NON-ABELIAN HOMOLOGICAL ALGEBRA
II. VARIETIES

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On the basis of the theory of derived functors and satellites, as developed in the preceding paper† (cf. (4)), we shall study now the functors and homomorphisms associated with varieties. The universal category $\mathfrak{C}$ is, as before, a complete category of (Λ, Σ)-groups, Λ being a near-ring generated by the distributive semigroup Σ. To each variety $\mathfrak{V}$ there belongs a subfunctor V and a quotient functor U of the identity functor‡ of $\mathfrak{C}$ (cf. I, § 1, and (3), § 2). After establishing in § 1 some general results on functor sequences, the basic properties of $L_n V$ and $L_n U$ will be derived in § 2.

Let T be an additive functor of $\mathfrak{C}$, which for the purposes of discussion we assume to be covariant,§ and let $\mathfrak{V}$ be a variety. Considering here for simplicity's sake only groups, rather than pairs, one obtains the $\mathfrak{V}$-derived functors $L_n^{\mathfrak{V}} T(U(A))$, in addition to the $\mathfrak{C}$-derived functors $L_n T(A)$, together with natural homomorphisms $L_n T(A) \to L_n^{\mathfrak{V}} T(U(A))$ (*reduction homomorphisms*). These, and related functors and homomorphisms, will be investigated in §§ 3 and 4.

The important varieties in this context are those which contain the variety $\mathfrak{A}$ of Λ'-modulus, Λ' being the residue ring of Λ modulo its commutator ideal. By replacing, if necessary, Λ by some quotient near-ring one can always place a given variety $\mathfrak{V}$ into this situation. If then $\mathfrak{V} \supseteq \mathfrak{A}$, the strongest results are obtained when T is $\mathfrak{V}$-compatible, i.e. when $T(A) = T(U(A))$, for all A. In this case the functors and homomorphisms, associated with the change $\mathfrak{C} \to \mathfrak{V}$ of variety, appear as the terms and mappings of one infinite exact sequence. As will later become clear, some of the results derived for particular situations have in fact, in a different context, already made their appearance in the Abelian theory.

In § 5 we consider the functor homomorphisms derived from the basic mapping

$$\omega_{A|A'}: A|0 \to A|A'$$

† Throughout referred to as I.

‡ The symbols V and U will be used in this connotation throughout.

§ Throughout the paper we shall, in the discussion and in proofs, always restrict ourselves to covariant functors.

associated with each pair. For $\mathfrak{A}$-compatible functors we shall obtain a complete description in terms of a single exact sequence.

Section 6 is concerned with the special topic of varieties derived from ideals and semigroups in Λ. Replacing Λ by a quotient near-ring amounts to a change of variety. The general theory of a change in near-ring will, however, not be dealt with here.

As an example we shall, for an arbitrary distributively generated near-ring Λ, determine the values of $L_n V(A)$ and $L_n U(A)$ for all varieties $\mathfrak{B}$ and for all n, when A is a quotient group of Λ (§ 6). In the two 'basic' special cases of our theory, namely when $\mathfrak{C}$ is a category of modules (cf. § 6), and when $\mathfrak{C}$ is the category of abstract groups (cf. § 2), we shall also derive expressions for the functors $L_n V$ and $L_n U$ with arbitrary argument in terms of known invariants.

In the module case a change of variety is the same as a transition from Λ to the residue ring modulo some ideal $\mathfrak{n}$, and the values of $L_n V$ and $L_n U$ can be expressed in terms of tensor products with $\Lambda/\mathfrak{n}$ and with $\mathfrak{n}$, and of their derived functors.

For abstract groups we obtain certain invariants associated with their representations as quotients of free groups. Moreover, the induced homomorphisms $L_n V(f|f'), L_n U(f|f')$, the connecting homomorphisms of the homology sequences, and the various functor homomorphisms defined in this context can all be seen to coincide with mappings induced in a natural manner by the structure of the representing free group.

As this discussion already indicates, there is here a considerable overlap with Baer's work (cf. (**1**)), as far as abstract groups are concerned. One can, in fact, express the whole theory of Baer invariants in homological terms and apply it to more general categories. This, however, will involve a number of new concepts which lie outside the scope of the present paper.

