One may insist on being aware only of the empirical, not of a formation and transformation of concepts which is independent of it. Everything seems to be 'in the service of the empirical'. We do not alter our form of thinking, we only seem to be 'fitting our thinking to experience' (*RFM* III:29). For a person who thinks in this way, to experience 2 + 2 is also to experience 4. He does not have any clear concept of what experience would correspond to the opposite, what it would be like for it to be otherwise. In the course of a mathematical proof, our way of seeing is remodelled; we formed our way of looking at 2 and 2 marbles which excludes the experience of one falling off or breaking, for should such things happen we would always insist on never having had the experience of 2 and 2 marbles in the first place. In the course of the proof, we formed our way of looking at the trisection of the angle which excludes a construction with ruler and compass (*RFM* III:30). The situation may be compared with a person being put inside a room and trying to push open the door, which never opens outwards but only inwards. The person *experiences* himself as being imprisoned in the room, he *experiences* the door as being locked in so far as this experience excludes the possibility of its opening inward (*RFM* III:37).

2. Mathematics as Experiencing Aspects

Remodelling experience or closing its flow into definite channels is a modified concept of experience itself. It is experiencing an aspect, rather than experiencing an object or its properties. Most examples of 'aspect-experience' or 'aspect-seeing' as given in *PI* are however non-mathematical. We shall start with a few of these examples before we pass on to the main point at issue.

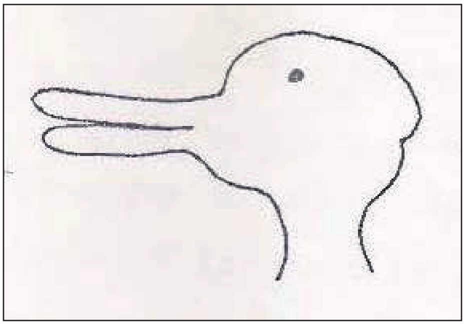
Seeing a face or two faces is experiencing an object, while seeing a similarity between the two is seeing an aspect (*PI* p. 193). Most of the examples of aspect-seeing employ pictures and not real objects, such as the illustration in Figure 4.1 that appears in several places of a textbook. Each time, the text supplies a different interpretation of the illustration.

Figure 4.1



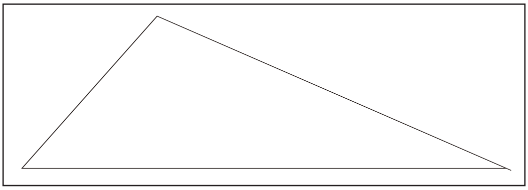
We see it in one aspect or another; here we see it as a glass cube, there an inverted open box, there a wire frame, or three boards forming a solid angle (*PI* p. 193). One can see Figure 4.2 in two aspects—either as a duck's head or as a rabbit's head (*PI* p. 194).

Figure 4.2



Another picture, Figure 4.3, can be seen as a triangular hole, as a solid, as a geometrical drawing, as standing on its base or hanging on its apex, as a mountain, as a wedge, as an overturned object meant to stand on the shorter side of the right angle and so on (*PI* p. 200).

Figure 4.3



Among many other examples, Wittgenstein (*PI* p. 203) also speaks of alternating aspects, like the convex and concave aspects of Figure 4.4, and also the aspects of figure 4.5, which one can see as a black cross on a white background or a white cross on a black background (*PI* p. 207).

Figure 4.4

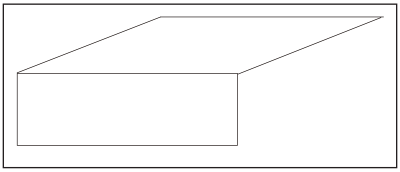
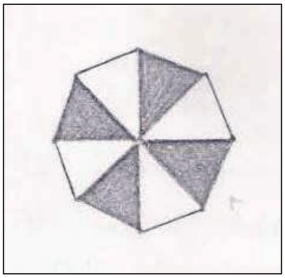


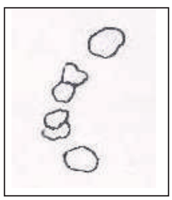
Figure 4.5



Can mathematical closure of experience be regarded as experiencing an aspect as contrasted to experiencing an object? Wittgenstein is never too explicit on this issue, and we have to shape our views from his scattered suggestions. He speaks of teaching a student of arithmetic how to see that certain things can be 'taken together', or that they 'go together' (*PI* p. 208). Perhaps we can think of a situation where a variety of dissimilar things are scattered on the table—say a book, a sugar cube, a glass of water, peas—and ask the child to count how many things there are on the table. Here we ask her to *see* these things in a new, 'countable' aspect, i.e., to *see* each object as correlated with each of her fingers, disregarding their individual differences. Like seeing a similarity between two faces, or seeing that certain things 'go together'; i.e., that two apparently dissimilar figures like 'Hands' and 'Pentacles'

(Figures 2.1.A and 2.1.B in chapter II) can 'go together' in an isomorphic correlation is a case in point. Wittgenstein also speaks of seeing a triangle in alternating aspects, now seeing *this* as apex and *that* as base, now *that* as apex and *this* as base (*PI* p. 208). Extending these suggestions, we can construct for ourselves a few more examples of mathematical aspect-seeing. One can see the picture in Figure 4.6 as 1 + 1 + 1 + 1, he cannot split each of the intermediate couples into two separate units. One can be trained to see the picture as 2 + 2 coming up to 4, and thus land on a new aspect, a new perception, where the old perception is re-identified with the new. And just as one can alternate between 2 + 2 and 4, similarly one can also alternate between 1 + 2 + 2 + 1 on the one hand and 4 on the other (now seeing the second and the third blobs as each split into two, now seeing each of the four blobs as a single unit).

Figure 4.6



It is natural to object to experiencing such a 'deviant' mathematical aspect. It may be argued that one only needs to look at Figure 4.7 to see that 2 + 2 = 4. Then, I only need to look at Figure 4.8 to see that 2 + 2 + 2 = 4 (*RFM* I:38, discussed in chapter II, section 2.5, 'Circle of Reason and Experience'). One person can alternate between 2 + 2 and 4, another between 2 + 2 + 2 and 6, and still another between 2 + 2 + 2 and 4. What happens here is that each of the lower pairs of crosses merges with those of another oval, horizontally overlapping the first two, thus allowing us to see the congruent pair of crosses either as distinguishable or as indistinguishable, thus allowing the entire picture to be seen in various numerical aspects. Wittgenstein cites a similar instance where one may be said to see the group I I I I I in an unusual aspect as the group I I ⅡI I, with the middle stroke seen as the fusion of two strokes, and should accordingly count the middle stroke twice (*RFM* I:168).

Figure 4.7

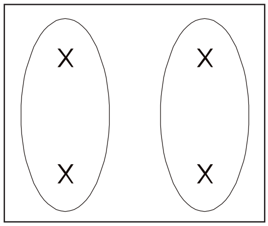
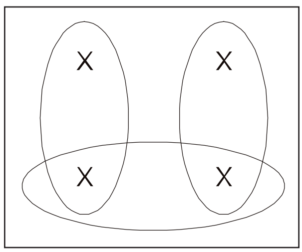


Figure 4.8



Seeing mathematical aspects is a matter of going through a cinematographic cycle of training, and a different training, i.e., a different cinematography, can lead one to see deviant aspects. One may take blobs of paint as units, film the process of putting five blobs in a row and then five more, chanting '1', '2', '3' as he goes on; and finally, freeze the very moment when the blobs merge into a single horizontal track, with the occasional protuberances still identifiable. The film finishes off with a flourishing highlight, and the vocal commentary dramatically concludes: 5 + 5 = 1. And a different strategy of film narrative might preserve the good old statement 5 + 5 = 10 even if the film is on blobs of paint.

'The expression of a change of aspect is the expression of a *new* perception and at the same time of the perception's being unchanged' (*PI* p. 196). A person first sees the Hands and Pentacles as two figures that are distinct and dissimilar; he sees a row of paint blobs as 5 + 5, but not yet as 10. When he sees the Hands and the Pentacles as isomorphically related, and the row of 5 and 5 blobs as also a row of 10, this new perception, this new aspect is not different content-wise from the old. Similar remarks would apply to the perceptions of 'normal' and 'deviant' aspects: two people see the same picture (Figure 4.8), yet one sees it as 2 + 2 = 4, another as 2 + 2 + 2 = 4. What one formerly saw as an ornament or a wallpaper design is later *seen* as a mathematical proof, where two configurations—the premises and the conclusion—are *seen* as leading stepwise to a unique one-to-one mapping. Here also it is the same picture, the same physical marks that correspond both to the old perception and the new aspect.

To be convinced of something as a changed aspect of a picture or an object is also to be convinced of having perceived it *already, in some manner or other*. And 'he has not any *clear* concept of what experience would correspond to the opposite' (*RFM* III:29). He has no concept of what it is to experience Figure 4.2 as a duck-elephant, or what it is to experience Figure 4.8 alternately as 2 + 2 and 6.

It is in this sense that closing or remodelling experience turns out to be a case of experiencing aspects. Mathematical relations are aspectual relations or formal relations as opposed to empirical relations (e.g., causal relations). In mathematics, we switch over to a new perception, a new aspect—which is also a new concept or a new technique—not because we tell ourselves that it will work this way too, but because we feel the new perception or technique is identical with the old one. '[B]ecause we have to give it the same sense, because we recognize it as the same just as we recognize this colour as green' (*RFM* III:36).

3. Empiricists on Aspect-Perception

Before moving any further, we need to pause and see what the empiricists have to say on aspect-experience and sensation. In other words, we need an exposition and a consequent dismissal of the empiricist theory for a proper appreciation of mathematical cognition as a kind of aspect-seeing.[2]

It is quite clear that the empiricist framework does not permit the possibility of *experiencing* a change of aspect. On this theory, experience is ultimately made out of discrete simple ideas, or more accurately, of discrete states of nervous excitement immediately produced by external objects and carried to discrete parts of the brain. These states are presumed to be given irrevocably, and apprehended immediately, prior to all interventions and modifications effected by language, description, knowledge, inference or behaviour. Representations of objects are compounded out of these pre-given bits through certain psychological laws of association, while, objectively speaking, such associations are always loose and contingent. How then can there be an aspectual transition from one experience to a new one, a transition which is necessary, internal and formal, as Wittgenstein claims, where the old experience is seen as identical with the new? How can one piece of sense-experience be *remodelled* into another, since experience is ultimately constituted of irrevocable, unalterable bits of sensation? Relations of identity can obtain, not between two different experiences, but between two *concepts* where *we* perform the feat of identification by defining one in terms of the other. Mathematical cognition for the empiricists is plainly not a case of aspect-experience; rather, it is completely expressed in a system of analytic propositions formulated by the mathematicians.

Let us get into some of the details of the empiricist theory of sensation and aspectual cognition. On this theory, colour sensations are prototypical of visual sensation, and we presume that sensations of taste like sweet or sour, sensations of touch like hot and cold, would be instances of other categories of sensation as well. These sensations, as mentioned earlier, are given irreversibly; they are not subject to change by a change of attitude or will, i.e., without a corresponding change of stimulus. Cognition of an object, of a form, or even of a shape or line, appears to change in a change of aspect. They are not *sensed*, but *inferred* through an unconscious memory of past perception of objects. When I claim to *see* my friend standing before me, patting me on my back and humming a tune, I actually infer it from a cluster of coloured plane surfaces, discrete notes and touches that are originally given in my sensations. Now, the empiricists claim that these unconscious inferences are inductive, i.e., inductive leaps from discrete sensations to full-fledged objects, and hence fallible. And since the shift from one aspectual perception to another, say from perceiving my friend patting my back to perceiving some configuration of limbs displacing air molecules, is nothing but a transition from one inductive leap to another, this transition does not give us an infallible cognition.

Coming to numbers, the empiricists might claim that single units are given as simple ideas in our sensations. If so, they would also be available for being compounded into complex ideas, which in their turn would be combined to form analytic judgements on complete or partial identity. Moreover, propositions like 2 + 2 + 2 = 4, or 5 + 5 = 1, would plainly be dismissed as self-contradictions in this model.

While numbers may be given in sensations, representation of full-fledged objects, e.g., representation of 2 and 2 marbles, would be an inductive inference according to Helmholtz.[3] When we claim to see 2 and 2 marbles as 4 marbles, we actually engage in two acts of inference, and the aspectual transition from 2 + 2 marbles to 4 marbles is a transition from one inference to the other. Within the empiricist model, there are two possible accounts of perceiving a change of numerical aspects. On one account, when 2 and 2 marbles are seen as 4 marbles, this transition from one inference to the other also involves a change of stimuli. In other words, it involves two different sense-impressions at two subsequent points of time. We perform two different acts of counting (one being 2 + 2 and another 4), i.e., two different acts of 1–1 correlation—where in both cases we correlate each marble with each fingertip, or with each numeral uttered orally. We perform an experiment on the given cluster of marbles to see what *effect* the second act of correlation would have on the original group of marbles. And a causal transformation, as the empiricists have taught us, is always a *loose* (though constant) conjunction of *distinct* impressions, and is never necessary. The other account will be in conformity with Helmholtz's theory, according to which change of aspect is effected by change of will or attitude and not by a change of stimulus (i.e., of impression). On both these accounts, the aspectual transition—

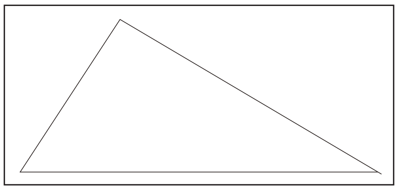
i.e., the transition from one inference to the other—is plainly contingent.

The crux of empiricism consists in splitting experience into discrete fragments—splitting the experience of 2 + 2 marbles from that of 4, the colour, heat and glow of fire from its burning sensation. When a substance in a beaker undergoes a chemical change, shows frothing and finally red crystals, the empiricists would split up the entire process into discrete sensations, specially focusing on the two distinct impressions before and after the frothing, the cause and the effect. When a group of signs, say OVER, is reversed to give rise to REVO, the empiricists would again insist on two distinct impressions—one of the process of reversal and another of the actual outcome, and (they claim) one can always speak of one impression not giving rise to the other.

Once we are entrenched in the empiricist model, we tend to impose the following constraints. (*a*) The numerical relations will be specific only to particulars situations, informing us that *this* cluster of marbles correlated with two and two fingers of *this* palm actually gives rise to another act of correlation with *these* four fingers. It gives us no idea of a general relation between *similar* groups of marbles, still less to speak of *similar* groups of other things. (*b*) Even with *this* group of marbles, one can always imagine that the first act of correlation does not give rise to the second; marbles fall off, disappear, etc. (*c*) Once the empiricists contrive an impossible fragmentation of perception, they are also led to contrive a kind of scepticism about units disappearing or doubling up—things that go undetected by an equally fragmentary memory. This makes all number-relations between concrete objects fallible. Such constraints are sought to be extended to geometrical applications as well.

To take a simple instance of a triangle ABC (Figure 4.9). Here we can see A as the apex and BC as the base, or C as the apex and AB as the base. In the empiricist model, the relation between the two alternating aspects is not necessary, for there would be two sets of sensations, two distinct states of inferences, where one may *not* give rise to the other. We can indulge in such suppositions like the line AB becoming curved when we try to hold the book or paper sideways, or AC and BC drifting away—all due to some strange quality of ink and paper.

Figure 4.9



In a nutshell, the empiricist theory of aspect-perception comes to this. First, sensations are given, while objects and properties of objects are always inferred from the sensations. The purported discreteness of given sensations always leaves gaps among them, and thus between the sensations and the inferred object.

Secondly, there cannot be a relation of identity between two clusters of sensation, still less between sensations of colour. Change of aspect is a transition from one inference to another, or rather from the conclusion of one inference to that of the other. This transition, as we have seen, is always contingent.

Thirdly, the jargon of irreversibility of each bit of given sensation debars one from talking about *seeing* or *sensing* a *change* in aspect. Aspect-perception is thus a matter of inference and not of seeing.

This entire framework of aspect-perception, with pre-lingual bits of sensations on the one hand, and description and inference on the other, is alien to Wittgenstein's conception. Nor does the notion of analyticity, the pretence of purely verbal stipulation made once for all, even get a foothold in his philosophy, where meaning is an organic complex, continually dissipated and lacerated. Wittgenstein can neither borrow the notion of analytic necessity, nor the empiricist theory of aspect-perception to graft it on mathematics. Though the empiricists clearly did not intend *their* theory of aspect-perception to suit *their* view of mathematics, Wittgenstein has to get rid of the theory before he starts spelling out his own view of mathematical cognition as a kind of aspect-seeing.

4. Wittgenstein's Critique of the Empiricist Theory of Aspect-Perception

For Wittgenstein, seeing an aspect presents an immensely rich variety of uses and behaviours which cannot be described by a uniform theory of 'inductive inference' or 'interpretation'. The empiricist epistemology feeds on a false dichotomy between a given cluster of discrete, pre-lingual sensations, and a higher process of description, interpretation or inference in the brain (which however cannot add anything new to the given cluster). Here we need a detailed critique of the empiricist theory of sensation and its relation to aspect-perception.

4.1 *Wittgenstein on Sensations*

Contrary to what empiricists thought, sensations for Wittgenstein are not self-identical chunks or cores, beyond all relations or aspectual transitions. They are not passive entities lying in our mind, to be picked up and clamped with word labels. Nor do they serve as grounds of inference in the way a signboard, physically isolated from what we infer, serves as one. One has to appreciate how sensation and description, inference and activities are all absorbed into a continuous and complex process.

Let us try to analyse a few of Wittgenstein's scattered comments on various kinds of sensation. When we think of the sensation of shuddering, the words 'It makes me shiver' themselves constitute such a shuddering reaction. '[I]f I hear and feel them as I utter them, this belongs among the rest of those sensations'. There is not a pre-lingual shuddering that is the ground of the verbal one (*PI* p. 174). Again, when our kinaesthetic sensations are claimed to advise us about the movement or position of our limbs, we cannot isolate or label a single sensation preceding our knowledge or description. When I let my index finger make a slight pendulum movement of small amplitude, I hardly feel it; only perhaps a slight tension at the fingertips and none at the joints. The empiricists would no doubt insist on a pre-linguistic chunk of sensation as a ground of my description or inference. They would say something like this: 'But after all, you must feel it, otherwise you wouldn't know (without looking) how your finger was moving' (*PI* p. 185). Wittgenstein points out that in this case, knowing the movement and position of our limbs is just being able to describe, and the kinaesthetic sensations are a part of the entire description (*PI* p. 185). Similar remarks would apply to situations where on hearing a sound, I am able to tell the direction, for it affects one ear more strongly than the other, and yet I do not feel this in my ears (*PI* p. 185). According to the traditional empiricist account of perception, each ear can register the stimulus-content that affects it, so that the comparative exercise that this ear is more strongly affected than that ear is a complex cognition that cannot be received in the ear itself. While for the empiricists we *infer* this complex description of the direction of the incoming sound from pre-given bits of sensation, for Wittgenstein, language and description do not trail behind pre-given bits of sensation via concepts and inference; they all forge an irreducible whole.

Wittgenstein further speaks of certain situations where sensation of pain advises us of the movement or position of the limbs or of the nature of the injury. Suppose one has just regained consciousness and does not know whether his arms are stretched out—he finds out only by a piercing pain in his elbow. Contrary to what the empiricists usually claim, here the pain in the elbow is not an isolated ground for inferring its position, in the sense the signs 'OPEN' hung in front of a shop is the ground for inferring that the shopkeeper is in. Rather, feeling the nature of injury and the position of the arms are two *aspects* of the same feeling of pain. The yellowish hue of the photograph is not a ground for inferring how old it is, but the oldness is rather seen as an aspect of its yellow colour and vice versa. One can give the command: See this paper (beside the other colours) as white, now (beside the lump of snow) as grey; see this white-hot kettle alternately as a lighter shade of its original brown (*qua* the dispersal of its molecules), now see it is as a darker shade of brown as concentrating all the heat, glow and the molecular force into itself. Just as it makes sense to instruct one (say in a music lesson) to hear this bar under two aspects—say in a particular key, or as an introduction—it also makes sense to instruct one to feel pain under two alternating aspects—say the exact location of the injury, *and* the extent to which other sensations (like itching, or a gentle touch of breeze on that area) are submerged by the pain (*PI* pp. 185, 186).

In fine, sensations can be said to be grounds of description, inference and activities, only as being already integrated into this complex. The need to demarcate between a pre-lingual, non-relational and non-aspectual block of sensation on the one hand, and language, description or inferential knowledge on the other, is rather a search for a grammatical distinction—the distinction between 'this' and 'so' in sentences like 'This feels so,' 'This looks so,' 'This tastes so,' and so on (*PI* p. 185).

The empiricists take colour to be prototypical of visual sensations, irrevocably given blocks sharply demarcated from and grounding inferences on changeable aspects. They would no doubt take a similar stand with regard to pain. Apparently, Wittgenstein too uses the notion of pain in a typical contrast with his own notion of aspect-seeing. Noting that 'the substratum of this experience is the mastery of a technique,' Wittgenstein observes: 'After all, you don't say that one only "has tooth-ache" if one is capable of doing such and such' (*PI* p. 208). Now, given this background, Wittgenstein's analysis of the *aspectual* character of both colour and pain that we have just noted (the yellow hue of the photograph, and the pain in the elbow) are particularly interesting. We need to devote two sections to colour and pain. Once we come to appreciate colour and pain to be aspectual, and perhaps not as far removed from the number and space of mathematics as is generally presumed, we should be in a better position to place mathematical cognition in a broader perspective.

4.2 *Wittgenstein on Colour*

Wittgenstein offers an immensely colourful variety of aspectual transitions of colour. In a picture in which a piece of white paper gets its lightness from the blue sky, the sky is lighter than the white paper. Placed on the palette on the other hand, white is lighter than blue (*ROC* I:2). A piece of paper which looks white beside red, blue or black, looks grey beside a piece of snow (*ROC* I:5). One normally sees white, brown and grey as different colours of which white is the lightest. But in certain situations, one can also see brown and grey as lighter shades of white. When a thing is heated up gradually, it first turns brown, then grey, and finally white. Here, brown-hot and grey-hot are seen as lighter shades of white-hot (*ROC* I:34). While watching a black-and-white movie we usually see the white screen as opaque, on which pictures are projected, and the varying shades of black and white are seen as a proxy for real colours like blue, red, green, etc. But one can also see the white screen as a transparent pane of glass and the events of the film as lying behind the glass. The glass would take the colour away from things and allow only white, grey and black to come through. And here we *see* white and black not as proxies, or (as the empiricists would say) as indicators for *inferring* real colours, but just as green, red or blue (*ROC* I:25).

We know that in the empiricist framework, the same colour cannot be *seen* or *sensed* under variant aspects. What are irrevocably seen or sensed are discrete colour patches lying universally under all variant aspectual transitions. When the same piece of paper is seen alternately as greyish and white, or the same surface is seen as an opaque white screen or a transparent pane of glass, we actually engage in two separate acts of inference. We infer or interpret the given colour patches as full-fledged objects, e.g., as a white paper, or a screen or a pane of glass, a hot glowing kettle, etc. We infer their depth and texture, their shade and highlights, the relations of lighter and darker shades, as well as their spatial relations (like ‘upon’, ‘front’ or ‘behind’) with other objects. When these properties or relations are seen to change into a new aspect (i.e., when the same white surface is seen as an opaque screen with objects projected *upon* it, or alternatively as a glass pane with objects *behind* it), we actually *infer* different properties or relations from the same stock of given sensations, i.e., from the primordially given colour patches. This model of explanation devised by the empiricists would enable them to dismiss the unusual modes of colour perception cited above as false inferences or false interpretations.

The best way to get out of the empiricist framework is to indulge in certain thought experiments with flat colour patches themselves, whereby their pre-lingual and pre-aspectual status falls to the ground. Wittgenstein ponders whether it is possible for someone to see a surface as a combination of red, white and blue (as in the French tricolour) but not as red. The person has no ability to separate the strips of colour. For her, there is only one colour adjective for red, white and blue, say 'bu'; and she is trained to report only 'bu' and 'non-bu', i.e., all colours other than 'bu' are reduced to a single chrome (*RFM* V:42). In *RFM* (V:43), Wittgenstein contrives another situation where such modes of colour perception become intelligible. Suppose the colour of the strips of a surface change every minute, all at the same time, as from Figure 4.10 A to Fig 4.10B, then again to 4.10 C and finally back to 4.10 A.

Figure 4.10

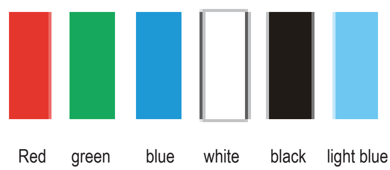


Figure 4.10.A

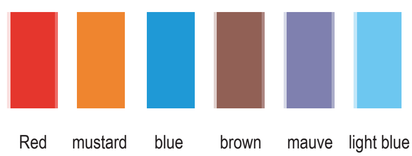


Figure 4.10.B

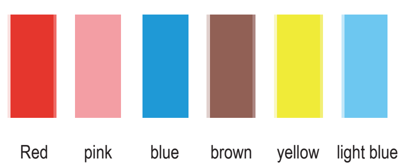


Figure 4.10.C

Here one observes (A) {(red . blue) Ɔ black} Ɔ white, and also (B) ~green Ɔ ~white. One does not merely observe the arrangement of colours and deduce the above propositions therefrom. For someone observing the surface may be quite preoccupied with the question of whether it is going to turn green or not-green; she need not be attentive to the particular colours the surface is. She does not observe Figures 4.10.A and 4.10.B as an array of discrete colour strips lying inertly beside one another, then to define them as ~green and ~white respectively, finally to infer (B), i.e., ~green Ɔ ~white. She simply *sees* (B), i.e., ~green Ɔ ~white and (A), in a way that she is taught to forget everything else, and only to look at the surface from this point of view, under *this* aspect. Wittgenstein says that if he observes A and B, then he can also *observe* and not *infer* (C) i.e., ~green Ɔ {(red . blue) . ~black}. Wittgenstein says that it is possible for this third observation not to agree with the logical conclusion of the first two, say with

(E) {(red . blue) Ɔ black} Ɔ green, or

(F) ~green Ɔ [(~white v red) . (~white v blue) . (~white v ~black)]

Perhaps she has no power of separating red from blue or to see what green is in isolation. It is like someone who observes the surface of a flag as a combination of red and black, but if she now sets herself to see one of the two halves, she sees blue instead of red. Wittgenstein compares this with cases where someone looking at a groups of apples always seen it as two groups of two apples each, but as soon as he tries to take the whole lot in at a glance, they seem to him to be five. The situation is more akin to the + 2 series where ‘2’ and ‘+’ assumed different identities with each consecutive range of 1,000 (see chapter III of this work; *PI* 185). To give a further analogy with geometry: a rhombus, seen as a diamond, does not look like a parallelogram; the parallelism does not strike us (*RFM* V:43).

It gradually emerges that just like the talk of *ideal* reified units, or ideal space, the talk of pure colours severed from real objects, without depth or dimension, highlight or shade, glow or texture, beyond relations or aspectual transitions, is not meaningful. The empiricists would however insist on flat, saturated, pure colours given irrevocably in experience, and their availability in forging analytic statements. For the empiricists, the perception of a red and black surface implies the perception of black by the simple rule of logic, viz., (A.B) Ɔ A. A person who sees red and black in combination, but sees the black strip as blue in isolation, would either be contradicting himself, or be labouring under a perceptual illusion, which (for many empiricists like Helmholtz) is a false inference. So would the person who holds (C) and yet does not agree to (E) or (F).[4]

Empiricists do not appreciate that the flat, opaque pigments that we put on a palette, or the paper colour samples, are not primordial units of experience. They are constructed in the same fashion as mathematicians freeze experience into flat pictures. And just as mathematical pictures are not extracts of pure reason, the flat pigments are not pure elements of experience. The mathematical pictures that on the one hand freeze the experience of 2 + 2 into 4, open it up on the other. They link up many dissimilar pictures or dissimilar experiences (like tables, apples, fingers, drops of water, rays of light, beats of pulse) through transitions of newer aspects, newer patterns of isomorphic correlation. Likewise, a flat, opaque pigment, say yellow, too effects a sweeping aspectual adventure among objects as dissimilar as sunflowers, grains of sand, chicks, the sun and its rays.

We do not carry an objective chunk of colour that repeats itself, every time we repeat a colour word, say 'blue'. We might take this as an occasion to remember the various language-games played with the word 'blue' (*PI* 33, discussed earlier in this work). We have already suggested that pain language and pain behaviour are an extension of the pain sensation, and not an external label or manifestation of an inner private feeling. So are our various language-games and activities involved in the perception of colour. Just as the words 'It makes me shudder' incorporates the sensation of shuddering, the words 'The sky is so blu-u-u-e' is an extension of our sensation of blue. Otherwise we could have said 'The sky is red' and *mean* 'The sky is blue.'[5] Just as we *see* the black-and-white photographic paper as coloured, we often *hear* the sound of the colour word as the colour itself. Suppose we see a boy in a photograph with slicked-back blonde hair, standing in front of a kind of lathe made partly of iron castings painted black, partly of smooth axles, gear, etc., and next to it a grating made of light galvanised wire. Here I *see* the boy's hair *as* blonde, the grating *as* zinc coloured, the finished iron surfaces *as* iron coloured. I do not *infer* them from lighter and darker tones of photographic paper. Similarly the word 'blonde' does not act as a label to clamp on the actual colour sensation, it *sounds* blond (*ROC* I:63, 64, 65).

Since colours are not isolated and repeatable chunks, there is no criterion of sameness, of exact reproduction of a particular shade, or hue, or any other colour adjective. The fact that I can say 'This place in my visual field is grey-green' does not mean that I should know what should be called an exact reproduction of this shade of colour. For the tracks of family resemblances, the transitional links of fibres through which one forms his colour concepts are as intractable and wayward as numbers. One may know the concept of intermediate colours, she might have learnt to mix a colour that is more yellowish, more whitish or more reddish than a given shade. Suppose she is asked now to show us a reddish green. She may not simply understand this order or unhesitatingly point to a colour sample (say what we should call blackish brown) (*ROC* I:10). There could be people who didn't understand our way of saying that orange is rather reddish yellow, but have no difficulty in understanding reddish green. All these examples irresistibly remind us of continuing a number series where the fact that one has calculated 'normally' up to a certain point does not predict any of her subsequent moves.

As we have seen, Wittgenstein's account of mathematical knowledge, its non-foundational, paradigmatic and aspectual character, derives largely from an analogy with colour. Freezing experience into flat patterns was compared with freezing the swirling colours at a particular shade, fixing the exact mixture of colours, giving a name and putting it in the archives (see the opening pages of this chapter). Once again, when these frozen pictures are seen to be a remodelled experience or aspectual experience, Wittgenstein falls back on colour: 'When white turns black some people say "Essentially it is still the same"; and others, when the colour turns a shade darker: "It is *completely* different"' (*RFM* III:38). The first relation is a mathematical relation, like that between 2 + 2 and 4, where we switch over to a new perception, a new technique or a new aspect, and yet we feel the new aspect is identical with the old. '[B]ecause we have to give it the same sense, because we recognize it as the same just as we recognize this colour as green' (*RFM* III:36, quoted earlier).

What we need to emphasise is that the relation between number and space on the one hand, and colour on the other, is more than just analogical. Numbers themselves are coloured and colours themselves are numbered. There is nothing like pure, colourless, non-sensual numbers, and there is nothing like pure colours without a numerical aspect. The indeterminacy of number and space is not *like* that of colour; rather, both kinds of indeterminacies are different aspects of the same object. Units written in flowing colours dissipate, Euclidean diagrams drawn partly in white on a black background, and partly in the reverse, lose a unique interpretation. It is again the relative definiteness of colours that is used to demarcate units or a cluster of units, i.e., they get identified in terms of colours. Here one cannot split either the identity of units from the colours, or the identity of colours from numbers.

4.3 *Wittgenstein on Pain*

Aspect-seeing, in so far as it is a capacity for doing something, or mastering a technique, is typically unlike the experience of pain. To quote once again (from *PI* p. 208): 'After all, you don't say that one only "has toothache" if one is capable of doing such and such.' This does not imply that experience of pain is 'non-aspectual' in the sense of there being pre-lingual and pre-behavioural chunks of pain in the mind waiting to be picked up or described by pain language. Such a picture, as we know, would once again be an Augustinian hangover. In this model, the pain sensations are supposed to have no spatial dimension, no conceptual complexity, nor even any temporal structure beyond their momentary presence. Obviously Wittgenstein finds this an impossible position to uphold. Just as a colour sensation, say that of red, has various aspects of duration, intensity and protensity, so does the sensation of pain. Each of these categories again stretches out into various options. For instance, does the protensity of pain signify a feeling that starts from a centre point of my arm and gradually spreads towards the periphery? Or is it how the new pain sensation submerges the old sensation of itching that occurred in the same area? We have seen that it is meaningful to command a person to feel his pain in these two aspects, say as the exact location of the injury, *and* the extent to which other sensations are submerged by the pain. Such commands seem to stand on a par with instructing someone to hear a bar under two aspects, say in a particular key and as an introduction, or to see a triangle under three alternative combinations of apex and base.

When a child hurts himself and cries out in pain, we teach him new pain behaviours—e.g., exclamations like 'oh!', 'ouch', putting his hands on the sore place; and later, pain languages like 'stubbing one's toes', 'itching', 'toothache', etc.[6] Teaching pain language is teaching him a new kind of pain behaviour; it is not supplying him with a label to stick to the pre-lingual and pre-behavioural pain residing in his mind. Teaching new (non-linguistic) pain behaviours too is not teaching him to give new signboard indicators for his internal and private pain sensations. Teaching a child a new cluster of pain behaviours (linguistic and non-linguistic) is not the end of the language-game, but rather its beginning. It is the beginning of a process of forming and expanding the concept of pain. And the concept of pain, like all other concepts, is spun along the transitional links of family resemblances—from burning eyes to throbbing temples, gnawing in the stomach to toothache, from puckering up one's face to writhing or the twisting of limbs. These entire complexes of linguistic and non-linguistic behaviours are the brute facts that leave no residual question about an entity referred to or inferred from pain language or pain behaviours.

While learning pain language or new pain behaviours may initially be a repetition (or substitution) of primitive pain expression, something more is involved in the expansion of the concept of pain, or mastering the use of pain language. This largely consists in learning to see others' behaviour in a particular aspect, i.e., as taking an interest in my hurt, being ready to attend to or nurse my wound. However, though mastering the use of first-person pain language *involves* seeing other people's behaviour in a particular aspect, experiencing pain, or learning new pain behaviours does not *consist in* this aspect-experience.

We might think of examples where our pain sensations *do* consist in seeing our own behaviour in a particular aspect. We all face situations where we are morally obliged to feel grief, we are morally obliged to bring up an image of the deceased person in our minds, seek hard to bring tears to our eyes, gulp hard to form a lump in our throat. Feeling pain here consists in seeing our own behaviour *as* pain behaviour. On the other hand, a person who prides herself on being cold and emotionless may in a particular situation find tears welling up in her eyes, a lump forming in her throat. Ironically, her feeling of pain here consists in seeing her own behaviour as pretence or shamming.

Using pain language in the second or third person virtually amounts to seeing that person's behaviour in a particular aspect. It is extremely important to realise that while in Wittgenstein's philosophy, reality is cashed out in a flow of linguistic and non-linguistic behaviours, behaviours themselves cannot be allowed to remain as a self-interpretive chunk, beyond relations, language and further behaviours. Behaviours too present a picture that undergoes aspectual transitions. To *see* a man tossing about in bed with a puckered face, groaning, twitching his hands, *as* a man in pain, presupposes an expansive participation in a particular form of life. One can imagine people under different forms of life seeing this behaviour under different aspects, say as a kind of dance or exercise, a way to toss up the bed linens, testing the noises of the joints or the strength of the bedstead. Seeing a behaviour as shamming, which is also a case of aspect-seeing, should not be assimilated to any of the above instances; to see a behaviour as shamming, one should also be able to see it as genuine pain behaviour, both of which belong to the same form of life.

Thus, Wittgenstein's view of pain rejects two theories of inference traditionally popularised by the empiricists. In the first place, first-person sensation of pain is presumed to be given as pre-lingual and pre-aspectual chunks, while aspectual transitions (if there are any) are to be inferred from them. Secondly, second- and third-person cognition of pain is an inference from the behaviour of the person in question. Wittgenstein holds that experience of pain, whether it is in the second or third person, or even in the first, is aspectual. And with first-person experience of pain, one does not *infer* the aspects from pre-aspectual bits of pain sensations, just as one does not infer depth, smoothness and highlights from flat, saturated colour patches, or a particular number of marbles from corresponding bits of sensation. Secondly, seeing a person we care for writhing, groaning or bleeding, we *feel* pain, we do not *infer* it from his behaviour, and here our feeling of pain *consists in* seeing his behaviour in a particular aspect. The behaviour does not act as a signboard indicator for inferring his pain; rather, the pain and pain behaviours form an organic complex.

We are now in a better position to assess how exactly mathematical aspect-seeing contrasts with pain. Let us consider the entire range of pain expressions—the primitive ejaculations of neonates, babies and small children, those of adults exhibiting various shades of training and cultivation, with an interesting mix of verbal and non-verbal content, and finally the unexpressed pain of Stoics and super-Spartans (the latter case discussed by Hilary Putnam[7]). None of them has a pre-behavioural, pre-descriptional and unanalysable status—and in this sense all of them, in their complex configurational character, lend themselves to the phenomenon of various aspectual interplays. Further, we have seen that to expand the concept of pain in the first person or in the second or third, we need to learn to see others' behaviours and even sometimes our own under new aspects. And just as seeing a triangle in different apex–base combinations or seeing the proof of $2 + 2 = 4$ as a restructuring of one numeral into the other involves a technique or acquiring a capacity, so do the aspectual interchanges within the feelings of pain.

But to push the analogy between this aspectual character of pain and that of mathematical cognition beyond a certain point would put considerable strain on our imagination. Mathematical activities, unlike pain behaviours, are often confined to specific classes and lifestyles, or at least to specific spans or regions of life with well-defined objectives and goals, or to specific needs and interests. While learning mathematics is as much a form of living as feeling pain,[8] the operations of activating the aspectual character of pain sensations, or converting them into institutional practices, has not gained prominence in our style of living, except perhaps in specific genres of literature, psychiatry, psychoanalysis, etc. But can we effectively say that while mathematical cognition involves conscious manoeuvres to direct experience into particular channels, thus thinning down reality into a physiognomic circle, pain, on the other hand, is too real, irresistibly and frightfully real to be reduced to flat pictures and their aspectual interlocking? The rhetorical force of such comments misleads us in two directions: on the one hand, it obscures the creative and form-of-life-character of mathematics, and on the other hand it turns pain experiences back to their pre-conceptual, insular and 'private' status, a notion that Wittgenstein is at such pains to dismiss.

We may wind up this section with reminders relevant for our present purpose. Sensations are not pre-behavioural, pre-lingual or pre-aspectual grounds for inferring aspects, as the empiricists took them to be. Rather, sensations of colour (and even sometimes pain) are themselves aspectual. Aspect-perception is an unimaginably complex phenomenon which cannot be grounded on given fragments of sensation and thus be reduced to a uniform structure of inference. Sensations, inference, description, behaviour—all go to form the complex body of aspect-perception.

5. The Empiricist Game of Fragments

We know that the empiricist flair for fragmenting experience at various levels—first into discrete bits of simple sensations, then assembled into progressively complex (or rather, compound) ideas through well-demarcated stages—was primarily a reaction against rationalism, or what they thought was a dogmatic excess of reason. However, in the process, the empiricists virtually turned experience into a dogma. They strenuously attempted to break up the seamless complex of experience into impossible fragments of sensation occurring along a series of equally fragmentary moments of time. We have seen how they strived to wrench out a pure and pristine state of sensation prior to all language, description, inference and action. We have come across their hysterical insistence on a pre-lingual shudder as the ground of our verbal statement ('It makes me shudder'), and also on kinaesthetic sensation as a pre-lingual ground for describing the exact movement of our fingers. They invoked plane, coloured surfaces as pre-given grounds for inferring coloured objects—their depth and texture, glow and highlights—all their aspectual changes. They postulated pre-behavioural pain as a ground for *inferring* the position of the limb, or the degree to which it submerges other sensations, etc. When the empiricists denied universals or enduring substances (material or spiritual) behind qualities, they were consciously reacting against the grammatical invasions in philosophy. However, in splitting experience in the way they did, they themselves were befuddled by grammatical categories. Insistence on pre-lingual sensations behind all descriptions is rather like a grammatical trick to interlock the predicate with the subject, a noun with a verb, or more ironically, interlocking 2 + 2 with 4 in a single circular motion. One cannot explain the criterion of *this* sensation giving rise to *this* colour and shape, unless one *cashes out* or *describes* that sensation in precisely *those* terms.

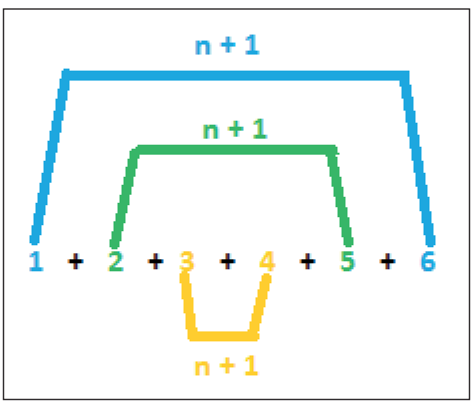
And if one wants to keep her sensations clear of this circularity, she has to phrase it as something 'special and indefinable' (*PI* p. 185). The rationalist doctrine of supersensible entities like ideal number, or the ideal triangle in the third realm, reifies these concepts, shelving them up and away from their actual language-games, their actual 'homes'.[9] But this empirical notion of a subtle and indefinable split, an extensionless needle point, between frothing and crystallisation, between the reversal of OVER and outcome of REVO, and between pre-lingual sensations and their consequent descriptions, is no less a reification of the notions of sensation and experience themselves.

Our previous talk of freezing experience into flat patterns may have suggested a primordial, free-flowing experience beyond language and activity, which was later twisted and channelised into desired patterns. But we have learnt to appreciate in the course of our discussion that this free flow of experience, in so far as it actually consists in a strenuous fragmentation, does not hold any privileged position in Wittgenstein's philosophy. The primordial bits of sensations, i.e., the flat colours, or the pre-behavioural chunks of pain, are as much *contrived* as the flat and frozen patterns of the mathematicians. When white turns a shade darker, some people say it is essentially the same colour, while some others say it is completely different. Neither of these games of identification and differentiation can be privileged; one cannot say one is *real*, and the other *constructed*.

