FOURIER EXPANSIONS
A Collection of Formulas

FRITZ OBERHETTINGER

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A COLLECTION OF FORMULAS

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PREFACE

The purpose of this monograph is to give a collection of Fourier series. Its limited scope made a number of compromises necessary. The question regarding the choice and organization of the material to be included posed certain problems. In order to preserve some consistency it seemed best to stay within the framework of what one could call the "classical" Fourier series, i.e., those of the trigonometric and their simplest generalization the Fourier–Bessel series. Thus results relating to Fourier series of generalized functions* or such series arising from Sturm–Liouville eigenvalue or integral equation problems are not included here. It was felt that such topics should be the subject of a separate treatment. An important question was which should be placed first, the Fourier series or the sum it represents. After some deliberation it was decided to opt for the first alternative. The material presented here is subdivided into five sections:

I. Series with elementary coefficients representing elementary functions
II. Series with elementary coefficients representing higher functions
III. Series with higher function coefficients representing elementary functions
IV. Series with higher function coefficients representing higher functions
V. Exponential Fourier and Fourier–Bessel series

* A few examples are given in an appendix to I.

This arrangement should be helpful in equally balancing the task of either establishing the sum function of a given Fourier series or finding the Fourier expansion of a given function. It seems apparent that a sizable amount of attention centers around results involving higher functions.

The author is not aware of the possible existence of a presentation of this subject on a similar scale and it is hoped that the contribution here will meet the requirements so often needed in applied mathematics, physics, and engineering. Since there is no lack of excellent texts concerning the subject of Fourier series no references have been given. Most of the material displayed here is widely scattered over the literature and a sizable number of results seem not to have been available before.

LIST OF NOTATION

Special Symbols

$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$
$n = 1, 2, 3, \cdots$
$(a)_0 = 1$
$\epsilon_n =$ Neumann's number, $\epsilon_0 = 1$, $\epsilon_n = 2$,
$n = 1, 2, 3, \cdots$

$\gamma =$ Euler's constant
$\tau_{\nu,n} = n$th positive root of $J_\nu(x) = 0$
$k =$ modulus of the elliptic integrals,
$k' = (1-k^2)^{1/2}$

List of Functions*

$\mathrm{am}(z, k)$ Jacobian amplitude function of argument z and modulus k (see 2.1)

B_0 Bernoulli numbers (see 1.37)
$B_n(x)$ Bernoulli polynomials (see 1.37)

$$C(x) = (2\pi)^{-1/2} \int_0^x t^{-1/2} \cos t\, dt \qquad \text{Fresnel's integral}$$

$$\mathrm{Ci}(x) = -\int_x^\infty t^{-1} \cos t\, dt \qquad \text{cosine integral}$$

* The definitions are the same as in A. Erdélyi *et al.*, "Higher Transcendental Functions," 3 vols., McGraw-Hill, New York, 1953.

$\mathrm{cn}(z, k)$ Jacobian elliptic function cosine amplitude of modulus k and argument z

$C_n^\nu(z)$ Gegenbauer polynomials

$dn(z, k)$ Jacobian elliptic function delta amplitude of argument z and modulus k

$D_\nu(z)$ parabolic cylinder function

$E(k)$ complete elliptic integral of the second kind of modulus k

E_n Euler numbers (see 1.37)

$E_n(x)$ Euler polynomials

$$\mathrm{Erf}(z) = 2\pi^{-1/2} \int_0^z \exp(-t^2)\, dt = 1 - \mathrm{Erfc}(z)$$

error integrals

$$\mathrm{Erfc}(z) = 2\pi^{-1/2} \int_z^\infty \exp(-t^2)\, dt = 1 - \mathrm{Erf}(z)$$

$E_\nu(z)$ Weber's function of order ν

${}_2F_1(a, b; c; z)$ hypergeometric function

$\Gamma(z)$ Gamma function

$H_\nu^{(1),(2)}(z)$ Hankel functions of order ν

$\mathbf{H}_\nu(z)$ Struve's function of order ν

$I_\nu(z)$ modified Bessel function of order ν

$J_\nu(z)$ Bessel function of order ν

$\mathrm{J}_\nu(z)$ Anger's function of order ν

$K(k)$ complete elliptic integral of the first kind of modulus k

$K_\nu(z)$ modified Hankel function of order ν

$\psi(z)$ Euler's psi function

$P_n(x)$ Legendre's polynomials

$P_\nu^\mu(x)$ associated Legendre functions of the first kind of argument x with $-1<x<1$ and $x>1$, respectively

$Q_\nu^\mu(x)$ associated Legendre function of the second kind of argument x with $-1<x<1$ and $x>1$, respectively

$$S(x) = (2\pi)^{-1/2} \int_0^x t^{-1/2} \sin t\, dt \qquad \text{Fresnel's integral}$$

$$\mathrm{Si}(x) = \int_0^x t^{-1/2} \sin t\, dt \qquad \text{sine integral}$$

$\mathrm{sn}(z, k)$ Jacobian elliptic function sine amplitude of argument z and modulus k

$s_{\mu,\nu}(z)$ Lommel's function

$T_n(x) = \cos(n \arccos x)$ Chebyshev's polynomials of the first kind

$U_n(x) = (1-x^2)^{-1/2} \sin[(n+1) \arccos x]$ Chebyshev's polynomials of the second kind

$\vartheta_l(x)$ elliptic theta functions $l=0, 1, 2, 3$ (see 2.26–2.29)

$Y_\mu(z)$ Neumann function of order ν

$\mathrm{zn}(z, k)$ Jacobi zeta function of argument z and modulus k (see 2.25)

$\zeta(z, a) = \sum_1^\infty (n+a)^{-z}$ Hurwitz zeta function (see 2.51, 2.52)

INTRODUCTION

1. The Fourier Series

Let $f(x)$ be a function defined and bounded in the range $a \leq x \leq b$ satisfying the conditions

(a) $f(x)$ has only a finite number of maxima and minima in (a, b), and

(b) $f(x)$ has only a finite number of (finite) discontinuities in (a, b) and outside this range $f(x)$ is defined by the relation $f[x \pm (b-a)] = f(x)$, i.e., $f(x)$ is a periodic function of period $b-a$. If two sets of coefficients a_n and b_n (the Fourier coefficients) are defined by

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{2\pi n x}{b-a}\right) dx, \qquad n = 0, 1, 2, \ldots \tag{1}$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin\left(\frac{2\pi n x}{b-a}\right) dx, \qquad b_n = 1, 2, 3, \ldots \tag{2}$$

then the series

$$\tfrac{1}{2}a_0 + \sum_1^\infty \left[a_n \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \sin\left(\frac{2n\pi x}{b-a}\right) \right] \tag{3}$$

is called the Fourier series of $f(x)$. It converges at a point x_0 $(a \leq x_0 \leq b)$ to the sum

$$\tfrac{1}{2}[f(x_0+0)+f(x_0-0)]; \qquad f(x_0 \pm 0) = \lim_{h\to 0} f(x_0 \pm h)$$

If $b=l$ and $a=-l$, then

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty \{a_n \cos[n\pi(x/l)] + b_n \sin[n\pi(x/l)]\} \qquad \text{(period } 2l) \tag{4}$$

with

$$a_n = (1/l) \int_{-l}^{l} f(x) \cos[n\pi(x/l)]\, dx \tag{5}$$

$$b_n = (1/l) \int_{-l}^{l} f(x) \sin[n\pi(x/l)]\, dx \tag{6}$$

If $f(x)$ is an *even* function, then all $b_n = 0$ and

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos[n\pi(x/l)]; \qquad a_n = (2/l) \int_0^l f(x) \cos[n\pi(x/l)]\, dx \tag{7}$$

If $f(x)$ is *odd*, then all $a_n = 0$ and

$$f(x) = \sum_1^\infty b_n \sin[n\pi(x/l)]; \qquad b_n = (2/l) \int_0^l f(x) \sin[n\pi(x/l)]\, dx \tag{8}$$

The period in (7) and (8) is $2l$.

Also in exponential form

$$f(x) = \sum_{-\infty}^{\infty} c_n \exp[i2\pi nx/(b-a)]; \qquad \text{period } b-a \tag{9}$$

$$c_n = (b-a)^{-1} \int_a^b f(x) \exp[-i2\pi nx/(b-a)]\, dx \tag{10}$$

Again, if $a=-l$ and $b=l$ (period $2l$), then

$$f(x) = \sum_{-\infty}^{\infty} c_n \exp[in\pi(x/l)] \tag{11}$$

$$c_n = (2l)^{-1} \int_{-l}^{l} f(x) \exp[-in\pi(x/l)]\, dx \tag{12}$$

The case of an even or odd function $f(x)$ leads to formulas (7) and (8), respectively.