Since the completion of this work, the author has found a whole range of structures which give rise to non-Abelian categories, and to which our homology theory and Baer's theory apply (cf. (**5**)). An interesting example is the category $\mathfrak{C}_\Lambda$ of algebras (not necessarily with identity) over a fixed commutative ring Λ with identity. For $\Lambda = Z$ one gets in particular the category of rings. In general $\mathfrak{C}_\Lambda$ is isomorphic with the category of supplemented algebras in the sense of (**2**); one only has to associate with each supplemented algebra its augmentation ideal. $\mathfrak{C}_\Lambda$ contains again the category of Λ-modules, viewed as algebras with trivial multiplication. A pair $A \mid A'$ is now an algebra A together with an ideal A' annihilated by A.

We shall continue to formulate everything in terms of (Λ, Σ)-groups. It is left to the reader to convince himself that the theory applies in fact

to Ω-groups in the sense of Higgins, and in particular to categories $\mathfrak{C}_\Lambda$. A homological discussion of Baer-invariants in $\mathfrak{C}_\Lambda$, which again applies in principle to Ω-groups in general, will be given elsewhere.

The general conventions and notations of I will be taken over here, without further reference.

1. Exact functor sequences

Let, throughout this section,

$$0 \to T_1 \to T_2 \to T_3 \to 0 \tag{1.1}$$

be a sequence of additive functors of $\mathfrak{C}$ with the property that the sequence

$$0 \to T_1(A) \to T_2(A) \to T_3(A) \to 0 \tag{1.2}$$

is exact whenever $A \mid A'$ is $\mathfrak{C}$-projective for some A'.

Assume T_i $(i = 1, 2, 3)$ to be covariant. Let $A \mid A'$ be any pair in $\mathfrak{C}$ and let $[\mathbf{A} \to A]$ be a $\mathfrak{C}$-resolution of $A \mid A'$. We obtain an exact sequence

$$0 \to T_1(\mathbf{A}) \to T_2(\mathbf{A}) \to T_3(\mathbf{A}) \to 0 \tag{1.3}$$

of complexes. As shown in I, § 4, we derive a sequence

$$\left.\begin{array}{l} \ldots \to L_{n+1} T_1(A \mid A') \to L_{n+1} T_2(A \mid A') \to L_{n+1} T_3(A \mid A') \\ \to L_n T_1(A \mid A') \to \ldots \qquad \ldots \to L_0 T_3(A \mid A') \to 0 \end{array}\right\} \tag{1.4}$$

with connecting homomorphisms δ_n. Let $f \mid f' : A \mid A' \to B \mid B'$ be a homomorphism of pairs, let $[\mathbf{B} \to B]$ be a $\mathfrak{C}$-resolution of $B \mid B'$ and $F : \mathbf{A} \to \mathbf{B}$ a homomorphism over f. Then the diagram

$$\begin{array}{ccccc} T_1(\mathbf{A}) & \to & T_2(\mathbf{A}) & \to & T_3(\mathbf{A}) \\ \downarrow & & \downarrow & & \downarrow \\ T_1(\mathbf{B}) & \to & T_2(\mathbf{B}) & \to & T_3(\mathbf{B}) \end{array}$$

will commute, and hence so will the diagram

$$\begin{array}{ccc} L_{n+1} T_3(A \mid A') & \to & L_n T_1(A \mid A') \\ \downarrow & & \downarrow \\ L_{n+1} T_3(B \mid B') & \to & L_n T_1(B \mid B'). \end{array}$$

This implies in the first place that the connecting homomorphisms δ_n in (1.4) are in fact independent of the choice of $\mathfrak{C}$-resolution and in the second place that they give rise to functor homomorphisms.

By (I (4.6), (4.7)) we obtain:

1.1. THEOREM. *With every sequence* (1.1) *which is exact on $\mathfrak{C}$-projective pairs there are, for all $n \geq 1$, associated connecting homomorphisms of functors*

$$L_n T_3 \to L_{n-1} T_1.$$

The composition of any two consecutive homomorphisms in the sequence

$$\text{(a)} \qquad \ldots \to L_{n+1} T_3 \to L_n T_1 \to L_n T_2 \to L_n T_3 \to \ldots \to L_0 T_3 \to 0$$

is null. The sequences

$$L_n T_1 \to L_n T_2 \to L_n T_3,$$

$$L_0 T_1 \to L_0 T_2 \to L_0 T_3 \to 0$$

are exact.