6. Gauss's Proof about the First n Positive Integers as a Case of Aspect-Seeing

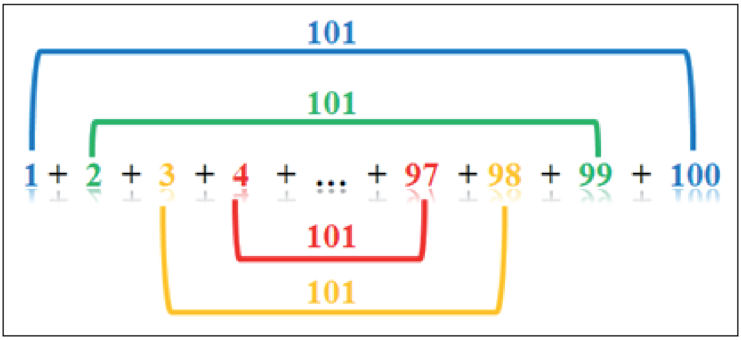
Let us consider the well-known proof of Gauss, viz., the sum of the first n positive integers as being equal to ½ n × (n + 1). To start with numbers of a smaller range, i.e., n = 6, this proof is usually laid out as in Figure 4.11.[10]

Figure 4.11



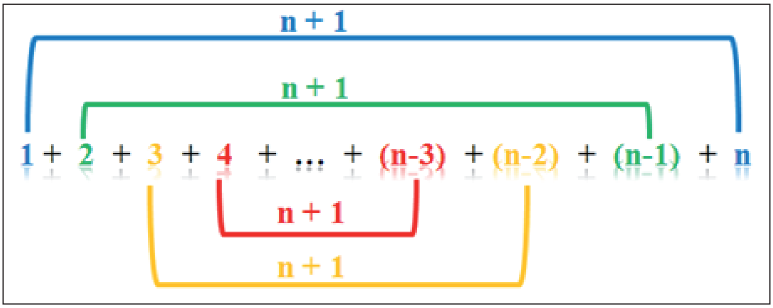
We see that we can pair the first term with the last, and the second term with the last but one, the third term with the last but two. The sums of all these pairs have to be equal, because while the sum of 6 and 1 = n + 1, the sum of the other pairs will be 2 + (n − 1), 3 + (n − 2), i.e., progressively adding 1 to the first addendum and subtracting 1 from the second, thus preserving their identity. We can go on increasing n to 100 as shown in Figure 4.12.

Figure 4.12



This process can further be recast as in Figure 4.13. As the number of pairs will be ½ n and each pair comes up to n + 1, we can say: the sum of the first n positive integers (when n is even) = n + 1 occurring ½ n times, i.e., ½ n × (n + 1).

Figure 4.13



This will also be applicable to cases where n is odd. In that case there will be an x, falling in between 1 and n, having an equal number of pairs on both sides, but itself not paired with any number. Let y be the total number of pairs in 1 + 2 + 3 + 4 + . . . n. Now, as x falls in between having no pair, having an equal number of pairs on both sides, the number on each side comes to ½ y, x = ½ (n + 1).

Thus, the sum of n positive integers (when n is odd) is:

$$= (y \times (n + 1)) + x$$
$$= (y \times (n + 1)) + \tfrac{1}{2} (n + 1)$$
$$= (y + \tfrac{1}{2}) \times (n + 1)$$

Now, y = ½ (n – 1), as n – 1 is even, because n is odd. So y + ½ = ½ n. So the sum of n positive integers (when n is odd) = ½ n × n + 1.

Now, passing over from mathematics to the philosophy of mathematics, let us not bypass the fact that the rules for generating the series of natural numbers as well as the principles of addition, all exemplifying this phenomenon of aspect-perception, are presumed in this proof. Secondly, we see that the one–one correlation between 1 + 2 + 3 + 4 + 5 + 6 and ½ of 6 × (6 + 1) has been laid out in visual dynamism (Figure 4.11), highlighting the relevant pairs in the same colour. Had the equation between these two numbers been self-explanatory or fully logicised, the pictorial operations would not have been necessary. The further question arises as to whether the picture is self-interpretive as a still shot, or whether it requires a mobilisation in a particular direction, which will have to be frozen into grammatical definitions. Does it, like the Hands–Pentacles example (chapter II), leave several options of one–one collation? Will the verbal captions of this proof be infected by several layers of indeterminacies—the ongoing series of internal ruptures, which perhaps are starting to sound hackneyed by now? We just need to recall that however endlessly tiresome these ruptures may seem, they will continue to bother us if we insist on cutting up the proof as a still shot, externalised to and based securely on its verbal explanation. The meaning of the terms or descriptions incorporated in the verbal proof cash out in and through the pictorial constructions; it is not a detached foundation lying underneath. The exercise of seeing the still shot *as* a still shot, fully equipped with the familiar rules of addition in the familiar symbolism, is itself a dynamic exercise. There are no pre-given simples either in the verbal narrative or in the proof-picture, which we arrange into complex structures, and then equate the two into self-stipulated analytic propositions. For the empiricists, what we cannot do is to start with aspectual experience as a whole, a whole that is emancipated from the constraints of the pre-given building blocks, and navigate amongst the several options of aspectual exchange, organisation and reorganisation in a radically holistic fashion. And this is exactly what Wittgenstein claims to obtain in mathematical proofs—where the verbal and non-verbal content blend into a seamless complex of mutual exchange and interplay.

More importantly, when Gauss's proof goes on to larger numbers in decimalic notation with the customary triple dots (. . .) in between (Figures 4.12 and 4.13 above), let us repeat that the decimalic notation does not condense the unwieldy non-decimalic notation (already at hand) into a manageable capsule. When logic claims to abbreviate long, unsurveyable series of numerals (consisting of a few thousand signs) into short, surveyable ones through its definitions or rules of equivalence, it is producing a proof pattern where there was none before (*RFM* II:2). The logical or pictorial stance of abbreviation by decimal notation, whether for small numbers or for larger ones, whether putting '2' for the marks '. .' or 10,000 for the series '. . . . .' (a chain of dots a mile long), gives a new criterion of identity to the old signs. It is not squeezing a spatter of granules into a capsule or trailing a shadow of the original longer notations. Rather, we now see the old set of signs in a different way, just as the light rays coming from various sources, once made to go through the optical instrument, were seen to fall into a different pattern (*RFM* III:33). When I say, this picture leads to that, or these light rays form this pattern by this rule, or Figure 4.11, the first figure of Gauss's proof, leads to Figure 4.12 and then to Figure 4.13, I am thereby making a transition from 'it is like this' or 'it will be like this,' to 'it must be like this.' And it is in this sense that we are making something into a criterion of identity, we are recasting our concept of identity (*RFM* III:29). But even though we have made this transition from one mode of concept-formation to another, the old concept is still in the background (*RFM* III:30). We make this transition from the old notation of {((1 + 1) + 1) + 1} or a chain of strokes like | | | | | | | | | | a mile long, not because we tell ourselves that the new technique will work too, but because we feel that the old notation is identical with the new decimal notation. It is *as if* the old picture—the old site or the set of signs | | | | | | | | | | | | |—is now purged of its inessential elements and the hidden identity with the new picture (i.e., the decimal notation) is brought to light.

But let us note that this phenomenon of aspect-seeing is impressively different from that of associating and reassociating two surveyable pictures (like the one involved in the proof of 2 + 2 = 4 in the Q system of Robinson arithmetic). It is rather like ending Figure 4.12 halfway after 4 with a blurred edge or triple dots, and a picture of a dark zone to follow, a blank and black space where no one steps in; and then finally another picture emerges from 97, from where we are magically transported to another space where we take up the trail we had lost before, the trail from 97 leading us to the final number 100. (*RFM* IV:27) But it is not an aspectual transition between a surveyable and an unsurveyable picture, but rather the unsurveyable pictures of Figures 4.12 and 4.13 given a new criterion of identity in terms of the surveyable one.

7. Necessity and empiricality: assigning different roles

The proposition 2 + 2 = 4 may be regarded either as a mathematical proposition or as an empirical one, and in neither case does one (or should one) delve into one's own mind to find out how exactly the knowledge came about. That is, one can regard the opposite of 2 + 2 = 4 as being inconceivable or conceivable, but in neither case does one actually track down the actual causal process through which the knowledge of such inconceivability (or conceivability) came into being. To take the parallel axiom, for instance, we have not made experiments and found out that in reality only *one* straight line through a given point fails to cut another. One does not try to experiment or experience the opposite, and we find it impossible to do so. The proposition describes a picture like that in Figure 4.14. Similarly, the proposition 2 + 2 = 1 + 1 + 1 + 1 describes a picture such as in Figure 4.15.

Figure 4.14

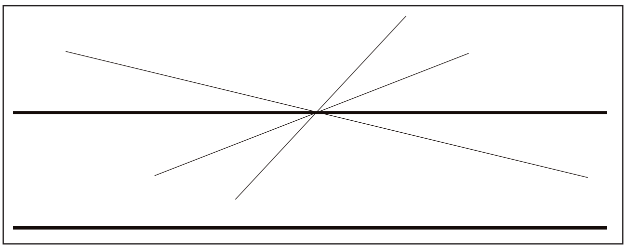
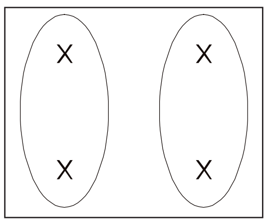


Figure 4.15



We simply find such pictures or such physiognomies acceptable. When we accept propositions on large numbers like 10,000 = 100 × 100, we do not get into an actual process of making 100 groups, and counting whether each group contains 100 items. '[W]e find it acceptable to indicate our rough knowledge of a number by rounding it off at a multiple of 10' (*RFM* III:2). Our reference to experience in all these cases is a routine affair, or a token ceremony, where we have already agreed to concur that there can be no experience of the opposite. What does it mean to say that the propositions of mathematics are self-evident? Suppose I venture to suggest, 'This is how I find it easiest to imagine.' Here imagining is not a mental process during which one usually shuts one's eyes or covers them with one's hand, and tries to form a mental picture of the axiom or its opposite (*RFM* III:1). To say that mathematical axioms are self-evident is to say that one has already chosen a definite kind of employment without realising it. It is not our finding it to be self-evident but our making the self-evidence count, that makes it into a mathematical proposition (*RFM* III:3).

On the other hand, to state that a proposition, say 2 + 2 = 4, can be imagined to be otherwise is to ascribe a different role, viz., that of an empirical proposition. It is *not* our *actually* tracking down two distinct impressions—one corresponding to 2 + 2 and the other to 4, one corresponding to frothing and the other to crystals—that constitutes the empiricality or contingency of a proposition. It is rather a commitment to the very *concept* of experience as consisting of discrete ideas; it is a conceptual obligation to postulate indefinable splits corresponding to each word of an empirical proposition.

> And don't I have to admit that sentences are often used on the border line between logic and the empirical, so that their meaning shifts back and forth and they are now expressions of norms, now treated as expressions of experience?
>
> For it is not the 'thought' (an accompanying mental phenomenon) but its use (something that surrounds it), that distinguishes the logical proposition [and we can add mathematical proposition] from the empirical one. (*ROC* III:19)

This shift of meaning is a grammatical shift. This difference between the roles assigned to necessary and empirical propositions is that between different grammatical roles assigned, for instance, to different parts of speech. 'An axiom, I should like to say, is a different part of speech.' Also, 'We give an axiom a different kind of acknowledgement from an empirical proposition. And by this I do not mean that the "mental act of acknowledgement" is a different one' (*RFM* III:5). The difference between the roles assigned to a noun and to a verb consists not in the 'thought', not in a special mental act of acknowledgement, but in their respective uses, their respective placement in a sentence, their mode of occurrence with other parts of speech, kinds of manoeuvres permitted with one and not with the other—all of which 'surround' the word.[11]

Both the rationalists and the empiricists question how mathematical knowledge is *caused*—a causal approach which is at the same time a primarily Augustinian one. Both the theories believe in a pre-established harmony between language and reality, and a transparent bridge between the two. However, reality is conceptualised differently by each party and so is the nature of ostension. For the rationalists, mathematical reality being a supersensible realm of abstract objects (number, set, perfect triangle, etc.), the bridge must be an immediate flash of rational intuition. When both language and reality are analysed down to their simplest elements, and the metaphysical harmony is brought to light, this immediate flash of intuition captures the essential nature of the object and its necessary and synthetic relations of identity with other objects (see chapter II, section 1). Since there is no such mathematical reality for the empiricist, the need to postulate a flash of intuition to represent mathematical identities does not arise. Dismissing mathematical propositions as analytic, they are more interested in showing how numerical and spatial properties of concrete objects, known in experience, are fragmentary and fallible. While the Augustinian model of isomorphic correlation motivates rationalists to postulate flashes of mathematical identities, the same model (i.e., the model of isomorphic correlation between a fragmented reality and an equally fragmentary language) propels the empiricists to postulate flashes of difference, of indefinable splits hidden behind an apparently continuous complex. These splits cut through any proposed experience of identity between numerical or spatial properties of concrete objects.

Thus this causal-cum-Augustinian model encumbers both rationalists and empiricists with the burden of tracking down a transparent flash of identity or of difference, i.e., either a flash of necessity or one of contingency. Construing Wittgenstein's view of mathematical cognition as perceiving aspects, or 'seeing aspects' as he calls it, will help us break away from such constraints and yet preserve a meaning of mathematical necessity. This shall be further clarified in the next chapter.

Notes

1. This distinction between 'causal connection' and 'connection

between patterns' will be problematised and clarified in due course.

2. I have relied on Wayne H. Stromberg, 'Wittgenstein and the Nativism–Empiricism Controversy', *Philosophy and Phenomenological Research*, vol. 41, nos 1–2, 1980, for the empiricist account of aspect-perception.
3. Ibid., pp. 127–28.
4. It must be mentioned that the empiricists feel quite embarrassed with statements on 'red', 'blue', 'yellow', etc., words which are simple and verbally indefinable by their own admission. Statements like 'Nothing can be red and blue all over' cannot be shown to be analytic in their scheme.
5. We may make an experiment of *saying* 'It's cold here' and *meaning* 'It is warm here,' but this experiment, if made at all, would consist in a different cluster of activities and not in the single acts of substituting a different set of labels in place of the former (*PI* 510). It is often claimed that since words are arbitrary signs, we can put 'a', 'b', 'c', 'd' in place of each word of the sentence 'The weather is fine.' But when we read it, we cannot connect it straightaway with the above sense. It is not because we are not used to the association between 'a' and 'the', 'b' and the 'weather', and so on, but because we do not play this language-game of seeing each word piecemeal, and as substitutable by any arbitrary sign (*PI* 508). 'Can I say "bububu" and mean "If it doesn't rain I shall go for a walk?"—It is only in a language that I mean something by something' (*PI* p. 18, footnote).
6. I am greatly indebted to C. E. M. Dunlop's 'Wittgenstein on Sensation and Seeing As', *Synthese*, vol. 60, no. 3, 1984, for the following account on pain.
7. Hilary Putnam, 'Brains and Behaviour', in John Heil (ed.), *A Philosophy of Mind: A Guide and Anthology* (Oxford: Oxford University Press, 2004).
8. The form-of-life character of mathematics will be discussed fully in the final chapter.
9. It is with reference to words like 'knowledge', 'being', 'object', 'I', etc. etc., that Wittgenstein observes that philosophers do not ever ask themselves: 'is the word ever actually used in this way in the language-game which is its original home?' (*PI* 116).
10. The following proof and diagrams are chiefly borrowed from 'Young Gauss and the Sum of the First n Positive Integers', available at: http://mathandmultimedia.com/2010/09/15/sum-first-n-positive-integers/ (accessed 16 July 2016).

11. The talk of 'assigning roles' primarily occurs with reference to actors on stage or in film, where each role obviously consists in behaviours, movements and manoeuvres with other persons or objects in the scene. A 'role' means everything that surrounds the actor, and cannot be encapsulated in a special state of his or her mind.

CHAPTER V

A Conceptual Approach to Aspect-Seeing and Mathematical Cognition

Once Wittgenstein has got rid of different forms of foundationalism in mathematics—Platonism, intuitionism and logicism—he has to be careful that mathematical cognition as aspect-seeing does not fall back into psychological foundationalism. Wittgenstein is careful to point out that, like the difference between necessary and empirical propositions, the difference between aspect-seeing and 'ordinary' (what we may call 'non-aspectual') seeing will likewise turn out to be a matter of assigning different grammatical roles. We cannot track down a special cognitive state of mind, or a distinctive ontical entity, or a transparent mental image lying beneath a changing aspect. The notion of aspect-seeing or 'seeing as' consists in its contrastive uses with regard to other related notions (like 'seeing', 'regarding as', 'interpreting', 'knowing'). We are not interested in the cause of noticing an aspect, but 'in the concept and its place among the different concepts' (*PI* p. 193). As we noted, the best way to understand what a 'conceptual' investigation is, as contrasted to causal, epistemological or ontological ones, is to note the kind of approach we commonly take to our grammatical categories, grammatical rules and moves. Teaching or learning to identify a particular part of speech, say the *noun*, in contrast to others is not to engage in a special mental state characteristically different from others. Nor does one normally suppose that there are real entities corresponding to each part of speech or each level of an adjective. The notion of a noun or of a superlative adjective gets its meaning from its uses, in the network in which it is placed with other parts of speech, with other levels of adjectives.

1. Against a Foundational Theory of Aspect-Seeing

Wittgenstein's conceptual investigation into the notion of aspect-seeing centres around the following issues:

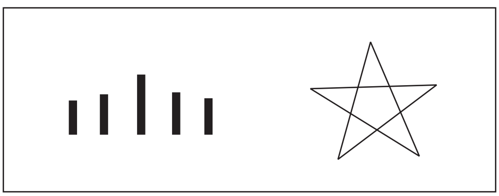
1. There is no unique state of mind (like a special impression, feeling or a transparent mental image) nor a characteristic physiological state (a typical brain pattern or a typical movement of eyes or muscles) corresponding to a new aspect.
2. The difference between seeing an object, or its colour or shape, and seeing an aspect is not ontical or psychological, but lies in different clusters of uses.
3. The difference between the causal and the reason paradigm lies in two different games.
4. 'Seeing' has no essence that clearly marks it off from cases where one can be said to go beyond purely visual (or sensual) reference.
5. 'Seeing an aspect', to repeat, gets its meaning through its uses, specially its contrastive uses with other related notions—viz., 'seeing an object or its properties', 'regarding as', 'interpreting as', 'knowing that it is supposed to be so', etc.

The first point will be clarified in two parts below, where the first part (section 1.1) is directed against the theory of a foundational mental state, and section 1.2 attacks the theory of physiological foundation. The rest of the points will be developed as extensions of this theme.

1.1 *Against Special Impression, Special State of Mind*

There is no special impression or a special state of mind behind a new aspect. We see an object in its likeness to another; we see that it has not changed and yet we see it differently. We have a new perception which is at the same time recognised as identical with the old (*PI* p. 193). Thus, 'the expression of a change of aspect is the expression of a new perception and at the same time the perception's being unchanged' (*PI* p. 196, quoted in the previous chapter). When I am simply shown a picture-rabbit and asked what it is, I say 'It is a rabbit' and not 'Now it is a rabbit.' But shown a duck-rabbit, I may say 'It is a duck-rabbit,' or I may react to the question differently, saying 'Now it is a rabbit' (or 'Now it is a duck'). 'It is a rabbit' is a report of my perception, while 'Now it is a rabbit' is an expression of a new aspect. But in none of these cases can we say that the picture is altogether different now. '[W]hat is different: my impression? my point of view?—Can I say?' (*PI* p. 195). Take the instance of a triangle ABC, where we usually report that A is the apex and BC is the base. Once we learn to see the triangle in a new aspect, we report our new perception as 'Now B is the apex and AC is the base.' Let us take our familiar figures of the 'Hands' and the 'Pentacles' (Figure 5.1).

Figure 5.1



While one person sees them as quite dissimilar, another sees them as isomorphically connected with each other. The second person sees each figure in a new aspect that the first person does not, and yet the difference between their respective perceptions does not consist in a special image or mental state enjoyed by the second person and missed by the first. Given two faces, A sees a likeness between the two, which B does not, though B can make an accurate drawing of the two faces (*PI* p. 193). Forming concepts in mathematics, say that of number five, is moving from one object to another, each time seeing it in a new aspect—from the petals of a flower, to the fingers of our palms, to the toes of an animals, to a star. Writing out a logico-mathematical proof is, as we have seen, actually drawing out a picture,[1] and understanding the proof is seeing the picture in a new aspect. What was before a mere 'wallpaper design' or an 'ornament' (*RFM* I:28) emerges as a new pattern, where the transition from each step to the next is a transition to a new aspect. Going back to Gauss's proof, for instance, suppose someone is shown the picture in Figure 5.2.[2]

Figure 5.2

1 + 2 + 3 = 6

1 + 2 + 3 + 4 = 10

1 + 2 + 3 + 4 + 5 = 15

1 + 2 + 3 + 4 + 5 + 6 = 21

1 + 2 + 3 + 4 + 5 + 6 + 7 = 28

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36

What first appeared to him as a random distribution of colours gradually recasts itself as monochromatic pairs each adding up to n + 1. Take the logicist technique of generation of natural numbers: what looked like a single unit i.e., **O** is looked at in a different aspect after it is made to move to **OO**, juxtaposed with its complementary set which was further intersected with **OO** in which **O** re-identifies itself (chapter III, section 5). None of these aspect transitions involves a special image or impression. On the whole, understanding or working out a proof, which is usually characterised as *writing* out the premises in terms of the conclusion, is virtually *drawing* out the premises pattern in the *shape* of the conclusion, i.e., seeing each pattern in a new aspect, in their new isomorphic transition to the other. And again, one who does not understand the proof, i.e., fails to *see* the proof pattern in its required aspect, is often able to produce its exact replica. Common instances (outside the technical discourse of mathematics) of producing an exact copy of a picture, and yet not being able to see it in a new aspect, are scattered throughout Wittgenstein's writings on the issue. How would seeing a face *as* smiling contrast with not being able to see it to be so? The difference would be suggested in *mimicking* the face differently, while both appreciate the same drawing as being its copy (*PI* p. 198). Someone might be able to describe an unfamiliar shape that appeared before him (say the picture of a rabbit), just as *accurately* as I do, to whom it is familiar. His description would run differently—while I say 'The animal has long ears,' he would say 'There were two long appendages' (*PI* p. 197). Take another instance, where I meet someone whom I have not seen for years. I see her clearly, but I fail to know her. Suddenly I know her, I see the old face in the altered one. I believe that I should do a different portrait of her if I could paint, different from the one I would make, had I failed to recognise her (*PI* p. 197).

Reproducing a picture in its exact facsimile is merely producing certain colours and shapes, while seeing a new aspect is recognising a particular organisation. Let us take the instance of solving a picture puzzle, i.e., seeing a tangle of branches as a human figure, or seeing a proof-picture (a design of marks and signs) as one structure isomorphically connected to another. In all these usual cases of aspect-seeing, our visual impression can be said to change in a particular sense, for now we recognise it has not only colour and shape, but, to repeat, a 'particular organisation' (*PI* p. 196). When we see the tangle of branches as a human figure, or a duck's head also as a rabbit's head, we not only see certain colours and shapes (which we give in details) but also do something else, like pointing out to pictures of different human figures, or of different rabbits of different sizes and postures. To take another instance: the picture of a schematic cube, which I know has different aspects (like that of a glass case, an open box, a wire frame, a book, etc.) (*PI* p. 193). I want to know what someone else sees (non-aspectually). I get him to draw a copy, and even a model of what he sees. But when a changing aspect is concerned, the case is altered. Now the only possible expression of our experience is what before had seemed, or even was, a useless specification once we had the copy or the model, i.e., the flat or solid colours and shapes (*PI* p. 196). If one sees the cube as a book, one has to (in the similar fashion as above) point to different books of different sizes and shapes, stand them up and then put them down, with the leafed portions showing outward, and so on. In other words, seeing an aspect involves mobilising the static picture, spreading it out in different uses and activities. And this is particularly relevant for mathematics, in seeing new mathematical aspects, like a new apex–base combination of a triangle, or seeing a proof as two patterns woven in an isomorphic correlation. One has to draw triangles of different kinds, draw the same triangle in different positions, hold the page upside down, identify the same triangle drawn in an unusual position and hidden in a jumble of different shapes. One has to draw Gauss's proof in the shape of a staircase with progressively larger numbers (see Figure 5.2), draw the lines of correlation of each pair of n + 1. Understanding a mathematical proof, i.e., seeing the design of marks as a new isomorphic pattern, is not merely reduplicating the proof-picture, but recognising the particular organisation. This would include the ability to identify the same proof when written in a different handwriting, with a different ink, different notation, a different order in which the premises may be numbered or the steps carried out, and also recognising the same rules and definitions when used in another proof. This would consist not only in galvanising the stagnant content of the picture into a physiognomic cycle, and freezing up this cycle into a definition, but also in identifying and re-identifying this cycle in the same proof with a different sensual content.

One who is obsessed with a unique image or state behind a new aspect will tend to insist on an inner picture behind the outer physical drawing. Wittgenstein observes that the duality between an inner and outer picture is as absurd as that between (ideal) numbers and (concrete) numerals (*PI* p. 196). The remark carries a few interesting suggestions. First, the impossible 'splits' in experience forged by the empiricists are placed on the same footing with the rationalist hypostatisation of abstract identities (i.e., ideal number and space), over and above the concrete particulars. Secondly, this view of mathematical cognition as seeing *new* aspects cannot be reduced to a synthetic and necessary cognition about real, abstract mathematical entities. Nor would it incur a theory of rational intuition, an immediate flash of the identical essence supposedly represented in mathematical statements. Lastly, the absurdity of an inner picture behind a new aspect and that of an ideal number behind concrete numerals, when compared together, clearly show the non-foundational and aspectual character of mathematical cognition.

The myth of a unique foundation behind a new aspect has too strong a hold, and Wittgenstein has to grapple with it on various occasions, from various perspectives, and with various examples.

It is important to note in the first place that while aspect transition involves a switchover to a new perception, the first-person report of aspect-seeing never has this *split* between *what* is seen and *what* it is seen *as*. It is always the third-person report of aspect change that exhibits this duality—'He is seeing X as Y.' When I see the duck-rabbit in a new aspect, I do not report my perception as 'I see it as a rabbit.' What we usually say is, 'It is a rabbit,' 'I see a rabbit,' 'Now I see that it is a rabbit,' or sometimes we exclaim, 'Oh! Now it's a *rabbit*!' Nevertheless, someone else could have said, 'He is seeing the figure as a picture-rabbit' (*PI* p. 195). We cannot specify what is different with a changing aspect: '. . . my impression? my point of view?—Can I say? I *describe* the alteration like a perception; quite as if the object had altered before my eyes' (*PI* p. 195).

> [T]he expression in one's voice and gestures is the same as if the object had altered and had ended by *becoming* this or that.
>
> I have a theme played to me several times and each time in a slower tempo. In the end I say '*Now* it is right,' or '*Now* at last it's a march,' 'Now at last it's a dance'—The same tone of voice expresses the dawning of an aspect. (*PI* p. 206)

Mathematical transition from the premises to the conclusion is, as we have noted, a transition to a new aspect, where the conclusion (whether in the form of a distinct image, or a distinct state of mind) never sticks out from the proof, but is rather absorbed into it. A new aspect presents a new face, a new physiognomy which is then copied, memorised and frozen (*RFM* III:47). That is, we imitate it, and then accept it without imitating it, and then we are no longer struck by the new aspect, the (new) conclusion, or the (new) 1–1 correlation. One may be said to observe the likeness between X and X's father for a few minutes, and then no longer. It can mean that after a few minutes she stopped being struck by the likeness (*PI* p. 210). What 'dawned' as a new aspect is then 'continually seen' as an aspect.[3]

That a proof presents a new physiognomy has important suggestions for us to work out. 'If someone splits up four marbles into two and two, puts them together again, splits them up again and so on, what is he shewing us? He is impressing a physiognomy, a typical alteration of physiognomy on us' (*RFM* I:78). First, almost none of the new aspects, whether mathematical or non-mathematical, whether those of a duck-rabbit or of a schematic cube (as a glass box, wire frame, book, etc., etc.), or the apex–base of a triangle, presents a concrete reality in all its dimensions and interactions. A real rabbit, i.e., a fully living organism, or a real triangular object (and not a shape or model), or real marbles with their physical and chemical properties, are usually not represented as aspects. Even when seeing an object alternately as two-dimensional or three-dimensional, the latter aspect truncates reality at its outer layer, its physiognomy. Secondly, as we have already seen, the mechanism of reducing reality to a physiognomy, or rather, a cyclic alternation of physiognomy, compares with the repetitive strategy of cinematographic picture (see chapter II). The cyclic motion of aspect-seeing debars the possibility of any putative image, or a mental or physical state, jutting out from the entire cycle.

Aspect-perception *consists in* this cyclic usage and activity; it does not imply an inner foundation lying deep underneath, be it a special image, a special psychological state or physiological occurrence, or a characteristic brain pattern. We have seen that an image has a role to play in aspect-seeing, like in seeing the aspects of a triangle (a mountain, a wedge or a triangular hole); it is 'as if an *image* comes into contact and for a time remained in contact with the visual impression' (*PI* p. 207).[4] But it is precisely this coming and passing away that gives it significance, and not a transparent foundational presence.

1.2 *Against Physiological Foundations*

How would Wittgenstein react when a special physiological state, a special nervous excitement, a new movement of eyes and muscles is offered as underlying the representation of a new aspect? The way to tackle such theories is well indicated in *PI* (p. 212), where he asks us to imagine a physiological explanation of an experience.

> When we look at a figure, our eyes scan it repeatedly, always following a particular path. The path corresponds to a particular pattern of oscillation of the eyeballs in the act of looking. It is possible to jump from one such pattern to another and for the two to alternate. (PI p. 212)

Wittgenstein cites the instance of the alternating aspects of the double cross (*PI* p. 207), to which we may add the convex–concave (*PI* p. 203), the duck-rabbit, the apex–base, $o'' + o'' = (o'')'$ and other similar instances from the mathematical proofs we have considered in the previous chapters. But one should be aware that our description of the physiology is itself a kind of seeing, and ' this can screen the old problem from view, but not solve it'. Whenever a physiological explanation is offered, '[t]he psychological concept hangs out of reach of this explanation' (*PI* p. 212).

The Gestalt psychologists[5] revolutionised the 'brick and mortar' perception of the empiricists, only to reinstate a new kind of foundation, a unique brain pattern underlying every perception, whether aspectual or non-aspectual. All the discrete stimuli, the moment they enter the brain (which in Gestalt theory is virtually a dynamic electric field), interact and fall into a pattern.[6] All brain patterns or 'organisations' have a universal character; they shape up certain stimuli into a three-dimensional figure, protruding from the rest, while the rest of the stimuli form a flat, loose and receding background. While our perception of 'this chair', or 'this table' are 'stable' organisations, aspect changes are founded on 'unstable' organisations, where the distribution of 'figure' and 'ground' alternates. The Gestalt theorists speak of various factors and forces of such organisations, of which we can mention a few: (*a*) proximity, similarity, etc., supposed to be present in the stimuli; (*b*) familiarity and attitude present in the perceiving organism; and (*c*) pregnance, good figure, which are the reinforcing factors. For Wolfgang Köhler and Kurt Koffka, every perception is founded on a unique brain pattern, and a new reorganisation of 'figure' and 'ground' underlies each representation of a new aspect. There is a one-to-one mapping between each perceived pattern and the corresponding organisation in the cerebral cortex.[7]

Wittgenstein's remarks in *PI* (p. 212, quoted above) show again a clear direction to steer away from the Gestalt theory. First of all, aspect-perceptions consist in varied kinds of activity. They do not share a universal feature of figure–ground reorganisation. Many cases of aspect-seeing, like the duck-rabbit, 'double cross', apex–base combinations of a triangle, the alternating numerical aspects we have encountered in $o'' + o'' = (o''')'$, Gauss's proof, and those between a standard and deviant aspect like 2 + 2 + 2 = 6 and 2 + 2 + 2 = 4 (see chapter IV, Figure 4.8), do not involve a three-dimensional reshuffle.

But apart from this, Wittgenstein can raise more substantial objections that would show that the Gestalt theory is mistaken in its basic principle, as well as in every detail at every juncture. While constantly drawing upon brain dynamics, brain pattern and features of the stimuli, it never addresses the question as to how the nature of the cerebral cortex itself is to be perceived, how brain patterns themselves are represented, how the 'primitive' features of the stimuli are known. To explain them by prior Gestalt is only to push the problem one step backward. If perception of motion itself is caused by an actual motion in the brain, how can one perceive the latter, as well as the common motion of the stimuli, without a prior Gestalt? Even if the Gestalt theorists pass this problem of perceiving the Gestalts themselves to the second level, they will have to face other predicaments. Since the notion of similarity always involves similarity in *perceived* qualities like shape and colour, how do the Gestalt theorists talk of similarity among certain stimuli, since stimuli are claimed to be conceptually independent of perception? Whatever might be the nature of the stimuli, they themselves cannot perceive the proximity, continuity and (particularly) the similarity amongst themselves to readily fall into the desired whole. Besides, once the Gestalt theorists take factors like 'familiarity' and 'attitude' as lying in the organism, and 'closure' and 'pregnance' as reinforcing factors, they are also obliged to let in the factors of past experience and different socio-cultural settings that would familiarise their members to certain objects, alienating them from others. Thus, the supposedly universal principles of 'familiarity', 'set' or 'closure' may generate completely different Gestalts for people of different societies. And by the same force of logic, we should also admit certain societies where people are constantly trained to see certain colours like red, blue and white together; their nerve currents are constantly trained to put certain stimuli together and put all other colour in a uniform background. The phenomenon of seeing light blue and dark blue as completely different can also be explained in terms of repeated rehearsals compelling the subjects to assimilate light blue to cream and white on the one hand, and dark blue to black on the other. To come to mathematical instances, an environmental setting where things regularly fuse in their edges, or coalesce into one, may constrain people to perceive deviant numerical aspects in deviant Gestalts, like seeing 2 + 2 as 3 or 2 + 2 as 1.

It is quite doubtful whether the Gestalt theorists would be interested in grafting their theory of aspect-perception onto mathematical knowledge. And if they do, they would again 'screen the old problem from view, but not solve it'.(*PI* p 212, quoted earlier) We know that the Gestalt theorists speak of an equal *number* of stimuli as forming compact figures, they also speak of a *one–one* mapping between patterns that we see or hear and the corresponding patterns in the brain. When we hear a sound grow softer and die away, the cortical process has a pattern of decreasing intensity. When we see a figure standing out from its background, some part of the occipital lobe also stands out from its surroundings. Every new mathematical aspect, say seeing a figure as either 2 + 2 or as 4, would be grounded on a unique cortical pattern. In the first case, there would be two parts of the occipital lobe protruding as two compact areas; in each area two units would be identifiable. In the second case *four* units of the lobe would be jutting out, while the two previous figures flatten out into the background. Thus, given this construal, the foundations of mathematics would be shifted to physiology. The *number* of stimuli, their *spatial* properties (like symmetry, continuity and proximity), *units* of the cortex, protrusion and flattening of grey matter would be posed as self-interpretive foundations of mathematics, taking the place of Platonic entities, logical symbolism, or a priori forms of intuition!

It emerges that in the process of displacing the pre-given bits of sense-data, the Gestalt theorists bring back 'different organisations', different 'visual objects', reifying them into private mental entities with ineffable features. For Wittgenstein on the other hand, the change in the organisation is a change in activity that cannot be specified in terms of a given datum, whether a special visual impression, a specific angle or a different pattern in the brain. Reports of aspect dawning are not descriptions of inner experiences accompanying ordinary perceptions. They are avowals, spontaneous reactions and changes in attitudes, which are fully exhausted in how we react, what we do with it.[8]

1.3 *Seeing an Object and Seeing an Aspect: Different Clusters of Uses*

The difference between seeing an object or its colour or shape, and seeing an aspect, is not ontical or psychological, but lies in different clusters of uses. Our previous statements on change of organisation in spite of the same colour and shape might involve some misleading suggestions. We have seen in the previous chapter that colour (and shape) does not have a primordial and unchangeable status, lying beyond language and behaviour. There are no flat colour patches that commonly underlie all change of aspects, from the duck to the rabbit, from one apex–base combination to the other, from an array of marks to the isomorphic configuration of a proof. Two things need to be noted in this connection. First, a person who does not appreciate the changes of aspect mentioned above may however see an aspectual transition in colour and shape. That is, a person who sees an array of signs simply as a design and not also as an isomorphic proof pattern, may see the design as undergoing an aspectual transition in its colour and shape, just as the same piece of paper alternately looks grey and white, or the same array of flat colour strips assumes different aspects (chapter IV). Even if we produce an exact facsimile of the pictures mentioned above (viz., of the duck-rabbit, or of the proof pattern), this person may be unable to see the same colour and shape reduplicating in both the pictures. Secondly, as we have already noted, the person who *does* appreciate the aspect changes mentioned above (like the duck-rabbit, apex–base combinations) is no doubt able to see the same colour and shape repeating itself in spite of the change of aspects. But seeing the common colour and shape underlying aspectual changes does not spill over her uses and behaviours—like her producing a second copy of the pictures, or pointing to or outlining the 'common' colour and shape. The common colour patches and shapes do not have a primordial givenness that is typically absent in the representation of changing aspects. To put it more accurately, the repetitive identity of colour and shape is a constructed fixity against which another construction, that of aspectual transition between two objects, becomes intelligible.

Many other observations tend to show how mathematical cognition as a kind of aspect-seeing detracts from ordinary seeing or the usual kind of sense-impressions.

> The colour of the visual impression corresponds to the colour of the object . . . the shape of the visual impression to the shape of the object . . . but what I perceive in the dawning of an aspect is not a property of the object, but an internal relation between it and other objects. (PI p. 212)

The blotting paper looks pink to us and is pink, or it looks rectangular to us and is rectangular. It is a different situation when we look at a face whose similarity with another face makes a striking impression on us and then passes away. Or when children play a game with a chest, seeing it as a house, interpreting it as a house in every detail (*PI* p. 206). Let us also remind ourselves of Figure 4.3 in the previous chapter, which can be seen as a triangular hole, a mountain, a wedge, or an upturned object. In none of these cases can we say that what we see is in the object or in the picture. What we see as an aspect is an internal relation of this picture with other pictures, or other objects. Seeing the same object under varying aspects, describing each with a different phrase would, in the Augustinian model, bring in such absurdities as squeezing all these aspects into a single form in which they do not fit. The model of one–one correlation between a word and its designation would imply that a triangular hole, a mountain and a wedge were all somehow squeezed into the figure. 'But no squeezing, no forcing took place here.' 'When it looks as if there were no room for such a form between other ones you have to look for it in another dimension.' If there is no room in the object (here the figure), there is room in another dimension of the object, in its internal relations with other objects (*PI* p. 200).

Wittgenstein says that in this sense too, there is no room for imaginary numbers like $\sqrt{-1}$ in the continuum of real numbers. To *see* imaginary numbers *as* similar to real numbers, it is not enough to have a 'look' at the calculation, but we should rather get down to the actual applications. And then the concept of imaginary numbers finds a different place, one which so to speak one never dreamed of. Similarly, to see the triangle as an upturned object, one has to mobilise the figure, i.e., draw a series of pictures, showing the intermediate position through which the original picture gets inverted, and also draw several upturned figures of different colours, shapes and sizes. It is only then that the new aspect dawns upon us (*PI* p. 201).

Seeing an aspect is learning an activity, mastering a technique. There is, to repeat, no new sense-impression or a new state of cognition corresponding to the representation of a new aspect. Now, we have already noted that seeing an object or its properties (like colour or shape) does not involve learning a 'technique' in the same sense as that involved in seeing an aspect. This however does not imply that, as opposed to aspectual relations, objects and their properties are non-relational chunks of entities, waiting to be represented by a single state of cognition (a single sense-impression or a single state of inference). Experiences of full-fledged objects, like a full duck or a rabbit, marbles and their red colour, as we know, are shaped by a transitional flow of similarity relations extended over an indefinite span of time. Of course, one can ostend to two different objects—a duck and a rabbit—in two different acts of ostension, while one cannot in the same way ostend to the duck aspect as distinguished from the rabbit aspect. One can be said to have two visual impressions, one of a duck and another of a rabbit, while one cannot be said to have two characteristic visual impressions, each corresponding to each aspect.

But at the same time, we must note that to point to an object, say a duck, or to speak of a characteristic sense-impression of the duck, a lot of stage-setting has to be accomplished. One must have seen a considerable variety of birds and animals, real ones as well as pictures, read about their behaviours or food habits, and above all, must have been fairly exposed to the institution of pointing, uttering ostensive phrases and responding to them. It is only in this richly variegated background that one can engage in two acts of ostension, one for the duck and another for the rabbit, or speak of their characteristic visual impressions. Thus, ordinary non-aspectual seeing may not consist in a technique or capacity like the one involved in mathematical (or other non-mathematical) instances of aspect-seeing, but neither is it a transparent act of picking out an external object lying out there. There is nothing like a characteristic visual impression of 'blue', recurring every time we utter the word (see chapter I, section 5, 'Opacity of Acts of Ostension'). Meaning and understanding one property—say the colour of an object rather than its shape—does not consist in a self-interpretive and unique sense-impression, but in a different cluster of uses and behaviours. Of course, when I claim to see a new aspect or ask somebody to see it—say a duck's head also as a rabbit's head—I not only describe the colour and shape in detail, but also show her different pictures of rabbits, in different postures and positions, train her to see them from different angles. While I need not do this simply to point to a rabbit, a massive cluster of activities is present in the background, which are not consciously brought to the surface.

Aspect-seeing involves finding a new physiognomy and re-identifying the old physiognomy with the new. On the other hand, seeing an 'object' or its 'properties', as the very terms suggest, involves a fuller representation of the object. But it would be absurd to suggest that there is a specific range of entities allotted to each kind of seeing—a non-relational full chunk of objects allotted for object-seeing, and truncated surface features or physiognomic relations reserved for aspect-seeing. In a way, aspect-seeing for Wittgenstein is integral to all perceptions. It will perhaps be more accurate to say that objects and their properties consist in a different level of relations, which can be strategically set against aspectual or physiognomic relations encountered in the stock instances of aspect-seeing. Perception of objects and properties may be characterised as 'continuous seeing of an aspect' as contrasted to 'dawning of an aspect'. While the former has no duration, and is liable to errors, the latter is clockable and unavailable to corrections. For the former kind, Wittgenstein gives examples of the perception of pictures, where there is no need for a contrast between seeing the picture as blotches of colours and shapes and seeing it as going out of itself to represent a three-dimensional reality.