2. The Fourier Integral

The limiting case $l \to \infty$ in (11) and (12) leads to the representation of a function $f(x)$ by a Fourier integral

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(t) \exp[iy(x-t)]\, dt \tag{13}$$

This is equivalent to the pair of inversion formulas (exponential Fourier transform)

$$\begin{aligned} g_e(y) &= \int_{-\infty}^{\infty} f(x) e^{ixy}\, dx \\ f(x) &= (1/2\pi) \int_{-\infty}^{\infty} g_e(y) e^{-ixy}\, dy \end{aligned} \tag{14}$$

Again, if $f(x)$ is an even or an odd function, respectively, it follows from (14) that

$$g_c(y) = \int_0^{\infty} f(x) \cos(xy)\, dx; \qquad f(x) = (2/\pi) \int_0^{\infty} g_c(y) \cos(xy)\, dy \tag{15}$$

$$g_s(y) = \int_0^{\infty} f(x) \sin(xy)\, dx; \qquad f(x) = (2/\pi) \int_0^{\infty} g_s(y) \sin(xy)\, dy \tag{16}$$

Here $g_c(y)$ and $g_s(y)$ denote the Fourier cosine and sine transform, respectively.

3. Fourier–Bessel Series

The Fourier–Bessel series represent a generalization of the trigonometric Fourier series. They are series involving Bessel functions as terms. Let $J_\nu(x)$ be the Bessel function of the order ν and the argument x and let $\nu > -1$. Then the zeros of $J_\nu(x)$ are real and of equal absolute value. If $\tau_{\nu,n}$ denotes the nth positive root of $J_\nu(x) = 0$, then a function $f(x)$ defined in $(0, a)$ and vanishing at $x = a$ admits under certain conditions the expansion

$$f(x) = \sum_1^{\infty} a_n J_\nu[\tau_{\nu,n}(x/a)], \tag{17}$$

with

$$a_n = 2[aJ_{\nu+1}(\tau_{\nu,n})]^{-2} \int_0^a xf(x) J_\nu[\tau_{\nu,n}(x/a)]\, dx \tag{18}$$

For the special values $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$ one has

$$J_{-\frac{1}{2}}(x)=(\tfrac{1}{2}\pi x)^{-\frac{1}{2}}\cos x \qquad \text{with the zeros } \tau_{\nu,n}=(n-\tfrac{1}{2})\pi, \qquad n=1,2,3,\ldots \tag{19}$$

$$J_{\frac{1}{2}}(x)=(\tfrac{1}{2}\pi x)^{-\frac{1}{2}}\sin x \qquad \text{with the zeros } \tau_{\nu,n}=n\pi, \tag{20}$$

The substitution $f(x)=(\frac{1}{2}\pi x)^{-\frac{1}{2}}F(x)$; $a_n=(\tau_{\nu,n}/a)^{\frac{1}{2}}A_n$ in (17) and (18) leads to

$$F(x)=\sum_1^\infty A_n\cos[(n-\tfrac{1}{2})\pi(x/a)] \qquad \text{with} \quad A_n=(2/a)\int_0^a F(x)\cos[(n-\tfrac{1}{2})\pi(x/a)]\,dx \tag{21}$$

$$F(x)=\sum_1^\infty A_n\sin[n\pi(x/a)] \qquad \text{with} \quad A_n=(2/a)\int_0^a F(x)\sin[n\pi(x/a)]\,dx \tag{22}$$

The periods of $F(x)$ as represented in (21) and (22) are $4a$ and $2a$, respectively. Obviously the series (21) and (22) are of the form (7) and (8).

4. The Bessel Transform

The expression of an arbitrary function $F(x)$ by means of a double integral similar to (13) but involving Bessel functions is given by Hankel's formula

$$F(x)=\int_0^\infty J_\nu(tx)t\,dt\int_0^\infty F(u)J_\nu(ut)u\,du \tag{23}$$

Equivalent to (23) is the pair of inversion formulas

$$h_\nu(y)=\int_0^\infty (xy)^{\frac{1}{2}}f(x)J_\nu(xy)\,dx; \qquad f(x)=\int_0^\infty (xy)^{\frac{1}{2}}h_\nu(y)J_\nu(xy)\,dy \tag{24}$$

which for the special cases $\nu=\pm\frac{1}{2}$ lead to formulas (15) and (16). The property $h_\nu(y)$ is called the Hankel transform of the order ν of a function $f(x)$. Also used is

$$H_\nu(y)=\int_0^\infty F(x)J_\nu(xy)\,dx; \qquad F(x)=x\int_0^\infty yH_\nu(y)J_\nu(xy)\,dy \tag{25}$$

5. Generation of Fourier Series by Means of Integral Transforms

The integrals for $g_c(y)$ and $g_s(y)$ in (15) and (16) taken over a finite interval (a, b) of integration are called the *finite cosine and sine Fourier transforms* of $f(x)$

$$g_c(y)=\int_0^a f(x)\cos(xy)\,dx; \qquad g_s(y)=\int_0^a f(x)\sin(xy)\,dy$$

If $f(x)$ is regarded as an even function of x, then by (7)

$$|f(x)| = (1/a) \sum_{0}^{\infty} \epsilon_n g_c[n(\pi/a)] \cos[n\pi(x/a)], \qquad -a<x<a \tag{26}$$

Similarly when $f(x)$ is regarded as an odd function

$$\operatorname{sgn}x\, f(x) = (2/a) \sum_{1}^{\infty} g_s[n(\pi/a)] \sin[n\pi(x/a)], \qquad -a<x<a \tag{27}$$

These formulas represent the Fourier expansion in $-a<x<a$ of a function $f(x)$ when its finite Fourier transforms are known. Similar considerations can be applied to *Poisson's summation formula*

$$\sum_{n=-\infty}^{\infty} e^{inx} G(y+nd) = (1/d) \sum_{m=-\infty}^{\infty} g_e\left(\frac{2\pi m+x}{d}\right) \exp[-i(2\pi m+x)y/d] \tag{28}$$

with

$$g_e(t) = \int_{-\infty}^{\infty} G(u) e^{iut}\, du \tag{29}$$

Here $G(u)$ is a given function and $g_e(t)$ represents its exponential Fourier transform (14); x, y, and d are real parameters.

If $y=0$, then

$$\sum_{n=-\infty}^{\infty} e^{inx} G(nd) = (1/d) \sum_{m=-\infty}^{\infty} g_e[(2\pi m+x)/d] \tag{30}$$

If $x=0$, then

$$\sum_{n=-\infty}^{\infty} G(y+nd) = (1/d) \sum_{m=-\infty}^{\infty} g_e[2\pi(m/d)] \exp[-i2\pi m(y/d)] \tag{31}$$

The left side of (31) represents a periodic function in y of period d while the right side represents its Fourier expansion in exponential form (9). Suppose now that the function $G(u)$ occurring in the left side summation of (28) is such that its exponential Fourier transform $g_e(t)$ in (29) vanishes for $a<t<b$ [i.e., $g_e(t)$ is different from zero in a finite interval only]. The sum at the right side of (28) reduces to a finite sum and

$$\sum_{n=-\infty}^{\infty} e^{inx} G(y+nd) = (1/d) \sum_{m=m_1}^{m_2} g_e[(2\pi m+x)/d] \exp[-i(y/d)(2\pi m+x)] \tag{32}$$

The limits m_1 and m_2 are then defined by

$$m_1 \geq (ad-x)/2\pi; \qquad m_2 = (bd-x)/2\pi \tag{33}$$

If b, a, d, and x are such that the equal sign holds then one half of the corresponding value in the right side summation has to be taken.

The expressions (17) and (18) for the Fourier–Bessel expansion can also be written as

$$x^{-\frac{1}{2}}f(x)=2\sum_{1}^{\infty}\tau_{\nu,n}^{-\frac{1}{2}}[h_\nu(\tau_{\nu,n})/J_{\nu+1}^2(\tau_{\nu,n})]J_\nu(\tau_{\nu,n}x), \qquad 0<x<1 \tag{34}$$

with

$$h_\nu(y)=\int_0^1 f(t)\,(ty)^{\frac{1}{2}}J_\nu(ty)\,dt \tag{35}$$

This is the finite Hankel transform of $f(x)$.

Finally the Jacobi–Anger relations from the theory of the Bessel functions are used:

$$\cos(x\cos z)=\sum_{0}^{\infty}\epsilon_n(-1)^nJ_{2n}(x)\cos(2nz) \tag{36}$$

$$\sin(x\cos z)=2\sum_{0}^{\infty}(-1)^nJ_{2n+1}(x)\cos[(2n+1)z] \tag{37}$$

Upon multiplying these relations by a suitable function of x and integrating over x from zero to infinity the results are

$$\sum_{0}^{\infty}(-1)^n\epsilon_nH_{2n}(1)\cos(2nz)=g_c(\cos z) \tag{38}$$

$$2\sum_{0}^{\infty}(-1)^nH_{2n+1}(1)\cos[(2n+1)z]=g_s(\cos z) \tag{39}$$

where g_c, g_s, and H_ν are defined in (15), (16), and (25).