If T_1, T_2, T_3 are weakly normal, the sequence (a) *is exact.*

Assume now that (1.1) is an exact sequence. By considering $\mathfrak{C}$-representations of length $n > 0$ rather than $\mathfrak{C}$-resolutions, we obtain

1.2. Theorem. *With every exact sequence* (1.1) *there are, for all $n \geqq 1$, associated connecting homomorphisms of functors*

$$S_n T_3 \to L_{n-1} T_1.$$

The composition of any two consecutive homomorphisms in the sequence

$$\text{(b)} \qquad 0 \to S_n T_1 \to S_n T_2 \to S_n T_3 \to L_{n-1} T_1 \to \ldots \to L_0 T_3 \to 0$$

is null.

The sequence

$$0 \to S_n T_1 \to S_n T_2 \to S_n T_3$$

is exact.

If T_1, T_2, T_3 are weakly normal, the sequence (b) *is exact.*

The 'main part' $L_{n-1} T_1 \to \ldots \to L_0 T_3 \to 0$ of (b) coincides by (I, 7.1) with the corresponding part of the sequence (a) in (1.1). Moreover, we have

1.3. *If* (1.1) *is an exact sequence, then the diagram*

$$\begin{array}{ccccccccc} \ldots \to & L_n T_1 & \to & L_n T_2 & \to & L_n T_3 & \to & L_{n-1} T_1 & \to \ldots, \\ & \downarrow & & \downarrow & & \downarrow & & \| & \\ 0 \to & S_n T_1 & \to & S_n T_2 & \to & S_n T_3 & \to & L_{n-1} T_1 & \to \ldots \end{array}$$

commutes, the vertical mappings $L_n T_i \to S_n T_i$ being given by σ_n (cf. I, 7.5). *Also the diagram*

$$\begin{array}{cccccc} \ldots \to & L_0 T_1 & \to & L_0 T_2 & \to & L_0 T_3 \\ & \downarrow & & \downarrow & & \downarrow \\ & T_1 & \to & T_2 & \to & T_3 \end{array}$$

commutes.

From 1.3 we get in particular for all $n > 1$ a commutative diagram

$$\begin{array}{ccccccccccc} 0 \to S_n T_1 \to S_n T_2 \to & S_n T_3 & \to & L_{n-1} T_1 & \to & L_{n-1} T_2 & \to & L_{n-1} T_3 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & 0 & \to & S_{n-1} T_1 & \to & S_{n-1} T_2 & \to & S_{n-1} T_3. \end{array}$$

By I, 9.4, by 1.2 and 1.3 we now have

1.4. *Let the sequence* (1.1) *be exact. If the T_i are weakly normal and T_2, T_3 are covariant, weakly right exact, then the sequence*

$$0 \to S_n T_1 \to S_n T_2 \to S_n T_3 \to L_{n-1} T_1 \to S_{n-1} T_1 \to 0$$

is exact.

From the definitions we verify easily

1.5. *Homomorphisms of sequences* (1.1) *induce homomorphisms of sequences* (a), (b).

1.6. *If $\mathfrak{V}$ is a complete subcategory of $\mathfrak{C}$, then the diagram*

$$\begin{array}{ccc} \mathrm{P}L_n T_3 & \to & \mathrm{P}L_{n-1} T_1 \\ \downarrow & & \downarrow \\ L_n^{\mathfrak{V}} \mathrm{P}T_3 & \to & L_{n-1}^{\mathfrak{V}} \mathrm{P}T_1 \end{array}$$

(and the analogous diagram with L_n replaced by S_n) of connecting homomorphisms and restriction homomorphisms commutes, for all n.

1.6, of course, presupposes that the conditions for the definition of both horizontal mappings are satisfied. Thus for L_n we should require (1.1) to be exact on $\mathfrak{V}$-projective pairs. As the restriction homomorphisms are absolute, it follows now that they induce homomorphisms of sequences (a), (b).