Going by the trend of Wittgenstein's writing, we can perhaps extend his notion of 'continuous seeing of an aspect' to the so-called perception of objects and properties as well. There are circumstances where a person can oscillate between seeing the picture as flat, non-representational patches of colour, or as standing for an external object; between hearing sounds as *mere* sounds or as words; between seeing certain movements of limbs or seeing it *as* human behaviours. And what one person sees as an aspect can be seen to be an object by another, and vice versa. While it is true that a person who already knows a thing to be X cannot meaningfully be said to *see* it *as* X (e.g., one cannot see a fork and spoon *as* cutlery, or a leaf *as* green [*PI* p. 195], this does not mean that X is debarred from aspectual representation. A community where dried leaves play an expansive role in the people's lives (and fresh green leaves do not), may suddenly come to see the green leaves *as* green, in a new aspect. Again, when children play the game of seeing the chest as a house, and of seeing cutlery as human figures, it is meaningful to train them to see the cutlery *as* cutlery and the chest *as* a chest, and above all to see a *spade* as a *spade*. All this suggests that the difference between aspect-seeing and object-seeing lies in different clusters of uses or different styles of activities.

1.4 *The Conceptual Distinction between the Causal Paradigm and the Reason Paradigm*

Before we move further, we would do well to elucidate Wittgenstein's notion of cause to bring out the exact significance of the practice of freezing causes into reasons.[9] Like all other words in general, causal expressions too are not labels to be clamped onto chunks of events, one preceding the other and tied in a real bond. Like other cases, causal language-games too are sophisticated extensions of our primitive behaviours.

Wittgenstein mentions some prototypical occasions from which our causal expressions take off—collision of billiard balls, pulling a string (traction), clockworks which combine both collisions and tractions, human reactions on being hit physically or emotionally, and lastly, occasions of Humean succession. It is important to realise that when we see physical collisions and tractions, or react to being hit, or hold others responsible, such events do not contain the real essence of causation which we passively represent in our cognition, to be further expressible in language and to be followed up by suitable actions. On the contrary, all these expressions like 'collision', 'impact', 'generation', 'action and reaction', 'tit for tat', 'you hit me so I hit back,' 'so', 'therefore', etc., are shaped by our primitive and spontaneous actions. Causal propositions are as much paradigms of description as are the 'reason' paradigms with respect to mathematics or logic. One however must be careful not to construe either of these paradigms as an a priori human category schematising the raw, uninterpreted manifold in the Kantian fashion.

Now, while causal propositions are grammatical paradigms of linking things together, unlike the case of the reason paradigm, the following gaps in the link are built into the causal paradigm itself. Firstly, there must be an epistemological uncertainty in knowing one or more links in the causal mechanism—that we might *not* know the cause of an effect or the effect of a cause is a part of the paradigm. Secondly, there may be an uncaused cause; thirdly, a cause might not be necessitating, i.e., the same causes may produce different effects, whereas the same effects may also be produced by different causes. Wittgenstein's example of two identical seeds A and B giving rise to different plants[10] is a typical example of the first option of the third feature of the causal paradigm. Lastly, the chain of causes goes on ad infinitum, but reasons ultimately peter out. All these features pertain to the mutual externality of cause and effect which is built into the paradigm. The cause and the effect do not constitute each other's identity: one cannot read the cause into the effect and vice versa. Interestingly this anomalous behaviour of causation is sought to be accommodated in the paradigm, not by forcibly packing a hidden, unexplored difference in the apparently identical seeds, but by stretching out their differences in their respective histories, i.e., in stretching out the identities of seeds A and B in their being produced respectively from A-type plant and B-type plant. So the grammar of causal expressions is not to create a path, not to coalesce the cause and the effect together. A cause moves to the effect, but the effect does not move back to the cause. When a cause does not produce its usual effect or different causes produce the same effect, we take it as a digestible shock. But when the sphere of reason shows up these exceptions and anomalies, when 2 + 2 sometimes leads to 4 and sometimes to 6, we do not stretch out these differences in the histories of 2 + 2; we settle the anomaly within the ahistorical path of reason. We either say 'I have miscalculated,' or 'There was no 2 + 2 in the first place,' or 'The ideal 4 units are hidden there beyond the empirical process.'

It emerges that when we say that mathematics turns causation or causal experiments into reasoned definitions, or a 'real process' into a flat physiognomic cycle, we must be wary of the misleading suggestion. We must not think that while causation is grounded on full-blooded reality constrained by its given richness, mathematics only turns this real process into two-dimensional fragments. In point of fact, both causal language and reason -language are paradigm of description. The openness and intractability of the causation is very much a part of the contrived paradigm, as is the feature of inviolability with respect to the 'rational' game of mathematics.

1.5 *Against an Exclusive Essence of Seeing or Sensing*

Seeing has no essence that clearly marks it off from cases where one goes beyond purely visual (or sensual) reference. Seeing or sensing does not have an exclusive essence that marks it off from aspect-perception, or what empiricists call 'inference' or 'interpretation', or from other psychological concepts like 'knowing', 'understanding', 'recalling', etc. There are no pre-lingual, pre-aspectual blocks of stimuli that underlie our description and inference of objects and our subsequent behaviour. Rather, the sense stimuli are integrated into a complex and continuous phenomenon. What we have seen in the previous chapter is that our sensations themselves (including those of colour and pain) can be said to be aspectual representation.

One usually demands a criterion for deciding whether a representation is a purely visual or a purely sensual one. Can the 'representation of what is seen' be placed as the required criterion? Wittgenstein says: 'The concept of a representation of what is seen, like that of a copy, is very elastic, and so *together with it* is the concept of what is seen' (*PI* p. 198; italics in original).

How does one say that human beings *see* three-dimensionally? Suppose we ask someone about the lie of the land over there of which he has a view. We ask him, 'Is not it like this?' and show him with our hands. He answers in the affirmative, he does not give any reasons for his surmise, no criterion of seeing. He simply says, 'It is not misty, I see it quite clearly' (*PI* p. 198).

One is apt to say that what one really sees must be produced in her by the influence of the object. And, Wittgenstein says, 'Then what is produced in me is a sort of copy, something that in its turn can be looked at, can be before one, almost something like a *materialization*' (*PI* p. 199). And since materialisation is something spatial, what is seen—that is, what materialises after the real object so to speak—must be describable in purely spatial terms. One can see a face smiling, but on this account one cannot see it as friendly, for while the former can be described in spatial terms, the latter cannot. This puts undue restrictions on the concept of seeing (*PI* p. 199).

Thus, the concept of 'seeing' makes a 'tangled impression' precisely because it *is* tangled. 'I look at the landscape, my gaze ranges over it, I see all sorts of distinct and indistinct movements; *this* impresses itself sharply on me, *that* is quite hazy. After all, how completely ragged what we see can appear' (*PI* p. 200). What is *seen*, i.e., the object of seeing, does not have a smooth, minimal essence, and the description of what is seen consists in these ragged bits and pieces, laying one bit of fibre on another. 'There is not *one genuine* proper case of such description—the rest being just vague, something which awaits clarification,' accidental properties of seeing 'which must just be swept aside as rubbish' (*PI* p. 200).

Here we are in 'an enormous danger of wanting to make fine distinctions', the danger of reifying language into a 'use' less circle. One tries to define a material object as 'what is really seen', and on the other hand, tries to define the concept of seeing as that which picks out a pure material object, an unrelated core, over and above aspectual transition (*PI* p. 200).

1.6 *Seeing an Aspect: Its Contrastive Uses*

'Seeing an aspect' gets Its meaning through uses, especially through contrastive uses against other related notions of interpreting, inferring, understanding, etc.'Seeing', as well as 'seeing as', does not consist in a privileged encounter with an immaculately given object, property or relation. Characterising a representation as 'seeing', 'inferring' or 'interpreting' consists in its surrounding uses and behaviours, just as classifying a word as 'verb' would comprise its placement, its mode of occurrence, its contrast with other parts of speech. What does it mean to say that I see the sphere floating in the picture (and not merely lying flat)? The question is how *seeing* the sphere as floating contrasts with merely *understanding* the picture in this way, or knowing what it is *supposed* to be. (How does a verb contrast with a noun, adverb, pronoun or an interjection?) Seeing the sphere to be floating is expressed by 'The sphere seems to float,' 'You see it floating,' or again, in a special tone of voice, 'It floats' (*PI* p. 201). Putting a word in a particular grammatical category likewise involves different clusters of expression.

Wittgenstein cites many other examples on various occasions to clarify his point. ‘When I see the picture of a galloping horse—do I merely *know* that this is the kind of movement meant? Is it a superstition to think that I *see* the horse galloping in the picture?’ And to say that it is a kind of seeing, one does not have to invoke a corresponding physiology, one does not have to say that one’s visual impression gallops too (*PI* p. 202).

To take another instance, when we see a silhouetted picture of an animal transfixed by an arrow, we ask, ‘Do you genuinely *see* the arrow?’ The question is akin to ‘Is it (genuinely) a noun?’ ‘Is it (genuinely) a superlative degree of adjective?’ The question makes sense in the background of its grammatical contrasts: the other parts of speech, the other levels of adjective. The question whether one genuinely *sees* the arrow also brings with it its grammatical contrasts. Now the question has to be analysed as, ‘Do you see the arrow, or you merely *know* that these two bits are *supposed* to represent parts of an arrow?’ (*PI* p. 203).

‘Our problem is not a causal but a conceptual one’ (*PI* p. 203). What decides the particular grammatical category of a concept (whether it is seeing, or knowing or inferring) is the kind of uses, the surroundings in which it is placed, and particularly the style of representation. What decides that I see the animal as being transfixed by an arrow, or that I see the two hexagons as interpenetrating?

> If [these pictures] . . . were shewn to me just for a moment and then I had to describe it, *that* would be my description; if I had to draw it I should certainly produce a very faulty copy, but it would shew some sort of animal transfixed by an arrow, or two hexagons interpenetrating. That is to say: there are certain mistakes that I should *not* make. (*PI* pp. 203–4)

I would not, for instance, forget to draw the arrow in the first place, or place the arrow behind, or in front of the animal’s body. Such variations are likely to occur when I merely look upon the animal as *supposed* to be transfixed by an arrow, or read a picture as a blueprint or a working drawing, and not *see* it as the picture of the object depicted (*PI* p. 204). The general reaction one gets by showing these pictures are, ‘I saw it at once as two hexagons. And that's the *whole of what* I saw' (*PI* p. 204). One does not treat it as one among the several possibilities. The difference between seeing and supposing, or between seeing and reading the picture like a blueprint, would not consist in bumping against a definite pre-lingual entity in one case, absent in the other. It would consist in a difference of 'fine shades of behaviour', which have 'important consequences' (*PI* p. 204). For instance, the differences between recognising a face in a crowd (i.e., seeing it as that of one's old friend) and not recognising it, imply painting a different portrait of one's friend (if he *could* draw). When one sees a picture-face as smiling, and another person does not see it as such, they would be mimicking it differently (*PI* pp. 197, 198; also mentioned earlier). Suppose again we have a musical theme played to us several times, each time in a slower tempo. We *sense* it, we *hear* it each time in a new aspect which is expressed by saying, 'Now it is right,' 'Now at last it is a march,' 'Now at last it's a dance.' What would be the difference between actually hearing (sensing) the music as a march, and merely knowing that it is supposed to be so? It would be using a different tone of voice, or more importantly, whistling it in the correct expression. All these are again instances of fine shades of behaviour (*PI* pp. 206, 207).

To consider further instances, like seeing a schematic cube as three-dimensional, and merely reading it as a working drawing. Or when looking at a working drawing in descriptive geometry, I say, 'I know that this line appears again, but I can't *see* it like that' (PI p. 203). Parallely, I may know that the old formula or the old definition that I had learnt appears again in this proof, but I can't *see* it like that. What is the difference between *seeing* the formula and merely knowing that it is *supposed* to be there? It involves 'familiarity', 'knowing one's way about', certain gestures of confidence with which one performs the calculations (*PI* p. 203). Mathematical cognition, or mathematical aspect-seeing, would thus consist in these 'fine shades of behaviour'.

Thus it is possible to characterise a perception as both 'seeing' and 'not-seeing'. It is possible to give both a 'conceptual justification' (*PI* p. 203). And to insist that something is genuinely a case of seeing or not, one needs simply to specify certain language-games, behaviours and reactions. To invoke concepts like absolute velocity, absolute spatial boundary or absolute identity, one simply needs to specify certain applications, construct certain pictures. When one constructs an outer space in a computer, designs different planets and stars moving in different orbits, shows how each planet calculates the velocity of the other from its own relative standpoint, one can even contrive a supreme and static planet at the end of this space, calculating real, i.e., absolute velocities of all other bodies (chapter II, section 2.3). All this contrivance is a way to make the talk of absolute velocity meaningful. To talk meaningfully of absolute measurement or absolute spatial identity is also to specify a motley of contrastive uses, a series of measuring devices with different levels of accuracy, or a juxtaposition of different objects (like apples, jelly, mercury, oil, water, gas, etc.) with contrastive boundaries and stability. Similarly, to speak of a perception as genuinely a case of seeing (or sensing) is to set it against other related notions of 'knowing what it is supposed to be', 'reading it as a blueprint', 'taking it as a working drawing', 'interpreting it to be so and so', and to spread out all these notions in finely distinguished shades of behaviour.

It is in a quite similar fashion that the talk of mathematical rules and formulae as 'necessarily determining unique conclusions' has to be cashed out in terms of contrastive uses. How is the expression 'The steps are determined by the formula . . .' used?

> We may refer perhaps to the fact that people are brought by their education (training) so to use the formula $y = x^2$, that they all work out the same value for y when they substitute the same number for x. Or we may say: 'These people are so trained that they all take the same step at the same point when they receive the order 'add 3'. . . . for these people the order 'add 3' completely determines every step from one number to the next. (*In contrast with* other people who do not know what they are to do on receiving this order, or who react to it with perfect certainty, but each one in a different way.) (*RFM* I:1; italics mine)

It must be added that what constitutes the same number for x, or the same value for y, or taking the same step at the same point, consists in the actual uses, the actual moves made, and not any pre-applicational meaning of *sameness*. One may work out a 'different' value of y, one may take a 'different' step and yet claim to be 'doing the same' (*PI* 185). However, the fact that we can and do set certain instances as doing the 'same' as against those that are 'different' makes these words usable, just as the fact that everything in this world does not melt, pop out, or double up erratically makes the proposition '$2 + 2 = 4$' usable. The meanings of expressions like 'a formula determining every step' are to be made usable through contrastive applications.

More interesting employments of such expressions are worked out by Wittgenstein. We can contrast such formulae which 'uniquely and necessarily determine a number y for a given value of x' with formulae of another kind, those that 'do not determine the number y for a given value of x'. $y = x^2 + 1$ would be of the first kind, while $y = x^2 \pm 1$, $y = x^2 + z$ would be of the second kind.[11] One might put such expressions in the form of a question: 'Does the formula $y = x^2$ determine y for a given value of x?' and address it to a pupil in order to test whether she understands the use of the word 'to determine'. Or it might be posed as a mathematical problem to work out whether there was only one variable on the right-hand side of the formula, e.g., in the case $y = (x^2 + z)^2 - z(2x^2 + z)$ (*RFM* I:1).

2. Novelty and Necessity in Mathematical Cognition

The remodelled notion of seeing, i.e., 'seeing-as', enables Wittgenstein to give a non-foundational account of mathematical necessity, freed from the Platonic theory of intuition, the ontology of the third realm, or the entire Kantian package. It embodies a notion of 'novelty' and 'necessity' of mathematical knowledge, a notion that is remarkably light, that has shirked off its classical burdens. To take the question of novelty in the first place: Plato and the traditional rationalists invoked a special state of knowledge (an instantaneous flash of intuition) that reveals each mathematical entity (like set, triangle, circle) in all its necessary connections with other such entities. Representation of novelty thus requires a novel state of cognition, distinct in its quality and content from the lower states, those that can at most represent numerical and spatial relations between empirical objects, and not mathematical properties and relations.

As Kant came up with his theory of analytic/synthetic dichotomy, the theory of mathematical novelty turned out to be more complicated. We know that for Kant, by adding homogeneous units, say 7 and 5 dots, one *after* the other, we generate the pure and a priori form of time itself. And we create the same form of time whether we just take 7 units and 5 units and know that they have to be added together, or whether we have *actually* added them together and arrived at 12 units. However, the concept of 7 + 5 (i.e., 7 units and 5 units are to be added together) is different from the concept of '12'. Kant suggests that the concepts '7 + 5' and '12' are denotationally the same (for they refer to the same groups of objects underlying which is the same form of time), but connotationally different. And it is only the connotation that determines novelty or syntheticity.[12] It is again a novel and unique state of cognition (of 12 units) in which this new connotation, or new mode of presentation, is represented.

We know that while the empiricists do not acknowledge any novelty in mathematical cognition, they surely do so with their numerical and spatial relations between empirical objects. And here, as already explained, they insist on a distinct sensation, or a distinct state of inference, underlying the representation of a new aspect. Aspect transition in the Gestalt theory too would involve a distinct brain pattern.

For Wittgenstein, on the other hand, transition to a new aspect is a new cluster of uses, a new style of representation. There is no new state of mind, or a new image or impression that occurs singly, solitarily, at a well-defined point of time, which can be singled out as the unique and self-interpretive ground of the new aspect. Nor can one point to a new point of view, a new angle of vision, different from the old, which would *by itself* explain the new aspectual representation. What happens when one fails to see the new apex–base combination of the triangle or the new isomorphic aspect of the old array of signs? While in all these cases the person can reduplicate the inert colour and shape, he cannot mobilise it in a new 'organisation'. Like he cannot draw the triangle in different positions, or recognise it when hidden in an unusual posture in a jumble of shapes. He cannot copy out the proof in a different handwriting, different symbolism, change the order of steps, employ some of the premises and rules in another proof and so on. (This has been explained in section 1.1 of this chapter.) And it is quite important to appreciate at this juncture that whichever new image or viewpoint one may try to serve up as an underlying foundation of the new aspect, and yet isolated from it, has actually to be incorporated in the new cluster of uses, in a new style of organisation. Talking about the various aspects of a triangle (a triangular hole, a solid, a half-parallelogram, etc.), Wittgenstein remarks, 'You can think now of *this*, now of *this* as you look at it, can regard now as *this* now as *this*, and then you will see it now *this* way, now *this*. *What* way? There is no further qualification' (*PI* p. 200). In other words, there are no *two* states of cognition—one state of *thinking* or *regarding* as this, which justifies or underlies another state of *seeing* it as this and this way. Any such purported duality has to be dispersed into a continuous cluster of uses.

It is more important to sum up the non-foundational character of mathematical *necessity*, already indicated in our foregoing account. We have noted that the first-person report of aspect-seeing is never of the form 'I see x as y,' i.e., it never exhibits the split between *what* is seen, and what it is seen *as*. The inability to show a split, a different image, or a different viewpoint is entrenched in one's form of life—a subject we shall dwell upon in the final chapter. Let us however repeat for the present, that this inability is non-foundational, or grammatical—a notion that will purge necessity of its traditional loads and liabilities. Firstly, this inability is not really a failed attempt to conjure an image of '4' as distinct from 2 + 2. Secondly, it is exactly the same kind of inability to conceive the opposite of necessary proposition, i.e., the inability to see 2 + 2 as 5. Likewise, it is not an actually failed attempt to construct a sentence, say with an interjection qualifying an adjective, that lies behind grammatical correctness or otherwise. We do not really try and find out that we are unable to use the interjection in such and such way; rather, we make that inability *count*, give it a special role to play, viz., that of fixing it at the beginning of the sentence, which would keep other parts of speech in their proper placement. Similarly, the professed inconceivability of 2 + 2 = 5 or of a split between 2 + 2 and 4 is a ploy to mark some propositions as incontrovertible in contrast to others.

Necessity and contingency are grammatical correlates, and each notion consists in the respective inability to split or to merge two concepts, and both these inabilities get exhausted in different clusters of applications. To consider a situation where one is shown a milk-white colour slide and a greyish one immediately after it. One person claims the second shade as essentially the same as the first, for he sees the second shade as a changing aspect of the first. He would express his cognition as 'Oh now it is dark' or 'Now it is greyish.' A second person says that it is completely different and agrees to *interpret* the second shade as the *same* as the first one, while conceding the possibility of its being false, or there being other modes of interpretation. A third person treats the first shade as the *working drawing* of the second one, while a fourth one sees that the first *causally* transforms into the second. Now when instructed to bring objects of the same shade as the second slide, the first person would carry out the instructions differently from the others. While he would tend to bring something nearer to the milk-white shade, the other perceivers would prefer objects of a greyish colour. Asked to try out a variation of the second shade, the first person would have a tendency to revert to the original white shade, while the others may mix different colours to make it greyish pink, or greyish orange, greyish blue. The first person does not enjoy a pre-linguistic flash of identity (between the two colours), which implies that the other perceivers do not enjoy any such flash of difference between two colours, each flat, frozen and saturated. The necessity and contingency of their respective cognitions consist in different styles of usage.

One sees the two familiar figures—the 'Hands' and the 'Pentacles', or o″ + o″ and o″″—as related by a unique isomorphism, another sees a unique non-isomorphism, still another may perceive various contingent modes of correlation. It is absurd to postulate a characteristic ground behind each of these cognitions, a distinctive flash of identity or of difference, as the case requires.

The novelty and necessity of mathematical cognition consists in leading one experience to a new one and letting the new experience revert back to the old. In this way, the old experience is seen under a new criterion of identity. The rays of light coming from different sources are made to fall into a pattern, only to revert back to the original rays. Counting 2 and 2 marbles that leads to the experience of 4 marbles is again led back to the experience of 2 and 2. The most powerful analogy that Wittgenstein uses in this connection is that of experiencing white light dispersing into seven colours when passed through a prism, and reverting back to the original white light when passed through a second prism placed in an inverted position against the first (*RFM* III:42). With the two crossing prisms we do not actually *see* either the dispersal or the reversal, we only see the white light going through the first and passing out through the second. We do not perceive the split between the operation and the result, i.e., the dispersal and the reversal, and yet in a sense we *see* it, we see white light as necessarily splitting into seven colours and reverting back to itself, we *see* white colour as identical with its seven components and vice versa. The substratum of this experience—i.e., experiencing light in the aspect of its seven constituent colours—is 'the mastery of a technique' (*PI* p. 208). 'It is only if someone *can do*, has learnt, is master of, such-and-such, that it makes sense to say he has had *this* experience' (*PI* p. 209). This experience is grounded on the repetitive technique of passing white light through one prism and then through crossing prisms.

All mathematical relations are phenomena of dispersal and reversal, and it is in this sense they are both synthetic and a priori. Wittgenstein holds that 'the synthetic character of the propositions of mathematics appears most obviously in the unpredictable occurrence of the prime number' (*RFM* III:42). But the way they are synthesised in experience makes them a priori. In every occurrence of prime numbers, the concept of a prime (i.e., divisibility only by 1 and by itself) gets dispersed and reverted. Every prime number, like 1, 2, 3, 7, 11, etc., consists of equivalence relations (between itself and the multiple of 1 and itself, like 7 = 7 × 1), and thus in this sense every prime number splits itself and reverts back to itself.

> [Mathematical propositions] being synthetic (in this sense) does not make them any the less a priori. They could be said . . . not to be got out of their concepts by some kind of analysis, but really to determine a concept by synthesis, e.g. as crossing prisms can be made to determine a body. (*RFM* III:42)

Mathematical aspect-seeing is not a closure of experience, but creating a cycle of experience, where one experience moves to another and then reverts back to the old.

How does this syntheticity and a-priority of mathematics differ from Kant's? For Kant, all men in all their mathematical calculations operate with the unique form of space and time. With Wittgenstein, on the other hand, whether it is moving from one number to the successive ones, converting non-decimalic notation to the decimalic one, or moving from one proof to the next—are all moving through different aspectual transitions—different cycles of experience, through different but related techniques.

Inability to make aspectual transitions, or what Wittgenstein terms 'aspect-blindness' (in mathematics as well as in other areas), 'would be *akin* to the lack of a "musical ear"' (*PI* p. 214). Such a person cannot hear the same theme in a different tempo, or in different scales, she cannot hear it now as a march, or now as a dance (*PI* p. 206). She cannot *see* an isomorphic connection between the pattern of drum beats and that of a dancer's footwork. She cannot see the aspects of the double cross (Figure 4.5, chapter IV) change from one to the other, or see the schematic cube as a cube. However, asked to show a figure containing a black cross from other such figures, she should be able to follow the instructions. She should also be able to recognise the schematic cube as a representation or a working drawing, or even *take* it as a cube as we do in certain circumstances. She might after repeated training be able to *interpret* or *suppose* one musical theme as reverting to this, or *take* two numerical or spatial patterns as being isomorphically related (*PI* pp. 213, 214).

This modified notion of seeing (i.e., aspect-seeing), grounded on a technique or activity, is 'necessary to get rid of a feeling of dizziness in mathematics' (*PI* p. 209). It is the feeling of there being nothing below our feet except the ethereal entities of Plato, or the equally rarefied, a priori forms of space and time of Kant and the intuitionists, or the empty verbal stipulations of the empiricists. With Wittgenstein we get a foothold in experience, in concrete objects, not so much in the experience of objects, but experience of pictures and physiognomies. This experience is what we create, not once for all, as a transparent self-interpretive content, but in an ongoing flow of related techniques. This new concept of experience, i.e., aspect-experience, gets rid of traditional dogmas, whether it is supersensible entities, or primordial bits of sensations, or unique brain patterns. Lastly, it preserves a meaning for necessity and novelty in mathematics, by creating newer and newer cycles of dispersal and reversal, newer and newer techniques of identification and re-identification.

3. The Tension between Mathematical Cognition, Aspect-Seeing and Mathematical Action

All forms of foundationalism in mathematics have been trying to track down some form of reservoir—whether the Platonic entities in the third realm, a priori forms of mind, logical primitives, definitions and rules—independent of and yet spurting forth infinite applications and practices from its reserve. These reservoirs are supposed to manage their dignified status somehow, never allowing themselves to merge with the applications. By assimilating Wittgenstein's notion of mathematical cognition to that of aspect-experience, we have attempted to handle another form of foundationalism—of the epistemological and psychological variety—which seeks to trace mathematical operations to their causal antecedents. These causes as we have noted are proposed to be either a special impression, or a special angle or point of view (as a positional objectivity), or pre-given bits of sensations, or structures and motions in the brain. The modern explosions in neurology will no doubt come up with patterns of neural firings claiming isomorphic correspondence with mathematical aspect-perception. Wittgenstein's resistance to these causal foundations brings out a richer dimension of his critique of foundationalism, and a more clinching argument in favour of blending language, cognition and action. Mathematical cognitions, if looked upon as passively insular mental states to be exhaustively explained in terms of causal antecedents—whether pre-given sense-data or brain physiology—can never account for the undeniably creative, active and volitional character of mathematical operations. The intention and purpose in these operations is exclusively identified in the course of the action itself; it cannot be pushed back to the deep mysteries of cortical patterns or neural firings. When a mathematician restructures $o'' + o''$ as $(o□)'$, when he generates the succeeding numbers from the predecessors through a conscious manipulation of advancement and retreat, when he gives the non-decimalic numbers a new criterion of identity on the base of 10, he is performing all these voluntarily and intentionally. When he articulates his intentions in writing out Gauss's proof in the shape of schematic figures, he is not simply undergoing an outburst of his hidden brain dynamics—something of which he is usually unconscious, ignorant, or that he can very well mis-describe. Such factors can play no role in the constitutive identity of the voluntary mathematical operation.

Now one may both try to preserve the 'actional' status of mathematics as well as consolidate its physiological foundation against Wittgenstein. For this we would have to start with a split between cognition, wish, will and action, specifically between mathematical cognition, the mathematician's wish and will, and the actual mathematical operations. We would have to insist on brain patterns or neural firings as the underlying cause of mathematical cognition, which in its turn gives rise to a wish to perform the actual operation of the proof, further generating the mathematician's will to carry out the implementation of the wish. This account might also be claimed to be in keeping with Wittgenstein's suggestion that aspect-seeing, unlike most cases of seeing, is subject to the will; it always makes sense to try to float an aspectual transition or to keep it in focus.[13]

Let us see how Wittgenstein would handle the proposed splits between cognition, wish, will and action on which the above physiological construction of mathematics rests. First we need to recall that the Augustinians wanted to forge a bridge between ostensive definition, rules of meaning on the one hand and the object meant on the other, through a self-interpretive phenomenon of *meaning* (mental imagery, silent thought, etc.). This will help us appreciate that the above account of action is also ensconced in the Augustinian model. Here it projects the phenomenon of willing as the precise connecting point (between the wish to do ϕ and the precise way to do ϕ), the self-interpretive bridge to close the gap between the wish and the action, and yet holding itself apart from both (*PI* 617). This account when applied to mathematics would sound weird. The mathematical cognition caused by physiological events would have to generate a wish to work out the actual proof and yet leave a gap between the wish and the precise way to carry out the *actual* one–one correlation required for the specific proof and calculation. This gap would be bridged by the mathematician's will that touches the precise connecting point between the wish and the action. Whether this account sounds palatable or not, one who wishes to uphold both the physiological foundation of mathematics and its active, operational and volitional status, would be obliged to adhere to something of this kind.

For Wittgenstein, no action of any sort, whether mathematical or non-mathematical, can be grafted on this model. As one makes a dichotomy between the willing subject and the acting subject, will becomes the primordial origin, totally purified of any movement or action—pure mover without being moved. While all physical bodies in so far as they are acted upon by other bodies are governed by the principle of passivity and inertia, will is conceived as a pure mover without being moved, something which always acts, and is itself never acted upon. Wittgenstein points out that if will is thus segregated as the 'pre-actional' foundation of action, doing itself seems to be bereft of any volume of experience. Whatever we feel in the course of the action would be pushed out of its boundaries and projected either as some contingent accompaniments or as its external consequences. In other words, if will is conceived as the source of action in its consummated fullness and yet neatly segregated from the action itself, then there is nothing left as the content of *action* or *doing*. Doing turns into an extensionless point of a needle.

That there has to be a will as a pure foundation of actions is merely a grammatical point, like the need for an 'it' to complete the sense of 'it is raining,' or the need for 'sweetest' as a grammatical contrast to 'sweet' and 'sweeter'. Similarly it is our *language-game* with 'action' and 'will' that falls back on the further game as 'I cannot fail to will.' In the same vein, it is also connected with the language-game that 'one cannot try to will,' as it becomes a grammatical requirement that the will is not determined by anything, say an attempt or effort (*PI* 618–20).

Wittgenstein takes to his characteristic style of actual survey of cases where terms like 'willing' and 'intending' are actually used (*PI* 588). Obviously his project is to dissipate such myths of their sharing a common, self-identical character in the shape of a special mental undertone that can be retrieved through introspection. Let us consider the following: (*a*) I am revoking my decision to leave tomorrow. (*b*) Your arguments do not convince me, I stick to my previous decision. (*c*) Asked how long I am going to stay, I say 'Tomorrow my holiday ends.' (*d*) At the end of a quarrel I say 'OK. I decide to leave tomorrow.' There is no characteristic, typical experience of 'tending towards something' underlying all these diverse phenomena. Intention to say something does not consist in opening one's mouth, drawing one's breath and letting it out again, for such things can happen in a completely different situation to feed a completely different concept (*PI* 591). On the whole, the dimension of 'depth' in cases of genuine volition or intention as contrasted to faked ones consists in a flattening out of this depth in painstaking descriptions of humdrum uses (*PI* 594).

For Wittgenstein, a wish too does not enjoy the status of a purely mental state of feeling, ontically different from and giving rise to the action. He tries to lay out how the content of hopes and expectations spills over to imbibe the precedents and consequents of the situation, resisting anything like an insular phenomenological quality of the present (*PI* 584). Suppose the entire morning I have been hoping that N.N. will come and bring me some money. If one minute is cut off from this context, 'will it not be hope?' The question can be answered sensibly only if we realize that whether we cut off a chunk of one minute or five hours from the stretch, hoping cannot preserve a purely mental status if the words do not belong to the language-game, if the 'feeling' of hope is displaced from the entire institution of moneylending in which it is situated. No doubt a similar analysis would apply to wishing as well.

Wittgenstein is emphatic on the point that if a will is to be distinguished from a wish, the former cannot stop short of the action. On the other hand, when we say that a wish stops short of a will and thus of the action, this is not to say that the causal nexus stops at the wish, *qua* a purely mental antecedent of the action, but rather that the wish as embossed in a plethora of actions stops short of extending to further actions (*PI* 639–49).

We know that Davidson has a more sophisticated theory of human actions,[14] though it is extremely doubtful whether Davidson would be willing to apply his theory of actions to mathematical operations. Still less would he agree to the proposal of there being a unique cortical pattern or unique neural firing corresponding to each mathematical cognition. But the point I wish to emphases is this: *If* one attempts to borrow some elements of Davidson's theory of actions to give a physiological account of mathematical operations, that attempt is bound to be a failure, both in the dominant theories of mathematics as well as from the Wittgensteinian perspective. We can, however, consciously attempt to start with some points or questions that sound congenial to our project. Firstly we can ask, when Davidson says that it is not merely intellectual or cognitive states but also pro-attitudes (wish, desire, urges, promptings, ethical convictions, public or private goals) that are required for actions,[15] does he suggest that just having a rule for a mathematical proof is not sufficient for the actual operation of the proof? Would he concede that the mathematician needs to have the further pro-attitude and belief to plunge voluntarily into the actual implementation of the proof? Secondly, let us also recall that for Davidson, having the pro-attitude and the belief pertaining to some properties of the action may be superseded by other con-attitudes to certain other properties of the same action. Hence, he goes on to invoke an extra judgement—an all-out unconditional judgement for a satisfactory explanation of the action.[16] Now again it would perhaps be a considerably strenuous exercise to give a convincing application of these agendas to mathematics. Above all we must also repeat that for Davidson, the causal story sought to be applied to mathematics would have to be normative, holistic and non-nomological, in order to preserve the volitional character of mathematical operations.[17]

In fine, whether or not Davidson would have approved of this account of mathematics, it is clear that such an account, even with its added sophistications and enriched conceptual repertoire (pro-attitudes, beliefs, an unconventional notion of causation, all-out unconditional judgement in lieu of the traditional act of will), cannot be accommodated in Wittgenstein's philosophy. If mathematical cognition is to stop at pure perceptions, inviting pro-attitudes, beliefs and a special judgement to implement the cognition, the required operation will never come into being. Whatever may be proposed as a causal antecedent of actions, however finely chiselled it might be, whatever shape it takes (brain patterns, pro-attitudes, etc., etc.), so far as it is kept external to the action, will fall back on an endless series of interpretations, never being adequate to explain the required mathematical operation.

This consideration has led externalists like Kripke to suggest that all our usages of 'natural kind' terms or of mathematical expressions are inherently geared to the rigid identities of their referents, even if we attach wrong descriptions to them, or make counterfactual claims about them.[18] For Kripke, whoever is performing a mathematical operation, say a one–one correlation between '1 + 2 + . . . + n' and '½ n × (n + 1)', even if she lapses into a wrong calculation or gives a wrong description of any of the numbers, is non-descriptionally and incorrigibly related to the unique referents on each side of the identity or one–one correlation.[19] Now it is clear that Wittgenstein, who rejects both internal foundations and external transworld identities, steers clear of both internalist and externalist accounts of language in general, and mathematical language in particular. For him it is the mathematical operations themselves, the cyclic rehearsal of dispersal and rehearsal, that create the identity of units and their isomorphic correlations.

In view of certain dominant objections—that Wittgenstein lagged behind the epoch-breaking explosions in neurology seen in the latest century and thus naïvely put the entire onus of meaning on common usage—I am tempted to push his point to a provocative extreme. Even if future technology is able to exhibit a connection between every subtle operation of the mathematician and a neurological correlate, even if an isomorphic connection between our speech, action and our nervous system is made available to the common man, made ready to be absorbed in common day-to-day uses, even then the meanings of our words, including our mathematical vocabulary, would not be shifted to neural events. It would at most be shifted to the *'common' usage of correlating meaning with neural happenings*! Meaning would continue to be cashed out in uses, in common day-to-day uses—though the character of *commonality* of *common* uses may go on changing through the overlapping and criss-crossing fibres of family resemblances. This is the dominant message of reducing all foundations to actions—it is the actions that give meaning to the putative primacy of reference (as opposed to descriptions or conceptions), it is actions that activate the supposedly primitive notion of 'one' or a single unit in mathematics or logic, rather than the reverse process of the pre-given referential identities governing the actions, as it is popularly supposed to be.

NOTES

1. This issue has been discussed at length in chapter III.
2. This diagram has also been borrowed from the website Math and Multimedia, available at: http://mathandmultimedia.com/2010/09/15/sum-first-n-positive-integers/ (accessed 17 July 2016).
3. See *PI* (p. 194) where Wittgenstein mentions this distinction between 'continuous seeing' of an aspect and the 'dawning' of an aspect.
4. Wittgenstein takes care to point out that images may not figure in the same way in many other cases of aspect-seeing. He cites the

examples of convex and concave aspects, and the double cross to this effect. As he points out, 'the description of alternating aspects are of a different kind in each case' (*PI* p. 207).

5. The following account of Gestalt theory is derived largely from Robert S. Woodworth, *The Contemporary Schools of Psychology* (London: Methuen and Co., 1965), pp. 215–27, and also from Stromberg, 'Wittgenstein and the Nativism–Empiricism Controversy'.
6. For the Gestalt theorists, all perceptions, be they of objects, colours, shapes, lines, tones or tunes, are responses to the entire pattern in the brain, and thus, contrary to the empiricist assumption, are physiologically on the same footing. They are all physiologically *real*, we *see* objects, aspects as much as we see colour, we *hear* tunes as much as we hear tones. The organisation is a sensory quality, a given sensory fact in the brain and not (as the empiricists claimed) superimposed on the stimuli by a higher act of interpretation or inference. See Woodworth, *The Contemporary Schools of Psychology*, note 5.
7. K Koffka (1886-1941) and W. Köhler (1887- 1967) were the pioneers in developing the Gestalt theory of perception. One can mention Koffka's works like *Perception: An Introduction to Gestalt Theory* (1922) and also *Principles of Gestalt Psychology* (1935). As for Köhler, works like *Gestalt Psychology*, *The Task of Gestalt Psychology* are noteworthy. However, I have only relied on Woodworth's work for the minimal exposition of their theories.
8. I rely substantially on Glock's account of Wittgenstein's notion of aspect-seeing. See H. J. Glock, *A Wittgenstein Dictionary* (Cambridge, MA: Blackwell, 2005).
9. See ibid. for a very comprehensive account of Wittgenstein's notion of causation.
10. Ludwig Wittgenstein, 'Cause and Effect: An Intuitive Awareness', *Philosophia*, vol. 6, nos 3–4, 1976, pp. 409–25.
11. Following this line of thought, we can say that the meaning of such sentences as 'The symbol o″ + o″ *necessarily determines* the meaning of o″″,' or 'The sum of n positive integers *necessarily* equals ½ n × (n + 1)' too consists in juxtaposing internal contrasts with imaginary numbers, negative numbers or fractions.
12. See Kant's letter to Johan Schultz, 25 November 1788, in *Kant: Philosophical Correspondence 1759–99*, ed. and trans. A. Zweig (Chicago: Chicago University Press, 1967), pp. 128–31. Here Kant states that when we form the concept of one and the same quantity (say 12) by means of different additions and subtractions (say 7

+ 5, 3 × 4, 16 – 4, etc.), ' . . . objectively, the concepts I form are identical (as in every equation)'. But these processes of addition, subtraction or multiplication, which are also processes of synthesis, are 'subjectively . . . very different'. 'So at any rate my judgement goes beyond the concept I get from the synthesis, in that the judgement substitutes *another concept* (simpler and more appropriate to the construction) in place of the first concept, though it determines the same object' (ibid., p. 129; italics mine).

13. It is interesting to note that both Helmholtz and Wittgenstein, philosophers with two very different agendas, acknowledge the role of will in aspect-perception. Helmholtz's view on the role of will, based on a comprehensive account given by Stromberg ('Wittgenstein and the Nativism–Empiricism Controversy'), has been mentioned in chapter IV of this work. On the other hand, Glock (*A Wittgenstein Dictionary*) also points out that for Wittgenstein, aspect-seeing is more akin to *action* than to *seeing* in so far as it is subject to the will. Though we may not always succeed in noticing an aspect or keeping it in focus, it always makes sense *to make an effort* to do so. We shall see that that while Helmholtz's will in the empiricist model is different from and causally antecedes aspect-seeing (the latter being an act of inference from pre-given sensational bits), for Wittgenstein, the will forms the very constitutive identity of an action.

    Let me also mention that I would like to differ from Glock's reading of a cleavage between Wittgenstein's notion of object-seeing and aspect-seeing, and bring the activity-oriented character of the former into full focus. Doesn't it make sense to instruct a person to *try* to see a crocodile floating still on the water *as* a crocodile and *not* as a dead tree trunk; doesn't it make sense to command him to keep it in focus? Glock's observations are resourcefully based on texts like *PI* (II, p. 212), *RPP* (sections 27, 169 and 544–5 in vols I and II respectively); and also *LPP* (sections 451, 488, 612)—which perhaps have to be given a different treatment.
14. Donald Davidson, *Essays on Actions and Events* (Oxford: Clarendon Press, 2001).
15. Ibid., Essay I.
16. Ibid., Essay 5.
17. Ibid., Essay 11.
18. Saul Kripke, *Naming and Necessity* (Cambridge, MA: Harvard University Press, 2001).
19. This would surely follow from his theory of mathematical terms as 'strongly rigid designators'.

CHAPTER VI

Wittgenstein's Charge of Circularity in the Frege-Russell Definition of Number

This chapter shall be exclusively devoted to the charge of circularity that Wittgenstein directly levelled against the Frege-Russell definition of 'number'. In the first section, I lay out a preliminary account of the objection, principally following de Bruin's summary of the same,[1] backed up by his own reading of the way Wittgenstein treated the tension between one–one correlation and cardinality. In the following sections, I juxtapose my own assessment of the charge against that of de Bruin, trying to situate this issue against the wider backdrop of my anti-foundationalist reading of family resemblances, aspect-seeing and action.

1. De Bruin's Reading of the Charge of Circularity

It has been noted that Frege's definition of number runs somewhat as follows:

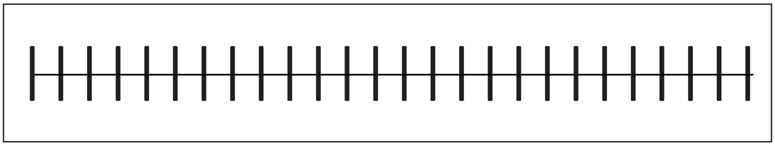
> The number of objects falling under the concept F is identical to the number of objects falling under the concept G if there is a one–one correlation between F and G.[2]

De Bruin finds it convenient to rely on Marion's[3] presentation of Wittgenstein's charge against the above definition, to which he adds his own qualifications. Marion reads Wittgenstein as posing a two-pronged attack against Frege. According to the first prong, Frege cannot retain his notion of one–one correlation, say between cups and spoons, as an *actual* one. The reason is simple: if the notion of one–one correlation involves the notion of 'as many', then this notion cannot be captured in the actual correlation of putting one spoon inside each cup, but rather in the notion that one *can* perform such an operation. Put in a different way, such a correlation may be impracticable, i.e., when they (cups and spoons) are locked up in separate places; or to push this point of impracticability to empirical impossibility, one may be posed with the question of effecting one–one correlations with whiffs of smoke, drops of water or camphor. Under the pressures of such considerations, Frege would be obliged to construe this one–one correlation as a *possible* correlation. And to say that one *can* correlate each cup with each spoon, one must already have conceived of them as *numbered*, a conception not necessarily involved in the *actual* operation of putting one spoon into each cup. Possible correlation means an awareness of one's capacity to do this, and such an awareness involves an awareness of 'the right number of spoons'. Hence, Frege cannot regard the notion of one–one correlation as conceptually prior to that of number and devise a non-circular definition of the latter in terms of the former. Thus, the strategy of Wittgenstein's attack on Frege's definition is first to corner him into accepting a distinction between actual and possible correlation, push him to abandon the first option in favour of the second, and finally explode the second option as already presupposing the notion of cardinality.