I

FOURIER SERIES WITH ELEMENTARY COEFFICIENTS REPRESENTING ELEMENTARY FUNCTIONS

1.1 $$\sin^{2l}x = 2^{-2l}(2l)!\sum_{0}^{l}(-1)^n\epsilon_n\cos(2nx)/[(l+n)!(l-n)!]$$

1.2 $$\sin^{2l+1}x = 2^{-2l}(2l+1)!\sum_{0}^{l}(-1)^n\sin[(2n+1)x]/[(l+1+n)!(l-n)!]$$

1.3 $$\cos^{2l}x = 2^{-2l}(2l)!\sum_{0}^{l}[\epsilon_n\cos(2nx)]/[(l+n)!(l-n)!]$$

1.4 $$\cos^{2l+1}x = 2^{-2l}(2l+1)!\sum_{0}^{l}\cos[(2n+1)x]/[(l+1+n)!(l-n)!]$$

1.5 $$\sum_{1}^{N}\cos(nx) = \frac{\sin(\frac{1}{2}Nx)}{\sin(\frac{1}{2}x)}\cos[(N+1)\tfrac{1}{2}x] = \frac{1}{2}\left\{\frac{\sin[\frac{1}{2}(2N+1)x]}{\sin(\frac{1}{2}x)}-1\right\}$$

1.6 $$\sum_{1}^{N}\sin(nx) = [\sin(\tfrac{1}{2}Nx)/\sin(\tfrac{1}{2}x)]\sin[\tfrac{1}{2}(N+1)x]$$
$$= \tfrac{1}{2}\{\cos(\tfrac{1}{2}x)-\cos[\tfrac{1}{2}(2N+1)x]\}/\sin(\tfrac{1}{2}x)$$

1.7 $\sum_{0}^{N} \cos[(2n+1)x] = \{\sin[(N+1)x]/\sin x\} \cos[(N+1)x]$

1.8 $\sum_{0}^{N} \sin[(2n+1)x] = \{\sin[(N+1)x]/\sin x\} \sin[(N+1)x]$

1.9 $[\sin(Nx)/\sin x]^2 = \sum_{0}^{N-1} \{\sin[(2n+1)x]/\sin x\}$

$$= N + 2(N-1)\cos(2x) + 2(N-2)\cos(4x) + \cdots + 2\cos[2(N-1)x]$$

1.10 $\sum_{0}^{N-1} \sin(x+ny) = \sin[x+\frac{1}{2}(N-1)y][\sin(\frac{1}{2}Ny)/\sin(\frac{1}{2}y)]$

1.11 $\sum_{0}^{N-1} \cos(x+ny) = \cos[x+\frac{1}{2}(N-1)y][\sin(\frac{1}{2}Ny)/\sin(\frac{1}{2}y)]$

1.12 $\sum_{1}^{\infty} n^{-1}\sin(nx) = \frac{1}{2}\pi - \frac{1}{2}x, \quad 0<x<2\pi$

1.13 $\sum_{1}^{\infty} (-1)^n n^{-1}\sin(nx) = -\frac{1}{2}x, \quad -\pi<x<\pi$

1.14 $\sum_{1}^{\infty} n^{-1}\cos(nx) = -\log|2\sin(\frac{1}{2}x)|$

1.15 $\sum_{1}^{\infty} (-1)^n n^{-1}\cos(nx) = -\log|2\cos(\frac{1}{2}x)|$

1.16 $\sum_{0}^{\infty} (2n+1)^{-1}\sin[(2n+1)x] = \frac{1}{4}\pi \operatorname{sgn} x, \quad -\pi<x<\pi$

1.17 $\sum_{0}^{\infty} (-1)^n(2n+1)^{-1}\sin[(2n+1)x] = \frac{1}{2}\log|[\cos(\frac{1}{2}x)+\sin(\frac{1}{2}x)]/[\cos(\frac{1}{2}x)-\sin(\frac{1}{2}x)]|$

1.18 $\sum_{0}^{\infty} (2n+1)^{-1}\cos[(2n+1)x] = \frac{1}{2}\log|\cot(\frac{1}{2}x)|$

1.19 $\sum_{0}^{\infty} (-1)^n (2n+1)^{-1} \cos[(2n+1)x] = \begin{cases} \frac{1}{4}\pi, & 0<x<\frac{1}{2}\pi \\ -\frac{1}{4}\pi, & \frac{1}{2}\pi<x<\frac{3}{2}\pi \end{cases}$

1.20 $\sum_{1}^{\infty} (n+p)^{-1} \sin(nx) = (\frac{1}{2}\pi - \frac{1}{2}x) \cos(px) - \cos(px) \sum_{1}^{p} n^{-1} \sin(nx)$
$+\sin(px) \{\log[2 \sin(\frac{1}{2}x)] + \sum_{1}^{p} n^{-1} \cos(nx)\}, \quad p=1, 2, 3, \ldots; \quad 0<x<2\pi$

1.21 $\sum_{1}^{\infty} (n+p)^{-1} \cos(nx) = -\cos(px) \{\log[2 \sin(\frac{1}{2}x)] + \sum_{1}^{p} n^{-1} \cos(nx)\}$
$+\sin(px) [\frac{1}{2}\pi - \frac{1}{2}x - \sum_{1}^{p} n^{-1} \sin(nx)], \quad p=1, 2, 3, \ldots; \quad 0<x<2\pi$

1.22 $\sum_{1}^{\infty} (n+p)^{-1} (-1)^n \sin(nx) = (-1)^{p+1} \cos(px) [\sum_{1}^{p} (-1)^n n^{-1} \sin(nx) + \frac{1}{2}x]$
$+(-1)^p \sin(px) \{\log[2 \cos(\frac{1}{2}x)] + \sum_{1}^{p} (-1)^n n^{-1} \cos(nx)\}, \quad p=1,2,3,\ldots; \quad -\pi<x<\pi$

1.23 $\sum_{1}^{\infty} (n+p)^{-1} (-1)^n \cos(nx) = (-1)^{p+1} \cos(px) \{\log[2 \cos(\frac{1}{2}x)] + \sum_{1}^{p} (-1)^n n^{-1} \cos(nx)\}$
$-(-1)^p \sin(px) [\frac{1}{2}x + \sum_{1}^{p} (-1)^n n^{-1} \sin(nx)], \quad p=1, 2, 3, \ldots; \quad -\pi<x<\pi$

1.24 $\frac{4}{3}A\pi^2 + B\pi + C + 4A \sum_{1}^{\infty} n^{-2} \cos(nx) - (4\pi A + 2B) \sum_{1}^{\infty} n^{-1} \sin(nx)$
$= Ax^2 + Bx + C, \quad 0<x<2\pi, \quad A, B, C = \text{constants}$

1.25 $\sum_{1}^{\infty} n^{-2} \sin(nx) = -\int_0^x \log[2 \sin(\frac{1}{2}t)]\, dt, \quad 0 \le x \le 2\pi$

1.26 $\sum_{1}^{\infty} (-1)^n n^{-2} \sin(nx) = -\int_0^x \log[2 \cos(\frac{1}{2}t)]\, dt, \quad -\pi \le x \le \pi$

1.27 $\sum_{1}^{\infty} n^{-2} \cos(nx) = (1/12)(3x^2 - 6\pi x + 2\pi^2), \quad 0 \le x \le 2\pi$

1.28 $\sum_{1}^{\infty} (-1)^n n^{-2} \cos(nx) = (1/12)(3x^2 - \pi^2), \quad -\pi \le x \le \pi$

1.29 $\sum_{0}^{\infty} (2n+1)^{-2} \sin(nx) = -\frac{1}{2}\int_0^x \log|\tan(\frac{1}{2}t)|\,dt, \quad -\pi\le x\le\pi$

1.30 $\sum_{0}^{\infty} (2n+1)^{-2} \cos[(2n+1)x] = \frac{1}{8}(\pi^2 - 2\pi|x|), \quad -\pi\le x\le\pi$

1.31 $\sum_{0}^{\infty} (-1)^n(2n+1)^{-2} \sin[(2n+1)x] = \frac{1}{4}\pi x, \quad -\frac{1}{2}\pi\le x\le\frac{1}{2}\pi$

1.32 $\sum_{0}^{\infty} (-1)^n(2n+1)^{-2} \cos[(2n+1)x] = -\frac{1}{2}\int_0^{\frac{1}{2}\pi-x} \log[\tan(\frac{1}{2}t)]\,dt, \quad -\frac{1}{2}\pi\le x\le\frac{1}{2}\pi$

1.33 $2(-1)^{l+1}(2l)!\sum_{1}^{\infty} (2\pi n)^{-2l} \cos(2\pi nx) = B_{2l}(x), \quad 0\le x\le 1, \quad l=1, 2, 3, \ldots$