Results analogous, and dual to those stated in this section, hold for contravariant functors.

2. Variety functors and quotient functors

We consider a variety $\mathfrak{V}$ in $\mathfrak{C}$ with associated variety functor V and quotient functor U. Denoting by I the identity functor of $\mathfrak{C}$, we have an exact sequence

$$0 \to V \to I \to U \to 0. \tag{2.1}$$

V is a normal subfunctor of I (cf. (**3**), § 1) and is thus additive, covariant, weakly normal, and preserves monomorphisms; by definition it preserves epimorphisms. U is additive, covariant, right exact, and normal. By I, 7.7, 9.4,

$$\left.\begin{array}{l} S_n V = 0, \quad S_n U = L_n U \quad (n > 0), \\ L_0 U = U. \end{array}\right\} \tag{2.2}$$

I being exact, we obtain by I, 7.6, and from 1.1, 1.4,

2.1. Theorem. *The sequences*

$$0 \to L_1 U \to L_0 V \to I \to U \to 0,$$
$$0 \to L_1 U \to L_0 V \to V \to 0$$

are exact, and for $n > 1$,

$$L_n U \cong L_{n-1} V.$$

In terms of the functor homomorphisms of 2.1, we get immediately

2.2. $L_1 U(A\,|\,A') = 0$ *if, and only if,* $L_0 V(A\,|\,A') \cong V(A)$.

$V(A) = 0$ *if, and only if,* $L_1 U(A\,|\,A') \cong L_0 V(A\,|\,A')$.

For fixed A the last isomorphism is thus valid either for all pairs $A\,|\,A'$ or for no such pair.

U being weakly normal and right exact we have, by I, 9.3, 9.4,

2.3. *The connected sequence* $\{L_n, \mu_n\}$ *of functors associates with every central sequence*

$$0 \to C\,|\,C \to B\,|\,B' \to A\,|\,A' \to 0$$

an exact sequence

$$\ldots \to L_{n+1} U(A\,|\,A') \to L_n^{*} U(C) \to L_n U(B\,|\,B') \to L_n U(A\,|\,A') \to \ldots$$

$$\ldots \to L_1 U(A\,|\,A') \to U(C) \to U(B) \to U(A) \to 0.$$

For any complex $\mathbf{A}$ of groups in $\mathfrak{C}$, $U(\mathbf{A})$ and so also $H(U(\mathbf{A}))$ will be a complex in $\mathfrak{B}$. Hence

2.4. *The groups* $L_n U(A\,|\,A')$ *lie in* $\mathfrak{B}$.

Assume now $\mathfrak{C}$ to be the category of abstract groups. We represent a given group B as a quotient group of a free group F with kernel R, i.e. we identify

$$B = F/R. \tag{2.3}$$

If B' is a central subgroup of B, let F' be the inverse image of B' in F. Then $(F, F') \subseteq R$ and

$$\ldots \to 0 \to \frac{(F,F')}{(R,R)} \to \frac{R}{(R,R)} \to \frac{F}{(F,F')} \to B \to 0 \tag{2.4}$$

is a $\mathfrak{C}$-resolution of $B\,|\,B'$. In particular, for $F' = R$ we obtain a $\mathfrak{C}$-resolution of B. The groups $V(B)$, $U(B)$, $L_n U(B\,|\,B')$, $L_n V(B\,|\,B')$ are, with identification (2.3), given† by forming quotients of normal subgroups of F. Thus

$$\left.\begin{aligned} L_0 V(B\,|\,B') &= \frac{V(F)+(F,F')}{V(R)+(F,F')}, \\ L_1 U(B\,|\,B') &= \frac{[R \cap V(F)]+(F,F')}{V(R)+(F,F')}, \\ L_2 U(B\,|\,B') &= \frac{[(F,F') \cap V(R)]+(R,R)}{V((F,F'))+(R,R)}. \end{aligned}\right\} \tag{2.5}$$

Let M_i, N_i $(i = 1, 2)$ be normal subgroups of F, with $M_1 \subseteq M_2$, $N_1 \subseteq N_2$, $N_i \subseteq M_i$. The mapping $x + N_1 \to x + N_2$ $(x \in M_1)$ is a homomorphism $M_1/N_1 \to M_2/N_2$, which we say is *induced by* F. We now have

† Recall that $L_n T = 0$ for $n > 2$ (cf. I, 8.9).