It emerges that for Wittgenstein the notion of *possible* one–one correlation does not hold ground. As Marion observes, there is nothing like a one–one correlation before one has actually correlated the Fs and Gs. Friedrich Waisemann also reports Wittgenstein as saying: ' A correlation is there only when I actually correlate the sets, i.e., as soon as I specify a definitive relation.'[4]

For Wittgenstein, while the notion of one–one correlation already involves the notion of 'as many', it does not necessarily involve the notion of 'how many'. Here we need to quote at length from *PR* (118): 'Can I know there are as many . . . without knowing how many? And what is meant by not knowing how many? And how can I find out how many? Surely by counting.' He further states that diagrams like the one in Figure 6.1 (one horizontal line separating or connecting two sets of vertical lines) show that 'It is obvious that you can discover that there are the same number by correlation, without counting the classes' (*PR* 118).

Figure 6.1



De Bruin goes on to claim that Wittgenstein's distinction between 'as many' and 'how many' has to be construed in terms of knowledge, particularly the distinction between *de dicto* and *de re* knowledge. Knowing the sameness of numbers via the knowledge of exact cardinality is *de re* knowledge, while knowing the sameness of number without the knowledge of their precise cardinality is the knowledge of the *de dicto* variety. While in the former case one is able to give a name to the cardinality, this is not possible in the latter. Following de Bruin, we can rehearse the symbolic formulation of this distinction drawn customarily in terms of knowledge *of* and knowledge *that*:

a. ∃x K Px (*de re* knowledge with respect to an object x having the property P)
b. K ∃xPx (*de dicto* knowledge with respect to an object x having the property P)
c. K ∃xPx & ~ ∃x K Px (merely *de dicto* knowledge with respect to an object x having the property P)

So when Wittgenstein speaks of knowing as many without exactly knowing how many, he is speaking of merely *de dicto* knowledge about cardinal equivalence. Here one knows that there exists some number n that is the cardinality of both F and G, but one is not able to name this cardinality. With *de re* knowledge, the knower is able to single out that specific number as the unique cardinality of both F and G. Substituting R for P, where R stands for the relational property of one–one correlation shared by F and

G, we can easily get the following symbolic versions of *de re* and *de dicto* knowledge:

$$\exists R\ K\ (F\ 1\text{–}1_R\ \ G)$$
$$K\ \exists R\ (F\ 1\text{–}1_R\ \ G)$$

According to de Bruin, Wittgenstein's imputation of the charge of circularity comes mainly to this: It is the merely *de dicto* knowledge of one–one correlation that in some sense presupposes the *de re* knowledge of cardinality. What this means is one cannot stop at the mere possibility of one–one correlation without actually correlating them each to each. Thus, the so called *de dicto* knowledge of one–one correlation, in a way, breaks through its putative boundaries to explode into the *de re* knowledge of actual cardinality. It is for this reason that Frege's definition of number in terms of one–one correlation is circular—it presupposes both 'as many' and 'how many'. But for Wittgenstein, to have *de re* knowledge of one–one correlation between F and G, i.e., to perform the actual act of correlating them, one need not be geared to their actual cardinality.

Construing Marion's reading of Wittgenstein's charge of circularity against Frege in terms of this distinction between *de re* and *de dicto* knowledge, de Bruin goes on to report that Marion is not too sure whether this dependence of *de dicto* knowledge of one–one correlation on the *de re* knowledge of cardinality is strong enough to build a full-fledged charge of circularity against Frege. However, by demonstrating Frege's definition of number as falling back on actual cardinality, Marion thinks that Wittgenstein is able to expose the Platonist assumption underlying logicism.

Let us pause briefly at this juncture to take note of an anxiety that might have been growing all through the above account of de Bruin. Does not this construal in terms of *de re* and *de dicto* knowledge turn Wittgenstein's view of mathematics in a realist direction—the direction we have been trying to resist throughout? Is there not an unmistakeable suggestion that whether we are engaged in one–one correlation or are trying to conceive (*per impossibile*) its mere possibility in pure thought experiments, we are fully constrained by the extra-linguistic and extra-conceptual reality of numbers, the inexorable givenness of *how many*, whether we know it or not? However, in spite of this burgeoning anxiety, we prefer to complete de Bruin's reading of Wittgenstein before we try out our own programme of resisting and diverting its realist inclinations.

De Bruin claims to track down three interrelated epistemic principles underlying Wittgenstein's approach to one–one correlation and cardinality:

1. For something to exist means that it must be constructed.
2. Existence of a one–one correlation means that it must be actually constructed. We have seen that this epistemological principle charts out the path through which Frege's definition falls into an evitable circularity.
3. On the strength of (1), any *de dicto* knowledge of one–one correlation must fall back on *some de re* knowledge. But according to de Bruin, this does not imply that the *de dicto* knowledge of one–one correlation has to be exploded to *de re* knowledge about actual cardinality. We can preserve both the *de dicto* knowledge within its boundary of possibility and principle (2), provided we concede that the *de dicto* knowledge of one–one correlation rests on *some* piece of *de re* knowledge which is entirely different from particular one–one correlation or particular cardinality.
4. It follows that there are two mutually independent ways of obtaining knowledge about one–one correlation between two concepts: one involving a non-cardinal or non-counting method, and the other involving the usual cardinal method of counting.

It is now clear how the dilemma designed by Marion is recast by de Bruin into a new form. Firstly, as noted earlier, to avoid the restrictions of actuality, Frege is obliged to construe his one–one correlation in terms of possibility. It is here that de Bruin opens up two options of possibility: (*a*) it is unable to hold itself as conceptually prior to and distinct from actuality and virtually inflates into the latter; (*b*) the only way it can preserve its purely *de dicto* status with an unspecified cardinality is by holding itself independent of and basically distinct from cardinality. With (*a*), Frege slides down the familiar path of circularity. With (*b*), the mutual independence of one–one correlation and cardinality debars him from defining the latter in terms of the former.

The objection against principle (3) is inevitable and predictable. Seen in the reverse direction, the process of counting involves a process of effecting one–one correlation between the target set and the set of our fingers—where the first finger is taken as the set of one, the first and second are taken as a set of duos, and so on. Thus, if *de re* knowledge about cardinality established by counting depends upon *de re* knowledge about one–one correlation, it is doubtful whether one can delink merely *de dicto* knowledge of one–one correlation from the *de re* knowledge about cardinality and resist Frege's definition on that count.

De Bruin replies that to claim that there are two independent methods of obtaining one–one correlation—one cardinal and the other non-cardinal or non-counting—is not to claim that they constitute two different practices. On the contrary, with the first six natural numbers we immediately 'take in' both their cardinality as well as their one–one correlation without counting or setting up a *procedure* of one–one correlation. This phenomenon of 'subitisation' gives us immediate perception of numbers independent of any actual practice, any *de re* knowledge of one–one correlation. Thus, de Bruin winds up his construal of Wittgenstein's charge of circularity with a reminder. Even if one unjustifiably limits the notions of one–one correlation and cardinality to the contingent factor of human practices, one cannot but recognise the essential difference between these two notions at least with the first six natural numbers. And this fact is a perfectly adequate reason to retain the charge of circularity designed by Wittgenstein in full measure.

2. Revisiting Wittgenstein's Charge of Circularity against Logicism

I shall now embark on my own construal of de Bruin's presentation to see whether we can make the necessary adjustments to match it up with the anti-foundationalist track we have been following throughout. While agreeing summarily with Marion's and de Bruin's presentation of the charge of circularity, I would like to differ on the following counts.

Firstly, perhaps de Bruin did not quite appreciate how, for the later Wittgenstein, a concept underdetermines its extension, or rather how it underdetermines what constitutes each individual instance of its extension. As a result, he opens up certain digressive tracks in his narrative which go against the basic anti-foundationalist spirit of the later Wittgenstein. The second note of dissent (as already mentioned earlier) is that the way de Bruin recasts the distinction between actual and possible one–one correlation in terms of *de re* and *de dicto* knowledge reads a strain of realism or foundationalism into the later Wittgenstein's view of mathematics. As a result, the third point of disagreement pertains to de Bruin's not quite appreciating the family resemblance character of numbers, at least not in its decided resistance against the putative recursion of numerical identity along the different contexts. He does effectively deploy the fluid character of such phrases as 'having the same length' and 'having the same number' in the different cases of subitisation in order to delink the notion of cardinality from the various non-practitional incidences of one–one correlation. But he achieves this only at the cost of according an insular, psychological transparency to the subitised perception of numbers. This strategy, while apparently constructing a full-fledged critique of logicism, inclines the later Wittgenstein towards a psychological or physiological foundationalism.

2.1 *A Different Reading of Actual and Possible Correlation*

I suggest that the *de dicto* knowledge of one–one correlation is a preliminary rule or plan, or a rudimentary stage to start with the minimal grammar of units. It is an architectonic starting point with 'a short bit of a handrail' that fleshes out in and through the actual one–one correlations (see chapter III, section 7 on rule-following and also chapter VII). As a rule does not have a pre-applicational content, there cannot be a programme of designing a definitional equivalence between the two, or a prospect of extracting a definite mode of cardinal equivalence or an exact cardinality from this preliminary plan of one–one correlation.

It seems that de Bruin does not give due importance to Wittgenstein's repeated insistence on the different ways of one–one collation that internally rupture the descriptive content of a rule or concept. While he actually cites *RFM* (I:25–40) in favour of his introduction of *de re* knowledge,[5] I would insist that these passages rather point in an anti-realist and anti-foundationalist direction (see chapter II, section 2.1, 'Failure of Ostension in Mathematics'). Two people may seem to start with the same picture of one–one correlation and yet work out the actual lines of correlation in different ways. Person A takes each stroke as a series of points placed vertically and thus operates a one–one correlation among these points. Person B may be sensitive to the magnified breadth of each stroke and work out the one–one correlation accordingly. Person C can take each stroke as a set of radiations which need to be put into one–one correlations with each other, and lastly and least adventurously, person D works out the correlations between two sets of positions or directions—upward and downward. We have seen that any attempt to confine these multiple fulgurations of meaning into a strict concept of 'strokes in an array' only proliferates layers and layers of indeterminacies. Thus, in a nutshell, Frege's fallacy lay in his commitment to the objectivity and unique predictive power of concepts. His statements are worth quoting in this respect:

> The concept has a power of collecting together far superior to the unifying power of synthetic apperception. By means of the latter it would not be possible to join the inhabitants of Germany together into a whole; but we can certainly bring them all under the concept 'inhabitant of Germany' and number them.[6]

Frege further claims that while a concept like 'red' does not ensure a unique way of procuring the numerical identity of each instance, most concepts isolate their respective instances in a definite manner. Thus, it is this power of concepts that explains the wide applicability of number across all kinds of phenomena—

across apples, drops of water, whiffs of smoke, images and dreams.[7] As for Wittgenstein's reaction to these claims, we find one of the most finely tuned and graphic articulations against this fallacy of circularity in *PG* (Part II, section 21, pp. 335–36). Here Wittgenstein effectually states that Frege and Russell's definition of number 'seduces us' into imagining correlation as a check of sameness of number. We get bogged down by a mythical distinction between being correlated and being connected by a relation, and then think that correlation is a geometrical straight line already sketched out in advance by logic, so that when we connect them in reality we merely trace it out. 'Correlation turns out to be a possibility conceived as shadowy activity.' It is this myth that we need to fight with extreme caution for an honest reading of Wittgenstein's critique of a logicist theory of mathematics. Thus, to repeat, it is in Frege's claim that the concepts F and G neatly carve out its extension, forge a unique one–one correlation among its instances and a unique cardinality, that the fallacy of circularity lies.

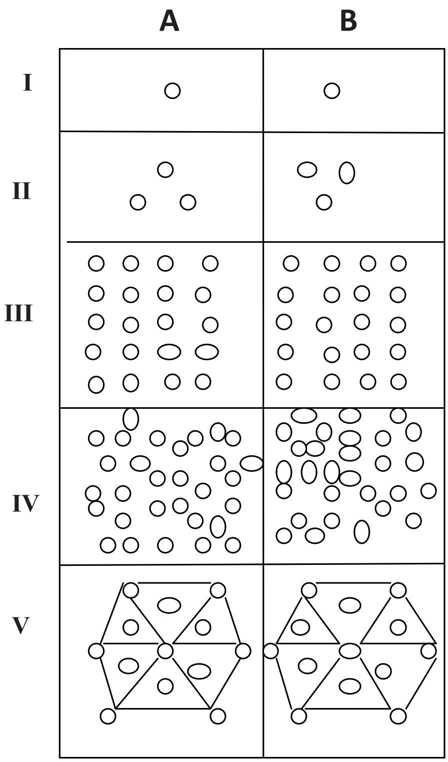
How to respond to de Bruin's claim that for Wittgenstein, the two methods of obtaining isomorphism—by actually effecting the non-cardinal method of correlation and the cardinal method of counting—are independent of each other? Here let us remind ourselves that while a concept or a rule cannot be independent of its applications, a piece of action, behaviour or cognition can however be independent of its aspectual transition. We suggest that the difference between the *de re* knowledge of one–one correlation and *de re* knowledge of actual cardinality (by counting) is the difference between seeing two aspects of the same picture or the same operation. The duck-rabbit picture can alternately be seen as a duck and as a rabbit; an action of uttering words may also be seen as one of displacing air molecules. In both these cases, the aspectual transitions are optional, i.e., the alternating aspects are mutually independent. Similarly, effecting a one–one correlation may not transit to the other mathematical aspect of counting. The difference between seeing a series as merely a one–one correlation and seeing it as having a precise cardinality is solely a difference between two practices or activities. We have seen that the difference between the two aspects of a duck-rabbit, or that between a foreground and background, does not involve a characteristic difference between two different visual impressions or between two different psychological data. Nor does it fall back on a self-identical neutral referent underlying the alternative descriptions of 'duck' and 'rabbit'. There is no action with a pre-descriptional status or with a simpler content that is logically prior to its receiving alternating characterisations like 'uttering words' and 'displacing air molecules'. Similar remarks apply to the aspectual transitions between non-cardinal and cardinal methods of performing one–one correlations.

This gives us a fresh occasion to ruminate once more on the exact tension between object-seeing and aspect-seeing and relate it with the issue of reference and description. Aspect-seeing can be said to fall back on object-seeing only in the sense in which referring games are prioritised over descriptional games. For example, we have seen that putting pieces on the board is a referring game that can be contrasted with the descriptive games of playing out the actual moves of the game. But the material content of the pieces of the board does not figure as the foundation of the subsequent moves undertaken, as particularly evident in cases where we pose certain pieces as having special powers, treat certain positions on the board as invincible, or when a pawn's movement to the opposite side of the board turns it into a queen. The referring game of putting pieces on the board does not entail, but is spread in and through, the subsequent moves of the game. One can say that the game of referring principally consists in privileging a thing as an indivisible lump—in playing down its internal complexities, or in making the internal contrasts and connexions between its elements overshadow its relations and connections with the environment.[8] But this way of privileging the indivisibility of the referent, or posing a thing as a 'thing' jutting out from the environment, does not fall back on a supposedly pre-linguistic content of the referent. Rather, its 'indivisible' referential identity gradually sprawls out in and through each of the descriptive games in an ever incomplete process. Similarly, the different aspects floated by the non-cardinal and cardinal practices of one–one correlation fall back only on the ceremonial stance of sharing a common referential identity, of strategically playing up a relatively independent and self-identical facet. This projected identity too gets its actual content in and through the two different aspectual practices, i.e., through the non-cardinal and cardinal practices of one–one correlation. It is only in the light of this insight that one can appreciate how it is impossible that two people should share the same *de dicto* knowledge of one–one correlation and yet apply them in different modes of interpretations to arrive at different cardinalities. We can effectively place this beside one of the most important observations of Wittgenstein on rule-following (which we have already paraphrased in chapter III): two people cannot share the same formula, same rules and definitions, and yet produce different interpretations and applications.

2.2 *Reviewing Number as a Family Resemblance Notion*

It is at this juncture that the family resemblance character of words and expressions needs to be recounted with special reference to numbers. (We base the following account primarily on *PG* Part II, section 21.) Let us remind ourselves that for Wittgenstein, words flow through the routes of overlapping and criss-crossing similarities—similarities which do not base themselves on any foundational and non-relational 'respect', so to speak. (See the penultimate section of chapter I.) When we use the words 'same in number', it does not mean the same with respect to various cases like: (*a*) lines simultaneously present in the visual field; (*b*) apples in two boxes; (*c*) lengths in Euclidean space. Take the further cases of when we count the horses in a stall, and when we count different species of animals in a stall; or when we count the strokes in a line and the kinds of groups defined by different numbers of strokes. Obviously no self-identical essence of 'one' jumps neatly from one context to the next. We can parallely bring up the instance of colour, the case where we use the words 'the same green colour' with respect to the painted green glass on the canvas and the real green glass on the table, in spite of the fact that the respective pigments used for them are totally different. So here, 'the same green colour' means the colour impression and not the physical pigments. Wittgenstein brings up other instances where we ascribe the description 'the same length' to patches in the visual field seen simultaneously, both to the patches which are immediately adjacent to each other and to those which are distant from each other. This is a more interesting case, where the expression 'same length' palpably resists the logicist insistence on the 'same number of units' necessitating a neat recursion of numerical identity. Wittgenstein goes on to explore further this internal rupture in the notion of equinumerability or 'having the same number' with respect to the pictures in Figure 6.2 (*PG* p. 354).

Figure 6.2



Wittgenstein categorises I and II as having numbers that one immediately recognises; in III, we take in each line as having 5 elements and there being 4 such rows, and then we also take in that there are the same number of rows with the same number of elements. (One can perhaps say that here is a complex process of subitisation with three sub-phases.) In IV, we need to count both groups, and in V, we recognise the same number of elements in both pictures by recognising the same pattern. Here we have some more cases where the putatively recurrent identity of 'one' gets constantly invaded. Once we come to appreciate this, we shall also come to realise that the entire logicist endeavour to recast the notion of number first in terms of equinumerability and then into one–one correlation fails to get hold of numerical identities as transparently individual instances of transparent concepts that neatly close over their own range of values or extension.

2.3 *A Different Reading of Subitisation*

To take on the question we have already posed: does the phenomenon of subitisation—a process taking in both 'as many' and 'how many' without any actual operation of one–one correlation—betray a 'pre-practitional' reality of numbers? The fact that this phenomenon is observed in newborn infants, who are not initiated into conceptual learning nor into the game of freezing experiments into physiognomic pictures, not only goes against Frege's logicism but poses a threat to Wittgenstein himself. In order to address such anxieties, we need to look more closely into certain of his arguments (*PG* pp. 355–36) which display a neat format of a dilemma (or rather a 'trilemma') that is both interestingly similar to and different from the ones charted by Marion and de Bruin.

The subitised perception of number, i.e., the cardinal content of a perception taken in at a glance, is either simultaneous with the process of counting or there is a temporal gap between the two. In the first option, the subitisation and the counting process are enmeshed together; the former does not enjoy an independent content that distances itself from and yet entails a unique method of counting and a unique cardinality. In the second option, the temporal gap necessitates a conceptual gap as well, which cannot be bridged in a non-circular fashion. And as for the third option, where there is an unsurveyable and unsubitisable series of related units and a subsequent process of counting, the conceptual gap between the two and the imminent charge of concocting a circular definition becomes more palpable. Hence we cannot speak of either a subitised or an unsubitised content of one–one correlation that unfailingly determines a unique cardinality.

If this is the basic structure of Wittgenstein's charge of circularity levelled against Frege, we can try to fill it out with more convincing details by following the line of his arguments and illustrations. Let us look closely at Figure 6.2, first at the non-cardinal and non-practitional picture of one–one correlation in I, II and V, and then at the actual performance of drawing the lines of correlation and counting their cardinalities. Going by the later trends of Wittgenstein's treatment of mathematics in *PI* and *RFM* (specially I:25–40), one can say that the first picture of subitised equinumerability may not ensure a matching equinumerability by virtue of the process of counting. The perception may alter with a short or long passage of time; one can be unsure about the physical character of the second set of dots in I and II, and look upon the first picture as a projection on a square and the second picture as a square with a hole, where the one–one correlation between the dots gets totally obscured. One can look upon each dot in the second set of II as a fusion of two dots, in which case the second process of cardinal operation may run contrary to the first process of non-cardinal subitisation.

The only way to rescue the supposed sanctity of the subitised content and its consequent parity with the subsequent cardinal operation is to argue in the following predictable format: If it is the *same* concept of dots and squares and the *same* physiological foundation underlying the two processes, then they (i.e., the processes) would match via a conceptual or physiological link that sacrosanctly connects the two. Now the first claim of conceptual sameness takes us back to merely *de dicto* knowledge, i.e., knowledge of merely possible correlation independent of practices—i.e., to the moot issue of circularity in the logicist definition. The second option of the sameness of physiological foundation will push the conditions of possibility of voluntary mathematical operations into passive events in the brain—an option fraught with various problems of various dimensions.

It needs to be emphasised that the subitised perception of number (whether it is for infants or adults) does not have an insular psychological content detached from the wider purview of preceding and succeeding behaviours. Any attempt to read a pre-practitional cardinality into the subitised content will actually turn it into a reified possibility of one–one correlation. This observation needs to be reinstated specifically with respect to the specialist reports on experiments conducted on infants or adults supposedly undergoing a subitised perception of number. When such readings or interpretations of experiments tend to hinge the perception on some physiological foundation (neural firing, brain Gestalt, etc., springing from external stimuli), we just need to remind ourselves that even if there are such unconscious causes of numerical cognition, they cannot play any role in the actual practices—one–one correlation or counting—in the wide network of voluntary activities that mathematics virtually is.

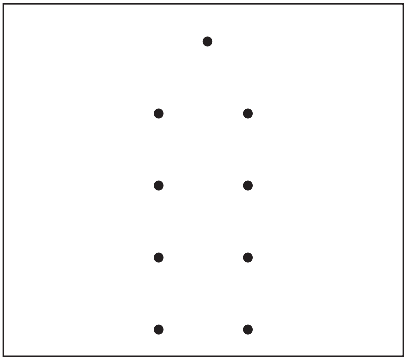
3. Placing the Charge of Circularity in a Wider Context

We may try to situate Wittgenstein's approach to this tension between number and one–one correlation within the broader purview of his general approach to language, concept-formation and family resemblances.

We have seen that our tools, like verbal definitions, physical or inner ostensions, cannot distance themselves from and yet capture their referents through a magical leap over an empty space, so to speak. We have sought to weave the same story of failure for all words like 'man', 'gold', 'green' and 'number'. To put it a bit otherwise, all self-identical and iterable essences—say the common property of manhood, or a neatly detachable chunk of nose commonly shared by all members of the Brown family, the same patch of green hovering over the green glass and its painted replica, or the same numerical identity jumping along lengths of visual fields and lengths of measuring rod—stand on the same level of absurdity. So what do we virtually say when we claim our verbal and non-verbal tools or concepts as independent of and yet carving out the exact boundary of their unbounded extension? And further, that they do so by representing such meaning-entities as the unitary manhood, the rarefied nose, or the floating colour patch? All such claims plainly amount to a decision not to allow any sense to people having the same nose or to things having the same green colour, except their being captured by *these* tools of definitions. And by the same logic, the only identity that I ascribe to my verbal or non-verbal definitions is that they capture *precisely these* cases that I wish them to capture. The inane obduracy of these moves starkly betrays the fact that there is no conceptual priority in our tools of definition or ostension that allow a genuine progression to the supposedly shared family features or the *common* green colour.

In the same vein, Wittgenstein argues that one can refuse to attribute any sense to the notion of sameness of number or its opposite to two groups—say to two groups of points, as in Figure 6.3—unless they are one–one correlated completely or partially. But by the same force of logic, one can refuse to attribute any sense to the notion of one–one correlation (or otherwise) to these two groups unless there is a sameness of numbers or its opposite. In neither of these cases is there a genuine advancement of thought from one to the other. Frege and Russell claimed that to attribute 'the number 5' to a concept or class F is to state its having the same extension with all classes that are one–one correlated with F: and from this they claimed to obtain the extension of all concepts or classes, viz., G, H, I, etc., that are equinumerous with F. Wittgenstein argues that once the proposed conceptual priority of one–one correlation is dismissed, nothing prevents one from reverting the game and deriving one–one correlation between F and G from their equinumerability. What underlies all these obscurantist mechanisms is nothing but an arbitrary stipulation to regard 'sameness of number' as synonymous with 'one–one correlated' in a piecemeal fashion; none of these notions would carry any alleged priority over the other, nor would they invoke any predictive power beyond specific pictures arbitrarily equated (*PG* Part II, section 21, p. 358).

Figure 6.3



4. Noting a Difference between Wittgenstein's Early and Later Views on Mathematics

Tracking down the routes through which Wittgenstein navigated from his early position on mathematics in *TLP*, via an intermediate phase in *PG* and *PR*, down to the thoughts of *RFM* and *PI*, is worth volumes of painstaking research. We shall satisfy ourselves by merely touching upon one chord of difference—the issue of *saying* versus *showing*—the one that Wittgenstein adored in the *TLP* and abandoned in his later writings. As in the later writings, the *TLP* too does not accord the mathematical propositions with any descriptive status—as descriptions of Platonic entities, mental workings or conceptual identities. They are said to be rules of equating signs that are equivalent, and these rules are claimed to govern iterable operations. Perhaps here in his early commitment to the overshadowing power of rules, we hit upon one of the primary notes of difference. But more importantly, in the process of denying that mathematical propositions are statements, Wittgenstein ended up saying that they *show* their logic, and in this seminal theory of saying versus showing, all the differences between his early and later views of mathematics are consummated.

A brief clarification of this *shown* status of the Tractarian and intermediate views of mathematics will hopefully explain why the later Wittgenstein abandoned this idea. We can perhaps start with some pointed observations in *PR* where he *does* seem to carry this distinction. Here Wittgenstein distinguishes between: (*a*) statement of number about an extension of a concept; and (*b*) statement about the range of a variable. He explains that the vital difference between the two is that while the former is a statement or a proposition, the second is not. The number of variables must show itself, it cannot be stated in the form of a proposition. I may try to capture the entire range of a variable by stating a material function that the values must satisfy. But then the sense of one proposition will depend upon the truth/falsity of another proposition—a process that would carry on ad infinitum. 'A statement of number about a variable consists in a transformation of the variable rendering the number of its values visible' (*PR*, section X, p. 133). Further, Wittgenstein goes on to state emphatically: 'No investigation of concepts, only direct insight can tell us that 3 + 2 = 5. . . . what enables us to tell that this expression is a tautology cannot itself be the result of an examination of concepts but must be immediately visible' *(PR* p. 129). As he further clarifies in the *TLP*, the substitutability of two signs flanked by an equation 'must be manifest in the two expressions themselves whether this is the case or not' (*TLP* 6.23). However, he qualifies that it is in the complex and embedded stages of calculation that the mathematical propositions come to show their senses (*TLP* 6.2331–6.241).

We can follow Juliet Floyd's analysis of the Tractarian shownness of the number statements as presented in de Bruin's paper. That the concepts F and G have 2 objects and that they are one–one correlated cannot be *said*, but *mirrored* or displayed in the internal structures of the following two existential propositions:

$$\exists x \exists y\ (Fx\ .\ Fy).\ \sim \exists x \exists y \exists z\ (Fx\ .\ Fy\ .\ Fz)$$
$$\exists x \exists y\ (Gx\ .\ Gy).\ \sim \exists x \exists y \exists z\ (Gx\ .\ Gy\ .\ Gz)$$

It is clear that in the *TLP*, what shows the possibility of one–one correlation between F and G also shows their cardinality, i.e., one–one correlation presupposes number. Thus in this framework too, Frege cannot uphold the conceptual priority of one–one correlation over cardinality.

What sets the Tractarian Wittgenstein apart from the one in *PI*, in spite of their markedly constructivist approach and their shared resistance to reified abstractions? We have already discussed this issue at length in terms of the later Wittgenstein's dismissal of a unique mode of analysis terminating into absolute simples. The present context persuades us to recast the difference in slightly different terms, rather through the strategy of playing up their similarity. Let us say that the early Wittgenstein in his ambition to demystify language and reality in terms of constructions ends up reifying constructions into an essential and self-interpretive identity. Though this constructivist stance apparently gives priority to human uses and applications, the usage is strictly determined within the cage of logical atomism. For the later Wittgenstein, on the other hand, language and reality cannot be chiselled into superfine constructions where they wear their meanings on their sleeves, where they speak for themselves. Whatever way one reduces complex abstractions into plain, simple and perspicuous constructions, the latter cannot hold itself back from practices; it has to be stretched out, enacted and enlivened within forms of living.

It needs further to be pointed out that the later Wittgenstein, in *PI* and *RFM*, seems to have broken through the traditional dichotomy between intensionalist and extensionalist positions. The growing insight that a concept underdetermines its extension does not leave him with the *only* option of a pre-linguistic extension constraining our usage. However, certain observations in *PR* do have an extensionalist leaning. 'If I have two objects, then I can of course, at least hypothetically, bring them under a class, and the concept encompassing it is still only a makeshift, a pretext.' And again, 'Numbers are pictures of the extensions of concepts' (*PR*, section X, pp. 123–24). Such remarks can be taken either as placing a special emphasis on the opacity and indeterminacy of concepts or as placing the notion of the extension as a referring game, or they may betray an intermediary phase with an extensionalist commitment before Wittgenstein finally emerged with his radically new thought adventure. For this final Wittgenstein, it is actions, behaviours and practices that shape language, meaning and reality—whether mathematical or non-mathematical.

Notes

1. Boudewijn de Bruin, 'Wittgenstein on Circularity in the Frege-Russell Definition of Cardinal Number', *Philosophia Mathematica* (III), vol. 16, no. 3, 2008, pp. 354–73.
2. Frege, *Foundations of Arithmetic*, section 63 (discussed earlier).
3. Mathieu Marion, *Wittgenstein, Finitism and the Foundations of Mathematics* (Oxford: Clarendon Press, 1998), cited in de Bruin, 'Wittgenstein on Circularity'.
4. De Bruin quotes this in pg 3 of the above-mentioned article. This quotation is from Waisemann Freidrich, *Wittgenstein and the Vienna Circle: Conversations Recorded by Friedrich Waismann,* 1979. (See de Bruin's article for the full reference.)
5. De Bruin, 'Wittgenstein on Circularity', p. 355n83.
6. Frege, *Foundations of Arithmetic*, section 48, p. 61.
7. Ibid., section 48, p. 61; section 54, p. 66.
8. Ironically it was Frege himself who treated this concept of indivisibility as a possible definition of 'one'. Obviously he would not be ready to deprive even such relatively indivisible objects of a pre-activational status.

CHAPTER VII

Mathematics, Language-Games and Forms of Life

As noted in the previous chapters, the difference between mathematical propositions and empirical ones does not lie in their respective epistemology or ontology. In other words, there is neither a special intuitive cognition nor a special realm of supersensible entities reserved for mathematics; nor is there a mosaic of discrete sensations and an equally fragmented reality allotted to empirical propositions. The celebrated distinction between these two kinds of propositions lies in the difference in our attitude and acknowledgement, the specific role they are assigned to play, and their respective modes of employment in relation to other propositions.

The talk of different roles and modes of employment attains its true philosophical significance only when we appreciate both mathematical and empirical propositions as being two different but mutually complementary language-games, embedded in a particular form of living. I shall start with a general account of language-games, place both empirical and mathematical games within this expansive purview, and thus attempt to get at the exact logic of their difference. In the course of this analysis, we shall be able to put both general language as well as mathematical language in a wider perspective, where all the related issues of language, its meaning, its foundations and forms of life, will be seen in the true Wittgensteinian spirit.

1. Language-Games and Forms of Living

We know that the analogy between language and games is doubly significant; it is not only that all the words of language like 'game', 'family', 'cat', 'gold', 'chair' have their meaning in a flow of family resemblances, but also that the word 'language' itself stands on the same footing. These two facets of the game analogy are logically related to one another. If there are no discrete meaning-entities to be named by each word of our language, the question of language having the essential functions of naming and describing does not arise. Language is now shorn of both its semantic and syntactic essence, its essential content *and* structure, which were presumed to run from sentence to sentence, i.e., through the entire length of language. We have seen that each use of language is a complex mechanism working in a distinctive style, and the parts of the mechanism have their meaning and function only with reference to that style. Each use of language is like playing a distinctive game, and the pieces of a game have their identity only in the manner of playing. There can be no question of a single part of a mechanism, say the handle of the crank being neatly cut off and joined to the brake (chapter I, section 8, 'Language as Tools or Levers'), or the same ball repeating itself in every ball game with an independent and self-identical essence. In point of fact, all the variegated language-games, like the various mechanisms and various games, are related by family resemblances, with no non-relational core recurring from one to the other.

In the Augustinian model, all words name their corresponding meaning-entities and all sentences (being combinations of names) describe the combinations of these meaning-entities. This descriptive essence of all sentences is traditionally formulated as 'S is P.' Thus, any question as to the possible *kinds* of sentences virtually reduces to the question as to the possible kinds of *species* that this 'descriptive' *genus* divides into. And the variety that is usually acknowledged is a fixed variety of grammatical types—the indicative, the interrogative, the injunctive, the exclamatory and the benedictive. While sentences are infinite in number, they are finite in kind, for the self-identical genus or the self-same descriptive essence is supposed to repeat itself in every possible usage; it is sometimes indicated, sometimes interrogated, sometimes commanded, sometimes exclaimed and sometimes benedicted. As Wittgenstein points out in *PI* 22, 'Frege's idea [was] that every assertion contains an assumption, which is the thing that is asserted.' He mistakenly thought that assertion consists of two actions, entertaining the sense or thought expressed by the sentence, and secondly, asserting or assigning the truth value. And we might add that on this mode of argument, one would naturally be tempted to postulate an identical thought-content running through all sentences, which is sometimes asserted, sometimes commanded, sometimes interrogated, and so on.

Wittgenstein says: 'There are *countless* kinds [of sentences]; countless different kinds of use of what we call "symbols", "words", "sentences"' (*PI* 23). He presents an impressive variety of language-games in this section of which we may mention a few: giving orders and obeying them, describing the appearance of an object or giving its measurement, constructing an object from a description or a drawing, reporting an event, speculating about an event, forming and testing a hypothesis, presenting the results of an experiment in tables and diagrams, making up a story and reading it, singing catches, guessing riddles, making a joke and telling it, translating from one language, praying, thanking, cursing, greeting. There cannot be a list of language-games; any such list is bound to be incomplete. Wittgenstein asks us to review the multiplicity of language-games in the above cluster and invites us to imagine many others.

'Here the term 'language-*game*' is meant to bring into prominence the fact that the *speaking* of language is part of an activity, or of a form of life' (*PI* 23). Each language-game is entrenched in a form of living and unless one participates in that, she cannot understand the game, though it might be phrased in a familiar vocabulary. A simple sentence like 'I know him' may be used to mean any of the following: 'I have met him once,' 'I have heard about him,' 'I am fairly acquainted with him,' 'I have seen that he has a vicious mind,' 'I recommend him,' 'You can rely on him,' 'You *cannot* rely on him,' 'He is capable of doing anything,' and so on.[1] It is not that the word 'know' is ambiguous between all these alternatives, each supposed to be definite and neatly recursible by virtue of its specific and well-stipulated meaning. Rather, in each case the meaning of the sentence consists in a different mode of activity set in a specific context of life.

As we have seen, Wittgenstein's trend of writing carries impressive suggestions to place the entire load of meaning on usage in the forms of life, and *not* on the specific word invested with a specific meaning figuring even as a minimal basis for the use to take off. If the supposedly common semantic content dissipates into use, it would have the same repercussions for the supposedly common syntactic structure of language as well. None of the expressions (words, or phrases) has an inherently simple or complex meaning to serve respectively as a referrer or a description. To take the above example of 'I know him,' the single word 'him' on the sheer strength of its use in different contexts allows itself to be stretched out into any of the above renditions without the aid of any other word(s) as elliptically synthesised with it. On the other hand, a different context allows the full sentence 'I know him' to perform a simple and rudimentary function of naming, i.e., to pose an individual at the start of a discourse without entering into any descriptive content about the referent.

Wittgenstein explains this with respect to the particular example 'block', 'pillar', 'slab', 'beam' in the context of the builder's game (*PI* 2, 6, 8). He points out that the single expression 'slab' is not the elliptical form of the longer sentence, 'Bring me a slab.' According to the Augustinians, the essence of speech is the composition of names, and a word uttered in isolation is just a name which cannot be *understood*. For it is only *meaning* and not *reference* that can be understood, and meaning emerges only at the level of syntactically complete composition of words, i.e., grammatically well-formed sentences. In that case, the word occurring in isolation if *understood* has to have an elliptical form of combination of words left unuttered due to laziness or some other philosophically uninteresting reason. For Wittgenstein, on the other hand, both reference and meaning are constructed in and through uses, the difference between the two lying in their respective modes of employment. He emphatically asserts that if 'slab' is to be considered as the *shortening* of 'Bring me a slab,' then the latter may well be considered as a *lengthening* of the former. One may just utter 'Slab' and *mean* 'Slab' (*PI* 19, 20). When a piece of wood is alternatively used as a pawn or a queen in the game of chess, its identity is shaped differently in the two different uses; it makes no sense to say that each use is elliptically contained in the piece of wood. The mode of using a word also lies in the manner in which we loosen it out in the surrounding expressions and behaviours in the stream of life, and not in bringing out the hidden elements packed into its insular content. '(In Russian, one says 'stone red' instead of 'the stone is red'; do they feel the copula to be missing in the sense, or attach it in *thought*?)' (*PI* 20).

While none of the above cluster of games with 'I know him' logically implies the other, we do happen to play all of them. In other words, all these language-games and forms of life are actually nested in a more pervasive pattern of living. One may, however, indulge in a more adventurous thought experiment, with a more alien form of living. We may suppose a community which never participates in any kind of imagination: people never make up stories or act in plays, children never walk on all fours and pretend they are lions. Language-games like singing catches, cracking jokes, play-acting, making suppositions, even though phrased in a known stock of words, would not be understood by the people of this community.[2]

One may take this occasion to recall the delightful play *Abak Jalpan*, roughly translatable as 'The Impossible Quench', written by Sukumar Ray,[3] where a thirsty traveller straying into a strange village puts a simple request for some drinking water successively to a number of persons, and each time his words were interpreted in a uniquely strange manner. Although he might have used exactly the same sentence, viz., 'I am looking for some drinking water,' it might be interpreted as: (*a*) a query for a popular book named *Some Drinking Water*; (*b*) an investigation into the exact taste and flavour of water of that particular locality in contrast with other localities; (*c*) a request for the possible ways in which these words, i.e., 'I am looking for some drinking water,' may be rhymed with other words and sentences; (*d*) a query about different locutions with the phrase 'some drinking water'; (*e*) an interrogation into the different kinds of speech acts that may be performed with the phrase; (*f*) an enquiry into the comparative chemistry of drinking water and that of polluted water, distilled water, etc.

This instance makes it quite clear that sympathising, sharing and caring too are significant forms of life, in the absence of which an array of familiar words and sentences may dismally fail to communicate. (Even if the traveller had put his request as explicitly as 'I am thirsty now, please give me some *real* water to drink,' in the absence of the required life pattern of give and take, requests and responses, the said request might be taken as a *statement* that the speaker's thirst can be exemplified in drinking a glass of water—thus starting off another style of discourse.) As Baker observes:

> Patterns of activity and response—following rules in the way we do, coping with the past, hoping for the future, caring for and educating the young, taking into account the interests and feelings of others—patterns so obvious as to escape notice, are constitutive of human life.[4]

One of the best appraisals of the exact significance of forms of life in its relation to language is perhaps found in Stanley Cavell, whose words are worth quoting:

> Nothing insures that this projection [of words from one context to another] will take place (in particular not the grasping of universals, nor the grasping of book of rules). . . . That on the whole we do is a matter of our sharing routes of interest and feeling, modes of response, sense of humor and of significance and of fulfillment, of what is outrageous, of what is similar to what else, . . . when an utterance is an assertion, when an appeal, when an explanation—all the whirl of organism Wittgenstein calls 'forms of life'. . . . Human speech and activity, sanity and community, rest upon nothing more, but nothing less, than this.[5]

When essentialists insist on a common stock of words and common descriptive essence shared by all language-games and forms of life, they are virtually turning language into inert ghostly entities. It is extremely important to realise that even if a lion uttered some familiar words in English, we would not understand him (*PI* p. 223). As the words cannot be absorbed into the entire being of the lion, his body and behaviour, his form of life, they would hang in the air like ghostly projections.[6] The essentialist endeavour compares with cutting the limbs off the entire organism, wrenching the different handles—those of the crank, the brake, the switch—from the entire mechanism, and displaying these severed parts with a great flourish, as the common underlying essence. Severed from the forms of life, words and sentences too are merely printed marks and sounds that are traditionally presented as the common essence of all language.

The picture of language that emerges is a dramatic contrast with the Augustinian model. According to most versions of the model, every word or language carries with it the whole of language. The *TLP* itself, which may be regarded as the most exemplary theory of essentialism, made explicit statements to this effect.[7] The author of the *TLP* had to traverse through real space (and not through 'logical space'), through real time (and not through time as a 'fixed form' of objects[8]). He had to play out newer and newer language-games, live through newer and newer forms of life to appreciate that language is as complex, as indeterminate and as unpredictable as life itself. With new forms of life, new language-games emerge, while old ones get obsolete or forgotten (*PI* 23). One cannot predict a new move of language any more than she can predict a new form of living, like the rise of feminism or non-objective painting.[9]

2. Mathematical Language-Games: Their Non-Revisionary Character

Once we appreciate asking, thanking, sympathising, cursing, greeting and praying as forms of life, we appreciate other games, like describing the appearance of an object, constructing an object from a description, testing and forming a hypothesis, presenting the results in an experiment, or predicting that certain things will happen, as forms of life too. (All these language-games are mentioned at *PI* 23.) And then it is also a short step to realising that mathematical games too, i.e., the games of freezing experiments, turning experience into paradigms, are as much woven into our ways of life.[10] It is true that unless one participates in such forms of life as experimenting, measuring concrete objects and forming hypotheses, one cannot freeze experiments, turn hypotheses into definitions. However, freezing an experiment is not freezing life or abstracting from it, but is a form of living itself. '[T]he mathematician, in so far as he really is "playing a game" *does not infer*. . . . And it would already be something *outside* [emphasis mine] the mere game for him to infer that he could act in this way according to the general rule' (*RFM* IV:1). A calculating machine does not itself calculate, it is always something outside the machine that does it. 'I want to say: it is essential to mathematics that its signs are also employed in *mufti*' (*RFM* IV:2). It is essential to the dignity of military uniforms that the soldiers be also seen in ordinary civilian clothes. 'It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics.' It is not a logical inference for us to make a change from one formation to another (say from one arrangement of chairs to another) unless these arrangements have a linguistic function apart from this transformation (*RFM* IV:2). Freezing experience into mathematical propositions becomes meaningful only in so far as the mathematical signs are also employed in the games of experience and experiment.