1.34 $2(-1)^{l+1}(2l+1)!\sum_{1}^{\infty} (2\pi n)^{-2l-1} \sin(2\pi nx) = B_{2l+1}(x)$

$0\le x\le 1, \quad l=1, 2, 3, \ldots; \quad 0<x<1, \quad l=0$

1.35 $4(-1)^{l}(2l)!\sum_{0}^{\infty} [(2n+1)\pi]^{-2l-1} \sin[(2n+1)\pi x] = E_{2l}(x)$

$0\le x\le 1, \quad l=1, 2, 3, \ldots; \quad 0<x<1, \quad l=0$

1.36 $4(-1)^{l+1}(2l+1)!\sum_{0}^{\infty} [(2n+1)\pi]^{-2l-2} \cos[(2n+1)\pi x] = E_{2l+1}(x)$

$0\le x\le 1, \quad l=0, 1, 2, \ldots$

1.37 The $B_k(x)$ and $E_k(x)$ are the Bernoulli and the Euler polynomials, respectively.

$$B_k(x) = \sum_{r=0}^{k} \binom{k}{r} B_r x^{k-r}$$

$$E_k(x) = \sum_{r=0}^{k} \binom{k}{r} 2^{-r}(x-\tfrac{1}{2})^{k-r} E_r$$

$B_0=1, \quad B_1=-\frac{1}{2}, \quad B_2=\frac{1}{6}, \quad B_4=-\frac{1}{30}, \ldots; \quad B_{2r+1}=0, \qquad r=1, 2, 3, \ldots$

$E_0=1, \quad E_2=-1, \quad E_4=5, \ldots; \quad E_{2r+1}=0, \qquad r=0, 1, 2, \ldots$

(Bernoulli's and Euler's numbers)

1.38 $\sum_{1}^{\infty} \{\cos[(n+1)x]/n(n+1)\} = (1-\cos x)\ \log[2\sin(\tfrac{1}{2}x)]$

$-(\tfrac{1}{2}\pi-\tfrac{1}{2}x)\ \sin x+\cos x, \quad 0\le x\le 2\pi$

1.39 $\sum_{1}^{\infty} \{\sin[(n+1)x]/n(n+1)\} = (\tfrac{1}{2}\pi-\tfrac{1}{2}x)(\cos x-1)$

$-\sin x\ \log[2\sin(\tfrac{1}{2}x)]+\sin x, \quad 0\le x\le 2\pi$

1.40 $\sum_{1}^{\infty} \{(-1)^n \cos[(n+1)x]/n(n+1)\} = -(1+\cos x)\ \log[2\cos(\tfrac{1}{2}x)]$

$+\tfrac{1}{2}x\ \sin x+\cos x, \quad -\pi\le x\le \pi$

1.41 $\sum_{1}^{\infty} \{(-1)^n \sin[(n+1)x]/n(n+1)\} = -\tfrac{1}{2}x(1+\cos x)$

$-\sin x\ \log[2\cos(\tfrac{1}{2}x)]+\sin x, \quad -\pi\le x\le \pi$

1.42 $\sum_{1}^{\infty} [(-1)^n \sin(nx)/n(n+1)] = -\sin x\ \log[2\sin(\tfrac{1}{2}x)]+(\tfrac{1}{2}\pi-\tfrac{1}{2}x)(1-\cos x), \quad 0\le x\le 2\pi$

1.43 $\sum_{1}^{\infty} [\cos(nx)/n(n+1)] = 1-(1-\cos x)\ \log[2\sin(\tfrac{1}{2}x)]-(\tfrac{1}{2}\pi-\tfrac{1}{2}x)\ \sin x, \quad 0\le x\le 2\pi$

1.44 $\sum_{1}^{\infty} [(-1)^n \sin(nx)/n(n+1)] = -\tfrac{1}{2}x(1+\cos x)+\sin x\ \log[2\cos(\tfrac{1}{2}x)], \quad -\pi\le x\le \pi$

1.45 $\sum_{1}^{\infty} [(-1)^n \cos(nx)/n(n+1)] = 1-(1+\cos x)\ \log[2\cos(\tfrac{1}{2}x)]-\tfrac{1}{2}x\ \sin x, \quad -\pi\le x\le \pi$

1.46 $(p-q)\sum_{1}^{\infty} [\cos(nx)/(n+p)(n+q)] = \cos(px)\sum_{1}^{p} n^{-1}\cos(nx)+\sin(px)\sum_{1}^{p} n^{-1}\sin(nx)$

$-\cos(qx)\sum_{1}^{q} n^{-1}\cos(nx)-\sin(qx)\sum_{1}^{q} n^{-1}\sin(nx)-2\sin[\tfrac{1}{2}x(p+q)]$

$\cdot\sin[\tfrac{1}{2}x(p-q)]\ \log[2\sin(\tfrac{1}{2}x)]-(\pi-x)\ \cos[\tfrac{1}{2}x(p+q)]\ \sin[\tfrac{1}{2}x(p-q)]$

$0\le x\le 2\pi, \quad p, q=0, 1, 2, \ldots; \quad \text{if} \quad p, q=0, \quad \sum_{1}^{0}(\ )=0$

1.47 $$(p-q)\sum_{1}^{\infty}[\sin(nx)/(n+p)(n+q)]=\cos(px)\sum_{1}^{p}n^{-1}\sin(nx)-\sin(px)\sum_{1}^{p}n^{-1}\cos(nx)$$
$$+\sin(qx)\sum_{1}^{q}n^{-1}\cos(nx)-\cos(qx)\sum_{1}^{q}n^{-1}\sin(nx)-2\cos[\tfrac{1}{2}x(p+q)]$$
$$\cdot\sin[\tfrac{1}{2}x(p-q)]\log(2\sin\tfrac{1}{2}x)+(\pi-x)\sin[\tfrac{1}{2}x(p+q)]\sin[\tfrac{1}{2}x(p-q)]$$
$$0\le x\le 2\pi,\quad p,q=0,1,2;\quad \text{if}\quad p,q=0,\quad \sum_{1}^{0}(\ )=0$$

1.48 $$(p-q)\sum_{1}^{\infty}[(-1)^n\cos(nx)/(n+p)(n+q)]=(-1)^p\sum_{1}^{p}(-1)^n n^{-1}\cos[x(p-n)]$$
$$-(-1)^q\sum_{1}^{q}(-1)^n n^{-1}\cos[x(q-n)]+\tfrac{1}{2}x[(-1)^p\sin(px)-(-1)^q\sin(qx)]$$
$$+\log(2\cos\tfrac{1}{2}x)[(-1)^p\cos(px)-(-1)^q\cos(qx)]$$
$$-\pi\le x\le\pi,\quad p,q=0,1,2,\ldots;\quad \text{if}\quad p,q=0,\quad \sum_{1}^{0}(\ )=0$$

1.49 $$(p-q)\sum_{1}^{\infty}[(-1)^n\sin(nx)/(n+p)(n+q)]=-(-1)^p\sum_{1}^{p}(-1)^n n^{-1}\sin[x(p-n)]$$
$$+(-1)^q\sum_{1}^{q}(-1)^n n^{-1}\sin[x(q-n)]+\tfrac{1}{2}x[(-1)^p\cos(px)-(-1)^q\cos(qx)]$$
$$-\log(2\cos\tfrac{1}{2}x)[(-1)^p\sin(px)-(-1)^q\sin(qx)]$$
$$-\pi\le x\le\pi,\quad p,q=0,1,2;\quad \text{if}\quad p,q=0,\quad \sum_{1}^{0}(\ )=0$$

1.50 $$(r-p)\sum_{1}^{\infty}\frac{(\pm 1)^n}{(n+p)(n+q)(n+r)}\begin{Bmatrix}\sin(nx)\\ \cos(nx)\end{Bmatrix},\quad p,q,r=0,1,2,\ldots$$

use $\dfrac{r-p}{(n+p)(n+q)(n+r)}=[(n+p)(n+q)]^{-1}-[(n+q)(n+r)]^{-1}$ and 1.46–1.49

1.51 $$\sum_{0}^{\infty}\frac{\epsilon_n}{b^2+n^2a^2}\cos\left(n\pi\frac{x}{l}\right)=\frac{\pi}{ab}\frac{\cosh[(\pi b/al)(l-x)]}{\sinh(\pi b/a)},\quad 0\le x\le 2l$$

1.52 $$\sum_{0}^{\infty}\frac{(-1)^n\epsilon_n}{b^2+n^2a^2}\cos\left(n\pi\frac{x}{l}\right)=\frac{\pi}{ab}\frac{\cosh[(\pi b/al)x]}{\sinh(\pi b/a)},\quad -l\le x\le l$$