2.5. *The mappings of the sequences*

$$0 \to L_1 U(B|B') \to L_0 V(B|B') \to B \to U(B) \to 0,$$

$$0 \to L_1 U(B|B') \to L_0 V(B|B') \to V(B) \to 0,$$

and the isomorphism

$$L_2 U(B|B') \cong L_1 V(B|B'),$$

as induced by the functor homomorphisms of 2.1, *coincide with the homomorphisms induced by* F.

A similar remark holds also for various other homomorphisms. Thus if $\mathfrak{V}_1$ is another variety with associated variety functor V_1 and if $V_1(B) = 0$, then on replacing F above by $F/V_1(F)$ we get the derived functors $L_n^{\mathfrak{V}_1} U, L_n^{\mathfrak{V}_1} V$, and again the restriction homomorphisms are induced by F. Next let

$$0 \to C|C \to B|B' \to A|A' \to 0$$

be a central sequence, and let G be the inverse image of C in F, considering C as embedded in B. We identify

$$C = G/R, \quad A = F/G, \tag{2.6}$$

and again express $L_n U(C|C)$, $L_n U(A|A')$ as quotients of normal subgroups of F. We then get

2.6. *The mappings of the sequence*

$$0 \to L_2 U(B|B') \to \dots \quad \dots \to U(B) \to U(A) \to 0,$$

as derived from the connected sequence $\{L_n, \mu_n\}$ *of functors, coincide with homomorphisms induced by* F.

We now consider as a particular case the variety functor K and the quotient functor J associated with the variety of Abelian groups; thus $K(B) = (B, B)$. We then have from (2.4)

$$\left.\begin{aligned} L_0 K(B|B') &= (F,F)/(F,F'), \\ L_1 J(B|B') &= R \cap (F,F)/(F,F'), \end{aligned}\right\} \tag{2.7}$$

and taking $B' = 0$,

$$\left.\begin{aligned} L_0 K(B) &= (F,F)/(F,R), \\ L_1 J(B) &= R \cap (F,F)/(F,R). \end{aligned}\right\} \tag{2.8}$$

Thus in the category of abstract groups $L_1 J(B)$ is the Schur multiplicator, as defined by representations of groups as quotient groups of a free group, and $L_0 K(B)$ is another well-known invariant of such representations (cf. (1)). $L_1 J(B)$ is thus isomorphic to the second integral homology group of B. We shall later obtain a generalization of this result.

The functor $L_1 J$ provides simple examples to show firstly that derived functors are non-additive, and secondly that for a central sequence

$0 \to C \to B \to A \to 0$ of groups the sequence $L_1 J(C) \to L_1 J(B) \to L_1 J(A)$ need not be exact. In fact, let B be the direct sum of two cyclic groups A and C of equal order $n > 1$. Then $L_1 J(A) = L_1 J(C) = 0$, and $L_1 J(B)$ is cyclic of order n, which shows that the above sequence is not exact, and that $L_1 J$ is not additive.

We now recall that if B is of finite order n, then $nL_1 J(B) = 0$. Comparing (2.7), (2.8), we also find $nL_1 J(B \mid B') = 0$.

Let now $\mathfrak{C}$ be again an arbitrary complete category of (Λ, Σ)-groups, Λ not necessarily the ring of integers, and let $\mathfrak{C}_Z$ be the category of abstract groups. Let $L_1 J$ denote the derived functor of J in $\mathfrak{C}$ and $L_1^Z J$ the derived functor of J in $\mathfrak{C}_Z$. Let $0 \to A_1 \to A_0 \to A \to 0$ be a $\mathfrak{C}$-representation of $A \mid A'$. We obtain a commutative diagram

$$\begin{array}{ccccc} L_1^Z J(A \mid A') & \to & J(A_1) & \to & J(A_0) \\ & & \| & & \| \\ 0 \to L_1 J(A \mid A') & \to & J(A_1) & \to & J(A_0) \end{array}$$

with exact rows, and so an epimorphism $L_1^Z J(A \mid A') \to L_1 J(A \mid A')$. Let now A be of finite order n. Then we conclude that some factor m of n annihilates $L_1 J(A \mid A')$.