We know that in *RFM*, mathematical propositions are repeatedly said not to be descriptions but 'frameworks of descriptions' (V:2). This is not to conceive mathematics as a stoppage of language-games, but rather as an emergence of a new game, a new technique, a new mode of activity. Mathematical aspect-seeing often consists in a repetitive pattern and exacting rehearsals, yet the very act of giving out and responding to this training, i.e., the very act of teaching and learning these techniques, is something new, something spontaneous. And '[s]omething new (spontaneous, "specific") is always a language-game' (*PI* p. 224).

The special character of mathematical propositions is, as we know, that of non-revisability in face of experience, i.e., its truth value being independent of any empirical fact or sense-impression. This non-revisionary character of mathematics is just another side of its language-game character. For to say that mathematics is non-revisionary is not to suggest that it blocks experience and usage, but rather that it channelises experience into newer and newer cycles of dispersal and reversal, newer modes of seeing, newer transition of aspects. Secondly, non-revisability means non-revisability in the face of empirical facts, not in the face of mathematical consideration, such as discovering a contradiction in the system, or the introduction of new kinds of numbers, new methods of axiomatisation. Newer mathematical games are always being added to the old corpus. As a matter of fact, Wittgenstein used mathematics as a privileged entry point into his philosophy of non-essentialism; he cited the concept of number at the very outset as a prime example of family resemblance (*PI* 67, 68; also mentioned in chapter I). His idea was to extend the family resemblance character of mathematical language to language in general, while we usually find it convenient to adopt the reverse order in our reading of Wittgenstein. Speaking of the continuous adding and dropping of fibres in the open corpus of language, Wittgenstein makes a parenthetical comment: 'We can get a *rough picture* of this from the changes in mathematics' (*PI* 23). On another occasion where he emphasises the inherent indeterminacy and incompleteness of language, he brings in the example of mathematics: 'ask yourself whether our language was complete—whether it was so before the symbolism of chemistry and infinitesimal calculus was incorporated into it' (*PI* 18). Thus mathematics is an open nexus of language-games, each of which is inexorable and non-revisionary in the face of experience.

In the light of the language-game character of mathematics, its feature of non-revisability, with which we are fairly acquainted through our previous discussion, will need some further elucidation from a slightly different perspective. Each language-game has its meaning in the transitional links of similarities and dissimilarities with other games. The non-revisionary character of mathematical games lies in its family resemblances with two other games, the revisionary games of experiment and prediction. One can play many different games with the same ball, the same net and on the same ground. One can play many different games with the same sentence, like 'I am looking for some drinking water,' or

'I know him.' Similarly, Wittgenstein points out that we can play three different games, different though related, with the same proposition, say '81 + 81 = 162'.[11]

First, it may be looked upon as an experiment, either as a physical or a psychological one. We can use the above proposition to test the *physical* quality of ink and paper, whether the figures disappear, double up, or change due to some magical quality, thus leading to a result different from 162. Alternatively we can set a student to do the calculation, to see what time he takes to calculate, or whether he remembers his tables correctly. The experimental character of this proposition '81 + 81 = 162' in both the physical and psychological facets is more easily appreciated in long and complicated operations.

Secondly, one can also use the proposition in order to predict. A teacher assigns a student the problem '81 + 81 = ?', and makes the prediction that if you add 81 and 81 you will get 162. The teacher who previously experimented to find out that 81 + 81 do come up to 162, now makes the student calculate in order to confirm the result of his previous experiment.

We know how '81 + 81 = 162' becomes a non-revisionary game of mathematics. Both the experimental and the predictive character of the proposition are revisable by experience, but the mathematical character is not. In calculation we do not experiment or predict the answer. When we get a different result in our experiment, when we see the digits double up or disappear due to some unusual circumstances, we can at best say that we consider ourselves crazy. When the electrolysis of a liquid does not produce its usual result we may no longer have any idea what to say (*RFM* I:76). But the possibility of an experiment yielding a different result is not hereby excluded. Similarly, with the game of prediction too, the very concept (i.e., of prediction) involves its fallibility—if the possibility that a prediction can go wrong is excluded, then calling it a prediction is merely a wheel of language sitting idle. On the other hand, if we get a different result from our game of calculation, we decide that we must have miscalculated. In other words, as we have already noted, the games of proving and calculating lock the process and result into a single circle, while on the other hand, with the games of experimenting and predicting, their conditions do not include their results. Experiments and predictions are of the form 'It is like this,' or 'It will be like this'—they choose between one possibility and another. Mathematical propositions state 'It *must* be like this'—they see only one possibility. Experience teaches us that we all find a calculation correct when we start ourselves off and get the result of the calculation. But in the non-revisionary game of mathematics, we are no longer interested in having under certain circumstances actually produced this result. We are interested rather in the pattern of our working; it interests us as a convincing and harmonious pattern, not as a result of the experiment but as a path. We say not 'that's how we go,' but 'that's how it goes' (*RFM* II:69).

3. The Charge of Conventionalism

As we have seen, the signs of mathematics have meaning only against the background of experiment and prediction, i.e., speaking more generally, only as woven into the broad stream of life. Within the games of mathematics too, a proof has meaning not in isolation, but in so far as it is worked over, reproduced, re-employed in generating other proofs. This new non-revisionary game of mathematics is itself a motley of several language-games, for there are always new kinds of numbers that are incorporated, new notations introduced, and new proofs worked out from the old ones—in short, each move of mathematics forms a new criterion of re-identifying the old with the new.

A failure to appreciate the form-of-life character of mathematics as well as other (non-mathematical) languages has resulted in reading Wittgenstein as a 'radical' or 'full-blooded' conventionalist.[12] On a standard reading, a 'conventionalist' theory of mathematics would be one which holds all mathematical propositions to be either direct registers of conventions, or indirect ones derived successively from the direct conventions.[13] While the first option is branded as 'radical', the second option receives the characterisation of being 'modified'. Michael Dummett suggests that the logical positivist theory of mathematics had a potential advantage—while rejecting Platonism, it could still have explained necessity by declaring mathematics as a system starting with explicit conventions (in the shape of axioms, definitions and rules of inference), from which the theorems of the system can be successively deduced without the need of further conventions. But modified conventionalism failed to exploit this advantage, for pressed with the question as to how the theorems are arrived at on the basis of axioms, it preferred to put forth simply the conventional status of the rules of inference, without appreciating the fact that once axioms and rules of inference are posed as conventions, the subsequent passage to theorems does not rest on a *further* convention. Thus, in a way, *modified conventionalism*, failing to deploy the 'non-conventional' implications of conventionalism, lapses into *radical conventionalism*, where every step is an adoption of a new convention. Now Dummett calls our attention to Wittgenstein's explicit statements that a rule or convention, made once for all, does not have the power to foreshadow a unique set of applications, thus ruling out any unique derivative relation between one convention and another. He reads Wittgenstein as holding every move in mathematics—be it an elementary addition, or continuing the '+ 2' series, or working out each step of inference within a proof—as a new convention, free-floating and independent of any other convention that might have come before or after. This according to him makes Wittgenstein a conventionalist of the starkly 'radical' or 'full-blooded' version, without bearing any strain vis-à-vis the 'modified' option.

The charge of radical conventionalism against Wittgenstein may be fleshed out with a few concrete examples. First, mathematics sets conventions for counting, i.e., the convention for setting a 1–1 correlation with a standard sequence, a sequence that is defined in terms of ideal units with definite positions. We know that for Wittgenstein the notion of a paradigmatic set or ideal unithood is vacuous; it has no power of determining a unique way of counting all possible clusters of concrete objects or numerals. Thus, on the conventionalist interpretation, one would be adopting new conventions every time we count. Secondly, Wittgenstein's talk of freezing experiments into new pictures or paradigms and applying them to similar pictures would also be housed in terms of conventions. We know that for Wittgenstein, simple paradigmatic equations like 5 + 7 = 12 are formed by freezing the experiment of counting 5 units and 7 units that actually add up to 12. When we count 5 boys and 7 girls in a class and then declare (without counting further) that there are 12 children altogether, we are adopting a new criterion of identity. Now on the conventionalist interpretation, this talk of adopting a new criterion of identity virtually amounts to that of adopting a new convention, different from the one we have adopted when we had previously counted them as 5 and 7. If there are two genuinely distinct criteria or conventions, one for counting 5 + 7 and one for counting 12, one for counting groups of marks on paper, and one for counting groups of children, they might clash; but the necessity of 5 + 7 = 12 consists just in this: we do not consider anything as a clash, if we count the children all together and get 11, we say 'I must have miscounted.'

Thirdly, with mathematical proofs as well, our adopting this set of conventions does not in itself guarantee a unique conclusion for each step of the proof. To take the simplest instance of a mathematical proof we have already discussed, viz., 1 + 2 = 3, our explicit assent to the law of Association, i.e., (x + y) + z = x + (y + z), does not by itself guarantee that we shall acknowledge (1 + 1) + 1 as a substitution instance of the left-hand side of the above equation, nor does it insure that we recognise 1 + (1 + 1) as a unique conclusion from the previous conventions, viz., (1 + 1) + 1 and the law of Association. In both the steps mentioned, i.e., (1 + 1) + 1 and 1 + (1 + 1), we have adopted new conventions; neither of them rests upon our having adopted the law of Association at the beginning of the proof. Similar remarks would apply to the continuation of + 2 series, for the conventions on the use of '+', '2', the rules of addition, do not guarantee that the person would calculate 1,000 + 2 as 1,002 and not 1,004. At each step of a mathematical operation, at each step of a proof, we make new conventions; 'there is nothing in the formulation of axioms and of the rules of inference and nothing in our minds which of itself shows whether we shall accept the proof or not.' If we accept the rules of elementary computation (say 5 + 7 = 12 or the multiplication table), or theorems (whether shallow or deep), 'we confer necessity' on them; 'we "put [them] in the archives" and will count nothing as telling against [them]. In doing so, we are making a new decision, not merely making explicit a decision we have already made explicitly.'

4. Coping with the Charge of Conventionalism

The conventionalist interpretation fails to appreciate the simple but significant distinction between adopting a new convention, and playing a new language-game or living a new form of life. Formulating and following rules, setting conventions and obeying them, are themselves language-games, themselves forms of life, and unless one knows how to participate in them, one cannot teach or learn anything by conventions. One can teach and learn through ostension only in so far as he has already been practising the ostensive techniques, responding to the institution of ostension as a custom, a concerted activity—a form of living.

If setting orders and following them are forms of living (*PI* 23), so are setting rules and obeying them. Wittgenstein holds that rules or conventions cannot compel us either through indirect derivation (as held in the 'modified' version of conventionalism) or as independent decisions at every stage (as held in the radical version). A rule does not compel me to act like this, but 'it makes it possible for me to hold by it and make it compel me' (*RFM* V:45). And it is an interesting fact, rather an interesting form of living that people set up rules for pleasure and then hold by them (*RFM* V:45). We demand rules and definitions 'for the sake not of their content, but of their form'. 'Our requirement is an architectural one; . . . a kind of ornamental coping that supports nothing' (*PI* 217). It is like constructing an ornamental gateway at the entrance of the building, or making a beginning caption of a film, or a decorative cover of a book, where we find pleasure in using a picture, the picture of the entrance structure or the opening credits somehow suggesting or even encapsulating the entire content of the building or the film. More precisely, a rule seems like a 'short bit of hand rail by means of which I am to let myself be guided further than the rail reaches' (*RFM* V:45). 'But', Wittgenstein adds, 'there *is* nothing there; but there isn't *nothing* there.' The need for a handrail, the architectonic need for an entrance structure and the '*deep* need' for setting conventions and obeying them (*RFM* I:74) are themselves not conventions. They are nothing short of our forms of living.

Suppose the uses of language *are* learned through agreement or conventions, either derivatively or radically. In that case one has to understand the meaning of statements like '*This* agrees with this convention, *that* doesn't,' or 'Let us agree that this agrees with this rule or convention and that doesn't,' *prior* to learning any usage through conventions. And one cannot be said to learn the meaning of these terms or phrases, like 'agreeing with', 'rules' or 'conventions', through prior conventions, on pain of infinite regress. In point of fact, one does not learn to obey a rule by first learning the meaning of the word 'agreement'; rather, one learns the meaning of 'agreement' by learning to follow a rule (*RFM* V:32). More precisely, if one does not participate in the form of life of setting conventions and obeying them, no attempt to teach or learn through conventions can even get off the ground. Talking about a mathematical demonstration that patently claims that 'This follows inexorably from that,' Wittgenstein observes: 'This is a demonstration for whoever acknowledges it as a demonstration. If anyone *doesn't* acknowledge it, doesn't go by it as a demonstration, then he has parted company with us even before it comes to talk' (*RFM* I:61). Unless one knows how to respond to the request of a thirsty person, adding words, more and more words (like 'I am thirsty, give me some water to drink,' etc.) will drift into even more devious routes of interpretation. The roads of the strange villagers on one side and our thirsty traveller on the other have parted 'even before it comes to traffic by means of this language' (i.e., the request for water in this case) (*RFM* I:66).

It is not agreement or consensus of opinions but consensus of actions and reactions that shapes language, both mathematical and non-mathematical. Actions are not the consequence of language; rather, language is an extension of this consensus of actions, of forms of life, in the same manner as pain language is an extension of pain behaviour. And it is not only the consensus of the actions of an individual at different times, but the consensus of the actions of an entire community.

An original philosophical insight like this is best appreciated in the words of the philosopher himself:

> It is not possible that there should have been only one occasion on which someone obeyed a rule. It is not possible that there should have been only one occasion on which a report was made, an order given or understood; and so on—To obey a rule, to make a report, to give an order, to play a game of chess, are *customs* (uses, institutions). (*PI* 199)

> [A] person goes by a sign-post only in so far as there exists a regular use of sign-posts, a custom. (*PI* 198)

> [If a proof] got ratification from one person and not from another, and they could not *come to any understanding*—would what we had here be calculation?
>
> So it is not the ratification by itself that makes it calculation but the agreement of ratifications. . . . The agreement of ratification is the precondition of our language-game, it is not affirmed by it. (*RFM* V:6)

Speaking against any misconception as to rules conferring truth on subsequent applications, Wittgenstein remarks:

> 'So you are saying that human agreement decides what is true and what is false?'—It is what human beings s*ay* that is true and false; and they agree in the *language* they use. That is not agreement in opinions but in forms of life. (*PI* 241)

Colour judgements, like mathematical judgements, too cannot be justified, and it is characteristic of this language-game as of the other that other people consent to it without question (*PI* pp. 226, 227). In this connection, it is relevant to point out that Dummett seems to make the mistake of basing conventions regarding the meanings of colour words on given realities of determinate colour samples and self-explanatory ostensive procedures. He labours under the supposition that there exist saturated exclusivities of red and green colour samples on the one hand, and the subliminal zone between blue and green, equally given as a self-interpretive reality on the other. It is indeed from these instances that his notion of conventionalism along with its dual versions takes off. According to him, the ostensive conventions that we make about the meanings of the words 'blue' and 'green' leave out that slippery region between these two colours, thus necessitating an extra stipulation (in the 'radical' manner) to rule out any colour to be named as 'blue-green'. On the other hand, the ostensive conventions resting on the mutually exclusive realities of red and green samples make it possible for such conventions to *entail non-conventionally* (*qua* the 'modified' version) the further convention that nothing is to be called 'red and green' at the same time. In response to Dummett, we need to recall Wittgenstein's observations on both the opacity of ostensive procedure as well as the wayward movement of colours. Neither the determinate colour samples nor the intermediary region between any two such samples is given out there as readily available for ostension. To repeat a few instances we have mentioned before: it is possible for one to observe the colours red, blue and white as a single indissoluble whole named 'bu' and assimilate all other colours to 'non-bu' (*RFM* V:42). Any attempt to train her in wrenching out each colour from the trio and assimilating them to their supposedly *natural* continuums, say red to orange, blue to green, white to grey, would give rise to more and more disruptive exercises. Again, a person may not be able to trace the intermediate region between blue and green, or light and darker shades of red, but may readily respond to the instruction of finding something reddish green, a shade that may well be termed as blackish brown (*ROC* I:10, mentioned in chapter IV). Dummett's construal of conventionalism, its dual versions and above all its imputation to Wittgenstein do not accord with the latter's arguments and philosophical temperament.

In the non-revisionary games of mathematics, as already mentioned, we are not interested in the experiment of actually producing this result, but in the path, in the convincing and harmonious pattern of experiment. But as Wittgenstein has repeatedly exhorted, there is nothing in the units of calculations, in the rules we make, nothing in our minds, which compels us to accept the proof; which compels us to be engulfed in the closed circle of premise and conclusion. It is nothing but a 'great . . . and interesting . . . agreement' (*RFM* I:35), an interesting form of life that we all accept *these* rules to have *these* applications, accept *these* patterns as proofs of *these* propositions, that we *all* allow ourselves to be engulfed in the same physiognomic cycle of aspectual transition. 'And how does it come about that the proof *compels* me? Well in the fact that once I have got it I go ahead in such-and-such way, and refuse any other path' (*RFM* I:34). '[T]his is simply what we *do*. This is use and custom among us, or a fact of our natural history' (*RFM* I:63).

Disputes do not break out among mathematicians as to whether a rule has been obeyed or not. There can be dispute over the correct result of a calculation, but such disputes are rare and short-lived, and can be decided with certainty. Mathematicians in general do not quarrel about the result of a calculation (*PI* p. 225). This is a part of the framework in which the working of our mathematical language, its peculiar inexorability, is meaningful. If, for instance, one mathematician was convinced that a figure had altered unperceived, or that her memory had deceived her, and so on, then our concept of mathematical certainty would not exist (*PI* p. 225). This pervasive agreement of actions is in its turn embedded in an environment consisting of a fair amount of regularity—a regularity in the behaviour of physical objects as well as in human nature. The paper and ink, i.e., the materials of calculation, do not behave unpredictably, the units do not disappear, coalesce, multiply at random. Lumps of cheese when put on a balance do not suddenly grow or shrink for no obvious reasons (*PI* 142). We who calculate do not usually forget to count the units, do not suffer from optical diseases resulting in a double vision. Now these factual regularities form the *background*, and not the foundation or content of mathematics. The fact that some units and the writing materials do change haphazardly is known by memory and comparison with other means of calculation. The question that inevitably arises is how these other means of calculation are to be tested in their turn (*PI* p. 226). Likewise, mathematics does not depend on the psychological characteristics of men, their faculty of memory, their conviction in mathematical proofs. Men may forget their convictions (*RFM* I:63), and any statement about the faculty of memory can be put to further tests. Thus Wittgenstein remarks:

> But am I trying to say some such thing as that the certainty of mathematics is based on the reliability of ink and paper? *No*. . . . I have not said *why* mathematicians do not quarrel but only *that* they do not. . . . What has to be accepted, the given, is—so one could say—*forms of life*. (*PI* p. 226)

Mathematics derives its *meaning*, not *truth value*, from these regularities. These regularities do not form the hypotheses from which we deduce the mathematical propositions as their consequences. For if such regularities turn out to be false, we would not have a different mathematics, but no mathematics at all. As Wittgenstein has further remarked at *PI* 142,

> if things were quite different from what they actually are—if there were for instance, no characteristic expression of pain, of fear, of joy, if rules became exception and exception rules, or if both became phenomena of roughly equal frequency—this would make our normal language-games lose their point.

False moves of mathematics can exist only as exceptions, for if what we now call 'exceptions' became the rule, the game in which there are false moves would have been abrogated (*PI* p. 227). Mathematical calculations as well as the usage of colour words like 'red', 'blue', etc., would lose their point if confusion supervened. But it seems nonsensical to say that a proposition of mathematics or any semantic definition of ordinary words of language *asserts* that there will be no confusion, or no anomalies in the physical and psychological facts. We cannot say that the use of colour words, say 'green', signifies that confusion will not supervene—'because then the use of the word "confusion" would have in its turn to assert just the same thing about *this* word' (*RFM* II:75). Practice depends on there being a regularity; if there were too much confusion and anomalies, there would be no practice, and hence no sense, no language. Confusion does not result in false mathematical propositions, but in non-sense.

While emphasising that mathematics cannot be descriptive of, or be founded upon, physical or psychological facts or conventions (as all these *facts* can be put to further tests or be measured), Wittgenstein affirms: '[M]athematics as such is always measure, not thing measured' (*RFM* II:75). And some thing can be claimed to be the ultimate measure not by its inherent quality, for any such purported quality can be put to further tests. The unalterable nature of writing materials can be questioned, the faculty of memory and the mental state of conviction invite measures for assessing their stability, and any convention claiming to have a privileged explanatory value would always involve a regress into further conventions. The peculiar unassailability of mathematics, its ultimate paradigmatic character, cannot be rested on a foundation; it is *lived* through a pervasive consensus of actions. We confront this concept of unassailability in many ordinary games—e.g., the 'post' of the children's game of 'chase' is considered invincible, the 'joker' in the game of cards is made all-powerful, certain spaces on the ludo board (those marked with a star) are made uncapturable. No doubt these are declared to be unassailable, not on the strength of a supposedly inherent property that they possess, but as conventions; however, the convention to regard them as 'unassailable' rests on 'a deep need for conventions', a deep need of setting conventions *about* ultimate measure, ultimate unassailability, a need which is nothing but a form of life.

5. Deviant Mathematics—Deviant Forms of Life

Wittgenstein's texts, specially *RFM*, abound in instances of strange or deviant mathematical operations and also deviant language-games in general, like those on colour. To handle the overwhelming variety of these deviant games, we need to categorise them; and the best way to do so is to track down in each case the specific pattern of living they are entrenched in, or rather the specific directions in which they deviate from *our* forms of living. We have already identified, in the course of our previous discussion, some pervasive practices that specially underline our mathematical operations. These are 'really remarks on the natural history of man: not curiosities . . . but rather observations on facts which no one has doubted and which have only gone unremarked because they are always before our eyes' (*RFM* I:141). Let us recall a few of them:

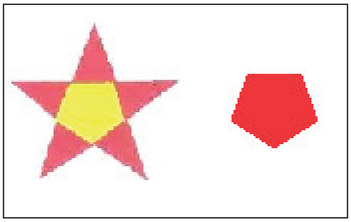
(a) We freeze experience into flat, inert paradigms abstracting from all real interactions and dimensions. We do not operate with units and figures that undergo unpredictable causal changes, we do not calculate with elastic rulers. The flat units and operations of arithmetic and the frozen properties of space are supposed to recur neatly from one operation to another.
(b) There is a wide consensus of actions among all individuals and in one individual at different times, as to what is the *same*, and as to what is the *same* rule yielding the *same* conclusion.
(c) Men agree in following the rules of counting and measuring in the *same* way, they agree in not leaving out units, not to count a unit twice, not to stop after coming to a certain number in a set and leaving the rest uncounted.
(d) The propositions of mathematics are abstract and general, and are logically prior to their concrete applications. While the concrete objects of calculation differ from one another, the principles of pure mathematics they embody are universal and invariant.
(e) Any perceptual illusion or memory lapses about the actual nature and number of units, if detected later, will overthrow that particular calculation. In other words, no epistemological lag is tolerated in mathematics.

These are, as already mentioned, not the rules or conventions we stipulate; they are rather the ways in which we live, and which form the background of our mathematical operations. To have a different mathematics is to live a different form of life. A rough picture of this may be sketched from Wittgenstein's texts, *in contrast* with *our* form of life, the specific trends of which have just been recounted. We take care to lay out this sketch of deviant mathematical life in that order of contrast.

(a)[1] It is imaginable that someone observing a coloured surface should see the combination red–black (say of a flag), but if he sets himself to see *one* of the two halves, he sees blue instead of black.

Similarly, one looks at a group of apples and always sees it as two groups each consisting of two apples, but as soon as she tries to take the whole lot at a glance they seem to her to be five (*RFM* V:43). Suppose a person is shown the picture in Figure 7.1.

Figure 7.1

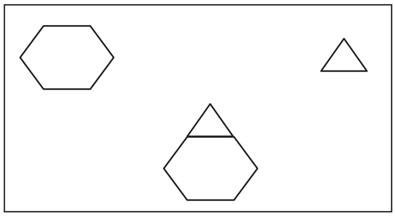


Asked whether he sees a red pentagon, he says 'yes,' and asked whether he sees a yellow pentagon, he says 'no.' It seemed to him that the colour in the star cannot be divided because the shapes cannot (*RFM* V:44). It is possible to see the complex formed of A and B without seeing A *or* B; it is even possible to call the complex 'the complex of A and B' and to think that this name points to some kind of kinship of this whole with A and with B. Thus it is possible to say that one is seeing a complex formed from A and B but neither A nor B, just as it is possible to say that one is seeing a reddish yellow but neither red nor yellow (*RFM* V:47). In other words, the mixed colours like reddish yellow or bluish green on the one hand, and what we think to be 'atomic', 'saturated' 'primary' colour samples or discrete symbols like 'A', 'B', 'C' on the other, stand on an equal footing; the latter do not derive their clean, atomic insularity from a pre-linguistic ontology. '[T]he observer is pre-occupied with a particular aspect,' the *kinship* of A and B; 'he has a special kind of paradigm before him'; he is engaged is a particular routine of application.' Thus, only A.B strikes him, and not A or B (*RFM* V:47). We have already dealt with these kinds of examples before, where a person was trained to observe that the surface was coloured red, blue and white (as in the French tricolour), but not to observe it as either red or blue or white. The colour adjective used for this tricolour is 'bu', and the person only knows how to report 'bu' or 'not-bu'. Let us recall the further instance of a person observing a surface composed of a number of strips changing their colour every moment. Being preoccupied with the question whether it is going to turn green or not-green, she sees it *simply* as not-green, she does not register any of the constituent colours that appear at that moment (see chapter IV). All these cases effectively bring out different modes of referring or identification, thus reinstating Wittgenstein's critique against pre-given identities as ultimate referents upheld in the theory of logical atomism.

These cases show that one may not be playing the game of splitting full pictures of coloured objects into flat isolated fragments, or be seeing these fragments as jumping from one picture to another. The person who sees simply the *kinship* of two numbers or two colours (i.e., what for us are two numbers or two colours) and not *either* of them individually, is not missing out pure, recursible colour essences that are really out there. That she responds to a different 'training', that she has a 'special paradigm before her', or is 'preoccupied with a different aspect'—all this talk cannot be reduced to her adopting different conventions. She is simply participating in a different custom, a different form of life. She is not adopting two different conventions—one for colours or numbers in unbreakable 'combination', and another for them in 'isolation'. Rather, at each stage she is playing a *new* language-game, just as we do at each stage of repeating our 'self-same' units from context to context. While she at each stage plays the new game of differentiation, we at each stage play new games of repetition or recursion. We play the games of freezing while she plays the game of not-freezing, and both these games may employ a vocabulary that is familiar to both of us. And yet one may not understand the other, just as the strange villagers failed to respond to a simple request for water phrased in familiar words. And if one of us can be trained to play the other game, that would not be the strength of a single saturated foundation, characteristically different for each case. It will simply be what we do.

Measuring with an elastic ruler (*RFM* I:5) is not simply adopting a new convention of measuring, for, as we have seen, such conventions become effective only in a form of life where such modes of measuring are deemed necessary. At *RFM* V:40, Wittgenstein cites an instance where shapes are added together in a way that their edges fuse, like in Figure 7.2.

Figure 7.2



Such modes of addition play a very small part in our lives. 'But if it were an *important* operation our ordinary concept of arithmetical addition would perhaps be different' (*RFM* V:40). Similarly, measuring with an elastic ruler and comparing the measurement with that of a steel or wooden ruler might be a very significant custom in a particular form of life. One can play yet another game of measuring objects with a ruler which expands to an extraordinary extent when slightly heated. Suppose we perceive the expansion with the naked eye and we ascribe the same numerical measure of length to bodies in rooms of different temperatures, if they measure the same by the ruler which is now longer, now shorter. It is natural to object that what we here call 'length', 'equal length', and 'measuring' are different from their normal usage. But Wittgenstein thinks that '[t]he use of these words is different from ours; but it is *akin* to it; and we too use these words in a variety of ways' (*RFM* I:5) Considering the fact that what we call 'normal' acts of measuring do not share a pre-applicational identity of the measuring scale, or ideal units of measurement, we can say that our acts of measuring are only *similar*, a similarity which can shade into a different mode of measuring, i.e., measuring the 'same' length with a ruler of changing sizes. Further, one cannot explain the strange uses of 'length', 'same length' or 'measuring' simply through new convention, for such conventions would have to be embedded in a form of life with a deep *need* for such conventions.

There will no doubt be a persistent resistance and indignation against Wittgenstein's attempted 'trivialisation' of mathematics, along with the impression that all these strained examples of deviances are actually petty and inconsequential. A fresh consideration of the phenomenon of FitzGerald contraction, i.e., the contraction of the measuring scale proportionate to the velocity of the planet in which it is placed, might be relevant to reorient ourselves to Wittgenstein's insights.[14] According to the theory of FitzGerald contraction, the measuring scale, which is nothing but a swarm of electrical particles with a volume delicately balanced between opposing forces, inevitably contracts in accordance with the velocity of the planet in which it is carried. This rod which was at first pointing transversely to its line of motion, in order to be placed against the side of the measurable object, has to be turned through a right angle to point along the line of its motion. If the velocity of the planet happens to be 161,000 miles per second, the FitzGerald contraction would be half its original length. With such an upheaval in the measuring system, all the conceptual and perceptual structures would be changed accordingly. We cannot rest comfortably in the supposition of being on a decently slow-moving planet (the earth's velocity being 19 km per second) where the F-contraction is negligible—only 2½ inches in the diameter of the earth—while an imagined planet, out in a spiral nebula far away in space, is moving at, say, 1,000 miles per second. Measurement of velocity is always relative to the position of the observer; so if we see them moving at a speed of 1,000 miles per second, they too will see us receding at the same speed. For the same reason, we cannot invoke another planet of another galaxy, with a different velocity, to arbitrate on the issue. There will be endless relativisations, without ever coming to the calculation of an absolute velocity. Thus there is no conceptual absurdity in holding that our measuring scale is undergoing a considerable contraction to half its length, whenever it is adjacent to its line of motion. This contraction will be systematically concealed by an exactly compensating contraction of any device that we may use to detect it—a second measuring scale, a theodolite, the retina of our eyes, and so on. Now, to finish up this story in favour of

Wittgenstein, we have to rule out the possibility of a supremely objective measuring system, a unique and absolute system of numbers which will be able to track down all these schemes of relativisation *non-relativistically*, holding itself away from space, away from velocity and the network of interrelations.

(b)[1] While emphasising that agreement of actions is a *precondition* of mathematical calculations, Wittgenstein says that by contrast, we can imagine another game, or another form of life where 'people were prompted by expressions (similar perhaps to general rules) to let sequences of signs come to them for particular practical purposes, i.e., *ad hoc*...' (*RFM* V: 7) For instance, they may be guided by one rule for measuring the floor area, another for counting the number of boxes to be loaded on the truck; and Wittgenstein observes that 'this even proved to pay.' (*RFM* V: 7) Further, the calculation of one person may not agree with that of another. At *RFM* V:36, Wittgenstein further remarks: 'it is not clear that the general agreement of people doing calculations is a characteristic mark of all that is called "calculating"'. He imagines a situation where people who have learnt to calculate, say under the influence of opium, begin to calculate differently from one another. They make effective use of their calculation and also claim such differences to be reasonable. It is like a group of musicians given the same musical score, where each musician plays it in a different way under the influence of opium, a difference which is considered acceptable. Here again the apparent triviality of such modes of deviance has to be checked against constant reminders that our standard units and modes of reference take off from a common practice, and from a unique reality beyond those practices.

(c)[1] One may count in a strange way—for instance, we might find it convenient to leave numbers in a set (*RFM* I:139), call two items by the same numeral, or one item by two, change the order of numerals by suddenly going back, or even break off after counting up to a particular number of items in a group as the rest does not matter to us. Now apparently such ways of counting can be dismissed straightaway by the traditional definition of counting as a strict *one–one* correlation with a standard sequence, a sequence with each unit having a definite identity, a sequence which must be a progression, that is, there must be a first unit but no last. The progression employed must also be recursive in the sense that for any pair of its elements, which of the pair belongs earlier in the sequence must be decidable. All these criteria are supposed to preclude the possibility of leaving units, tampering with the order (like counting 4, 3, 7, 6 and so on), or stopping short of the entire series, leaving the rest uncounted. Now as we know, the definition of counting and of the standard sequence does not determine a unique way of counting for all possible cases. More precisely, it is the strategy of mathematicians to enclose both the definition of counting and the actual operations in a single circle. And as already mentioned, carving out a circle and conforming to it is itself a form of life, which may not be lived or practised universally.

In the Frege-Russell scheme of mathematics, starting with the concepts of 0 and of 'immediate successor', one can generate an infinite sequence of numbers as 1, 1 + 1, (1 + 1) + 1; and the self-identity of '1' is supposed to recur along the entire series. In other words, each numeral has a logical gap within itself that logically invites its successors up to infinity, for in spite of their professed dismissal of Platonic realism, their language of numerals turns out to be internally geared to a reality containing infinite numbers. Stopping after 50, when there are 100 apples in a box, or leaving numbers within a series, is not only a disservice to the rule of counting but a disservice to reality. The later Wittgenstein would point out that the deviant ways of counting do not stop short of reality, but stop short at certain authoritative impositions practised in our 'normally standardised' society. They are just different language-games played in different forms of life. Counting in the normal way is deeply grafted onto the way of our life, 'and that is why we learn to count as we do: with endless practice, with merciless exactitude; that is why it is inexorably insisted that we shall all say "two" after "one", "three" after "two" and so on' (*RFM* I:4). And that is why, in counting the number of items of a particular set, it is inexorably insisted that we count up to the last member. The practices of leaving units, or merely seeing 50 out of 100 apples in a box, are comparable to playing a simplified game of chess (with a smaller number of pieces), or 'one-day' cricket matches instead of the traditional 'five-day' tests. Both the deviant games of counting noted above as well as the simplified games of cricket and chess are *new* games, they are not pieces chopped off the 'normal' wholes. And they do not have logical gaps waiting to be filled up by full normal games. And thus neither of them stops short of (or conflicts with) reality in any sense of the term.

(d)[1] Wittgenstein tries out a different concept of mathematics which teaches us experimental methods of investigation, or teaches us to formulate empirical questions. It can be questioned whether a body moves according to the equation of a parabola, or whether its path can be represented by the construction of an ellipse with two pegs and a string (*RFM* V:37). Here, we need to appreciate that this is a new game of mathematics—it is neither experiment, nor freezing experiment (as our mathematics traditionally is) nor making *applications* of general mathematical rules. The results of experiment (say of counting) are always formulated in general terms, and the practical applications of mathematics are *by definition* subordinated to the general mathematical rules in the background. This concept of mathematics on the other hand is *new* in the sense that it simply addresses questions about individual objects, and does not formulate or need any pure mathematical rules as its starting point. We have seen that the abstract paradigms of mathematics (that actually underdetermine the applications) serve the architectonic requirement of an ornamental starting point. Now we see that there can be people doing calculations without that architectonic or ceremonial requirement.

At *RFM* V:40, Wittgenstein further points out that one may learn mathematics without learning the distinction between mathematical and physical facts. When a child learns that a square piece of paper can be folded in such and such shapes, he is not conscious of the two kinds of possibilities—viz., the geometrical possibility and the physical possibility. He does not consider whether the results of folding are due to certain physical properties of the paper. Wittgenstein says that we can have a mathematics that does not imbibe the concept of a mathematical and a physical fact; we merely know that this is always the result when we take care and do what we have learnt (*RFM* V:40). This new game of mathematics is also different from Mill's concept of mathematics as inductive generalisation on the behaviour of empirical objects. For unlike inductions, which can be falsified by a negative instance, this mathematics of physical objects may yet be non-revisionary.

One may teach someone to build a house and at the same time how she is to obtain a sufficient quantity of materials, boards, etc., and for this purpose we may teach a technique of calculation. This technique of calculation is not backed up by a mathematical rule, it is part of the technique of house building (*RFM* I:142). People may have a particular technique of selling logs—they pile up logs and measure the length, breadth and height of the pile, multiply them together, and what comes out is the number of pence which has to be paid. People doing these calculations need not utter any propositions of arithmetic (*RFM* I:142). The multiplication tables or the measuring scales they might be using are constantly adjusted to the demands of the intractably varying specificities of the concrete situations.

At *RFM* I:148, Wittgenstein further imagines that such people pile timbers in arbitrary heaps and then sell it at a price proportionate to the ground area. Suppose someone spreads out a small heap containing a smaller number of timbers in an area more spacious than a heap containing a larger number. These people will say, 'Now it is more wood, and one must pay more.' Here they are not simply adopting a new convention whereby the expressions 'more wood' or 'less wood' mean something different, viz., 'more or less time and energy required to arrange the wood into a large or small area'. For, as we have pointed out on several occasions, such conventions are possible only where these practices of selling wood are already in vogue. There might still be similar practices of selling timber by weight and the time it took to fell the timber or, stranger still, they might hand it over to a buyer for any price that the buyers wish to pay, or may simply give away the wood. '[T]hey have found it possible to live like that' (*RFM* I:147).

Wittgenstein cites more instances of some curious transactions with coins (or, what look like our coins), where each person gives just what he pleases for the goods and receives any amount she wishes from the merchant (*RFM* I:152). Now if these practices are arbitrary customs or institutions, so is *our* custom of calculation. Our practice of counting units with 'merciless exactitude', our laws of punishing people who make mistakes in counting (public) money, our obsession with flat, repeatable fragments, are as much customs and institutions, and as arbitrary as those just noted. And considering the fact that often our custom of ruthless counting and calculation does not contribute to social health and justice, they seem to assume the vulgar pomp and pretentious dignity of ceremonies. One is tempted to cite a particular ceremony, viz., the coronation of the king. For Wittgenstein suggests that if the deviant practices are called purposeless or insane, so are our religions, actions and ceremonies like the coronation of the king (*RFM* I:152). Let us add to this a more provocative comment, 'So are our *normal* mathematical operations.'

The last two or three instances, if they are to preserve their genuinely deviant character, cannot be made intelligible on the basis of a common frame of reference or a common conception of units. That is, for both our standard practices with timber and coins, as well as for their deviant practices with what look like our timber and coins, it is the practices themselves that shape the units. Hence, the common charge that Wittgenstein is merely showing certain superficially weird manipulations with pre-existent chunks should be reframed and reconstituted to suit the philosopher's insights that constantly blend the seemingly external ruptures of meaning with the internal ones.

(e)[1] A devil might have been deceiving me in all my calculations, and I keep overlooking something however often I go over it step by step. Wittgenstein asserts: 'But what difference does it make for me to "assume" this? I might say: "Yes to be sure, the calculation is wrong, but this is how I calculate."' (*RFM* I:135). Once we truly appreciate the language-game character or the form-of-life character of mathematics, the ascription of right or wrong to its paradigms, or the threat of an epistemological lag (say a perceptual error or a memory lapse) is rendered pointless. This is as pointless as to be threatened by the sudden knowledge that we had so far been reading one face of the die as having seven dots and have been playing accordingly, while as a matter of fact there are *six* dots inscribed on it. It does not matter whether we have played one face as seven, for all the players have played like that. It does not matter whether it is wrong, it does not conflict with truth. Similarly, an epistemological lag in mathematical games does not conflict with truth.

The talk of the devil's deception in mathematics is designed in a deliberate contrast with Descartes. It should be clear at the very outset that for Descartes, mathematical properties and relations are revealed instantaneously through the 'natural light of reason', and hence cannot be subjected to the manipulation of the evil genius. In Descartes's philosophy, any possibility of doubt or deception in mathematical cognition may be occasioned in three different ways: (*a*) doubting remains a psychological state of mind unless and until the existence of the Perfect Being is proved, for as far as the order of knowledge (*ordo cognoscendi*) is concerned, the perfect correspondence between clear and distinct ideas on the one hand and mathematical reality on the other follows from this Perfection; (*b*) the doubt accrues only to the *memory* of the mathematical rules and axioms used in the later stages of the proof; (*c*) the doubt accrues only to our faculty of judgement, the faculty we extend *beyond* the range of clear and distinct perceptions (of pure mathematical properties in this case), to the numerical and spatial properties of empirical objects. Thus, Descartes' deployment of the devil's deception does not affect mathematical cognition; on the contrary, there is a strong reassurance of the perfect balance between mathematical cognition on the one hand, and mathematical reality on the other. For Wittgenstein too, any perceptual error or memory lapse contrived by the evil genius will not create a lag or gap in the harmony between mathematical language and reality. This is for the simple reason that there is no metaphysical harmony *out* there; rather, it is a harmony that we create through the closed physiognomic circles, which is again nothing but a custom, an institution. There might be a mathematical practice or mathematical custom that imbibes the so-called epistemological lag, while the mathematics *we* practise does not. This lag will not invalidate that mathematical custom or mathematical game; it will only turn out to be a different game, a different mathematics, a different form of life.

6. 'The Whirl of Organism': The Issue of Deviance

All theories that seek to *explain* or *justify* the use of language employ a two-tier mechanism. On one side there is language, and reality on the other, i.e., on one side there are names and compositions of names, while objects and facts are on the other side; there are acts of ostension on one side, and ostensible objects on the other. When language manifestly claims to be not about reality, but about the meaning of language itself, even then, language (i.e., rules, conventions, etc.) is positioned on one side, and the unique set of its applications is supposed to reside on the other. The theory of radical conventionalism foisted on Wittgenstein is a theory in which there is nothing in the real nature of objects, nothing in the general rules, and nothing in our mental images or thoughts that guarantees the recursion of words from context to context. Such an extreme theory of conventionalism too, like all theories of realism, is deeply entrenched in the Augustinian model. On this theory, reality is full of dissimilar objects, say a bottle, peg, stick, apple, etc. on one side, and a cluster of dissimilar labels, 'A', 'B', 'C', 'D', etc. on the other. At each stage we are free to stick whatever label we wish on whatever object. At each stage we are free to name a particular coloured object as either 'red', 'blue', 'black', or whatever, for there is nothing in the nature of the coloured objects, nothing in the nature of ostension (say to the colour 'blue'), that guarantees that we shall agree in using the same word 'blue' for the same class of objects. With mathematics too, as already mentioned, there is nothing in the real nature of units, nothing in the nature of rules, which ensures that we work out each step in the same way. Here it is quite clear that the theory of radical conventionalism is set into the Augustinian model; on one side there are isolated and free-floating conventions at each stage, and the actual *applications* of these conventions or actual usage on the other. (At each stage we make conventions like: 'Let us agree to say that there are 7 units and 5 units,' 'Let us agree that 7 + 5 = 12,' 'Let us agree that 1 + (1 + 1) follows from *this* rule and *this* step.' Our actual *applications* of these conventions, i.e., our actual saying that 7 + 5 = 12, and 1 + (1 + 1), lie on the other side.)