1.53 $$\sum_{1}^{\infty} \frac{n}{b^2+n^2a^2} \sin\left(n\pi \frac{x}{l}\right) = \frac{\pi}{2a^2} \frac{\sinh[(\pi b/al)(l-x)]}{\sinh(\pi b/a)}, \quad 0<x<2l$$

1.54 $$\sum_{1}^{\infty} \frac{(-1)^n n}{b^2+n^2a^2} \sin\left(n\pi \frac{x}{l}\right) = -\frac{\pi}{2a^2} \frac{\sinh[(\pi b/al)x]}{\sinh(\pi b/a)}, \quad -l<x<l$$

1.55 $$\sum_{0}^{\infty} [b^2+(n+\tfrac{1}{2})^2a^2]^{-1} \cos\left[\left(n+\frac{1}{2}\right)\pi \frac{x}{l}\right] = \frac{\pi}{2ab} \frac{\sinh[(\pi b/al)(l-x)]}{\cosh(\pi b/a)}, \quad 0\leq x\leq 2l$$

1.56 $$\sum_{0}^{\infty} \frac{(n+\frac{1}{2})}{b^2+(n+\frac{1}{2})^2a^2} \sin\left[\left(n+\frac{1}{2}\right)\pi \frac{x}{l}\right] = \frac{\pi}{2a^2} \frac{\cosh[(\pi b/al)(l-x)]}{\cos(\pi b/a)}, \quad 0<x<2l$$

1.57 $$\sum_{0}^{\infty} \frac{(-1)^n}{b^2+(n+\frac{1}{2})^2a^2} \sin\left[\left(n+\frac{1}{2}\right)\pi \frac{x}{l}\right] = \frac{\pi}{2ab} \frac{\sinh[(\pi b/al)x]}{\cosh(\pi b/a)}, \quad -l\leq x\leq l$$

1.58 $$\sum_{0}^{\infty} \frac{(-1)^n(n+\frac{1}{2})}{b^2+(n+\frac{1}{2})^2a^2} \cos\left[\left(n+\frac{1}{2}\right)\pi \frac{x}{l}\right] = \frac{\pi}{2a^2} \frac{\cosh[(\pi b/al)x]}{\cosh(\pi b/a)}, \quad -l<x<l$$

1.59 $$\sum_{0}^{\infty} \frac{\epsilon_n}{n^2a^2-b^2} \cos\left(n\pi \frac{x}{l}\right) = -\frac{\pi}{ab} \frac{\cos[(\pi b/al)(l-x)]}{\sin(\pi b/a)}, \quad 0\leq x\leq 2l$$

1.60 $$\sum_{0}^{\infty} \frac{(-1)^n\epsilon_n}{n^2a^2-b^2} \cos\left(n\pi \frac{x}{l}\right) = -\frac{\pi}{ab} \frac{\cos[(\pi b/al)x]}{\sin(\pi b/a)}, \quad -l\leq x\leq l$$

1.61 $$\sum_{1}^{\infty} \frac{n}{n^2a^2-b^2} \sin\left(n\pi \frac{x}{l}\right) = \frac{\pi}{2a^2} \frac{\sin[(\pi b/al)(l-x)]}{\sin(\pi b/a)}, \quad 0<x<2l$$

1.62 $$\sum_{1}^{\infty} \frac{(-1)^n n}{n^2a^2-b^2} \sin\left(n\pi \frac{x}{l}\right) = -\frac{\pi}{2a^2} \frac{\sin[(\pi b/al)x]}{\sin(\pi b/a)}, \quad -l<x<l$$

1.63 $$\sum_{0}^{\infty} [(n+\tfrac{1}{2})^2a^2-b^2]^{-1} \cos\left[\left(n+\frac{1}{2}\right)\pi \frac{x}{l}\right] = \frac{\pi}{2ab} \frac{\sin[(\pi b/al)(l-x)]}{\cos(\pi b/a)}, \quad 0\leq x\leq 2l$$

1.64 $$\sum_{0}^{\infty} \frac{(n+\frac{1}{2})}{(n+\frac{1}{2})^2a^2-b^2} \sin\left[\left(n+\frac{1}{2}\right)\pi \frac{x}{l}\right] = \frac{\pi}{2a^2} \frac{\cos[(\pi b/al)(l-x)]}{\cos(\pi b/a)}, \quad 0<x<2l$$

1.65 $$\sum_{0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^2a^2-b^2} \sin\left[\left(n+\frac{1}{2}\right)\pi \frac{x}{l}\right] = \frac{\pi}{2ab} \frac{\sin[(\pi b/al)x]}{\cos(\pi b/a)}, \quad -l\leq x\leq l$$

1.66 $$\sum_0^\infty \frac{(-1)^n(n+\frac{1}{2})}{(n+\frac{1}{2})^2a^2-b^2}\cos\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]=\frac{\pi}{2a^2}\frac{\cos[(\pi b/al)x]}{\cos(\pi b/a)},\quad -l<x<l$$

1.67 $$\sum_0^\infty \frac{\epsilon_n(-1)^n}{b^2+n^2a^2}\left[\frac{b}{a}\cos\left(n\pi\frac{x}{l}\right)-n\sin\left(n\pi\frac{x}{l}\right)\right]=\frac{\pi}{a^2}\frac{\exp[(\pi b/al)x]}{\sinh[\pi(b/a)]},\quad -l<x<l$$

1.68 $$\sum_0^\infty \frac{\epsilon_n(-1)^n}{b^2+n^2a^2}\left[\frac{b}{a}\cos\left(n\pi\frac{x}{l}\right)+n\sin\left(n\pi\frac{x}{l}\right)\right]=\frac{\pi}{a^2}\frac{\exp[-(\pi b/al)x]}{\sinh[\pi(b/a)]},\quad -l<x<l$$

1.69 $$\sum_0^\infty \frac{(-1)^n}{b^2+(n+\frac{1}{2})^2a^2}\left\{\frac{b}{a}\sin\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]+\left(n+\frac{1}{2}\right)\cos\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]\right\}$$

$$=\frac{\pi}{2a^2}\frac{\exp[(\pi b/al)x]}{\cosh(\pi b/a)},\quad -l<x<l\quad (\text{period } 4l)$$

1.70 $$\sum_0^\infty \frac{(-1)^n}{b^2+(n+\frac{1}{2})^2a^2}\left\{\frac{b}{a}\sin\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]-\left(n+\frac{1}{2}\right)\cos\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]\right\}$$

$$=-\frac{\pi}{2a^2}\frac{\exp[-(\pi b/al)x]}{\cosh(\pi b/a)},\quad -l<x<l\quad (\text{period } 4l)$$

1.71 $$\sum_0^\infty \frac{\epsilon_n}{b^2+n^2a^2}\left[\frac{b}{a}\cos\left(n\pi\frac{x}{l}\right)+n\sin\left(n\pi\frac{x}{l}\right)\right]=\frac{\pi}{a^2}\frac{\exp[(\pi b/al)(l-x)]}{\sinh(\pi b/a)},\quad 0<x<2l$$

1.72 $$\sum_0^\infty \frac{\epsilon_n}{b^2+n^2a^2}\left[\frac{b}{a}\cos\left(n\pi\frac{x}{l}\right)-n\sin\left(n\pi\frac{x}{l}\right)\right]=\frac{\pi}{a^2}\frac{\exp[-(\pi b/al)(l-x)]}{\sinh(\pi b/a)},\quad 0<x<2l$$

1.73 $$\sum_0^\infty [b^2+(n+\tfrac{1}{2})^2a^2]^{-1}\left\{\frac{b}{a}\cos\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]+\left(n+\frac{1}{2}\right)\sin\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]\right\}$$

$$=\frac{\pi}{2a^2}\frac{\exp[(\pi b/al)(l-x)]}{\cosh(\pi b/a)},\quad 0<x<2l$$

1.74 $$\sum_0^\infty [b^2+(n+\tfrac{1}{2})^2a^2]^{-1}\left\{\frac{b}{a}\cos\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]-\left(n+\frac{1}{2}\right)\sin\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{l}\right]\right\}$$

$$=-\frac{\pi}{2a^2}\frac{\exp[-(\pi b/al)(l-x)]}{\cosh(\pi b/a)},\quad 0<x<2l$$

1.75 $\sum_{1}^{\infty} z^n \sin(nx) = z \sin x/(1-2z\cos x+z^2), \quad |z|<1$

1.76 $\sum_{0}^{\infty} z^n \cos(nx) = (1-z\cos x)/(1-2z\cos x+z^2), \quad |z|<1$

1.77 $\sum_{1}^{\infty} n^{-1} z^n \sin(nx) = \arctan[z\sin x/(1-z\cos x)], \quad |z|\leq 1$

1.78 $\sum_{1}^{\infty} n^{-1} z^n \cos(nx) = -\frac{1}{2}\log(1-2z\cos x+z^2), \quad |z|\leq 1$