Consider next a $\mathfrak{C}$-resolution, or $\mathfrak{C}$-representation

$$\cdots \to A_1 \xrightarrow{\alpha_1} A_0 \to A \to 0,$$

of $A \mid A'$, and write $\operatorname{Im} \alpha_1 = A_1^*$, so that $A_1^* \cap (A_0, A_0) = L_1 J(A \mid A')$. For $x, y \in A_0$ we have nx, ny, $n(x+y) \in A_1^*$, and $n(x+y) - ny - nx \in (A_0, A_0)$. The latter element thus lies in $A_1^* \cap (A_0, A_0)$ and so $mn(x+y) = mnx + mny$. Thus the mapping $\rho_0 \colon x \to mnx$ is an endomorphism of A_0. Trivially, $\rho \colon x \to mnx$ is an endomorphism of A, and as A_q is commutative for $q > 0$, $\rho_q \colon x \to mnx$ is an endomorphism of A_q. We obtain a commutative diagram

$$\begin{array}{ccccccc} \cdots \to & A_q & \to \cdots \to & A_0 & \to & A \\ & \downarrow \rho_q & & \downarrow \rho_0 & & \downarrow \rho \\ \cdots \to & A_q & \to \cdots \to & A_0 & \to & A. \end{array}$$

It follows that for $D = S_q, L_q$ $(q \geqq 0)$, and for any covariant functor T, $DT(\rho)$ is the mapping $x \to mnx$ of $DT(A \mid A')$ into itself. But ρ is null, and so $mnx = 0$ for $x \in DT(A \mid A')$. The same argument applies to contravariant functors (provided that they take only Abelian values). We conclude,

2.7. Theorem. *Let A be of finite order n. Then for every pair $A \mid A'$, and for every additive functor T, all the derived functors and satellites of T with argument $A \mid A'$ have exponent dividing n^2.*

3. The reduction homomorphism

Throughout this section $\mathfrak{V}$ is a fixed variety in $\mathfrak{C}$ with associated variety functor V and quotient functor U. π is the epimorphism $I \to U$.

As already mentioned in I, § 3, we associate with U a functor U^2 of pairs in C, given by the values

$$U^2(A \mid A') = U(A) \mid [A' + V(A)]/V(A). \tag{3.1}$$

U^2 is additive and covariant. With the usual identification $A \mid 0 = A$ we have $U^2(A) = U(A)$. The homomorphisms $A \mid A' \to U^2(A \mid A')$ give rise to a functor homomorphism $\pi^2 \colon I^2 \to U^2$, I^2 being the identity functor of pairs in $\mathfrak{C}$.

The right composition by U^2 (cf. I, 2.2) is a functor of functors. With each functor W of pairs in $\mathfrak{C}$ there is associated the composite functor WU^2, and with each homomorphism $\Psi \colon W_1 \to W_2$ of such functors there is associated the homomorphism

$$\Psi U^2 \colon W_1 U^2 \to W_2 U^2,$$

given by

$$(\Psi U^2)_{A \mid A'} = \Psi_{U^2(A \mid A')}.$$

If, in particular,

$$W(A \mid A') = W(A),$$

then

$$WU^2 = WU.$$

As $U^2(A \mid A')$ is always a pair in $\mathfrak{V}$, the functor WU^2 coincides essentially with the functor $(\mathrm{P}W)\,U^2$, P being the restriction to $\mathfrak{V}$. If now T is an additive covariant functor of $\mathfrak{C}$ then,† taking $W = L_n T$, we may identify $(L_n T)\,U^2$ with $(\mathrm{P}L_n T)\,U^2$. The restriction homomorphism (cf. I, § 8) $\mathrm{P}L_n T \to L_n^{\mathfrak{V}}\mathrm{P}T$ gives then rise to absolute functor homomorphisms $(\mathrm{P}L_n T)\,U^2 \to (L_n^{\mathfrak{V}}\mathrm{P}T)\,U^2$. Without danger of confusion we may omit the restriction symbol P in both expressions, as was already done in I. Applying the same argument in the other cases, we obtain absolute functor homomorphisms