Now Wittgenstein has pointed out that unless there is a consensus of actions—of setting rules and following them in the *same* way—one cannot even assume the stance of setting individual conventions at each stage and claiming each usage as following from that. Such suppositions seem to make sense only with a spurious dichotomy between language and reality, between rules and applications. Disagreements, i.e., making *different* agreements, or adopting different conventions, make sense only against the background of a pervasive consensus of action. And this consensus of actions is an organic, functional whole, a whole where ostension cannot be split from the ostended, language cannot be split from reality, and rules cannot be split from their applications. The nominalists with all their resistance to essentialism and absolutism fell into the Augustinian pitfall of interpreting all words as names, though arbitrary names. They failed to inject life into language, could not make language take off and blend into meaning and usage, and ended up giving only a truncated blueprint of language (*PI* 383). The entire practice of ostension and its fulfilment, i.e., the sense of achieving the ostended reality, is a custom, a common pattern of behaviour, *within* which one can apply alternative terminology or alternative modes of ostension. The entire practice of setting up rules and following them, the entire belief that the same rules will entail the same conclusions, and above all the firm conviction that language, if properly applied, will take us to a unique external reality—these are what Cavell calls the 'whirl of organism'. We may take this occasion to remind ourselves again of his words:

> That on the whole we do [i.e., project words from one context to another] is a matter of our sharing routes of interest and feeling . . . senses of humor and of significance and of fulfillment, . . . all the whirl of organism Wittgenstein calls 'forms of life'. (Quoted earlier in section 1 of this chapter)

It is within this whirl that one can make alternative conventions or variations like calling a particular shade as 'red', 'crimson' or 'peach', or describing a colour as 'bluish green' or 'greenish blue', or faithfully following all the colour words listed in the lexicon, or setting up an entire colour scheme under 'bu' or 'non-bu'. On the theory of radical conventionalism set in a false dichotomy, one can at each stage adopt a different convention to apply a different colour word; in which case the entire 'whirl of organism'—the entire cycle of actions that constitutes language—would be split into impossible fragments, nullifying all language, all actions, all 'sense' or 'meaning'. Similar remarks would apply to mathematical language if the false dichotomy between rule and applications and the spurious freedom to choose new conventions at each stage are allowed to persist. Differences or deviations are possible not by making deviant conventions, or hypostatising epistemological lapses, or deviant brain patterns; for every such programme operates in a false dichotomy tearing the immaculate whole of language into ghostly fragments. Differences are intelligible not by splitting the whole but by carving out a different whole, a different 'whirl of organism', a different 'style of painting', where all strangeness and deviances are woven into that seamless complex. 'I am not saying: if such and such facts of nature were different people would have different concepts (in the sense of a hypothesis)' (*PI* p. 230). This is because alternative hypotheses are possible only within the same form of life, the same whirl of organism. Rather,

> if anyone believes that certain concepts are absolutely the correct ones and that having different ones would mean not realising something that we realise—then let him imagine certain very general facts of nature to be different from what we are used to, and the *formation* of concepts different from the usual ones will become intelligible to him. (*PI* p. 230; italics mine)

That is, let him carve out a circle different from the usual one, let him imagine a different form of living, let him innovate a different style of painting. 'Compare a concept with a style of painting' (*PI* p. 230).

That disagreement is possible only against the background of a pervasive agreement, has virtually turned into a slogan which holds both the popular imagination and the philosopher's craving for submerging all differences in identity. It is important to be clear about how and in what sense Wittgenstein uses it in his philosophy if we do not wish to swerve onto the wrong track. During the past decade, this slogan has received a philosophical dressing at the hands of Davidson.[15] Unfortunately, the theory is often conceived as a threat to Wittgenstein's deviant language-games and deviant forms of life. The simplest and shortest possible sketch of Davidson's theory will hopefully show the implausibility of such proposals.

For Davidson, to speak or understand a language is to deduce, at each stage, a T-theorem (e.g., 'Snow is white' is true if and only if snow is white). And to deduce each T-theorem is to know its proof as to how it follows demonstratively from the words contained in it (i.e., 'snow', 'white' in this case), and how these words occur recursively in other contexts (i.e., 'Clouds are white,' 'Snow is cold,' 'White is a combination of seven colours,' etc.). That is, each T-theorem is deduced not in isolation but in a network of other T-theorems. Thus, a belief is identified not in isolation, but in a network of other beliefs, and this network will be uniform for all rational beings. Two people in order to hold different beliefs on the *same* subject need to share a common network of true beliefs (both logical and non-logical). Since the subject-matter of belief is identified through the uniform network, two people cannot have radically different beliefs on the same subject. That is, according to Davidson, we and the strange wood sellers who take a larger spread of logs to be 'more wood' than a larger *number* of logs condensed in a small pile would *not* be participating in the same network of beliefs. We would not be sharing the same beliefs about the identity of each piece of wood, about the method of 1–1 correlation, and that the number of logs does *not* increase or decrease without the actual addition or subtraction of units, and so on. Under such circumstances, our respective beliefs about 'more wood' or 'less wood', though coined in the same words, would not be about the *same* subject-matter. Likewise with the ancient belief about earth and sun (that the sun moves around the earth) and the modern view: it is clear that it is not the *same* sun and the *same* earth that they are talking about. Alternatively, if two people upholding two contradictory statements do happen to share a common set of true beliefs in the background, their mutual difference would soon be dissolved as being due to a terminological difference, malapropism, or certain perceptual errors.

Moreover, for Davidson, a person's actions too are explicable in this uniform network of belief and desire. The pro-attitude towards the general features of a kind of action and the belief that this action falls under the general kind usually entail the desirability of the particular action, from which stage the pro-attitude and the belief may go on to *cause* the action, though in a normative, holistic and a non-nomological way. However, on many occasions the pro-attitude and the belief towards a particular kind of action are overpowered by certain con-attitudes with respect to other features of the same action. In such a case there may arise an all-out unconditional judgement which again, in its turn, may or may not generate the relevant action. Within all these internal complexities, the predominant lesson that Davidson has to serve is rather simple. It would be a different set of pro-attitudes and beliefs of the wood seller, about that particular kind of his deviant action, which, with or without the help of the all-out judgement, cause the action itself. Thus to cut Davidson's long story short, all apparently deviant uses and actions are explicable through a *different* network of beliefs, and do not really conflict with the usual or standard ones. The mode of causation would of course be holistic and non-nomological, but the pattern of holism obtains within a universally intelligible framework. That is, the patterns of logical and causal connections holding deviant beliefs, uses and actions together are truly uniform in the sense that it enables us to attribute *this* deviant use, or *this* deviant action to *ourselves*, given the occurrence of the very *same* set of deviant beliefs and desires.

The picture depicted by Davidson has a certain affinity and yet offers an ironic contrast with the thoughts of Wittgenstein. It thus provides a fruitful occasion for refreshing and reinstating the latter's approach to deviance. It is not a common set of logical and non-logical beliefs enjoying an existence prior to language-games that would enable actual language-uses to follow from them. Rather, as already explained, a 'common set of true beliefs' dissipates in and through the common behaviour of mankind in the forms of living. Beliefs as to the desirability of a certain action cannot act as a *ground* (logical or causal) for doing such actions, for *that* we act in such and such ways makes the formulation of belief-statements and the construction of such a logical or causal model possible. It is always a continuous cycle of action, a whirl of organism that locks language with reality, rules with application, beliefs and desires with actions. Whether or not Davidson explicitly endorses the duality between language or knowledge on the one hand and an extra-linguistic reality on the other, he was certainly committed to intra-linguistic essences (i.e., the common set of true beliefs shared by all languages). He also seems to retain a mythical cleavage between beliefs and uses, beliefs and actions, whereby the beliefs enjoy a privileged status outside language and yet enable the actual uses of language and actions to follow from them. Thus for Davidson, while differences are to be submerged under a pre-linguistic identity, for Wittgenstein differences have to be played out in different whirls of organism, different cycles of activities, different forms of living.

Notes

1. This example is derived from Kalyan Sengupta, 'Bhashar Byabahar: Wittgenstein', in P. K. Sarkar (ed.), *Wittgensteiner Darshan* (Bangla) (Kolkata: Darshan o Samaj Trust, 1988), p. 173.
2. See Paul Edwards, *The Encyclopaedia of Philosophy*, vols 7–8, p. 335.
3. *Sukumar Samagra* (Bangla) (Calcutta: Reflect Publication, 1987).
4. L. R. Baker, 'On the Very Idea of a Form of Life', *Inquiry*, vol. 27, July 1984, p. 277.
5. Stanley Cavell, 'The Availability of Wittgenstein's Later Philosophy', in George Pitcher (ed.), *Wittgenstein: The Philosophical Investigations* (London: Macmillan, 1966), pp. 160–61.
6. In a lot of children's' films nowadays (like *Stuart Little*, *Babe*, etc.), words are skilfully grafted on animals' mouths, or are even matched with computerised movements of their lips and facial muscles. In spite of all this the speech looks thoroughly artificial and contrived; they do seem like ghostly projections. Cartoon films, on the other

hand, recreate animals to make them talk in a convincing manner rather than photographing live animals as non-cartoon films do. The cartoon films (like *The Lion King*, *101 Dalmatians*, and *Jungle Book*, made by the Walt Disney Co.) animate their animal characters (Simba, Mufasa, Baloo, Bagheera, Pongo, etc.) with a *new* body, a *new* function of organisation, a *new* pattern of behaviour, and thus with a *new* form of life. Watching such films, watching these characters talk, move, behave in a similar fashion as *we* do, provides a delightful occasion for some simple philosophical insights. Speech is not the vehicle or outer expression of an inner language or of disembodied thoughts, as some philosophers have taken it to be. Had it been so, one could perhaps attach speech to animals straightaway, without having to recreate their body and behaviours in a different manner. Animals, i.e., real animals and not the cartoon creations, do not talk, *not* because they cannot think, or lack the mental capacity. 'But—they simply do not talk' (*PI* 25). They simply do not play the games of commanding, questioning, recounting, chatting, etc., etc., as they play the games of walking, eating, drinking, playing, mating (*PI* 25). The issue will be further clarified in the course of this chapter.

7. 'If objects are given, then at the sane time we are given *all* objects.' (*TLP* 5.524). 'If all objects are given, then at the same time all *possible* states of affairs are also given (*TLP* 2.0124). Thus, each proposition carries with it the whole of 'logical space', i.e., the whole of language. See Paul Edwards, *The Encyclopaedia of Philosophy*, vol. 8, pp. 334–35.
8. 'Space, time . . . are forms of objects' (*TLP* 2.0251). 'There must be objects, if the world is to have an unalterable form.' (*TLP* 2.026).
9. Paul Edwards, *The Encyclopaedia of Philosophy*, vol. 8, p. 335.
10. See Steve Gerrard, 'A Philosophy of Mathematics between Two Camps', in H. Sluga and D. Stern (eds), *The Cambridge Companion to Wittgenstein* (Cambridge: Cambridge University Press, 1996). This article has been immensely helpful in writing out this and the following two sections.
11. Ibid., pp. 179–80.
12. This is how Michael Dummett reads Wittgenstein in his article 'Wittgenstein's Philosophy of Mathematics', in George Pitcher (ed.), *Wittgenstein*: *The Philosophical Investigations* (London: Macmillan, 1966). All quotations in this section are from this piece (ibid., pp. 424–27).

13. Russell cannot be categorised under any of these heads for reasons which will soon be clear.
14. I have relied on Eddington, *The Nature of the Physical World*, pp. 17–34, for the following account on FitzGerald contraction. We had occasion to refer to the same work in chapter III.
15. This idea has been developed in many of Davidson's writings, of which one may specially mention the essay 'On the Very Idea of a Conceptual Scheme', in *Inquiries into Truth and Interpretation* (Oxford: Clarendon Press, 1984). I have also relied upon Simon Evnine, *Donald Davidson* (Cambridge: Polity Press, 1991) (particularly chapters 3 and 5) for my exposition of Davidson's view.

CHAPTER VIII

In Lieu of a Conclusion

It is always difficult to cut out a smooth exit point from a philosophical discourse, particularly when it comes to issues like 'forms of life' and 'conditions of possibility' of meaning and language. One cannot wind up without addressing patent puzzlements about forms of life that usually occur at this juncture.[1]

Since forms of life are the conditions of possibility of language, they must likewise be presupposed in a philosophical language-game or a philosophical discourse. A philosopher has to identify and describe these forms of life, the common behaviour of mankind. She has to state that we do not *opine* that our fellow humans have souls but simply act as such, that we participate in customs and institutions like 'setting up rules for pleasure' and are engulfed in that circle, that we take into account the interests and feelings of others. A Wittgensteinian has to make statements to the effect that recursion of words is possible only due to the fact and extent of agreement, that people agree in their 'modes of response, sense of humor and of significance and of fulfillment, of what is outrageous, of what is similar to what else' (Cavell, quoted in the previous chapter). Such phrases and statements, as we have seen, had been used either by Wittgenstein or by his commentators. Now these philosophical statements about forms of life, like any other statement, in their turn would be embedded in other forms of living, logically prior to the first. While the description of one form of life logically invites another, forms of life *as a whole* would elude description for ever, they would eternally spill beyond language.

Philosophical language or philosophical discourse has never been accorded a privileged status in Wittgenstein's philosophy. Let us see how this fact has been made to lend support to the supposed ineffability of forms of life. For Wittgenstein, there is no isolated realm of philosophical reflection; it is not a self-contained activity providing insights into other activities that remain unaffected by philosophical reflections. The question is how to interpret such injunctions that philosophy must do away with all explanations and confine itself only to descriptions (*PI* 124). On the one hand, this injunction is intended to prevent philosophy from lapsing into the Augustinian model of description practised by the empiricists, a model which by its very nature leads into fragmentary causal explanations. On the other hand, the injunction is also directed against those philosophies that seek to explain language and meaning through justificatory foundations. By no means does the injunction suggest a special realm of 'philosophical' facts obscured by empirical anthropologists on the one hand and philosophers on the other. Indeed for Wittgenstein, a philosopher must speak the 'language of every day' (*PI* 120). The meanings of philosophical terms are shaped by the customs and practices in which language is used. '[I]f the words 'language', 'experience', 'world' have a use, it must be as humble a one as that of the words 'table', 'lamp', 'door' (*PI* 97). Now if like all philosophical terms and phrases 'forms of life' too has to be used in a form of life itself, the full meaning of this phrase can never be grasped through language: its ultimate residue would forever recede to silence.

Commentators resorting to this mode of argument are no doubt inspired by *TLP*, where the logical form of the picture could not be said or depicted, but only 'shown' or reflected in the propositions. The logical form of a proposition is the bare minimal essence that a proposition or picture should share with the fact in order that it depicts at all, whether truly or falsely. This logical form cannot be depicted or stated by a picture itself. The question is whether a second picture, a meta-picture can say that the first picture has the same structure in common with the situation depicted. According to *TLP*, the second picture can depict the specific representational form of the first picture; for instance, a line drawing of a table can depict the representational form of a hologram of the table. But the second picture cannot depict the logical form of the first picture, for it must itself have that form or structure, the structure which all pictures should commonly share with reality for it to be a picture at all. 'In order that you should have a language which can express or *say* everything that *can* be said, this language must have certain properties . . . and that it has them can no longer be said' (*NB* p. 107). In other words, there is a unique model of depiction or description, and no description can be outside that model. It is further implied that the sense of a proposition cannot be said but shown (*TLP* 4.121, 4.1212). The thought expressed by a proposition is a fact which by its own form represents reality; it is a logical picture par excellence, for its pictorial form is identical with the logical form. The arrangement of signs in the proposition *betrays* or *shows* (does not say) the sense of the thought.

Just as there cannot be a supreme meta-picture that would depict the logical form of all pictures, itself staying outside that form, similarly there cannot be any supreme meta-philosophy that would describe all philosophical discourse *on* forms of life, itself assuming a vantage point outside, itself not participating in any forms of living (*PI* 121). While one form of life may be attempted to be explained by another, forms of life *as a whole* cannot be explained. Deviant forms of living too can be identified and described only within our own, so any stance of carving out a deviant cycle of actions, a different style of painting, or a non-standard way of doing mathematics, is just an artificial construction or thought experiment, providing occasion to be reflectively aware of our own customs and practices. Such awareness, as many commentators would argue, ultimately involves a transcendental insight beyond the limits of language.[2]

To argue against the trend, we have to harp on the following points.

(a) When forms of life are characterised as what we 'do', or how we 'act', they are not intended to be pure actions, pure 'doings' so to speak, residing prior to language, and yet entailing the entire phenomena of language, meaning and discourse as their consequence. Such a picture would simply be a reversal to the classical Augustinian model where actions were supposed to be syllogistic conclusions of prior belief-statements. Wittgenstein's arguments against the traditional dichotomy between volition and action (*PI* 613–20), between the 'willing subject' and the 'acting subject' (*PI* 618), may be utilised in the present context. We have already noted his observations that if one splits the willing subject from the acting subject, acting or doing itself would seem not to have 'any volume of experience', like an 'extensionless point' of a needle (*PI* 610, 620). It would seem to be the real agent, pure action, and all the 'phenomenal happenings' would only be external consequences of this action (*PI* 620). 'I do' would be having an insular boundary severed from all experiences (*PI* 620). Now this mode of argument if pushed a bit further inevitably leads to forms of life as *pure* actions that would forever be beyond language, belief and desires, i.e., beyond all content, all volume of meaning and experience. Forms of life are not conditions of the possibility of language by way of residing prior to the same, but as a pervasive background of actions in which language emerges as an extension, or rather as a new form of living, in the same way as pain language emerges as a new kind of pain behaviour. Both language and forms of life are expressed in and through each other, they cannot be split into the 'condition' and the 'conditioned'.

(b) Thus, philosophical discourse is shaped within a form of life, or it may be said to be the emergence of a new form of life, a new language-game, bearing family resemblances with the ordinary ones. Philosophical discourse or philosophical forms of life may fruitfully be contrasted with empirical games or empirical forms of life. How would the empirical anthropologist's description of our language and practices differ from the philosopher's description of the same? The difference does not lie in self-reflection, for the anthropological account too may involve a tremendous amount of reflection on the subjects, as to why it is psychologically impossible for human beings not to believe in God, or what compels us to set rules and be engulfed into the closed circle of premises and conclusion. But none of these reflections would amount to philosophy, for while an empirical anthropologist would go on playing his games of explanation, explaining a practice by prior practices and prior beliefs, and beliefs through prior desires, a philosopher (particularly one of Wittgensteinian orientation) would have to recognise that explanations eventually come to an end. But this realisation does not imply that a philosopher bumps his head against the limits of language. Rather, recognition of this, as we have seen, would involve two very simple insights: (i) All explanations absorb the putative 'explanans' and the 'explanandum' into a single whole, where they cannot be mutually externalised to let the one follow from the other. (ii) These 'wholes of explanation' (if we may use such an expression) are integrated into a pervasive pattern of living, where the contrivance of such explanatory wholes makes sense. These insights are exercises *within* language; one would neither be confronting a transcendental realm of pure subjects (pure 'We' and 'Our Forms of Life'), nor would we be enjoying a special epistemological state of mind. While a philosopher may fully agree with every explanation given by the anthropologist, her 'philosophical' insight into what it *is* to give explanations would probably consist in seeing the same practices and the explanations of those practices in a *new* aspect. We know that for Wittgenstein, a philosopher must confine himself to describing the landscape, and not bridging gaps or erecting foundations. 'It is not that a new building has to be erected, or that a new bridge has to be built, but that the geography, *as it now is*, has to be judged' (*RFM* IV:52). In the light of such remarks, we say that while both the philosopher and the anthropologist may describe the same landscape, acknowledge the same bridges and foundations, recognise the same needs of the people to build such bridges, it is the philosopher who sees the foundations and the bridges as integrated into the landscape and into the people's way of life. Every bit of landscape added or every new structure built is seen not as externally based upon or bridged to the old, but as absorbed into the entire space and the people living out this space. We have already noted (chapter V) how the new upsurge in neurological explanation of language, meaning, grammar and also mathematics does not provide us with an unfailing bridge, but rather extends the old landscape into a new one—to absorb the new discoveries into the stream of our common usage into an unbroken continuum.

(c) Lear's suggestions as to the full nature and extent of 'We' or 'Our Forms of Life' as ever escaping language is a bit misleading, as is his talk of a 'transcendental insight' into the nature of the self. This is in spite of the fact that he is careful to point out that it is not Kant's noumenal self but the self living in space and time.[3] There may successively be many philosophies, each discoursing on the previous ones, but it does not make sense to talk of them as progressing towards the full nature of 'We' and 'Forms of Life'. There may be a new philosophy that undertakes to describe Wittgenstein's thoughts as shaped by *his* forms of life, but this philosophy would be nothing but a description; it would not achieve a justificatory explanation of the Wittgensteinian mode of doing philosophy. The new philosophy would itself be a philosophical form of life, or a new language-game, it would not unearth a pre-existing explanatory relation between Wittgenstein's philosophy and his forms of life. If Wittgenstein's philosophy gives us a landscape, the new philosophy may at most enrich that landscape by incorporating the life and activities of the philosopher *seeing* the landscape, by imbibing the space he traverses, by reckoning his needs and interests. But it cannot find an ultimate explanatory bridge between the old landscape and the new one. The 'We' and 'Our Forms of Life' do not spill over the landscape to recede into silence; rather, they may be further enriched, or made more vivid, by every new philosophical discourse.

In this sense, each new philosophy may *create* more and more space rather than *discovering* new routes that lead to new undiscovered regions. And it is this never-ending process of progressively enriching the meaning of 'We' and 'Our Forms of Life' that gives sense to expressions like 'limits of language', 'bedrock', 'the given', 'possibilities of phenomena'. It is this ongoing series of contrastive language-games that gives meaning to such statements as: '[O]ur investigation however is directed, not toward phenomena, but, as one might say, towards the "possibilities" of phenomena' (*PI* 90). 'The results of philosophy are the uncovering . . . of bumps that the understanding has got by running its head against the *limits of language*' (*PI* 119; italics mine). Further, 'In giving explanations I already have to use language full blown (not some sort of preparatory, provisional one); this by itself shows that I can adduce only exterior facts about language' (*PI* 120). 'If I have exhausted the justifications I have reached the bedrock, and my spade is turned. Then I am inclined to say "This is simply what I do"' (*PI* 217). And lastly, 'What has to be accepted, the given, is—so one could say—*forms of life*' (*PI* p. 226). Let me claim that none of these reflections points to ineffable forms of life beyond language; rather, they suggest that the distinctions between 'phenomena' and 'possibilities of phenomena', between 'language' and 'limits of language', between 'exterior' and 'interior' facts about language, between the 'given' and the 'constructed', and finally between 'bedrock' and 'outer layer', are all meaningful *within* language, within our description and discourse. It is in the same way that the gradual expansion of space and successive addition of each new planet over the existing network gives meaning to the distinction between 'relative' and 'absolute' velocity. It is in the same way that consecutive acts of measurement with different measuring scales (of different degrees of accuracy) make the talk of 'absolute identity' meaningful. The spade of language does not turn back in its effort to penetrate the bedrock. For speaking literally, even if 'bedrock' is taken to mean the inner core of earth which, as far as the story of science goes, pulls all the layers of earth together by its force of gravity, it is neither the end nor the beginning of earth, but a strategic device for scientific theorisation. And here also, it is the ongoing scientific explanations, one after another, that give meaning to 'inner core' or 'ultimate foundations of earth'.

In fine, philosophy cannot explain meaning by tracing it to anything ultimately given beyond language. The all-important task of a philosopher lies not in replacing one kind of universal for another, i.e., putting in a pre-linguistic essence of human behaviour for the traditional essences, but in shifting emphasis to something different, something simpler. One cannot trace meaning to something *outside* meaning or language, whether it is ontological universals, or pre-linguistic rules, fundamental brain patterns, neural firings or pure actions bereft of all meaning and experience. Instead, one should progressively attempt to find a simpler sphere to describe meaning, a sphere that is simpler and at the same time more vivid and enriching—the sphere of human life and forms of living.

Notes

1. One may refer to Jonathan Lear's 'Transcendental Anthropology', in Philip Pettit and John McDowell (eds), *Subject, Thought, and Context* (Oxford: Clarendon Press, 1986), which has provided valuable suggestions in writing this section.
2. L. R. Baker, in 'On the Very Idea of a Form of Life', states quite explicitly that forms of life make meaning possible, and hence little can be said about them meaningfully (p. 288). He also observes: 'it is doubtful that forms of life *per se*, as it were, can be the subject of meaningful discussion; as in the *Tractatus* not everything that can be shown can be said' (ibid., p. 282). Stanley Cavell too in 'The Availability of Wittgenstein's Later Philosophy' characterises the enquiry into forms of life as a 'transcendental' enquiry into the conditions of possibility of language (pp. 175–76).
3. See Lear, 'Transcendental Anthropology', pp. 291–92.

APPENDIX

Constructing a Wittgensteinian Critique of Formalism

Everything said and done, this entire journey of a philosopher steadfastly holding on to a non-foundationalist narration of mathematics may have to face the challenge of another seemingly non-foundational position—that of formalism. In fact, Wittgenstein's crucial claim that mathematical propositions are grammatical propositions about signs, or are about sign-geometry—the note with which we ended the third chapter, titled 'Critique of Rules and Logic'—may swerve into the theory of formalism, which claims a stand different from Platonism, intuitionism and even logicism. The theory starts with an initial stand that mathematical propositions are not about anything (abstract entities, intuitions of space and time, number and space as second-level concepts), but are simply about signs.[1] This is however only a rudimentary version of the theory, which sooner or later gives way to other versions with a more specific content. Before we delve into a routine investigation of the possible versions, let us try to be comfortable with the basic enunciation of this theory—its claim of mathematics being about signs, or rather consisting in blind techniques of manipulating with signs. We know that mathematics, starting with easily acceptable number-facts (like addition or multiplication rules with small positive integers), does not extend only to more complicated operations with larger numbers and fractions, but also those with non-standard (or irregular) dimensions like negative numbers, irrational numbers, imaginary numbers, etc., often without an adequate explanation of why the rule works. The pedagogical convenience of doing mathematics in terms of only man-made rules has a tendency to turn into a theoretical standpoint—the dismissive assertion that mathematics is nothing but blind techniques of transforming one sign-pattern to another. Often formalism prefers to allow an internal split into its own body: it takes an objectual or representative stance about natural numbers, while allowing itself to slide smoothly into an easy dismissal of any such status for the non-standard numerals, claiming them to stand for nothing, not even for second-level concepts or mental intuitions of spatio-temporal structures.

It is easy to see how the message of formalism gets a special booster from the negative numbers, irrational numbers, imaginary numbers and transcendental numbers. While positive integers and even fractions can be said to carry a representational stance, the other kind of numerals cannot be made to strain beyond themselves, beyond the brute physical content of their 'characters'. Negative numbers are sometimes accorded a seemingly applicational status with respect to, say, recording extremely cold temperatures, evaluation of answer scripts (negative marking), or bills with a negative quantity (estimated against an already paid amount). Here it needs to be reckoned that just as there is nothing like a negative fact or a void, and just as there is no space corresponding to numbers lesser than a void, there is no paper performance whose value can go beyond nothing, for a poor performance draws its identity from its positively deviant content. The negative markings are often pragmatic devices to cancel out some good performances in view of the poor ones; the bills with negative amounts show the reverse direction of payment, i.e., the amount of money that the billing offices owe their customers. In all these cases, the negative numbers just signify a mode of operating with the positive ones. Similarly, irrational numbers like '$\sqrt{2}$' have no meaning apart from the fact)that one can run some relations between it and other natural numbers in the form of equations like $\left(\sqrt{2}\right)^2 = 2$ or $\frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$. That these equations are possible is not because there is a reality corresponding to these relations on both sides of the equation; these operations are again characters generating further characters by a technique that responds only to figures *qua* characters and their spatial positions and configurations.

Formalism usually comes in two versions: 'term formalism' on the one hand, and 'rule formalism' or 'game formalism' on the other. According to the first version, mathematical propositions are about terms or signs and some syntactical operations among them, i.e., '2 + 2 = 4' talks only about itself and not anything else. According to rule formalism, mathematics does not talk about anything, not even about signs; there are only rules to operate with signs. While in the first version, mathematics consists of propositions with truth values, the second version confines it to regulations bereft of any truth value claim.

Once we try to thresh out the real implications of term formalism, its vital discrepancies and compromises leap to the eye. Let us be clear about the exact advantages that can be gained by turning the subject-matter of mathematics into signs and their mutual operations. The term formalist has to accommodate a distinction between sign-types and sign-tokens, say '2' as the universal sign and its innumerable instances as possible occurrences of the type. Not only the numerals, but the relational operators like '$\sqrt{2}$', '2^2', '+', 'etc., would have to be laid out in terms of type–token distinction. Unless this distinction is adopted, we cannot make sense of the self-identity of the signs, say '0 = 0', since '0' as two oval shapes or as two hunks of ink on both sides of the equation are plainly not identical; it is the sign-type '0' that is identical with itself. This brings with it the obvious repercussions for formalism—that this type–token distinction thrusts the load of essences or universals back upon formalism, a load that formalism had happily claimed to have shed.

However, formalism has a way to obviate such charges—a way that is apparently convincing. While in most cases the way in which essences or universals would instantiate themselves into particulars or tokens is in no way transparently contained in the types, the content of the sign-types bears a clear indication about its possible tokens or instantiations. There is no conceptual gap between the plain physical identity of the types and that of its tokens. On the other hand, logicism presents numbers as a set of sets—of units, dyads, triads, pentads—each with a hopelessly abstract content that has no clear indication as how to identify its instances. As we already noted in chapter III, a set of pentads cannot be said to contain such vastly distant collections—like the petals of a flower, the fingers of a human palm, the paws of an animal, points of a polygon, vertices of a star, term of office for the prime minister of a country—within itself.[2]

However, a little reflection will show that the type identity of a sign may not be as obvious as the formalists take it to be. To accommodate the fact of intertranslatability of signs across the different languages, the formalist has to go beyond the sign-types and admit same kind of rules or same patterns of configurations across different languages or different systems of signs. And the question as to what the same kind of configuration consists in is not easy to answer, specially within formalist parameters. A game of chess (admittedly constructed in the formalist model) has to go beyond the physicality of the chessboard, the specific set of chess pieces, in many ways that are thickly layered with all possible kinds of complexity. For example, what would constitute two-and-a-half moves of a knight on a surface that is spherical or jagged so as not to allow a change of direction; or if, more adventurously, the chess pieces and the board are constructed with a water jet, and the pieces have to be moved by manipulating the valves so that they shoot off the water jets in the required directions. All this shows that formalism, to be about signs, has to move beyond sign-tokens and even beyond sign-types to the most abstract structure—as contentious as the underlying metaphysical foundations of Platonism, intuitionism and logicism.

On the other hand, formalism has to forego its term formalistic version—along with both its seeming advantages as well as its disadvantages—and opt for rule formalism in certain spheres. The question is, how will a term formalist interpret a mathematical proposition like $6 + 5 = 4 + 7$? One cannot read the signs on both sides of the equation as denoting the same number, nor can it be read as an identity between two signs, either as tokens or as types.

Plainly, such statements cannot be about identity. Frege suggested that within the formalist scheme, what such propositions of the form A = B say is that the terms 'A' and 'B' are intersubstitutable in the mathematical context without any change in truth value.[3] But then, in this term formalistic version, the character of mathematical propositions moves beyond its professed parameters and assumes the status about rules—rules regulating their own truth or falsity. Does this not show that term formalism is unable to contain itself within the self-imposed boundary of signs and breaks forth into a more expansive notion of rules—rules that though initially confined to sign transactions were eventually to break away from this highly contrived limit?

We can see one other way in which term formalism cannot possibly restrict itself within these artificially manufactured frontiers. As already noted, this theory often chooses to take a logicist or realist approach to certain portions of mathematics, and finds it convenient to be a formalist only with respect to certain other regions—chiefly pertaining to the problematic zones like infinite points and non-standard numbers. This position is rather inconsistent and self-stultifying for the formalist, in so far as to implement its programme of demonstrating that some numerals are about nothing, she will have to take the help of certain others that *have* to be about something. (This will be demonstrated in greater detail in the forthcoming sections.) Formalism has to prove certain numbers to be irrational, some to be imaginary, some to be complex—and this task cannot be performed without assuming some logicist rules or theorems pertaining to the generation of natural numbers. The irony of formalism perhaps stands on the par with that of many-valued logic or fuzzy logic—where the latter, trying to chart out the fuzzy or borderline areas, or to lay out the proliferation of intermediate values between 'true' or 'false'—takes the help of rules that typically belong to bivalent logic grounded on an essentialist semantics. A formalist would be in an even more disadvantageous position in so far as formalism cannot split up its own body into two levels to adopt the strategy of deploying non-formalistic rules of a higher level to establish a formalist sign reality of mathematics in the first level.

It is vitally important to see that formalism—whether in its term formalistic or rule formalistic version—cannot afford to put itself on the same footing with a calculator. A calculator registers each digit as a spatial chunk occupying a spatial position and generating more spatial chunks out of a mechanical movement from one position to another. So what do we mean when we describe a calculator as being fed with the initial rules of addition and multiplication within the range of 10 digits; what are the implications of its being programmed to move to sets of subsequent digits and their operators; what is the purpose of its coming up with unique output in each operation? Overall it does not cognise numbers as concepts of concepts, or sets of sets, or as bundles of bundles; it is just programmed to move from a single digit to others or sets of other digits according to the rules already fed into it. In none of its operations does it conceptualise the spatial positions as different tokens of the general category of space—it is not sensitised to each specific position either as an instance of the general concept of space, or as different parts of a single space—a single whole. Analogically, can formalism ultimately sustain mathematics as simply as a blind, causally determined, spatial movement with signs (spatial chunks) in a non-conceptual manner, where all the signs and their mutual interactions hold all their physical content within themselves, moving not a step beyond in any representational capacity? I think that this brief outline gives a convincing indication that formalism cannot succeed in this impossible task, for it is its simultaneous efforts to be non-representational and yet conceptual that seem to bring it to its fatal downfall—it could not prevent its signs from bloating out into a signifiable reality, and thus is unable to break through the idealised foundations of Platonism, intuitionism and logicism.

Following Hilbert, the chief exponent of formalism, we shall navigate through his early views of mathematics as abstract, uninterpreted systems, gradually developing into his later insights about signs themselves posing as full-fledged reality—having properties, structures, relations—thus attaining a numerosity of their own, instead of being mere *symbols* of number. This theory of signs being given in pre-conceptual intuitions was constructed into finitary arithmetic, which was further extended into ideal mathematical systems that claimed to prove all the finitary statements about sign configurations as true. We shall see that while Wittgenstein's views on mathematics were greatly influenced by Hilbert, his (Wittgenstein's) commitment to proofs as being surveyable structures could not let it swell into a logicist or deductivist status, as happened with Hilbert's system of signs; such an exercise, according to Wittgenstein, would turn logic itself into a Platonic reality. The narration that follows is designed to contrast Hilbert's theory of mathematics as being about pre-conceptual signs with the insight of mathematics being grammatical paradigms of sign-geometry—the non-foundational position characteristic of Wittgenstein.

1. Formalism: From Its Rudimentary Versions to Hilbert's Theory

Starting with more precise descriptions of the unrevised versions of formalism, I shall move on to an overview of Hilbert's theory, noting the trajectory from his early theory of deductivism to the later development into finitary arithmetic grounding ideal arithmetic.

1.1 *Term Formalism*

Term formalism faces special problems with names, for most of the real numbers do not have any names. A statement about all real numbers will not get any foothold in formalism. Suppose the term formalist seeks to equate 'π' with its decimal expansion 3.14159. . . . Now this expansion is at least not a term; for Wittgenstein it would be an indefinite process, and theories that are prone to mathematical realism would postulate an infinite object corresponding to this sign. Suppose the term formalist introduces a theory of 'limits' of terminating symbols, and identifies that with the 'limit' of the symbols '3', '3.1', '3.14'. But neither much clarity nor advantage can be gained by this strategy. A limit of the above signs does not make sense unless it is interpreted as limits of rational numbers. And rational numbers *qua* —numbers have to be over and above signs, at least they have to be conceived as homogeneous units of space, repeating themselves identically; and each collection of units is further divided by another collection to yield a remainder of units—which are again extended to a base of 10 to avail of further division. It is this process of dividing numbers and extending the remainder to avail of further divisions that is sought to be brought to a limit, and this limit is equated with 'π'. It is all too evident that formalism, in order to interpret 'π' in this manner, would ironically lose its special character of formalism and merge with any of the non-formalist theories. In no way can term formalism interpret real numbers so as to avoid the concept of calculation—a concept which inalienably imbibes the notion of numbers.

Further, how would a term formalist interpret theorems—say on prime numbers, or those of a calculus? What would be the term formalistic interpretation of the theorem: 'For every natural number x there is a prime number such that y > x'? Formalism cannot make any headway by trying to read such propositions as being about signs, i.e., about one category of signs being associated with another category of signs, without infusing the moot notion of numbers—the repeatability of homogeneous units—into the professedly pure signs themselves.

1.2 *Game Formalism*

Game formalism is often sought to be understood in terms of a game of chess, the crucial point of analogy being that both chess and mathematics are not about anything. Chess is decidedly not about the chess pieces, the chessboard, or about number, spaces—or homogeneous spatio-temporal units.[4] Just as chess is constituted by the rules of moving the chess pieces in a certain manner, similarly mathematics amounts to rules of manipulating signs in certain ways—the rules of both chess and mathematics being devoid of any actual or potential truth value. In chess, the pawn stands only for the rule that it can move one step forward, either in a straight line or diagonally; it is not about the material content of the piece or its shape or of the squares on which it moves; 'x = 10' is neither about x, nor 10, nor about their identity, but simply the rule that one can write '8 + 2' instead of '10'. That is, if '10' is a permissible combination of signs, then one can deduce '8 + 2' as a permissible combination of signs. Within the prescribed format of game formalism, a mathematical system would look somewhat like this: It can propose primitives like ^ and *, prescribe all configurations preceded by ^ to be wffs, enunciate ^*** to be an axiom of the system, introduce an inference rule like: 'with any wff beginning with ^* and any other wff ending with ^* in that order, the second wff is entailed by the former.'[5] All theorems of mathematics would be fundamentally similar to this mode of transition. To infer 10 from 8 + 2, a game formalist cannot afford to construct equinumerous signs on either side, for in that case it would be to presuppose the very notion of number from which game formalism desists; and secondly, such constructions would not be possible with infinite numbers.

Though our spatial operations with the chess pieces do not need the geometrical perfection of straightness, angularity, flatness or two-dimensionality, yet there is a broad spectrum within which the spatial movements of the pieces are confined. These movements also require a type–token distinction within the chess pieces—an appreciation of the same type of movement obtaining across the different sets of chess pieces, across different boards and their possible variations of kinds, also perhaps accommodating the possibility of outgrowing the squareness of squares to a viable range of options. Above all, the game of chess has to retain the crucial notion of numbers—each genre of chess pieces (king, queen, rook, knight, etc.) having a definite number of instantiation for a particular game—and all these levels of abstraction have to be carried on through memory, from each move to the next. Thus the game of chess fails to dissipate the abstract spatial structure of signs and sign movements, the minimal identity of the medium of movement that would gloss over, say, water jets, light rays, or air gushes. And above all, the game of chess fails to explain away the numerical status of the pieces themselves. So if game formalism seeks to capitalise on the chess analogy, then its failure to outgrow the standard representational theories of mathematics becomes palpable from the very outset. Reducing itself to a world of signs is virtually to flare up another reality—the world of signs themselves—where all the patent indeterminacies pertaining to the physical qualities of the sign-pictures, the exact number of instantiation, the failure of rules in present or future predictions obtain in their full force.

Frege says[6] that the formalists may bring in some element of application in the mathematical propositions as a psychological aid or a heuristic tool, just as, while teaching the game of chess, the instructor may bring in the idea of a real battlefield where the warriors are divided into categories corresponding to each kind of chess piece, and the battle is conducted in exactly the same way that a game of chess proceeds. Just as this application is external to the game of chess, similarly for the formalists, application is external to mathematics—its significance draws not an iota from its applications.

The most natural objection against game formalism takes the form of the question: if mathematical propositions are not about anything, what makes the empirical sciences apply them so successfully to reality? Science is able to predict weather or eclipses on the basis of numerical calculation of wind movements or of velocity, apply geometrical concepts of triangles, circles and their several interrelations to real objects and their structures. On the other hand, chess rules cannot be applied to warfare, nor to any area of human behaviour and transactions. Further, within mathematics, the game of complex analysis is useful in the game of real analysis of arithmetic—a phenomenon that a formalist cannot explain. Shapiro points out[7] that to say that mathematical propositions are only abstract theoretical postulates designed to explain some empirical phenomena, just as electrons, though unobservable, can enable an intelligible account of macroscopic objects—their sensible qualities, size, shape and volume—is a defeatist position, leaving an inelegant gap between the concrete explanandum and the abstract explanans.

1.3 *Hilbert's Revised Version of Game Formalism (Deductivism)*

We shall have to see whether Hilbert's deductivism[8] can be considered to be a revised version of game formalism as narrated by Frege, and yet maintain its difference from classical logicism as endorsed by Frege and Russell. The idea is to construct a system that would only have signs and combination of signs as its primitives and axioms respectively, coupled with formation rules and rules of inference—the crucial requirement being that the signs themselves have to be left uninterpreted. The principal demand of the system is that, under any interpretation of the signs in which the axioms come true, so will the theorems. In other words, the formal systems will only contain logical constants—'if–then', 'not', 'and', 'all', etc., but will not embody any non-logical constants like 'number', 'triangle', 'point', 'line', 'set', etc. The logical terminology will be accepted with its normal meaning, while the non-logical terms will not be accorded any interpretation.