1.79 $\sum_{0}^{\infty} z^{2n+1} \sin[(2n+1)x] = z\sin x(1+z^2)/[(1+z^2)^2-4z^2\cos^2 x], \quad |z|<1$

1.80 $\sum_{0}^{\infty} z^{2n+1} \cos[(2n+1)x] = z\cos x(1-z^2)/[(1+z^2)^2-4z^2\cos^2 x], \quad |z|<1$

1.81 $\sum_{0}^{\infty} (-1)^n z^{2n+1} \sin[(2n+1)x] = z\sin x(1-z^2)/[(1+z^2)^2-4z^2\sin^2 x], \quad |z|<1$

1.82 $\sum_{0}^{\infty} (-1)^n z^{2n+1} \cos[(2n+1)x] = z\cos x(1+z^2)/[(1+z^2)^2-4z^2\sin^2 x], \quad |z|<1$

1.83 $\sum_{0}^{\infty} (2n+1)^{-1} z^{2n+1} \sin[(2n+1)x] = \frac{1}{2}\arctan[2z\sin x/(1-z^2)], \quad |z|\leq 1$

1.84 $\sum_{0}^{\infty} (2n+1)^{-1} z^{2n+1} \cos[(2n+1)x] = \frac{1}{4}\log[(1+2z\cos x+z^2)/(1-2z\cos x+z^2)], \quad |z|\leq 1$

1.85 $\sum_{0}^{\infty} (-1)^n (2n+1)^{-1} z^{2n+1} \sin[(2n+1)x] = \frac{1}{4}\log[(1+2z\sin x+z^2)/(1-2z\sin x+z^2)],$

$|z|\leq 1$

1.86 $\sum_{0}^{\infty} (-1)^n (2n+1)^{-1} z^{2n+1} \cos[(2n+1)x] = \frac{1}{2}\arctan[2z\cos x/(1-z^2)], \quad |z|\leq 1$

1.87 $\sum_{1}^{\infty} e^{-nt}\sin(nx)=\frac{1}{2}[\sin x/(\cosh t-\cos x)]$

1.88 $\sum_{1}^{\infty} (-1)^n e^{-nt}\sin(nx)=-\frac{1}{2}[\sin x/(\cosh t+\cos x)]$

1.89 $\sum_{0}^{\infty} \epsilon_n e^{-nt}\cos(nx)=\sinh t/(\cosh t-\cos x)$

1.90 $\sum_{0}^{\infty} (-1)^n\epsilon_n e^{-nt}\cos(nx)=\sinh t/(\cosh t+\cos x)$

1.91 $\sum_{0}^{\infty} \exp[-(2n+1)t]\sin[(2n+1)x]=\sin x\cosh t/[\cosh(2t)-\cos(2x)]$

1.92 $\sum_{0}^{\infty} (-1)^n\exp[-(2n+1)t]\sin[(2n+1)x]=\sin x\sinh t/[\cosh(2t)+\cos(2x)]$

1.93 $\sum_{0}^{\infty} \exp[-(2n+1)t]\cos[(2n+1)x]=\cos x\sinh t/[\cosh(2t)-\cos(2x)]$

1.94 $\sum_{0}^{\infty} (-1)^n\exp[-(2n+1)t]\cos[(2n+1)x]=\cos x\cosh t/[\cosh(2t)+\cos(2x)]$

1.95 $\sum_{1}^{\infty} n^{-1}e^{-nt}\sin(nx)=\arctan[\sin x/(e^t-\cos x)]$

1.96 $\sum_{1}^{\infty} n^{-1}e^{-nt}\cos(nx)=\frac{1}{2}t-\frac{1}{2}\log[2(\cosh t-\cos x)]$

1.97 $\sum_{1}^{\infty} (-1)^n n^{-1}e^{-nt}\sin(nx)=-\arctan[\sin x/(e^t+\cos x)]$

1.98 $\sum_{1}^{\infty} (-1)^n n^{-1}e^{-nt}\cos(nx)=\frac{1}{2}t-\frac{1}{2}\log[2(\cosh t+\cos x)]$

1.99 $\sum_{0}^{\infty} (2n+1)^{-1}\exp[-(2n+1)t]\sin[(2n+1)x]=\frac{1}{2}\arctan(\sin x/\sinh t)$

1.100 $\sum_{0}^{\infty} (2n+1)^{-1} \exp[-(2n+1)t] \cos[(2n+1)x] = \frac{1}{4} \log[(\cosh t + \cos x)/(\cosh t - \cos x)]$

1.101 $\sum_{0}^{\infty} (-1)^n (2n+1)^{-1} \exp[-(2n+1)t] \sin[(2n+1)x]$

$= \frac{1}{4} \log[(\cosh t + \sin x)/(\cosh t - \sin x)]$

1.102 $\sum_{0}^{\infty} (-1)^n (2n+1)^{-1} \exp[-(2n+1)t] \cos[(2n+1)x] = \frac{1}{2} \arctan(\cos x/\sinh t)$

1.103 $\sum_{1}^{\infty} (z^n/n!) \sin(nx) = \exp(z \cos x) \sin(z \sin x)$

1.104 $\sum_{0}^{\infty} (z^n/n!) \cos(nx) = \exp(z \cos x) \cos(z \sin x)$

1.105 $\sum_{0}^{\infty} [z^{2n+1}/(2n+1)!] \sin[(2n+1)x] = \sin(z \sin x) \cosh(z \cos x)$

1.106 $\sum_{1}^{\infty} [z^{2n}/(2n)!] \sin(2nx) = \sin(z \sin x) \sinh(z \cos x)$

1.107 $\sum_{0}^{\infty} [z^{2n+1}/(2n+1)!] \cos[(2n+1)x] = \cos(z \sin x) \sinh(z \cos x)$

1.108 $\sum_{0}^{\infty} [z^{2n}/(2n)!] \cos(2nx) = \cos(z \sin x) \cosh(z \cos x)$

1.109 $\sum_{0}^{\infty} (-1)^n [z^{2n+1}/(2n+1)!] \sin[(2n+1)x] = \sinh(z \sin x) \cosh(z \cos x)$

1.110 $\sum_{1}^{\infty} (-1)^n [z^{2n}/(2n)!] \sin(2nx) = -\sinh(z \sin x) \sin(z \cos x)$

1.111 $\sum_{0}^{\infty} (-1)^n [z^{2n+1}/(2n+1)!] \cos[(2n+1)x] = \cosh(z \sin x) \sin(z \cos x)$

1.112 $\sum_{0}^{\infty} (-1)^n[z^{2n}/(2n)!]\cos(2nx)=\cosh(z\sin x)\cos(z\cos x)$

1.113 $\sum_{1}^{\infty} n^{-1}\sin(nx)\sin(ny)e^{-nt}=\frac{1}{4}\log\{[\cosh t-\cos(x+y)]/[\cosh t-\cos(x-y)]\}$

1.114 $\sum_{1}^{\infty} n^{-1}\sin(nx)\cos(ny)e^{-nt}$

$$=\tfrac{1}{2}\arctan[(2e^t\sin x\cos y-2\sin x\cos x)/(e^{2t}-2e^t(\cos x\cos y+\cos 2x)]$$

1.115 $\sum_{1}^{\infty} n^{-1}\cos(nx)\cos(ny)e^{-nt}=\frac{1}{2}t-\frac{1}{4}\log\{4[\cosh t-\cos(x+y)][\cosh t-\cos(x-y)]\}$

Appendix: Some Results Involving Generalized Functions

The unit step function $U(t)$ and the "delta" function $\delta(t)$ are defined by

$$U(t)=0,\qquad t<0$$

$$U(t)=1,\qquad t>0$$

$$\delta(t)=U'(t)$$

Some formerly listed results and some of their generalizations can be expressed in terms of the unit step function $U(t)$. Also, some formal nonconvergent Fourier series can be interpreted as an infinite number of rows of delta functions. With the definition of two sets of four functions, $g_m(x)$, $h_m(x)$, $m=1, 2, 3, 4$:

$$g_1(x)=\sum_{0}^{\infty} U(x-2na+y)-\sum_{0}^{\infty} U(x-2na-y)$$

$$g_2(x)=\sum_{0}^{\infty} U(x-2na+y)+\sum_{0}^{\infty} U(x-2na-y)$$

$$g_3(x)=\sum_{0}^{\infty} (-1)^n U(x-2na+y)-\sum_{0}^{\infty} (-1)^n U(x-2na-y)$$

$$g_4(x)=\sum_{0}^{\infty} (-1)^n U(x-2na+y)+\sum_{0}^{\infty} (-1)^n U(x-2na-y)$$

$$h_1(x)=\sum_0^\infty \delta(x-2na+y)-\sum_0^\infty \delta(x-2na-y)$$

$$h_2(x)=\sum_0^\infty \delta(x-2na+y)+\sum_0^\infty \delta(x-2na-y)$$

$$h_3(x)=\sum_0^\infty (-1)^n\delta(x-2na+y)-\sum_0^\infty (-1)^n\delta(x-2na-y)$$

$$h_4(x)=\sum_0^\infty (-1)^n\delta(x-2na+y)+\sum_0^\infty (-1)^n\delta(x-2na-y)$$

[The $g_m(x)$ and $h_m(x)$ are regarded as functions of $x>0$ (this is no loss of generality since the terms in the Fourier series involved are either even or odd functions of x) with parameter y $(-a<y<a)$.] One has the following formulas.