$$\left.\begin{array}{ll}
\left.\begin{array}{l}(L_n T)\,U^2 \to (L_n^{\mathfrak{V}} T)\,U^2,\\ (S_n T)\,U^2 \to (S_n^{\mathfrak{V}} T)\,U^2,\end{array}\right\} & T \text{ covariant,}\\[2ex]
\left.\begin{array}{l}(R_{\mathfrak{V}}^n T)\,U^2 \to (R^n T)\,U^2,\\ (S_{\mathfrak{V}}^n T)\,U^2 \to (S^n T)\,U^2,\end{array}\right\} & T \text{ contravariant.}
\end{array}\right\} \tag{3.2}$$

On the other hand, the homomorphism $\pi^2 \colon I^2 \to U^2$ gives rise to absolute

† As in I, we omit the category index in the notation for derived functors and satellites when the underlying category is $\mathfrak{C}$. Thus $L_n T$ is the $\mathfrak{C}$-derived functor, while $L_n^{\mathfrak{V}}\,\mathrm{P}T$ is the $\mathfrak{V}$-derived functor of the restriction to $\mathfrak{V}$.

functor homomorphisms

$$\left.\begin{array}{ll}\left.\begin{array}{ll}L_n T & \to (L_n T)U^2, \\ S_n T & \to (S_n T)U^2,\end{array}\right\} & T \text{ covariant,} \\[2ex] \left.\begin{array}{ll}(R^n T)U^2 \to & R^n T, \\ (S^n T)U^2 \to & S^n T,\end{array}\right\} & T \text{ contravariant.}\end{array}\right\} \tag{3.3}$$

Composing the homomorphisms (3.2) and (3.3), we get absolute functor homomorphisms

$$\left.\begin{array}{ll}\left.\begin{array}{ll}L_n T & \to (L_n^{\mathfrak{B}} T)U^2, \\ S_n T & \to (S_n^{\mathfrak{B}} T)U^2,\end{array}\right\} & T \text{ covariant,} \\[2ex] \left.\begin{array}{ll}(R^n_{\mathfrak{B}} T)U^2 \to & R^n T, \\ (S^n_{\mathfrak{B}} T)U^2 \to & S^n T,\end{array}\right\} & T \text{ contravariant,}\end{array}\right\} \tag{3.4}$$

called the *reduction homomorphisms*,† and denoted by

$$r = r(\mathfrak{C}/\mathfrak{B}) = r(DT, \mathfrak{C}/\mathfrak{B})$$

for $D = L_n, S_n$, or $D = R^n, S^n$. We have then

3.1. *If $\mathfrak{B}_1$ is a subvariety of $\mathfrak{B}$, then*

$$r(\mathfrak{C}/\mathfrak{B}_1) = r(\mathfrak{B}/\mathfrak{B}_1) \circ r(\mathfrak{C}/\mathfrak{B}).$$

Explicitly, r is calculated as follows, taking as example derived functors. For any pair $A|A'$ in $\mathfrak{C}$ let $[\mathbf{A} \to A]$ be a $\mathfrak{C}$-resolution of $A|A'$, and $[\mathbf{A}^* \to U(A)]$ be a $\mathfrak{B}$-resolution of $U^2(A|A')$. Choose any homomorphism $F: \mathbf{A} \to \mathbf{A}^*$ over $\pi_A: A \to U(A)$. Then $H_n T(F)$ is the required homomorphism.

Let

$$0 \to A_1 \to A_0 \to A \to 0 \tag{3.5}$$

be a $\mathfrak{C}$-representation of $A|A'$ of length 1. We may consider A_1 as a subgroup‡ of A_0, and obtain a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1 \cap V(A_0) & \to & V(A_0) & \to & V(A) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1 & \to & A_0 & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1/A_1 \cap V(A_0) & \to & U(A_0) & \to & U(A) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.6}$$

† Note that the reduction homomorphism coincides with the restriction homomorphism (cf. I, §8) whenever the latter is defined, i.e. whenever $A \in \mathfrak{B}$. Our theory thus applies in particular to the restriction homomorphism.