Let us try to spell out the difference between this revised version of game formalism and its unrevised model on the one hand and logicism on the other. Let us first comprehend its basic difference from the game of chess, from which its divergence from the unrevised version will hopefully emerge. Chess rules (on a narrow interpretation) will obviously take terms like 'square', 'one square', 'black', 'white', forward movement, diagonal movement, horizontal movement, etc., as undefined primitives, and can go on to define each chess piece by its position on the board; present the rules of the movement of each kind of piece as the formation rules; and finally accept *Modus Ponens* as the required rule of inference. Now, the all-important question is what would constitute the axioms of the chess system. Evidently the axioms have to take the shape of some implicative propositions, where certain initial movements of the game are stated to imply a range of other possible movements and captures. The question is whether all the subsequent moves can be shown to be entailed as theorems, without resorting to our ordinary spatial intuitions. Besides, the game of chess, read in the narrow sense, does not admit of abstracting from its content altogether; perhaps it glosses over the squareness of the square, flatness of the board, colour of the pieces, but not over the number of the pieces and the squares, and the spatial properties (like line, straightness, angularity, diagonality) of the movements. So even if, with an effort, we construe chess as a revised version of game formalism, i.e., as an abstract uninterpreted system (glossing over the material specifities), it would fail on this vital account. Hiding the crucial mathematical notions of number and space already in its point of departure, the earlier version of game formalism could not accommodate the richer idea of axioms, whereby they can have an abstract and uninterpreted structure that yet has the power of taking any kind of content into its empty sockets (a content that is mathematical or non-mathematical), be enriched by the predicate of truth and thereby preserve the truth of its axioms.

Like chess, of course, the earlier version of game formalism can attempt to create a generality over its signs, some variations on the placement of signs (allowing an overlapping placement perhaps), some alterations in the linearity of the formulae (allowing a vertical mode of writing) instead of the standard horizontal mode. But can it seek to strain out a mode of presenting itself as an abstract, uninterpreted system preserving entailment? No, because given the confines of the system, game formalism in the old version can only use axioms and inference rules that are purely arbitrary, ruling out the possibility of developing into an abstract and uninterpreted system, yet capable of receiving alternative interpretations based on the sole power of '*if–then*'.

Let us delve into the exact claims of deductivism as the natural emergence of game formalism. Mortz Pasch is reported as saying[9] that geometry must be presented in a formal manner; the process of inference must be independent of the geometrical concepts, just as it is independent of the diagrams. It is only the relations specified in the propositions and definitions that have to be taken into account. Hilbert (at least in his early writings) marked an end to the role of intuition and observation; if they have any role to play, they are only motivational and heuristic. Once the axioms are formulated, the intuitions and observations are virtually banished.

Anything at all can play the role of undefined primitives—whether points, lines or planes—so long as the axioms are satisfied. In logic, the significance of the variables of different types—the individual variables, the predicate variables and the propositional variables—is that they can take in any constant of the appropriate category that preserves the range of signification. It is only the meaning of 'if–then' that has to be preserved through all possible values—whether the individual variables take microscopic objects like atoms or molecules, or macroscopic objects like tables or chairs, or psychological objects like perceptions, sensations and dreams, or mathematical objects like lines, points, planes or triangles. Formal systems are thus considered as conditions for what might be called a relational structure; logical reasoning is not a means of assisting intuitions in the study of spatial figures, but rather logical dependence is radically subject-neutral and studied for its own sake.

Further, as Shapiro points out,[10] deductivism, in spite of its content-neutral stance, has to be available for a meta-level discourse about its own status, i.e., it must be available to another system that will undertake to prove the independence, completeness and consistency of the first-level set of axioms. To this meta-mathematical system, the first-level system has to figure as *interpreted*—its axioms and theorems as being true of a particular content. Thus, ironically, the deductivist has to admit that while mathematics is neutralised of any subject-matter, meta-mathematics has to be geared to a content that is patently of mathematical objects. And it will be arbitrary to deny meta-mathematics the status of being a system of mathematics, for it has the same appearance and method as any other branch of mathematics. A deductivist has to say that while meta-mathematics may be about a specific mathematical content, the axioms by which it conducts the proofs of the (first-level) mathematical systems are themselves meaningless, bereft of any content (mathematical or non-mathematical). Arithmetic can be applied to counting, i.e., in deciding how many acts of counting are undertaken, or can address the relations of addition and subtraction among the numbers of these acts of counting themselves. But this application of arithmetic to counting itself does not necessarily load arithmetic with a specific content confined to counting only.

The exact significance and vulnerabilities of deductivism may hopefully be brought out in the tension between Frege and Hilbert on the layout of a deductivist mathematics. Frege questioned Hilbert's programme of treating axioms as definitions.[11] The traditional idea of a system is that there should be some primitive undefined terms, and axioms using these primitives, and some logical constants which are chosen for their self-evident status. Definitions form a different set of propositions that introduce some other terms by defining them with the help of the primitives. Definitions thus being arbitrary stipulations cannot take the place of the primitives, and being prescriptions devoid of truth value, they cannot also take the place of axioms, for axioms (under an interpretation, of course) are patently held to be true. Thus, Frege objects that Hilbert should not make his axioms play the role of definitions in his system. Besides, Frege complains that Hilbert's axioms sometimes fail in their definitional task. For instance, they cannot give a satisfactory mark to decide when the relation of 'between' holds.[12] The meanings of the words 'point', 'line', etc., are also not given, and are assumed to be known in advance. Frege further argues that while the term 'point' initially tallies with the Euclidean notion, soon it deflects from this meaning and is made to stand for a pair of numbers. Overall, Frege's objective is to expose Hilbert's deductivist programme in the shape of a dilemma:

> Either the axioms of the system should be laid out in terms of signs whose meanings are already specified, for otherwise the axioms will not have any truth value—that is they cannot qualify as axioms. And if the signs have meanings beforehand, those meanings cannot be laid down in terms of the axioms themselves, i.e., then the axioms cannot be definitions.

But what advantage would Hilbert gain by turning axioms into definitions? What would have been the harm if he had a separate set of definitions ascribing meanings to the terms, letting his axioms be laid out against the backdrop of those meanings? Hilbert's answer is that signs can be given their required mathematical interpretation, not in isolation by means of definitions, but only in a relational structure—in the shape of propositions—and these propositions have to be the axiomatic base of a particular interpretation of an uninterpreted system. This is what Hilbert calls implicit definitions of 'points', 'lines', 'planes'—which have no reference or meaning in isolation, but only in an integrated scheme. What the definitions are designed to do is accord a relational structure to each of these terms, where it is not the isolated content of points that entails truth about lines and planes, but a comprehensive web that synthesises all these elements together. Hilbert observes that any attempt to find real entities corresponding to geometrical terms like 'line', 'point', etc., degenerates into a game of 'hide and seek'. He says: 'it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relation to one another, and that the basic elements can be thought of in any way one likes.'[13]

This is a statement of deductivism, the supreme power of replacing mathematical realism by 'if–then-ism'. From Hilbert's standpoint, one can try to establish this supreme claim of logic in the more familiar rubric typically reserved for logic itself. We know that logical validity (of an argument) consists in a relation of inconsistency between the truth of the premises and the falsity of the conclusion, so that with respect to every valid argument there is an inconsistent statement constituted by the conjunction of the premises with the falsity of the conclusion. A statement is inconsistent when it ascribes a pair of incompatible predicates to the same subject at the same time. A definition of incompatible predicates would have to fall back on the notion of incompatibility range and that of neat exclusion; so that logic by defining each term carves out a range of exclusion, and thus with every definition arises a new pair of incompatible predicates and a new incidence of validity—a new case of implication or a new argument.[14] This power of logic to generate a new implication with each definition ranges over all possible content, thus making logic content-neutral. It is this power of logic that enables one to build up an abstract system of logical relations, and achieve different sets of interpretations—with different sets of primitives—{nerve, artery, cell, grey matter}, {perception, sensation, imagination}, {line, point, plane, triangle}.[15] The novel point about Hilbert's deductivism is that instead of taking certain terms as undefined primitives, he defines each of them in a relational network and presents them as the axioms of his system. Against Frege's discomfort that Hilbert does not give an adequate definition of 'point', Hilbert asserts that nothing stands in the way of defining points in terms of alternative sets of things (say a set of love relations, a set of laws, chimney sweeps), and in each case the elements are conceived as converging in a point.[16]

1.4 *Hilbert's Later View of Finitary Arithmetic: Expanded to Ideal Mathematical Systems*

Hilbert in his later writings revised his formalist position to what he called 'finitary arithmetic', which has a subject-matter and which can be extended to formal systems.[17] The formulae of finitary arithmetic include equations like $2 + 3 = 5$, long sums like $12{,}533 + 2{,}477 = 15{,}030$, $7 + 7 \neq 10$ or even $2^{10{,}000} + 1$ is prime. All these statements refer only to natural numbers, and all the properties and relations mentioned are effectively decidable in the sense that there is an algorithm (or computer program) that decides whether these properties and relations obtain. It is notable that statements with existential quantifiers can be both bounded and unbounded, and Hilbert held only sentences with bounded quantifiers to be finitary, while those with unbounded quantifiers were not. Consider the following sentence: (1) There is a number p which is greater than 100 and less than 100! + 2, such that p is prime. The existential quantifier in this sentence is bounded, for it quantifies over a finite number that is less than 101! + 2. Though 101! (being the result of multiplying 1, 2, 3 till 101 (i.e., $1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times 100$) is indeed very large, yet it is finite, and the numbers falling in between the range of greater than 100 and less than 101! + 2 are many but *finitely* many. But to consider another sentence: (2) There exists a number p that is greater than 100 such that both p and p + 2 are prime. The existential quantifier in this sentence has no limit; it ranges over *all* natural numbers above 100—an infinite collection. Thus the existential quantifier in (2) is unbounded, not finitary according to Hilbert.

It is only the combination of simple equations and sentences with bounded quantifiers that are decidable in the sense that there is an algorithm for computing whether they are true. When the bounded quantifiers range over a large number of instances, the proof has to involve some idealisation. However, there are only a finite number of cases to be considered—and thus the relevant statements can be proved through computation. But sentences with unbounded quantifiers cannot be so proved—cannot be decided by finite steps of algorithm—for here there is no principled limit to the number of cases to be considered. On the other hand, the sentences with bounded quantifiers can *ideally* range over a finitely large number of cases, without actually taking each of these steps in isolation.

Hilbert considered even universal statements like (3) $\partial + 100 = 100 + \partial$ as a finitary generalisation, for each of its instantiations is a finitary statement. What he implies is that while one can in principle decide the truth of (3) with respect to a finite number of its instances, (2) ranges over an infinitely large expanse; the number it seeks to hold on cannot be pinned down within a finite stretch.

Now while $3 + 5 \neq 8$ is a legitimate finitary statement, it is not clear what to make of such statements like (3) and also (4) ∂ + ß = ß + ∂. Hilbert insists that they are finitary, as they are in principle provable by finite steps of an algorithm (presumably with universal instantiation and universal generalisation with respect to arbitrarily selected individuals). But from this point of view, the negation of this universal but finitary statement is not finitary. For while the universal statement ideally covers each of its hypothetical instantiations, its negation cannot pertain to any individual instantiation. Nor can its negation be said to be a disjunction of several individual negatives (by the rule of DeMorgan) for the simple reason that the original positive formulation cannot be regarded as a conjunction of singular statements. Had it been a conjunction, it would have to be regarded as an infinite conjunction—and an infinite conjunction is not a conjunction at all. So in fine, the negation of a statement of generality which is finitary contains an unbounded quantifier, and hence is not finitary.

According to Hilbert, there is no serious epistemological issue concerning finitary statements containing a variable but not containing an explicit universal quantifier—he holds them to contain routine, if long, computation. He is not explicit about how we conduct finitary proofs with statements having letters of generality, but Shapiro explains that scholars agree upon a common proof technique of finitary arithmetic, and this technique corresponds to what is currently called 'primitive recursive arithmetic'.

As already noted, Hilbert in his later writings declares arithmetic—finitary arithmetic—to have a content, i.e., the natural numbers; and thus shifted from his earlier stance by declaring that the subject-matter of mathematics cannot solely be grounded on logic; and in this respect the attempts of Frege and Dedekind were a failure. Hilbert states that finitary arithmetic concerns what is in a sense a precondition of all thought—even logical deduction. 'Something must be given in conception, viz., certain extra-logical concrete objects which are intuited as directly experienced prior to all thinking.' [18] For logical deduction to be certain, we must be able to see every aspect of these objects—their properties, differences, sequences and contiguities must be given together with the objects themselves, as something which cannot be reduced to something else. Hilbert states that what constitutes the subject-matter of finitary arithmetic is 'the concrete symbols themselves whose structure is immediately clear and recognisable'. In finitary arithmetic, we identify the natural numbers with the numerical symbols:

| || |||

where each numerical symbol is intuitively recognisable by the fact that it contains only a specific number of |-s. It is based on this intuition that ||| is larger than || that we come to see the former symbol as containing the latter as its proper part, i.e., we come to know the proposition that 3 > 2 is true.

Shapiro points out that Hilbert grounds his enunciation of finitary arithmetic on pre-logical intuition in terms of term formalism. We see Hilbert taking the natural course whereby term formalism (being about mathematical terms) swells up into a reality of signs, and then into a version of game formalism whereby mathematics equates with rules of configuring the signs, i.e., mathematical terms or signs. At this stage, we realise that one should not use the word 'mathematical symbol' here to present either term formalism or game formalism. For now the mathematical signs become their own symbols—constituting a reality by themselves, they no longer stand for anything outside themselves; they declare themselves as the full-bodied reality with all properties, structures, directions, relations, spatiality, temporality, numerosity—with all their concreteness giving clear indication of the way of abstraction. Shapiro points out that though Hilbert uses the word 'concrete' for the mathematical signs (and 'concrete' usually means 'specific' as opposed to 'general' in mathematics, and philosophically 'concrete' means spatio-temporally localised), he intends the characters studied in finitary arithmetic to be understood as abstract types rather than as physical tokens. When | | is said to be a part of | | |, and | | and | | are said to add up to | | | |, it is not the physical hunk of the ink or the burnt toner to which these sentences are confined. Rather, they tend to move up to an abstract structure which is independent of the physical properties of the burnt toner, the linear shape of the stroke, their spatial location or spatial interrelations, the time in which they occur, etc.

Let us take note of how in Hilbert's later scheme, finitary arithmetic expands into a stretch more expansive than that of his early deductivism. Finitary arithmetic consists primarily of equations about numbers that are singular statements. All the following items go beyond the range of finitary arithmetic:

A. statements about natural numbers that contain unbounded quantifiers
B. negatives of finitary statements that contain letters of generality
C. real analysis, complex analysis, functional analysis and set theory

Hilbert named items under A, B, C—that are evidently beyond finitary arithmetic—as 'ideal mathematics', and this was claimed to be instrumental to finitary mathematics. As for the range of ideal mathematics, Hilbert retained his early position that their formulae mean nothing in themselves. Still, from these ideal formulae we can deduce other formulae to which we can ascribe meanings, viz., by interpreting them as finitary statements. Thus, mathematics comes to consist in two kinds of formulas:

a. those to which meaningful communications of finitary statements correspond;
b. other formulae which signify nothing and which are the ideal formulae of our theory.

Whether or not we follow Hilbert's new version of term formalism (i.e., mathematics being about pre-conceptual signs), there is an undeniable connection between numbers and symbols—because symbols cannot shirk off their own numerosity to represent the latter. Shapiro reports that this connection was later exploited by logicians and other mathematicians (e.g., Corcoran and others ever since).[19] Shapiro says that for the Hilbert programme, the identification of natural numbers with the character type allows finitary arithmetic to be applied to meta-mathematics as well. For as the formal systems are extensions of finitary arithmetic, and as the formal systems are uninterpreted systems (finitary arithmetic being its interpretation), it is in this way that finitary arithmetic comes to be applied to meta-mathematics. And it is in this way that formal systems come under the purview of finitary arithmetic. The abstract structure of signs is formalised under the system, and thus this abstract system develops into a mathematical proof-picture—a formalised proof which is like a numerical symbol, a concrete, visible object that we can describe completely.

If T is a formalised axiomatic system, then the statement that T is consistent is itself finitary—its formula would have a letter of generality (bounded quantifier). That T is consistent would have the form:

∂ is not derivable in T whose last line is $0 \neq 0$.

In order to use a theory of ideal mathematics, we have to formalise it and show how within the finitary arithmetic the theory is consistent. Once this is accomplished we will have achieved our epistemic goal, that is, that using T will not bring us to a contradiction, nor will it produce a false finitary statement. In other words, the concrete–abstract content given in intuition is further abstracted to forge the formal systems, and the latter in their turn should lead us back to the intuited content from which we started. The intuited content of | | | | = | | | | should be derivable from the formalised system.

2. The Non-standard Numbers

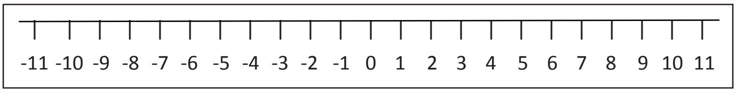
We have already noted in the beginning of this chapter that (unlike the natural numbers) the irregular numbers—viz., negative numbers, irrational numbers, imaginary numbers, real numbers, complex numbers and infinite numbers—cannot carry even a semblance of a representational stance of standing for something, that gives a special ground for formalism. Here, the formalist would insist that we have no option but to say that all statements about these irregular numbers are either about the signs or are rules for prescribing certain configurations (*qua* configurations and not on the strength of standing for an item of reality, or set, or an a priori structure of mind) as substitutable for certain others. In the pure formalist framework, we have seen these rules—say, $\frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)}$ should make no more claims than saying that the signs '**&^||' can be substituted by '&&#<<>'. But ironically, the formalist has to ground these claims of the presumption that the standard numbers do represent real entities, and it is only in a relational scheme with the regular numbers, *being about* something, that the irregular numbers, as *being not about* anything, can start to make any sense. This strategy of the formalist (in its game formalist version) to ground formalism with regard to certain numbers on the ground of non-formalism with regard to certain others is bound to be a failure. The early Hilbert's programme of deductivism can be read as an effort to prevent game formalism with regard to irregular numbers from lapsing into some kind of logicism or realism. But Hilbert's attempt to procure the statements about irregular numbers as interpretations of abstract uninterpreted systems—either in the shape of axioms (defined relationally) or as theorems provable from the axioms—falls back on some kind of logical essentialism or 'if–then-ism'; and this stands especially vulnerable to Wittgenstein's critique of rule-following (see chapter III of this work). Lastly, Hilbert can of course seek to salvage the irregular numbers as themselves being the represented realty—i.e., sign-reality given in our pre-conceptual intuition—and also as the appropriate fillers to go in the empty sockets of the abstract uninterpreted system, so to speak. But as we have noted, Wittgenstein's anti-foundationalist incisions have the full power to break through such purported reality of signs, given as a complete and pre-semantic content.

This section will be confined to the customary treatment of some non-standard numbers, like negative numbers, irrational numbers and imaginary numbers, while a detailed account of Wittgenstein's approach to some of these notionswill be taken up in the next section.. However a rough indication of the way he differs from the formalist management of these issues will run throughout the present section as well.

2.1 *Negative Numbers*

Time is popularly envisaged as a one-dimensional line progressing from moment to moment only in the forward direction, and patently debarred from moving backwards. The intrusion of negative numbers into the system of positive ones flares up another evocative imagery different from that of time—a line with the zero in the middle, where the positive numbers are made to lie on the right side of zero, while the negative numbers form a mirror-image to the left (see Figure A.1).[20] Both the positive and the negative integers lie as equally spaced points along the horizontal number-line.

Figure A.1



To attune ourselves to the dynamic relation between the positive and negative numbers, we need to rehearse the schoolbook rules once again:

a. To add a negative integer ($-m$) (to a positive or negative integer (n or $-n$) we need to move m spaces (from n or $-n$) to the left. For example: to add to −9 to 7, we have to take away 9 spaces to the left of 7 (i.e., to take away 9 spaces from 7), which reaches us to −2.
b. To add −9 to −7 by the same procedure, we have to move 9 spaces to the left of −7, which will take us to −16.
c. To subtract $-m$ (−9) from 7, we have to move m (9) spaces to the right of 7, which will take us to 16.
d. To subtract −9 from −7, we have to move m (9) spaces to the right of − 7, which will take us to 2.

Evidently, the rules feed on the direction and dynamism of space—to add $-m$ to any integer (positive or negative) is to take away m spaces on the left-hand side, and to subtract is to add m spaces on the right-hand side. It is this dynamism of spatial navigation that is frozen into the rules of a positive and negative yielding a negative, and two negatives adding up to a positive.

Multiplying a positive integer n by $-m$ will be to repeat n spaces m times to the left of 0. Multiplying a negative number $-n$ by $-m$ would be to repeat $-n$ (i.e., repeat n spaces to left of 0)m times to the right of 0. This last procedure would amount to releasing $-n$ from its negative status from the left of 0, and repeat its positive counterpart m times. While the first kind of movement is enunciated as the rule of the product of a positive and a negative number yielding a negative number, the second movement is captured by the rule of the product of two negatives yielding a positive.

Given the scenario that the operations with positive and negative numbers are to be carried out only in terms of space and number, it is extremely difficult for term formalists to insist that mathematical propositions are only about signs that are purified of any mathematical commitment to numbers. Can it safely transform itself into a game formalistic version relieving the rules of numerical operation from its numerosity? That is, can it say that the rule '7 – 9 = –2' merely prescribes a substitution of one spatial pattern for another—drawing nothing from the notion of counting, number or even equinumerability? Whether or not the game formalistic turn of rules is based on subitation, it is clear that without any guiding principle, it cannot go on even with its game of contriving rules of substitution. It is also clear that a game formalist cannot afford to maintain a non-formalist position about positive integers, while reducing negative numbers to mere sign transactions (independent of any mathematical content). Thus she can claim that the negative number, say 7 – 9 = –2, is as much a paradigm of sign-geometry—though of a slightly complex nature. A game of empirical observation or of experiment where two and two things are seen as coming up (or not coming up) to four things may enjoy a primacy vis-à-vis the posterior game of freezing them into mathematical definitions and proofs. But with negative numbers, it seems rather difficult to get a primary foothold either in some real dimensions of actual physical objects—say, the felt extremities of temperature, the intense shock of a bad performance, or a way to adjust our bills with an overcharging shopkeeper—and then to freeze these reverse dimensions of reality into rules of negative numbers. Rather, it is our positive construction of the positive number rules that are extended to the rules and proofs on negative numbers, where the two reverse dimensions on the number-line and the two opposed directions of manoeuvring with real objects fall in a balanced interplay.

In Hilbert's system, the proofs of equations involving negative numbers will be carried out in the same fashion as the equations involving positive integers (presumably in the model of the Q system of Robinson discussed in chapter III, section 6). It is a short step to applying Wittgenstein's critique to the proofs involving negative numbers in the same model as the positive.

2.2 *A General Account of Irrational Numbers*

Formalism in its most naïve version fails to see numbers as a paradigm of bundling up space into homogeneous units. It fails to see that the most obvious paradigm of doing this is to lay it out in the forward direction, i.e., as the series of positive integers, which soon needs to be extended into other dimensions and directions—activating other modes of fissure and fusion of space. Apart from cutting up space in the reverse direction of negative numbers, another demand of breaking up space into an endless recurrence of remainders comes up in the shape of irrational numbers. Irrational numbers are a subset of fractions, but unlike fractions which can be expressed as the ratio between two numbers, irrational numbers like $\sqrt{2}$ or π cannot be recast as a rational relation between the numerator and the denominator. Our initial division of π (i.e., $\frac{22}{7}$) as 22 parts divided by 7 leaves us with one unit which obviously cannot be divided by 7; hence we have to loosen out this space of one solitary unit which cannot be divided by 7. And the only way to do it is apparently to extend it to the base 10. Once this is done, it gets a remainder of 3 units, calling for another recast in terms of 30 units. Thus, space seems to get divided on and on—into more and more minuscule layers or levels of tens, hundreds, thousands—for instance, we know that gets expressed as a recurring decimal expansion of 3.142857142857. . . that never terminates.

It is this resistance of space against rational proportionalisation and its endless proliferation of remainders that easily prompts the formalist to say that these irrational numbers are not about space, nor about sets, nor a priori intuitions—but simply about signs or some blind techniques of operating with them. Some versions of formalism seem to hold that where spatial divisions cannot go on forever, there space itself fizzles out into a non-space or a non-entity, and the signs lose all their representational capacity (which they might have had with respect to the standard numerals) and turn into mere signs or rules for sign configuration—like the mere prescription to turn '**^^^' into '^**^*'—without any spatio-temporal commitment. The formalist fails to realise that when spatial operations face a resistance or recalcitrance, it cannot turn into a non-space; rather it becomes a different kind of space. This may be argued in different strands.

First, the irregularity of irrational numbers can be shown to be based on the regular behaviour of certain standard numbers, supplying a clear indication as to which practices have a terminating expansion and which do not. Often the decimal expansion of a fraction gets into a recurring pattern—like $\frac{3}{22}$ = .13636. . . with 36 parts repeating indefinitely. This shows that every fraction generates a recurring decimal—say $\frac{1}{2}$ generates .5000. . . (We know that the length of the recurring block in the recurring decimal expansion will be no larger than one less than the denominator. If the denominator is n, then the remainder after each step of division takes on one of the values 0, 1 . . . n–1) The expansion will terminate when the denominator is the product of the prime factors 2 and 5 of our base 10 (i.e., numbers like 40, 50, etc.). Thus, $\frac{11}{40}$ terminates in its equation with 0.275, but fractions like $\frac{1}{15}$ or $\frac{22}{7}$ do not. This neat relational scheme absorbing both rational and irrational numbers—a scheme that the formalist himself acknowledges—should prevent him from according a representational or non-formalist status to rational numbers and withdrawing it from the irrational ones at the same time.

Secondly, whether the decimal expansion of a fraction terminates or not is not just determined by that number in isolation, but also its relationship with the base in which we are operating. Suppose we worked on a ternary base—where space is bundled not into 10 or multiples of 10 spaces, but into 3 and multiples of 3 spaces, i.e., into 6, 9, etc. In that case the decimal expansion of a number that does not have a terminus in our standard base of 10 can however be shown to have a terminus in the ternary base. For instance $\frac{11}{3}$ has to be recast as $\frac{3332}{3}$, where the remainder of the division, i.e., 2, will have to be extended into a ternary base—i.e., to 6 (and not 20)—and will be divisible by 3 to yield a neat result of 2 without any remainder. Thus, finally $\frac{11}{3}$ will have to be recast as 111.2 in a ternary base. Again what emerges from this whole exercise is a serious censure against any formalist attempt to accord different statuses to the standard and non-standard numerals.

The process of turning non-terminating decimals back into integers, rational fractions and irrational fractions is made to fall into the same pattern of operation with integers or rational numbers. To consider the following operation:

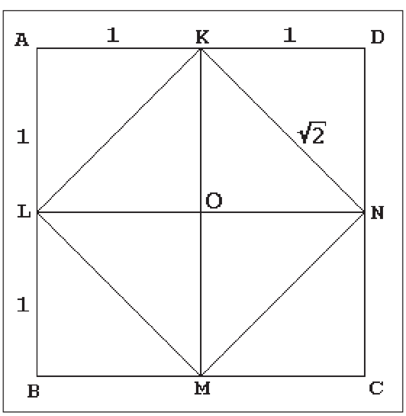
1.	Let a = 0.212121. . .	
2.	Then 100a = 21.212121. . .	(multiplying it by 100 on both sides)
3.	(99 + 1) a = 20 + (1.212121. . .)	(from 2)
4.	99a + a = 20 + (1.212121. . .)	(by applying distribution to the left-hand side)
5.	99a = 21	(by subtracting a from both sides of the equation)

What is noteworthy in the above procedure is how an irrational number in its relational product with other rational numbers yields another rational number. The formalists who concede this proof are labouring under an illusion—for them, 0.212121. . . bumps its head against a resistance of space that refuses to break up, thwarts all tools of rationalisation and thereby turns into a non-entity signifying nothing; and yet it lets itself be multiplied, distributed, subtracted from other rational numbers by standard procedures. A formalist cannot uphold a relation between a reality and unreality—between numerical operations and non-numerical techniques of sign configuration—and secure the grand, non-real status of the mathematical symbols as a result of this interaction. Hilbert's first step to revise formalism into a valid theory was to procure irrational numbers in a relational scheme with the rational ones (as the above equation) as either definitional axioms or provable theorems of a system. His later commitments to sign-reality, itself as the symbol and the symbolised, would perhaps have persuaded him to acknowledge irrational numbers as themselves embodying a different or deviant reality of space resistance—a reality given in pre-conceptual intuition—which in its turn would be available to rigorous formalisation. In both these routes he has to part ways with Wittgenstein.

2.3 *Treatment of* $\sqrt{2}$

Given the paradigms of multiplication and division, there is no number the square of which would equate to 2; however much we try to divide the space of 2 into neat units, extending its remainders to procure a neat scaffolding, the remainders persist as unmanageable excesses. But still (as we saw in the previous section), .212121. . . , which is actually an indefinite exercise, often with no specific rule of termination, is made to interact with another rational number to emerge as a multiple to yield another rational number. Now $\sqrt{2}$, though a misfit in the established paradigms of multiplication and division, yet comes to a have a foothold in more interesting ways than .212121. . . . It blends perfectly with the length of a line, when that line is opened out in a broad relational structure—when it squares out in all the four directions in a way that each of the sides has a unit length, and further, when the square formed thereby is embedded in a special manner in a larger square, double its size (see Figure A.2).

Figure A.2



Let the square KOND be of 4 feet (i.e., measuring up to 1 foot on each side). To this square we can add three others—ALOK, LBMO, and OMCN—each of them being 4 feet in area. While the whole square (ABCD) is four times the size of the first (KOND), our task is to get a square that is double the size of KOND. Now the line going from corner to corner (KN, LK, LM, MN) of each square cuts each of these squares in half. And as each of these lines cuts four equal squares into half, therefore each line is equal to the other. These four equal lines make a square MLKN, composed of four equal halves. These four equal halves taken together make the square MLKN that is half of the outer square ABCD. Hence, if the outside square has an area of 4 feet, the embedded square has an area of 2 feet. Hence each of its sides would be $\sqrt{2}$.

It further follows that since the number of halves in MLKN is four, i.e., double the halves in the original square KOND, therefore, since the latter is 4 feet, the former is 8 feet. And this 8-foot square is constructed on the base KN. Now, KN is the diagonal of the original square. Therefore, the square on the diagonal of the original square is double its size. It is from this aspect also that we can move to the matching dimension of $\sqrt{2}$. For if the area of the original square is half of the square on its diagonal, then the area of the latter will always be 2 in proportion to the area of 4 of the larger square. And this proves that the length of each side of a square with one unit length is $\sqrt{2}$. Alternatively, one can also arrive at this conclusion in a more formal and technical way—through the famous Pythagorean proof that the square on the hypotenuse of an isosceles triangle (with legs of one unit length) is equal to the sum of the squares on the other two sides. The insight behind the construction of these proofs lies in seeing KN in a double aspect—as the diagonal of an isosceles triangle and also as the side of a square that is embedded in another square double its size.[21]

Along similar lines, one can find a dimension answering to $\sqrt[3]{2}$, by presenting an altar that can be held to be a perfect cube, and then doubling it up, so that the larger cube can be said to be of volume 2 in relation to volume 1 of the original cube. The cube root of this larger cube will then equate with the desired number.[22]

We have tried to appreciate that for Wittgenstein there can be deviant modes of freezing or merging two stretches into a cinematographic cycle (see *RFM* I:38; discussed earlier in chapter II and chapter IV, section 2, Figure 4.8), which can be seen under two alternative aspects—viz., 2 + 2 + 2 = 6 and 2 + 2 + 2 = 4, i.e., either each set of two units staying apart or as the 2nd merging with the 5th, and the 4th with the 6th respectively. Analogically one may suggest that the recalcitrant nature of 2 refusing to break down into a square root is alternatively seen as expanding into a different dimension and direction—now as adding up equal squares in its sides, now as projecting its diagonal as the hypotenuse of an isosceles triangle, now as embedding its diagonal as the base of a larger square. Just as the frozen inequation between 2 + 2 + 2 and 4 could not reshuffle itself into an equation, as long as the units in the pairs could not let the coincident units merge into its body from a different direction; similarly so long as 2 remains confined within its crammed body of 1 + 1 or ½ + ½ + ½ + ½, it can find no way in which any of its standard parts can neatly square up to 2 itself—its full identity. It is only by activating itself in unexplored directions that the dead gap between $\sqrt{2}$ (i.e., 1.41429. . .) and 2 was filled up and its frozen irrationality cried out for a new route of dissolution.

2.4 *Complex and Imaginary Numbers*

Apart from the negative and irrational numbers, there are the complex and imaginary numbers that are of the form a + b*i*, where a and b are real numbers and *i* is the square root of −1. From a concrete point of view, we can quickly dismiss this number—since the square of any number is positive, $\sqrt{-1}$ is not a conceivable square root. Besides, comparing *i* with $\sqrt{2}$ one can say that while $\sqrt{2}$ has a decimal expansion that can be calculated with a desired accuracy, and can be matched with a specific dimension (viz., that of the diagonal of a unit square), nothing equivalent can be said about *i*. But still we can perform the following calculations with *i*:

1. $\frac{1}{i-1} = \frac{i+1}{(i-1)(i+1)}$ (the simple rule that the result of multiplying both

2. $\frac{i+1}{(i-1)(i+1)} = \frac{i+1}{(i^2 - i + i - 1)}$ (multiplying the numerator and the denominator of a fraction with the same number (here $i+1$) is equal to the original fraction) (applying the rule of distribution to the denominator of the right side of the equation)

3. $\frac{i+1}{(i^2 - i + i - 1)} = \frac{i+1}{-1-1}$ (substituting -1 for i^2 in 2)

4. $\frac{i+1}{-1-1} = \frac{i+1}{2}$ (from 3)

5. $\frac{i+1}{2} = \frac{1}{2}(i+1)$ (from 4)

The vacuous nature of *i* becomes all the more palpable when we see that this number does allow a distinction between its positive and negative counterpart: all true statements about *i* remain true when one replaces it with the corresponding statement about –*i*. For the formalists, operations with imaginary numbers like $\sqrt{-1}$ show that they are simply blind movements with signs[23]—purged of all spatio-numerical sensitivities pertaining to the reverse directions of positive and negative numbers, or to the recurring character of signs (irrespective of the fact that for Wittgenstein, there may be many modes mobilising this dynamic and recursive character of signs). However, several questions arise with respect to the above 'calculation' which clearly militate against the so-called 'formal' character of *i*. Why does the first step fall back on the *standard* rule of multiplication (that states that a fraction or a whole number remains the same when multiplied by a fraction that has the same digit for both the numerator and the denominator)? Again, why is i^2 blindly substitutable by –1, supposedly bereft of any spatio-numerical operation of squaring? And lastly, how can –1, bereft of all structural content, interact with another –1 to yield 2, in the reverse direction of the number-line? The formalist notion of numbers—specially irregular numbers—involves an impossible combination of space with non-space; no sooner than the numbers show any tendency to swell up to a paradigm of sign-geometry do they fizzle out into a non-space, into nothing.

3. Wittgenstein's Treatment of Irrational Numbers

While we have been working on a rough indication of the way Wittgenstein would be reacting to the realist as well as the formalist treatment of irrational numbers, this is the juncture at which we take a more detailed look at what special insights Wittgenstein has to offer on certain specific issues—Cantor's treatment of irrational numbers, the notion of infinity, the logicist notion of contradiction, and lastly a brief look at what Wittgenstein has to say on Gödel's approach to the arithmeticisation of meta-mathematical statements. The details of these insights will help us capture the exact nuance of the distance between Wittgenstein and the formalists in general, and Hilbert in particular.

3.1 *Wittgenstein on Cantor's Diagonal Proof: The Set of All Irrational Numbers is Non-Denumerable*

Wittgenstein probes into the question: what it is to say that the irrational numbers can or cannot be ordered in a series? Only this that we can order the roots and the algebraic numbers in a series—i.e., certain analogous formations which we can call by the common name 'series'. But from this it is by no means clear as how to construct a bridge that will take us to 'all real numbers'. Nor do we have a general method for seeing whether such and such a set 'can be ordered in a series'. Now we know that Cantor had a way of demonstrating that the set of real numbers is uncountable. To present it in the simplest possible terms: Cantor first takes the set that is greater than 0 and less than or equal to 1. Now, even if each distinct natural number is paired with a distinct real number (represented by a non-terminating decimal), each such pair would leave some real number unpaired. Any such pairing would begin like this:

1. 0.d d d d d d d d d. . . .
2. 0.d d d d d d d d d. . . .
3. 0.d d d d d d d d d. . . .

Let us note that d is some digit (from numeral 0 to 9). Now any numeral obtained by starting off with a 0, then a decimal point and then a digit different from the first d on the diagonal (but not 0), then a digit different from the second d of the diagonal (but not 0) and so on, would represent a real number greater than 0 and less than or equal to 1, that is not paired with any cardinal number. So the set of real numbers greater than 0 and less than or equal to 1 is uncountable. And this being a subset of the set of all real numbers, the latter is hereby proved to be uncountable.[24]

Wittgenstein says that all that the diagonal procedure shows is a difference in the variety of real numbers—the 'roots', algebraic number' on the one hand, and the diagonal numbers formed from the roots on the other. And Wittgenstein concedes that this applies to all series of real numbers. But this does not prove either that the set of all real numbers is uncountable or that it makes no sense to speak of 'all real numbers'. The meaning of the statement that the set of all real numbers cannot be ordered in a series is constituted in the very diagonal procedure—it does not have an independent content from which the diagonal procedure follows as a subsidiary tool. To say that 'this ordering cannot be done here,' it is not at all clear as to what it is that cannot be done here, until the diagonal procedure seeks to give it a foothold. But the power of the diagonal procedure does not explore infinite layers hidden inside the series of real numbers, thus demonstrating their non-denumerability. The vacuity of such a claim stands on the same footing as a particular method of counting a row of books—one that invokes the strategy of scraping up extra layers from them, in the shape of the row itself, or the top covers of each book as extra units, and saying that since the row of the books or the top cover is itself a book, hence the row of books is non-denumerable. Wittgenstein argues that the consideration of the diagonal procedure only shows that the *concept* real number has much less analogy with the concept cardinal number than we are normally inclined to believe. Or to say: 'If something is given as a real number then the expansion given by the diagonal procedure is also called a real number. And moreover said to be different from all members of the series.' But instead of making such good and honest sense, the diagonal proof goes on to do just the opposite—it pretends to compare the set of real numbers in magnitude with that of cardinal numbers. Here again it is clear that for Wittgenstein a proof does not go beyond its own physical content—whereas ironically it does within Hilbert's formalistic framework—on the ground that the signs are given in pre-conceptual intuitions and are extendable with respect to the non-standard numbers with the tool of 'ideal arithmetic'. What for Wittgenstein is the 'dangerous and deceptive thing' about the claim of non-denumerability of the set of all real numbers is the way in which a mere determination or a formation of concepts is made to look like a fact of nature. This invests the diagonal proof with a claim to prove more than its means allows us, what he characterises as ' a puffed up proof' (*RFM* Appendix II, 1–3).

3.2 *Wittgenstein's Approach to Infinity*

The proposition that there is no greatest cardinal number is not of the same category as 25 × 25 = 625, a fact that becomes palpable when both are put in their respective interrogative form. The latter comes with the range of possible answers, while the former is usually placed in a scenario where we have not yet carved out the logical space of its alternative answers. We can clarify the nature of the former question by drawing up a possible analogy: 'How do you know that space is three-dimensional?' This question does not make sense before a special discourse is created—the discourse marked by an internal contrast between two kinds or levels of experience—the first level supplying us only with two-dimensional patches and fragments, and the second level synthesising them together into full-fledged three-dimensional objects. Experience does not actually come in these two successive phases, each with a separate content; rather, it is this tension *within* experience that we have to invent in order to make the proposed question answerable. Similarly with regard to the question whether the series of cardinal numbers has a greatest number can be meaningful only in the backdrop of an invented contrast made between two rules or licences—the licence to terminate the series vis-à-vis the absence, or the unavailability of a rule or licence to stop the series. Before this contrast is activated, it makes no sense to frame the above question. This teaching that the licence to play language-games with the cardinal numbers does not terminate is a grammatical proposition of a different type than the proposition $25 \times 25 = 625$ (*RFM* Appendix II, 5). The latter is a rule about a specific number, which sometimes can be learnt by rote—though of course against the vast background of learning mathematics. But the rule about cardinal numbers is a generic rule, not a specific rule cast within, but rather constituting the very essence of an expansive form of living that mathematics is.

There are certain misleading pictures surrounding Cantor's proof of showing irrational numbers as different from *all the ones* in the system. It is as if the numbers were given as terminal points in space, but actually the points did not hit the limit of space; there are always hidden gaps in which a new number is lurking, and these new hidden numbers are to be hurled up through the diagonal route—the route that will go on extending as more and more numbers are hurled up along the route. Wittgenstein argues that Cantor does not show a way of hauling up a new number different from all those that are given on the surface, by showing a different route in which these *other* numbers are given. Rather, he just shows a different way to navigate some of the already given digits within each number in a different pattern. There may be many ways of expanding the system of real numbers; Cantor is not exploring the uniquely different way of expansion, he is only exercising one instantiation of this concept of difference. This is what is expressed in Wittgenstein's observation that Cantor is saying something about the multiplicity of the concept 'real numbers different from all the ones in the system'. There was no infinite system of irrational numbers that was already given within which Cantor discovers further folds of higher-order infinities. What was previously in place with the irrational numbers and what Cantor adds are both procedures which do not make provision for a licence to terminate the decimal expansion. It is only that Cantor's procedure of this non-termination was different from the first (*RFM* Appendix II:6–7). It is different in the sense that a row of books or the schematic diagram of their shape, size and configuration is different from each book. This difference is a difference of level, for the row or schema of books deals not with the books, but rather with the *concept* of books. And hence if one professes to dig up more and more invisible books hidden under the plainly visible stock in this way, what one has virtually done is simply engage in a higher-level concept of the books—higher in the sense that it surreptitiously feeds on the outline and the collection of the books, rather than the books themselves, and thereby generates an infinite stock of books magically from a finite body. Similarly, the diagonal procedure of cropping up a new number from the body of the old ones is just a second-level engagement to conceive the irrational numbers in a different way—so as to activate an infinite number of instantiations.

We know that Cantor also sought to generate these infinities within infinities by the diagonal procedure—another example of which comes in his exercise of generating an uncountable *power set* within the infinite set of cardinal numbers. The set of all subsets of a set A is known as the *power set* of A. The set of natural numbers is claimed to have its power set, i.e., there is a set that has for its members all subsets of the set of natural numbers and nothing else. Cantor first sought to show by the diagonal procedure that the set of all subsets of the set of natural numbers is uncountable. This is because it is always possible to define a subset of the set of natural numbers that does not occur in any alleged one–one pairing of the subset of natural numbers with the natural numbers. Secondly, he proved that the power set of a set has a greater cardinal number than the set itself,[25] i.e., to put it plainly, one infinity can contain another and the latter will be greater than the former. Given that these two infinite cardinalities are respectively called $2\aleph_0$ and $\aleph_0$, we have the dramatic discovery in the history of mathematics—viz., $2\aleph_0 > \aleph_0$.