1.116 $$\sum_1^\infty n^{-1}\sin[n\pi(x/a)]=-\tfrac{1}{2}\pi[1+(x/a)-2\sum_0^\infty U(x-2na)]$$

1.117 $$\sum_1^\infty (-1)^n n^{-1}\sin[n\pi(x/a)]=-\tfrac{1}{2}\pi\{(x/a)-2\sum_0^\infty U[x-(2n+1)a]\}$$

1.118 $$\sum_0^\infty (2n+1)^{-1}\sin[(2n+1)(\pi x/2a)]=-\tfrac{1}{2}\pi[\tfrac{1}{2}-\sum_0^\infty (-1)^n U(x-2na)]$$

1.119 $$\sum_0^\infty (-1)^n(2n+1)^{-1}\cos[(2n+1)(\pi x/2a)]=\tfrac{1}{2}\pi\{\tfrac{1}{2}-\sum_0^\infty (-1)^n U[x-(2n+1)a]\}$$

1.120 $$\sum_1^\infty n^{-1}\sin[n\pi(y/a)]\cos[n\pi(x/a)]=\tfrac{1}{2}\pi[g_1(x)-(y/a)]$$

1.121 $$\sum_1^\infty n^{-1}\cos[n\pi(y/a)]\sin[n\pi(x/a)]=\tfrac{1}{2}\pi[g_2(x)-1-(x/a)]$$

1.122 $$\sum_0^\infty (2n+1)^{-1}\sin[(2n+1)(\pi y/2a)]\cos[(2n+1)(\pi x/2a)]=\tfrac{1}{4}\pi g_3(x)$$

1.123 $$\sum_0^\infty (2n+1)^{-1}\cos[(2n+1)(\pi y/2a)]\sin[(2n+1)(\pi x/2a)]=\tfrac{1}{4}\pi[g_4(x)-1]$$

1.124 $\sum_{1}^{\infty} (-1)^n n^{-1} \sin[n\pi(y/a)] \cos[n\pi(x/a)] = \frac{1}{2}\pi[g_1(x-a) - (y/a)]$

1.125 $\sum_{1}^{\infty} (-1)^n n^{-1} \cos[n\pi(y/a)] \sin[n\pi(x/a)] = \frac{1}{2}\pi[g_2(x-a) - (x/a)]$

1.126 $\sum_{0}^{\infty} (-1)^n (2n+1)^{-1} \sin[(2n+1)(\pi y/2a)] \sin[(2n+1)(\pi x/2a)] = \frac{1}{4}\pi g_3(x-a)$

1.127 $\sum_{0}^{\infty} (-1)^n (2n+1)^{-1} \cos[(2n+1)(\pi y/2a)] \cos[(2n+1)(\pi x/2a)] = \frac{1}{4}\pi[1-g_4(x-a)]$

The formulas 1.116–1.119 are equivalent to 1.12, 1.13, 1.16, and 1.19. It should be pointed out that the periodicity in x (equal to $2a$) is preserved for both sides of the formulas before. Nonconvergent Fourier series representing an infinite number of rows of delta functions can be obtained by formally differentiating term by term the results 1.116–1.127.

1.128 $\sum_{0}^{\infty} \epsilon_n \cos[n\pi(x/a)] = 2a \sum_{0}^{\infty} \delta(x-2na)$

1.129 $\sum_{0}^{\infty} (-1)^n \epsilon_n \cos[n\pi(x/a)] = 2a \sum_{0}^{\infty} \delta[x-(2n+1)a]$

1.130 $\sum_{0}^{\infty} \cos[(2n+1)(\pi x/2a)] = a \sum_{0}^{\infty} (-1)^n \delta(x-2na)$

1.131 $\sum_{0}^{\infty} (-1)^n \sin[(2n+1)(\pi x/2a)] = a \sum_{0}^{\infty} (-1)^n \delta[x-(2n+1)a]$

1.132 $\sum_{1}^{\infty} \sin[n\pi(y/a)] \sin[n\pi(x/a)] = -\frac{1}{2}ah_1(x)$

1.133 $\sum_{1}^{\infty} \cos[n\pi(y/a)] \cos[n\pi(x/a)] = \frac{1}{2}a[h_2(x) - a^{-1}]$

1.134 $\sum_{0}^{\infty} \sin[(2n+1)(\pi y/2a)] \sin[(2n+1)(\pi x/2a)] = -\frac{1}{2}ah_3(x)$

1.135 $\sum_{0}^{\infty} \cos[(2n+1)(\pi y/2a)] \cos[(2n+1)(\pi x/2a)] = \frac{1}{2}ah_4(x)$

1.136 $\sum_{1}^{\infty} (-1)^n \sin[n\pi(y/a)] \sin[n\pi(x/a)] = -\frac{1}{2}ah_1(x-a)$

1.137 $\sum_{1}^{\infty} (-1)^n \cos[n\pi(y/a)] \cos[n\pi(x/a)] = \frac{1}{2}a[h_2(x-a) - a^{-1}]$

1.138 $\sum_{0}^{\infty} (-1)^n \sin[(2n+1)(\pi y/2a)] \cos[(2n+1)(\pi x/2a)] = \frac{1}{2}ah_3(x-a)$

1.139 $\sum_{0}^{\infty} (-1)^n \cos[(2n+1)(\pi y/2a)] \cos[(2n+1)(\pi x/2a)] = -\frac{1}{2}ah_4(x-a)$

II

FOURIER SERIES WITH ELEMENTARY COEFFICIENTS REPRESENTING HIGHER FUNCTIONS

Formulas 2.1–2.25 involve the Jacobian elliptic functions $\mathrm{am}(z, k)$, $\mathrm{sn}(z, k)$, $\mathrm{cn}(z, k)$, $\mathrm{dn}(z, k)$, and $\mathrm{zn}(z, k)$. The parameter k is omitted in these formulas, i.e., $\mathrm{sn}z = \mathrm{sn}(z, k)$, etc.

In 2.26–2.37 $\vartheta_l(z) = \vartheta_l(z, q)$ are the Jacobian theta functions defined in 2.26–2.29. Finally $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind of the modulus k, respectively; $k'^2 = 1 - k^2$. Also

$$q = \exp(i\pi\tau), \qquad \tau = i(K'/K); \qquad \mathrm{Im}\tau > 0; \qquad q = \exp[-\pi(K'/K)]$$

Furthermore

$$Q_0 = \prod_1^\infty (1-q^{2n}), \qquad \log Q_0 = -\sum_1^\infty [n^{-1}q^{2n}/(1-q^{2n})],$$

$$q^n/(1-q^{2n}) = \tfrac{1}{2}\{\sinh[n\pi(K'/K)]\}^{-1};$$

$$q^n/(1+q^{2n}) = \tfrac{1}{2}\{\cosh[n\pi(K'/K)]\}^{-1}, \qquad q^{n+\frac{1}{2}}/(1-q^{2n+1}) = \tfrac{1}{2}\{\sinh[(n+\tfrac{1}{2})\pi(K'/K)]\}^{-1};$$

$$[q^{n+\frac{1}{2}}/(1+q^{2n+1})] = \tfrac{1}{2}\{\cosh[(n+\tfrac{1}{2})(\pi K'/K)]\}^{-1}$$

Restrictions In formulas 2.1–2.4, 2.7, 2.9, 2.11, 2.14, 2.15, 2.17, 2.18, 2.20, 2.21, 2.23, 2.25, 2.30, 2.31

$$|\,\mathrm{Im}z\,| < \tfrac{1}{2}\pi\ \mathrm{Im}\tau$$

In 2.5, 2.6, 2.8, 2.10, 2.12, 2.13, 2.16, 2.22, 2.24, 2.32, 2.33

$$|\operatorname{Im} z| < \pi \operatorname{Im} \tau$$

In 2.19

$$|\operatorname{Im} z| < 2\pi \operatorname{Im} \tau$$

2.1 $$\sum_{1}^{\infty} [n^{-1}q^n/(1+q^{2n})] \sin(2nz) = \tfrac{1}{2} \operatorname{am}[2K(z/\pi)] - \tfrac{1}{2}z$$

2.2 $$\sum_{1}^{\infty} [q^{n-\frac{1}{2}}/(1-q^{2n-1})] \sin[(2n-1)z] = (kK/2\pi) \operatorname{sn}(2Kz/\pi)$$