‡ A similar convention will apply in various places in the sequel.

with central rows and exact columns. Denote by A_0' the inverse image of A' in A_0. By I, 3.9, $U^2(A_0|A_0')$ is $\mathfrak{B}$-projective, and so the bottom row of (3.6) is a $\mathfrak{B}$-representation of $U^2(A|A')$. We shall write

$$A_1^* = A_1 \cap V(A_0).$$

The left-hand column of (3.6) is a central sequence in $\mathfrak{A}$. The restriction of any additive functor T of $\mathfrak{C}$ to $\mathfrak{A}$ is weakly normal on $\mathfrak{A}$. If T is, say, covariant, we thus obtain from the sequence $\{L_n^{\mathfrak{A}}, \mu_n\}$ in $\mathfrak{A}$ (cf. I, § 9) an exact sequence

$$\left.\begin{array}{l} \dots \to L_{n+1}^{\mathfrak{A}} T(A_1/A_1^*) \to L_n^{\mathfrak{A}} T(A_1^*) \to L_n^{\mathfrak{A}} T(A_1) \\ \to L_n^{\mathfrak{A}} T(A_1/A_1^*) \to \dots \qquad \to L_0^{\mathfrak{A}} T(A_1/A_1^*) \to 0. \end{array}\right\} \tag{3.7}$$

Assume now that $\mathfrak{B} \supseteq \mathfrak{A}$. By I (8.2) we have

$$\left.\begin{array}{r} L_n^{\mathfrak{A}} T(A_1) = L_{n+1} T(A|A'), \\ L_n^{\mathfrak{A}} T(A_1/A_1^*) = L_{n+1}^{\mathfrak{B}} T(U^2(A|A')), \end{array}\right\} \tag{3.8}$$

and we verify that the mappings

$$L_{n+1} T(A|A') \to L_{n+1}^{\mathfrak{B}} T(U^2(A|A')),$$

given by (3.7) with identifications (3.8), are those induced by the reduction homomorphism. As $\mathfrak{B} \supseteq \mathfrak{A}$, we also have $V(A_1) = 0$, and so

$$A_1^* = \operatorname{Ker}[U(A_1) \to U(A_0)] = S_1 U(A|A').$$

By (2.2)

$$A_1^* = L_1 U(A|A'). \tag{3.9}$$

From (3.7) we thus derive mappings

$$\left.\begin{array}{l} L_{n+2}^{\mathfrak{B}} T(U^2(A|A')) \to L_n^{\mathfrak{A}} T(L_1 U(A|A')), \\ L_n^{\mathfrak{A}} T(L_1 U(A|A')) \to L_{n+1} T(A|A'). \end{array}\right\} \tag{3.10}$$

Homomorphisms of $\mathfrak{C}$-representations (3.5) induce homomorphisms of diagrams (3.6) and so of sequences (3.7). It follows that the mappings (3.10) are independent of the choice of the $\mathfrak{C}$-representation (3.5) and moreover that they give rise to functor homomorphisms. Homomorphisms of functors again induce homomorphisms of sequences (3.7). We thus have, including now also the contravariant case:

3.2. Theorem. *Let* $\mathfrak{B} \supseteq \mathfrak{A}$. *The homomorphisms of* (3.7) (*or of its dual*) *give rise to absolute functor homomorphisms*†

$$(L_{n+1}^{\mathfrak{A}} T)(L_1 U) \to L_{n+2} T \qquad (L_{n+2}^{\mathfrak{B}} T) U^2 \to (L_n^{\mathfrak{A}} T)(L_1 U)$$

or

$$R^{n+2} T \to (R_{\mathfrak{A}}^{n+1} T)(L_1 U) \qquad (R_{\mathfrak{A}}^n T)(L_1 U) \to (R_{\mathfrak{B}}^{n+2} T) U^2$$

for $n \geqq 0$.

† $L_1 U$ has Abelian values by I, 7.3, and $(L_n^{\mathfrak{A}} T)(L_1 U)$ is the composite functor with values $L_n^{\mathfrak{A}} T(L_1 U(A|A'))$, and similarly in the other case.