Wittgenstein thinks that this enigma about $2\aleph_0 > \aleph_0$ can be solved along the same line of argument explained above. We are not repeating the pattern of refutation with respect to this famous theorem of Cantor, except for noting Wittgenstein's remark that this mode of proliferating airy infinities 'is for the time being a piece of architecture which hangs in the air, and looks as if it were, . . . an architrave, but not supported by anything and supporting nothing.' It is as useless as to say '10^{10} souls fit into a cubic centimeter' (*RFM* Appendix II:8).

Wittgenstein makes a fine distinction between the terms 'infinity' or '$\aleph_0$', on the one hand, and 'infinite sequence' on the other—a distinction which can be rephrased as that between the number of the members of an infinite series and the number of the concept 'infinite series'. A 'series' in the mathematical sense is a method of constructing a series of numerical expressions. The term 'infinite series' has a use in so far as within the calculi of mathematics, we have a technique that can be described as '1–1 correlation of the members of two infinite series'. The justification of this description rests on the similarity it has with the one–one correlation of the members of two 'finite' classes. But from this it does not follow that we can meaningfully speak of the number of the concept of 'infinite series', or that here we have a use that compares with the use of a numeral. We only have the employment for a *kind* of numeral, which, as it were, gives the number of the members of an infinite series (by the absence of the licence of termination of the members), but from this it does not follow that the talk of the number of the concept 'infinite series' is also meaningful. A numerical sign and number are not the same—a numerical sign '$\aleph_0$' is used analogously with finite numerals to initiate a procedure of giving the number of members of an infinite series. But to talk of the number of the concept 'infinite series' is to claim the concept as having an infinite number of instantiations—which is a big leap from a mere procedure to an ontological position. When the word 'infinite series' is used for a concept—independent and distinct from a numeral—one must appreciate that no use for this concept as having infinite applications has yet been invented. And to repeat, for Wittgenstein, this has to be *invented*, not discovered. He further argues that to present such uses as 'the class of all classes' as being equinumerous with the class 'infinite series', and thereby giving the content of the crucial concept of infinity, will not hold ground. As all classes cannot be given as a complete content, a stance of presenting this unpresentable content is on the same footing as presenting something like the class of all angels who can get on this needle point. An attempt to lay one's hand on the possible instantiations of the concept of angels and then draw out a one–one correlation with those angels who can get on the needle point refuses even to take off; it fails to initiate any procedure of construction. Our notion of space refuses any kind of employment with respect to the angels as to what spatial dimensions, interactions, motion, velocity, inertia, weight or mass, or their absence may enable a specific number of angels to get on a particular needle point. Similarly, the concept of an infinite series having an infinite number of instantiations cannot possibly start up even a seemingly initial stage of an appropriate construction (*RFM* Appendix II:9).

Wittgenstein has strong reservations against the notion of an infinite set as being denumerable—i.e., as having a one–one correspondence with the set of natural numbers. To say that rational numbers cannot be counted, but one can count with them—as with the cardinal numbers—is a squint-eyed and pretentious way of projecting the finite in the same model as the infinite (*RFM* IV:13). He thinks that instead of saying 'denumerable' one should say 'numberable'—a safe terminology that confines oneself simply to the application of a concept, viz., that of assigning numbers to them, and not exploring a putatively long row of infinity already laid down. Wittgenstein characterises this realistic stance of mathematics—specially with respect to infinity—as mathematical alchemy, and yet observes that there is a 'solid core' to all these obscure and 'glistening concept-formations'. He thinks that the illusion of this solidity that turns mathematical issues into real productions has a special character—it is more complex than the superior mirage of the Fata Morgana. It is rather a dynamic alternation between two directions—what looks like a glistening and unsubstantial Fata Morgana from one direction looks to be a solid body from the other (*RFM* IV:16).

3.3 *Wittgenstein on Contradiction—With Special Reference to Decimal Expansion of Irrational Numbers*

At *RFM* V:8, with respect to Russell's contradiction expressed in the form of ~*f*(*f*), Wittgenstein reminds us that a proposition or a statement attains its so-called neutral status only in contrast with other non-statements—the various language-games that he has mentioned at *PI* 23. An identity proposition was shown to be associated with a play of imagination—the thing coming out and recoiling back into its own socket—or the phenomenon of the dispersal and reversal of white light through crossing prism, the example which has been repeated too often in this work. As a contrast point, ~*f*(*f*) can be said to be another game—another play of imagination that resists this operation of dispersal and reversal. In language, contradictions are useful, as it is useful to speak of an object in motion—that it exists and does not exist at that place. In general, change may effectively be described in terms of contradiction—minus the time factor. Often the fear and fastidiousness of avoiding contradiction takes the shape of dividing space and time regressively into finer and finer parts, with reference to which apparent contradictions (or apparently incompatible predicates) can be recast as belonging to two finely chiselled loci of space–time. In this way, the fear of contradiction lapses into the fear of there lurking an infinitesimal, impenetrable or extensionless point of time and space so to speak, on which, unbeknownst to us, the same thing with the two incompatible predicates may have been impaled. And for Wittgenstein, since this extensionless identity is a myth, so is the fear of contradiction—being both ϕ and not-ϕ—being attached to this identity. The fear itself becomes delinked from all language-games, resting on a despatialised space like a 'cancerous growth, seeming to have grown out of the normal body aimlessly and senselessly' (*RFM*

V:8). The claim that the provision of the calculus should be such that a mechanical application of the rules should not lead to a contradiction, uses the word 'mechanical' in a misleading fashion. For Wittgenstein, a calculus that does not make this provision and results in a contradiction is not a 'bad piece of mathematics', nor does it deserve the position of 'not being mathematics in the fullest sense'. That kind of calculus is like one which would allow the talk of motion as contradiction, not falling back on that underlying mythical identity that will eternally shove off contradiction (*RFM* V:8).

Wittgenstein goes on to argue that setting up a rule for avoiding contradiction is not a technique to improve bad mathematics, but to create a new bit of mathematics. He illustrates this in terms of the following cases—finding a way of constructing a . . . gon, which we were so far only able to draw by trial or error, or solving a third degree equation, or adopting a different strategy of assuring that '777' does not occur in the expansion of π, other than by replacing the 777 by 000. In the last case, since in our calculation of the places we are not allowed to look back at the earlier ones, we need another calculus so that we are assured in advance that it cannot yield 777. To take another simple instance from propositional logic, consider the two sentences: (A) It is not that either Crumm is innocent or Moriarty is guilty; and (B) Crumm and Moriarty are either both guilty or both innocent. Here, logic would obviously fall back upon the meaning of the sentences to show that they are mutually inconsistent. As logic wants to do away with this dependence, it demands a mechanical way to isolate each atomic proposition, each propositional connective, and thereby recast the above sentences into the format of p . ~p. This new technique of mechanical dismissal of contradiction does not rake up p . ~p which were as if hidden under a hollow, but activates a way to lay it out in terms of contradiction or non-contradiction—all the while presupposing the main issue, i.e., that p . ~p is a contradiction.

Wittgenstein further argues that ensuring the law of pure identity or avoiding contradiction is a way of scaffolding the world, where each object and each attribute is allotted a unique space–time locus, so that no two things can share the same locus and no two predicates of the same category (or same incompatibility range) can be predicated of the same object in the same space–time locus. One can go on with this scaffolding as long as it allows, but when we find the boundaries dissolving—or the so-called different things or predicates jostling into the same locus—this does not mean that the whole construction conducted hitherto has to be rejected on pain of 'contradiction' (*RFM* V:9–10). For finding a contradiction is not really to find an extensionless point (hitherto unexplored) that will lay bare the mutual exclusion of ϕ and $\sim\phi$, both erroneously projected on this point. It is not that this contradiction was previously obfuscated by our failure to chisel out the reality in terms of unique and exclusive points. Establishing non-contradiction or consistency in terms of finding exclusive spatial points works with a misguided meaning of 'consistency'; this meaning of 'consistency' makes as little sense as to speak of a consistent extension of a curve (*RFM* V:12).

The programme of cutting up space into atomic bits to avoid contradiction turns out to be invalid once we realise that what we are actually dealing with is an unsurveyable space or an unsurveyable series 'which starts to flicker at a distance' (*RFM* IV:12). The fact that all the members of a series—from the 1st till the 1,000th, then up to the 10^{10}th and so on, are so determined only implies *that it is not the case that so and so many-th are not determined*. And for Wittgenstein, this double negation does not bring us back to the positive, for there is no originary identity that is readily available for being wholly excluded by a subsequent negation. The negation only fleshes out through a flow of family resemblances in many directions—they only flare up newer and newer aspects of the putatively positive original; but the original was never a non-relational identity to compel a neatly reverse journey through the celebrated tool of double negation.

Thus, to say 'so and so is not non-determined' is not to compel a unique pattern. That the gradual division of 2 towards its square root will yield a particular dimension—the diagonal of a square of a particular kind—was not given in the line of division; one has to get into another dimension (as it were from a line to a surrounding plane) where the linear expansion of $\sqrt{2}$ was seen gradually as sprawling out into a network of triangles and squares, which were embedded into each other in a specific fashion (see section 2.3 of this Appendix). Wittgenstein obviously has to add that neither the line nor the expanded plane were given out there; they had to be invented in mathematics. And as these expanded dimensions are not given in the numbers, one cannot claim to capture this expansion in the law of the excluded middle. One cannot claim a generic essence of a pattern '. . .' as occurring in a particular decimal expansion: there is no such essence that goes *beyond* the examples actually determined in the very exercise of working out the pattern. As the examples cannot show a generic picture of what it is for the pattern to occur in the expansion, it cannot also show what the opposite means. Since we do not have a direction to construct the content of the pattern of expansion (p), the tautology p v ~p is just as shaky as the sense of q (a typically contingent proposition) (*RFM* IV:12).

Wittgenstein thinks that a simpler way to dissipate the law of the excluded middle is to recast the relevant mathematical propositions (about the pattern ϕ occurring in the series) as well as its negative counterpart (the pattern ϕ as *not* occurring the series) as a pair of positive and negative order—in which case the relevant law of the excluded middle would have to be rephrased as: 'Either you must do it or you must not do it.' Obviously a pair of positive and negative imperatives does not carve out a mutual exclusion—thus the seeming force of exclusion can easily be outgrown in a different language-game (*RFM* IV:17). There is nothing startling about a command's *commanding* that this must not occur in this series however far it is continued. It is perhaps this alternating aspect between an imperative game and a propositional game that shapes the oscillation between Fata Morgana and reality.

3.4 *Wittgenstein's Take on Gödel's Proof of Incompleteness of Arithmetic*

The following is perhaps one of the briefest and simplest introductions to Gödel's arguments[26] (steering clear of any entry gate to his proofs), hopefully enabling us to appreciate once again the non-foundational character of proofs in Wittgenstein's framework as revealed through his observations on Gödel's programme. Gödel proved that the arithmetical system is bound to be incomplete, i.e., given any consistent set of arithmetical axioms, there are true propositions that cannot be derived from that set. He designed his proof in the following way.

One has to consider a language (say English) in which the purely arithmetical properties of numbers can be formally defined. Such a system would consist of the following definitions—an integer is divisible by another, an integer is the product of two integers, etc. Each of these definitions will contain a finite number of words and thus a finite number of letters of the alphabet. On this ground, the definitions can be placed in a serial order—the definition with the smallest number of letters will be matched with number 1, the next definition in the series will correspond to number 2, and so on. Now, the property designated by a definition may or may not belong to the serial number allotted to it (the definition): for instance a definition having the serial number 17 and designating the property of prime will have achieved the match between these two aspects, while a definition having the serial number 15 and designating the property of the square of an integer clearly has not achieved a match. This mismatch between the property of the serial number assigned to the definition and that *stated* by the definition itself does not bring any anomaly or paradox, for the simple reason that statements *within* arithmetic do not include the serial number that pertains only to the letters embodied in the definitional expressions. The statements about the notation in which arithmetic is codified does not belong to arithmetic itself. However, for Gödel, this notation construction suggested that it may be possible to 'map' or 'mirror' meta-mathematical statements about a formal system within the body of the system itself. In this way, the complicated meta-mathematical statements about the formalised systems of arithmetic can be displayed or embodied in the arithmetical statements within the system itself. Logical relations will be transformed into their arithmetical counterparts or mirror-images, thus achieving better clarity and perspicuity. Gödel then showed that meta-mathematical statements about formal arithmetical calculus can be embodied or reflected in the arithmetical formulas *within* the calculus. This led him to devise a method by which neither the arithmetical formula corresponding to the meta-mathematical statement, nor the negation of the formula corresponding to the denial of the system, is demonstrable within the calculus. As one of these formulae must physically reflect or codify the arithmetical truth, and yet none of them can be derived from the axioms, this shows the axioms to be incomplete.

Let us follow Wittgenstein's representation of Gödel's approach to the incompleteness of mathematics. Let us have an arithmetical proposition saying that a particular number . . . cannot be obtained from the number . . . , . . . , by means of such and such operation. Let us call this proposition G. Three things are to be noted about G:

i. G can be translated by some translation-rule to the figures of the first number (i.e., the propositional sign-content of G can be translated into the figures of the first number).
ii. Axioms by which G is sought to be proved can be translated by some translation-rule to the figures of other numbers.
iii. The rules of inference by which G is sought to be proved is translated (by some translation-rule) into the operation mentioned above—the very operation that is crucial to show the unprovability of the particular number from other numbers.

So if we have derived G from the axioms according to the rules of inference, then by this means we should have demonstrated the underivability of the number and thus the derivability of G. But since the translation-rules transform the proposition into a propositional sign as embodying that purportedly underivable number, G—claiming the underivability of that number—is not derivable.

Wittgenstein lays bare the obvious points underlying this paradox. The proposition G constructed as a propositional sign-pattern constitutes the geometrical proof that this number can be got from this other number by means of these operations. Looked at this way, the so-called proposition and its proof are not propositions at all—they have nothing to do with logic—i.e., relations of implication that typically obtain between propositions and not propositional sign-patterns. Though the constructed proposition has the apparent form of a proposition, it cannot be compared with other propositions *saying* this or that—or as having sense.

What is crucially important in Wittgenstein's approach is that the constructed propositional sign does not carry the correlation between two numbers in its own body. Rather, this pattern has to be read as a mathematical proposition saying *that* it embodies or exhibits the correlation of the two numbers which are propositionally stated to be non-correlatable. In other words, a propositional sign or a pattern cannot non-propositionally contradict the original proposition; rather, the propositional pattern has to be *read as a proposition stating* the opposite of what we regard as proved by the original proposition G. Now as these two propositions are not about the same thing—one is about propositional relations between numbers and another is the relation between two sign-patterns—the question arises whether one can legitimately speak of an opposition between these two. Wittgenstein goes on to argue that it was not Gödel's agenda to exhibit that the sign-construction of a proposition necessarily leads to the opposite of what the original proposition purports to prove, or that the non-propositional body of a propositional sign necessarily contains the opposite of the propositional content. What Gödel wanted to say is simply that one must be able to trust a mathematical proposition when we want to conceive it as a practical demonstration of the construction of a propositional pattern. The opposition between the propositional and non-propositional aspect of G is not given as a necessary entity—as a limit of empiricism—i.e., an entity that cannot be contradicted by any experience. The limit of experience where we seem to be compelled to imbibe the opposition between the propositional and non-propositional aspects of G is neither unwarranted assumptions, nor intuitive revelations; they are just the 'ways in which we make comparisons and in which we act'. It is a matter of practice that we conceive the mathematical proposition also as a proposition of geometry which is also applicable to itself. The conviction of this opposition is neither contained in expression by voice or gesture, or in a feeling of satisfaction—its ratification consists in the use of what isproved. Gödel's projection of this necessary opposition, or his proof of the incompleteness of arithmetic, is simply a hypothetical projection of a situation purely meant to tease out our interest and persuade us to a possible response (*RFM* V:18–19). This is how Wittgenstein carries out his task—not of talking about Gödel's proof, but bypassing it (*RFM* V:16).

4. The Distance between Wittgenstein and the Formalists

While Wittgenstein is not a formalist, he along with all versions of this genre will be pitched against the same question—if there is nothing corresponding to numerals, how is it that certain numerals allow themselves to be added, divided, multiplied neatly to arrive at definite results, while certain other numbers do not? If there is no reality that figures as an external constraint to the behaviour of numerals, how is it that some fractions lend themselves to being recast as definite ratios between two spaces and some numbers do not, their ugly remainders always pushing up their heads through the process of division? Why does 4 have at least one definite square root and 2 does not?

I suggest that for Wittgenstein, just as statements and equations about regular numbers are constructed by freezing causal and experimental games (with real objects) into mathematical definitions, similarly the inequalities and resistances of certain other numbers—say the resistance of 22 to being neatly divisible by 7 (or the inequation between $\frac{22}{7}$ and 3) are also constructed by freezing both the actual equations and compatibilities as well as the actual inequations and incompatibilities between real objects or real space—into ideal inequalities or incompatibilities. In other words, just as the multifaceted variety of physical realities and their causal operations are glossed over in mathematical numeralisation with the regular numbers, similarly the empirical correspondences and non-correspondences between 22 and 7 parts, or the causal divisibility and non-divisibility of one part into seven segments, are turned into paradigmatic inequalities or incompatibilities. For once the causal and experimental games with real objects are frozen into mathematical definitions ($3 \times 7 = 21$ or $\frac{21}{7} = 3$) they cannot be permitted to thaw down into $\frac{22}{7}$ as being divisible by 3 or $\frac{1}{7}$ being a rational number. What emerges is that the irrational numbers do not pose any *actual* space resistance (or a deviant reality so to speak) that would throw a challenge to the formalist claim on that account; rather, the statements about irrational numbers freeze operations on the correspondences and compatibilities to turn them into non-correspondences and non-compatibilities. What spilled over as an anomalous excess in the addition of two and two drops of water, resisting all coercions to persist as four drops, again spills out in certain cases as a real correspondence of mutual rationalisation between 22 and 7 drops of water. In both the above cases, the real or causal operations are tailored down respectively into paradigmatic equalities and inequalities (i.e., into $2 + 2 = 4$ and $\frac{22}{7} \neq 3$). It must further be noted that for Wittgenstein, neither the so-called 'real' correspondences nor the 'real' incompatibilities holds a privileged pre-semantic status, marking the ground for the posterior games of mathematics to turn on. Rather, both these games—that of real causation, experience or experimentation vis-à-vis the mathematical contrivances—work in a correlative pattern, drawing their significance from each other. None of the versions of formalism has the provision of appreciating this dimension of mathematics—of its being an interplay between these two language-games, or rather these two forms of living.

We have seen that formalism, especially in its naïve versions, reckons irrational numbers as a stoppage of spatial activity that virtually lapses into blind techniques with signs, neutralised of any spatial or numerical dimension. Wittgenstein has displayed them as a different kind of spatial activity, where the actual but indefinite operations of division are frozen into a rule of decimal expansion, into an equation with a particular irrational number, stipulated as the limit of expansion. A cinematographic picture is constructed *within* the world of signs, by carving out a physiognomic cycle between two sign-spaces—the progressive division and expansion on the one hand, and the limit of the particular irrational number on the other. This is all there is to the grammar of irrational numbers, which is virtually the grammar of their sign-geometry exhibited in a surveyable number of steps. This expansion of decimals need not involve a commitment to infinite division of space, and hence does not stand in the further need for invoking formalism as a saviour of mathematics against such an inflationary ontology.

I would like to wind up this appendix by spelling out the differences between Wittgenstein and Hilbert in more explicit terms with respect to some specific issues, viz., the ontology of signs, the notion of infinity, the commitment to deductivism, and lastly, with respect to the possible impact that Gödel's proof can cast on these two styles of doing philosophy of mathematics.

4.1 *Contrasting Wittgenstein and Hilbert on the Nature of Signs*

Let us take a closer look at Hilbert's commitments to the nature of signs, as to what he means by 'concrete symbols' being given in conception and at the same time intuited or directly experienced prior to all thinking.[27] The apparent tension between mathematical signs being *conceived* on the one hand, and being *pre-conceptually given* on the other, can best be handled in the following way. Concepts as distinguished from passive intuitions are marked by an autonomy, which works principally in four ways.[28] Firstly, one can exercise the option of taking a conceptual experience as veridical or not. This is true not only with, say, the Muller-Lyer illusion, but also with secondary qualities like colour, shape and pain. I have the freedom to exercise the option whether the colour I see is really red or not, or whether the pain that I feel is really due to an external physical cause. There might be cases where the subject thoroughly programmed to expect a strong pain stimulus actually misconceives a pleasure stimulus to be pain. Our experience of mathematical signs is conceptual in so far as we can entertain the possibility of holding the colour, distance, material medium, texture, or even the number of the signs as non-veridical—being caused under certain deviant conditions generating faults in registering not only sensible qualities like colour, shape or texture, but also spatio-numerical properties like location, distance and number of the signs.

Secondly, conceptual experience, by its very nature, is not confined to a specific case, but is logically repeatable to other occasions. To have typically passive experience like those of the secondary qualities, viz., colour or shape, means that one is able to recognise that colour or shape in another object at a different space and time. Mathematical signs too are given in our conception, in so far as we can clearly understand what it is for a sign-type to repeat in various tokens, what it is to conceive internal variations within the different instances (tokens) falling under a sign-type without deflecting the latter. Thirdly, conceptual experience is embedded in judgements which in their turn are linked in logical or rational relations with other judgements. To understand that | | | is larger than | | is also to know the proposition that $3 > 2$ is true, and thereby to be potentially related with other judgements like $3 > 1$.

Lastly, all conceptual experience must involve a sense of representing reality—a sensitivity to the kind of states of affairs in the world. To experience a colour is to relate it with a surface, with the boundary of an object, recognising it in another object—and all these objects are situated in the wider reality. Here we face an interesting way in which the conception of mathematical signs stands apart from other conceptual experiences. In the first place, signs do involve a sense of representing reality, but unlike other experiences the sign-experience creates a split within its own body—the split between the sign and the signified. It is at the same time the experience of the representing sign as well as the represented reality. What emerges to be the crucial question is how far mathematical experience of signs, though in a sense full-bodied and replete with properties, direction, distance, contiguity, etc., can—like other experiences of sticks and stones, tables and chairs, rivers and mountains, shape, colour, etc.—undergo a dynamic recursion of repeatable features that can be integrated with the larger landscape of reality. This is exactly the point at issue, one that sets mathematics apart from other sciences, and this is exactly the common point of departure from which the widely divergent theories of mathematics—realism, logicism, intuitionism and formalism—have taken off.

Now, given that mathematical experience of sign-reality is conceptual, in what sense are these signs given in pre-conceptual intuitions? Hilbert would most probably insist that the autonomy, spontaneity and multiple options that are patently exercised in conceptual experience, even in the experience of mathematical signs, comes to a terminus at the pre-conceptual structure of space and number itself. Hilbert would no doubt insist that this unique structure pre-exists all conceptual thought experiments with numbers coalescing or fragmenting, distances crumbling or dissipating, to make such counterfactual luxuries possible. In this sense, mathematical signs can be given minimally in conception, in so far as this conception does not outstrip the given. And in this sense the mathematical judgements too are fully warranted by the given, because being warranted by the given constitutes the very identity of the concepts.

Shall we say that for Hilbert there is a complete congruence between intuition and concepts at the minimal content of the sign-reality, or would he, like Kant, still insist on that enigmatic split between intuition and concepts—the split that is magically overcome in our experience which is inherently conceptual? If (like Shapiro) we take Hilbert as envisaging mathematics in the same way as Kant—as being about a priori intuitions of space and time[29]—he, like Kant, would be at a loss to explain how pre-conceptual intuitions can upgrade themselves to sign-types that are logically repeatable in instances, or how these pre-judgemental intuitions can yet justify other judgements. For a type–token relation can only obtain between two levels that are equally conceptual, and nothing short of a judgement can justify another judgement. To avoid this lapse into a Kantian dichotomy between intuition and concepts, Hilbert has to take the first alternative (generally recommended by McDowell); and it is precisely here that the discord with Wittgenstein's anti-foundationalist approach flares up. Wittgenstein would never digest the idea of a concept being what it is, or the possibility of its being defined by a unique normative relation with the so-called *given*, once for all. For Wittgenstein, the proof is not totally bogged down by the specificity of the signs, nor is it overridden by a non-temporal or non-spatial abstraction. What needs to be appreciated is the relational tension between the spatial and sensual specificity of the signs and its possible fulgurations—the practices through which the specificity gets dissolved in the single cinematographic picture. Wittgenstein does admit a memorable picture attaching to each rule of configuration of signs, but that picture does not hover as an ethereal canopy over the signs; rather, it is a picture in so far as it is *enacted* as the proof operation. And there are many modes of this enaction—there are many ways in which our concepts can claim a normative relation with the so-called given, many grammatical paradigms through which it can describe the mathematical reality—i.e., the sign-reality. Contrary to Hilbert's scheme, there are many ways of coalescing the so-called receptivity of sign-intuitions and the autonomy of the conceptual paradigms used by mathematics to turn this sign-reality into systems.

4.2 *Wittgenstein and Hilbert Differing on Infinity*

Wittgenstein is quite clear in his claim that if some versions of formalism impose a meaning of the word 'infinite' on the calculus, instead of deriving its meaning from the technique itself, that approach has to be discarded. Similarly, if a formalist asserts that there is nothing infinite in the calculus, that statement is equally clumsy and unacceptable—as if one expected to see something 'enormously big', but could not find it (*RFM* Appendix II:17). Wittgenstein says that finitism and behaviourism go wrong when they formulate their view in the shape of such a sentence as 'There is no content or a subject-matter (say infinity) here'—thus projecting the possibility of a describable vacuum in a particular locus (*RFM* Appendix II:18). And Wittgenstein clearly suggests that he is not doing a finitary mathematics; he does not have the special goal of ousting the word 'infinite' from the vocabulary—but rather to survey all aspects of its employment. Indeed, Hilbert's theory of infinity too is far from a naïve dismissal of an infinite content of space, but his system of ideal arithmetic betrays his commitment to the propositions about infinity and non-denumerability of sets—admitted within Cantor's scheme—as logically deducible in the system of ideal arithmetic. In this connection, we need to take note of certain observations of Hilbert, where he considers real and complex analysis to be the most 'aesthetic and delicately erected structure of mathematics', or declares that 'mathematical analysis is the symphony of the infinite'.[30] He holds that certain antinomies like Russell's paradox and Cantor's inconsistent multitude (a collection of sets that were too big to be collected together in one set) are not threats to mathematics—they can all be resolved by useful application of definition and deduction. 'No one shall drive us out of the paradise that Cantor has created for us.'[31] On the other hand, what we have already noted with Wittgenstein—the proposition 'fractions cannot be arranged in a series in the order of magnitude'—is simply to be looked upon as a technique of calculating fractions, i.e., as a technique of continuous interpolation of fractions, without which the phrase 'the next greatest fraction' has no meaning. The technique itself is not non-ending in the sense that our actions are invested with an infinite positive content, but it is simply that the technique lacks the institution of an end, or rather it is invested with an additional rule of not letting it terminate. Wittgenstein gives further examples of a sentence without a period, or of a playing field where the rules of the game prescribes it as not having any boundaries (*RFM* Appendix II:11) . This technique of constant interpolation of signs within any two fractions cannot be cast into an axiomatic model constructed by Hilbert. The axiomatic definitions of Hilbert have to be invested with the *institution that there has to be a lack of institution—a form of life embedding the axiomatic definitions*. Hilbert's formalism probably does not have, nor does it want to have, the informal notion of forms of living as incorporable into his system.

For Hilbert, the irregular behaviours of certain sign-spaces or sign-intuitions can be smoothed out in an integrated whole with the regular intuitions; and this integration will be secured as an interpretation of an abstract system, or as fillers of the empty sockets of the latter, coupled with the tools of deductivism. But for Wittgenstein all equations with irrational numbers are operations of carving out single physiognomic cycles with the sign-realities, or constructions of cinematographic pictures with a surveyable content. For him, this integration is an exercise, not of the logical machine with pre-applicational axioms and definitions, but that of merging the two sign-realities—or rather the two procedures with the two kinds of sign-realities (viz., the uninterpreted sign-variables and the interpreted sign-values) in a single proof—by a single circular motion.

4.3 *Wittgenstein and Hilbert Differing on Deductivism*

For both Wittgenstein and Hilbert, there is nothing corresponding to the terms 'point', line', etc., in isolation, by virtue of which these terms can either be posed as primitives, or be formulated in the shape of discrete definitions, which in its turn may be available for the construction of axioms. For Wittgenstein, however, this does not persuade us to adopt Hilbert's alternative, that of constructing a relational structure of these terms in the shape of axioms, once for all, that would unfailingly yield a unique set of conclusion. We may say that Hilbert substitutes relational essences for property essences, not quite realising that a relational essence too does not foreshadow its instances—it spreads out bit by bit through each of its new instantiations. Deductivism embodies an intriguing combination: it insists on its axioms as abstracting from any specific content, and yet containing an exact and complete description of the relations subsisting between the elementary ideas of that particular science, so that these ideas can smoothly fill in the empty placeholders of the axiomatic structure. Thus the axioms that are set up in Hilbert's system are claimed to be the definitions of those elementary ideas at the same time. However, it is indeed difficult to understand how the axioms can set off only as an abstract relational structure and yet be fleshed out with all possible alternative materials, how they can define specific notions and yet conveniently shed them off when required to take in an alternative set of materials. Can one prepare a skeleton without pre-establishing the range of materials that are supposed to fill it in?

What Hilbert takes to be the immaculate relational structure, ready to absorb all possible content, is actually to be achieved by means of concrete constructions—not by putting values in the empty slots and procuring the interpreted axioms thereby. Putting values in the axiomatic structures and deducing unique theorems is not based on the powers of 'if–then', but on the surveyable constructions of the proof. We have already noted that for Hilbert, mathematical statements with an unbounded variable or a letter of generality (but with no explicit universal quantifier) can be handled by 'primitive recursive arithmetic', which is commonly agreed upon as a finitary proof technique. Wittgenstein's possible reaction to this proof technique (as explained in chapter III, section 6 of this work) is hopefully adequate to mark this route of divergence between the two philosophers.

In this connection, one is tempted to ask as to what exactly is the significance of Hilbert's claim that any attempt to find real entities corresponding to geometrical terms like 'line', 'point', etc., degenerates into a game of 'hide and seek'. The best way to understand Hilbert's realist undertone beneath his seemingly anti-realist profession is to put it against some powerful imageries of Wittgenstein—those that exhibit the seeming essences, whether in the shape of a property or a relation, as being essentially essenceless, ruptured and incomplete. We have seen that for him, the talk of all abstract entities—'absolute space', absolute quantity', 'extensionless points', 'breadthless lengths'—draw their significance from a series of progressive contrasts, where this gradual shedding of the empirical content from the so-called pure dimensions is exercised in a relational structure. (This has been explained in chapter II, section titled 'The Talk of Definite or Absolute Identity'.) And it is crucially important to note that this exercise derives its significance entirely from the *incompleteness* of the relational structure itself. For once we take this structure to be complete, we have entered into the fruitless game of 'hiding and seeking' that absolute limit of space—eluding us forever. While Hilbert's formalism rests on the denial of real entities corresponding to mathematical terms, he invokes a complete and immaculate structure in the shape of axioms and rules, and invests them with the full implicative power to stretch their long spectral arms reaching out to an infinity. Hilbert in his endeavour to upgrade formalism to deductivism and shirk off realism virtually ends up in another version of realism.

The sign ontology in Hilbert's system deepens further when finitary mathematics is claimed to extend to ideal mathematics containing not only statements about natural numbers with unbounded quantifiers, but also meta-mathematical statements about real numbers, complex numbers, set theory, etc. Let us remind ourselves that Hilbert regards ideal mathematics as instrumental to finitary mathematics. That is to say, while the formulae of ideal mathematics mean nothing by themselves, we can deduce meaningful finitary statements from these formulae as their interpretations. In Hilbert's finitary arithmetic, the natural numbers are identified with their names or character types, and obviously each of its specific instances is identified with 'tokens'—a strategy which enables him to recast the finitary statements as meta-mathematical statements about numbers. Seen in the reverse direction, it enables him to procure finitary arithmetic as an interpretation of ideal arithmetic, and broadly speaking, seems to obtain a method of recasting sign-intuitions in terms of meta-mathematical statements about irrational numbers, imaginary numbers, complex numbers, infinite numbers, etc., so that they may be derived as theorems of his uninterpreted abstract system. It is this inflation of sign ontology or sign mysticism, as the strange blend of the pre-conceptual and the conceptual, that Wittgenstein wants to resist. If signs allow a type/token structure within their body, then they are countable, computable or numerable—in one way or another—and if they are numerable, they are conceptual. And if signs are conceptual, they must allow spontaneous options of interpretation and alternative ways of recursion—i.e., alternative ways of being integrated with other signs and sometimes with the larger scheme of reality.

This is what Wittgenstein has been trying to convince us all through—in his constant insistence on the opacity of proof-pictures, on the failure of logicism, on the different modes of aspectual transitions between mathematical definitions and causal or empirical games, on the treatment of irrational numbers in different directions and dimensions of space, on the lack of necessity in the Gödelian arithmeticisation of meta-mathematical statements. Hilbert's system of finitary arithmetic—aided by the further instrument of ideal arithmetic—lies precisely in blocking this space of multiple interpretations of sign-reality. As already noted, for him signs are conceptual in the sense that they come with a fully consummated character—the conceptual operations cannot go beyond their putatively given content. The mathematical judgements cannot outstrip the given—and it is this given that is carried on in the infinitary claims of ideal arithmetic. Once committed to the givenness of signs, Hilbert's system becomes vulnerable to Gödel's attack—that every true meta-mathematical statement will allow its own transformation into a numerical mirror-image that will exhibit the exact opposite of what it (i.e., the original meta- mathematical statement) purports to say—all this fatally showing its non-demonstrability. If, according to Hilbert, the signs are given with a unique way of conception, perhaps he cannot resist Gödel's arithmeticisation of the meta-mathematical statements and the subsequent consequences. But Wittgenstein would look upon Gödel's programme as a pattern of geometrical transformation of meta-mathematical propositions—and this grammar of the pattern does not have an inbuilt necessity to persuade us to accept the fatally opposite character of the number (as Gödel claims)—it only projects an interesting situation to elicit our reaction to the proposed transformation of the statements (*RFM* V:19; also see section 3.4 of this Appendix).

It would perhaps be appropriate to end with a suggestive dissonance between Wittgenstein and Gödel—a dissonance lying beneath an apparent proximity. We have seen Wittgenstein argue extensively in favour of the opacity or multiple interpretability of the proof-picture or the verbal rule, and in the course of this he has also conceded that a proof or a rule can be read in a way that demonstrates precisely the opposite of its so-called standard meaning. While at first blush this is an attractive point to build possible parallels between these two philosophers, it is a short step to outgrowing the force of this analogy. For Wittgenstein, just as we cannot lay our hand on a normal pre-applicational content of the rule or the proof-picture, a content that would spurt forth the uniquely normal conclusion from its body—working over a distance—similar remarks apply to the purportedly deviant content, delinked from and yet shooting off a uniquely deviant application. A supposedly deviant foundation—severed from the conclusion—can also erupt in multiple directions and conclusions. If we are still under the sway of foundationalism, we may think that Gödel has discovered a hitherto unexplored content of the meta-mathematical formula (G), and an *out of the world* translation-rule whereby G can be transformed to a number—ironically just *that* number stated to be non-demonstrable by G. In other words, we may think that Gödel has unearthed a revolutionary interpretation of G, which when attached to the latter will show it to contain precisely its own opposite. Now Wittgenstein has repeatedly warned us against thinking that an interpretation is a fixture to be attached to a rule to produce its application, and thus against looking upon a different (or deviant) application as produced by a different fixture attached to the rule (see chapter III, section 7 of the present work). What Gödel has invented is a way to mobilise G into a cycle that encircles the translation-rule and the non-demonstrable number into a single cinematographic picture. The statement that '*This* formula "*G*" can be translated by *this* translation-rule to *these figures* of *this first number*' (*RFM* V:19) can make sense only when 'this formula G' is understood to make no sense apart from the translation-rule; the expression 'this translation-rule' makes no sense apart from the un-demonstrable number. So while Hilbert's system was trapped by the Gödelian discovery of the hidden content of G, Wittgenstein shows us the way to dissolve the split between the trapper and the trapped—thereby creating a latitude 'not to talk about . . . Gödel's proof, but to pass it by'. (*RFM* V:16)

Notes

1. The following exposition of formalism is chiefly drawn from Stewart Shapiro, *Thinking about Mathematics* (New York: Oxford University Press, 2000), chapter 6.
2. The formalists can take the help of possible variations within the universal/particular schema: (*a*) property exemplification (say the relation between manness and a particular human); (*b*) type/token, i.e., the relation between coins, currency and words figuring both as type and token; and (*c*) determinant/determinable—for instance, the relation between a colour sample and the specific variations pertaining to its hue, shade, etc. In the second and the third cases, the universal is also a particular—the coin type and colour sample are also individual coins or colours—but manness, if there is anything like it, is usually held not to be a particular. The wide conceptual gap between the universal and the particular that is supposed to exist between a property and its exemplification closes down in the other two—specially in the second case—apparently taking all unwanted metaphysical load off the formalist's shoulder.
3. Shapiro, *Thinking about Mathematics*, refers to G. Frege, *Grundgesetze der Arithmetik* (1) (Olms: Hildescheim, 1893), sections 86–137.
4. This is what chess ultimately has to commit itself to—its being at least about spatio-temporal units and structure and their logical implications.
5. Hunter gives these as typical examples of axioms, theorems, inference rules of an abstract uninterpreted system. See G. Hunter, *Metalogic* (London: Macmillan, 1971).
6. G. Frege G, *Grundgesetze der Arithmetik*, vol. 2, in P. Geach and M. Black (eds), *Translations from the Philosophical Writings of Gottlob Frege* (Oxford: Basil Blackwell, 1974), section 90. This is explained by Shapiro, *Thinking about Mathematics*, p. 146.
7. See Shapiro, *Thinking about Mathematics*, pp. 146–47.
8. Hilbert expounds this version of formalism in *Grundlagen der Geometrie* (Foundations of Geometry), trans. E. Townsend (La Salle, IL: Open Court, 1959). See Shapiro, *Thinking about Mathematics*, pp. 148–57.
9. See Shapiro, *Thinking about Mathematics*, p. 151.
10. Ibid., pp. 153–54.
11. See Gottlob Frege, *Philosophical and Mathematical Correspondence*,

ed. Gabriel Gottfried, trans. Kal Hans (Chicago: University of Chicago Press, 1980), pp. 31–52. This issue is also explained by Shapiro, *Thinking about Mathematics*, at pp. 154–55.

12. Frege, *Philosophical and Mathematical Correspondence*, p. 39.
13. Ibid., p. 51; also quoted in Shapiro, *Thinking about Mathematics*, p. 156.
14. See P. F. Strawson, *Introduction to Logical Theory* (New Delhi: B.I. Publications, 1976), chapter I, for a neat presentation of this basic claim of logic.
15. Strawson, however, is not so confident about logic attaining this kind of semantic supremacy, nor of its power of creating an immaculate scheme–content dichotomy that would automatically take in any kind of content. See ibid., chapters I and II.
16. Frege, *Philosophical and Mathematical Correspondence*, p. 51.
17. Hilbert expounded his finitary arithmetic chiefly in 'On the Infinite', in Jean van Heijenoort, *From Frege to Godel: A Source Book in Mathematical Logic, 1879–1931* (Cambridge, MA: Harvard University Press, 1967); also in P. Benacerraf and H. Putnam (eds), *Philosophy of Mathematics* (Cambridge: Cambridge University Press, 1983). See Shapiro, *Thinking about Mathematics*, pp. 158–65.
18. David Hilbert, 'On the Infinite', in P Benacerraf and H.Putnam (ed) *Philosophy of Mathematics*, CUP 1983, p 192. Shapiro quotes this statement in p 161 of his work.
19. See ibid., p. 164. But unfortunately Shapiro does not mention Wittgenstein, who was unmistakably influenced by Hilbert, without taking the mathematical sign-reality either to deductivism, intuitionism or Ideal Mathematics. Shapiro reckons on two occasions (ibid., p. 145n2 and p. 161) that Wittgenstein's view of rules as underdetermining their applications would thwart a formalist dependence on rules; and perhaps it was for this reason that he refused to be impressed by any proximity between their respective approaches to mathematics.
20. One can refer to Peter M. Higgins, *Numbers: A Very Short Introduction* (New York: Oxford University Press, 2011), for a lucid, comprehensive and philosophically unloaded account of numbers. It is this work (see especially chapter 6) that I have used for the required exposition of negative numbers and irrational numbers.
21. I have relied on the following websites for the presentation of this proof: A. Bogomolny, 'The length of the diagonal of the unit square

equals the square root of 2', Interactive Mathematics Miscellany and Puzzles, available at: http://www.cut-the-knot.org/do_you_know/SqRtOf2.shtml (accessed on 20 July 2016); see also 'Halving a Square', Cut the Knot, available at: http://www.cut-the-knot.org/proofs/half_sq.shtml (accessed on 20 July 2016); 'Square root of 2', available at: https://en.wikipedia.org/wiki/Square_root_of_2 (accessed on 20 July 2016).

22. See Higgins, *Numbers: A Very Short Introduction*, p. 78.
23. For this account of the nature of and operations on $\sqrt{-1}$ as well as its formalist overtones, I have drawn on Timothy Gowers, *Mathematics: A Very Short Introduction* (Oxford: Oxford University Press, 2002), pp. 29–31.
24. I have borrowed Hunter's presentation of this proof of Cantor. See Hunter, *Metalogic*, pp. 31–32.
25. As both these proofs are too technical, I have opted for the easier exercise of painting Wittgenstein's reservations about these proofs with a broad brush—justifying this by the assurance that Wittgenstein himself exercised this style. For a lucid and comprehensive presentation of these proofs, one can again refer to Hunter, ibid., pp. 21–25.
26. My comprehension and presentation of Gödel rests entirely on the wonderfully lucid exposition of E. Nagel and J. R. Newman, *Gödel's Proof* (London: Routledge, 1958). To avoid too much technical complication, I have used only chapter 6 of this book.
27. Shapiro explains this point by quoting from Hilbert, 'On the Infinite', in *Thinking about Mathematics*, p. 161.
28. I have derived these principles of conceptual operation from John McDowell, *Mind and World* (Cambridge: Harvard University Press, 1996), pp. 11–13. I have sought to give these general principles a special orientation to suit mathematical experience.
29. See Shapiro, *Thinking about Mathematics*, p. 162.
30. David Hilbert, 'On the Infinite', p 187, also quoted by Shapiro.*s*.
31. Ibid. p 191.

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