2.3 $$\sum_{1}^{\infty} [q^{n-\frac{1}{2}}/(1+q^{2n-1})] \cos[(2n-1)z] = (kK/2\pi) \operatorname{cn}(2Kz/\pi)$$

2.4 $$\sum_{1}^{\infty} [q^n/(1+q^{2n})] \cos(2nz) = (K/2\pi) \operatorname{dn}(2Kz/\pi) - \tfrac{1}{4}$$

2.5 $$\sum_{1}^{\infty} [q^{2n-1}/(1-q^{2n-1})] \sin[(2n-1)z] = \frac{K}{2\pi \operatorname{sn}(2Kz/\pi)} - (4 \sin z)^{-1}$$

2.6 $$\sum_{1}^{\infty} [(-1)^n q^{2n-1}/(1+q^{2n-1})] \cos[(2n-1)z] = \frac{k'K}{2\pi \operatorname{cn}(2Kz/\pi)} - (4 \cos z)^{-1}$$

2.7 $$\sum_{1}^{\infty} [(-1)^n q^n/(1+q^{2n})] \cos(2nz) = \frac{k'K}{2\pi \operatorname{dn}(2Kz/\pi)} - \frac{1}{4}$$

2.8 $$\sum_{1}^{\infty} [(-1)^n q^{2n}/(1+q^{2n})] \sin(2nz) = \frac{k'K \operatorname{sn}(2Kz/\pi)}{2\pi \operatorname{cn}(2Kz/\pi)} - \frac{1}{4} \tan z$$

2.9 $$\sum_{1}^{\infty} [(-1)^n q^{n-\frac{1}{2}}/(1+q^{2n-1})] \sin[(2n-1)z] = -\frac{kk'K \operatorname{sn}(2Kz/\pi)}{2\pi \operatorname{dn}(2Kz/\pi)}$$

2.10 $$\sum_{1}^{\infty} [q^{2n}/(1+q^{2n})] \sin(2nz) = -\frac{K \operatorname{cn}(2Kz/\pi)}{2\pi \operatorname{sn}(2Kz/\pi)} + \frac{1}{4} \cot z$$

2.11 $$\sum_{1}^{\infty} [(-1)^n q^{n-\frac{1}{2}}/(1-q^{2n-1})] \cos[(2n-1)z] = -\frac{kK \operatorname{sn}(2Kz/\pi)}{2\pi \operatorname{dn}(2Kz/\pi)}$$

2.12 $$\sum_{1}^{\infty} [q^{2n-1}/(1+q^{2n-1})] \sin[(2n-1)z] = -\frac{K\,\mathrm{dn}(2Kz/\pi)}{2\pi\,\mathrm{sn}(2Kz/\pi)} + (4\sin z)^{-1}$$

2.13 $$\sum_{1}^{\infty} [(-1)^n q^{2n-1}/(1-q^{2n-1})] \cos[(2n-1)z] = -\frac{K\,\mathrm{dn}(2Kz/\pi)}{2\pi\,\mathrm{cn}(2Kz/\pi)} + (4\cos z)^{-1}$$

2.14 $$\sum_{1}^{\infty} [q^n/(1+q^n)] \sin(2nz) = -\frac{K\,\mathrm{cn}(2Kz/\pi)\,\mathrm{dn}(2Kz/\pi)}{2\pi\,\mathrm{sn}(2Kz/\pi)} + \frac{1}{4}\cot z$$

2.15 $$\sum_{1}^{\infty} \{q^n/[1+(-1)^n q^n]\} \sin(2nz) = \frac{K\,\mathrm{sn}(2Kz/\pi)\,\mathrm{dn}(2Kz/\pi)}{2\pi\,\mathrm{cn}(2Kz/\pi)} - \frac{1}{4}\tan z$$

2.16 $$\sum_{1}^{\infty} [q^{2n-1}/(1-q^{4n-2})] \sin[(2n-1)z] = \frac{k^2 K\,\mathrm{sn}(Kz/\pi)\,\mathrm{cn}(Kz/\pi)}{4\pi\,\mathrm{dn}(Kz/\pi)}$$

2.17 $$\sum_{1}^{\infty} [(-1)^n q^n/(1-q^n)] \sin(2nz) = \frac{k'^2 K\,\mathrm{sn}(2Kz/\pi)}{2\pi\,\mathrm{cn}(2Kz/\pi)\,\mathrm{dn}(2Kz/\pi)} - \frac{1}{4}\tan z$$

2.18 $$\sum_{1}^{\infty} \{(-1)^n q^n/[1+(-1)^n q^n]\} \sin(2nz) = -\frac{K\,\mathrm{cn}(2Kz/\pi)}{2\pi\,\mathrm{sn}(2Kz/\pi)\,\mathrm{dn}(2Kz/\pi)} + \frac{1}{4}\cot z$$

2.19 $$\sum_{1}^{\infty} [q^{4n-2}/(1-q^{4n-2})] \sin[(2n-1)z] = \frac{K\,\mathrm{dn}(Kz/\pi)}{4\pi\,\mathrm{sn}(Kz/\pi)\,\mathrm{cn}(Kz/\pi)} - \frac{1}{4\sin z}$$

2.20 $$\sum_{1}^{\infty} [n^{-1}q^n/(1+q^n)] \cos(2nz) = \tfrac{1}{2}\log \mathrm{sn}(2Kz/\pi) - \tfrac{1}{2}\log(2q^{\frac{1}{4}}k^{-\frac{1}{2}}\sin z)$$

2.21 $$\sum_{1}^{\infty} \{n^{-1}q^n/[1+(-q)^n]\} \cos(2nz) = \tfrac{1}{2}\log \mathrm{cn}(2Kz/\pi) - \tfrac{1}{2}\log[2q^{\frac{1}{4}}(k'/k)^{\frac{1}{2}}\cos z]$$

2.22 $$\sum_{1}^{\infty} [(2n-1)^{-1}q^{2n-1}/(1-q^{4n-2})] \cos[(4n-2)z] = -\tfrac{1}{8}\log k' + \tfrac{1}{4}\log \mathrm{dn}(2Kz/\pi)$$

2.23 $$\sum_{1}^{\infty} [nq^n/(1-q^{2n})] \cos(2nz) = (K^2/2\pi^2)\,\mathrm{dn}^2(2Kz/\pi) - (KE/2\pi^2)$$

2.24 $$\sum_{1}^{\infty} [nq^{2n}/(1-q^{2n})] \cos(2nz) = -\frac{K^2}{2\pi^2\,\mathrm{sn}^2(2Kz/\pi)} + (8\sin^2 z)^{-1} + \frac{K(K-E)}{2\pi^2}$$

2.25 $\sum_{1}^{\infty} [q^n/(1-q^{2n})] \sin(2nz) = (K/2\pi)\ \mathrm{zn}[(2K/\pi)z]$

2.26 $\sum_{0}^{\infty} (-1)^n \epsilon_n q^{n^2} \cos(2nz) = \vartheta_0(z/\pi)$

2.27 $\sum_{0}^{\infty} \epsilon_n q^{n^2} \cos(2nz) = \vartheta_3(z/\pi)$

2.28 $\sum_{0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin[(2n+1)z] = \frac{1}{2}\vartheta_1(z/\pi)$

2.29 $\sum_{0}^{\infty} q^{(n+\frac{1}{2})^2} \cos[(2n+1)z] = \frac{1}{2}\vartheta_2(z/\pi)$

2.30 $\sum_{1}^{\infty} [n^{-1}q^n/(1-q^{2n})] \cos(2nz) = \frac{1}{2}\log Q_0 - \frac{1}{2}\log\vartheta_0(z/\pi)$

2.31 $\sum_{1}^{\infty} [n^{-1}q^n(-1)^n/(1-q^{2n})] \cos(2nz) = \frac{1}{2}\log Q_0 - \frac{1}{2}\log\vartheta_3(z/\pi)$

2.32 $\sum_{1}^{\infty} [n^{-1}q^{2n}/(1-q^{2n})] \cos(2nz) = \frac{1}{2}\log(2Q_0) + \frac{1}{8}\log q + \frac{1}{2}\log\sin z - \frac{1}{2}\log\vartheta_1(z/\pi)$

2.33 $\sum_{1}^{\infty} [(-1)^n n^{-1}q^{2n}/(1-q^{2n})] \cos(2nz) = \frac{1}{2}\log(2Q_0) + \frac{1}{8}\log q + \frac{1}{2}\log\cos z - \frac{1}{2}\log\vartheta_2(z/\pi)$

2.34 $\sum_{1}^{\infty} [n^{-1}q^{2n}/(1-q^{2n})] \sin(2nz_1)\sin(2nz_2) = \frac{1}{4}\log\left\{\frac{\vartheta_0[(z_1+z_2)/\pi]}{\vartheta_0[(z_1-z_2)/\pi]}\right\}$

$|\mathrm{Im} z_1| + |\mathrm{Im} z_2| < \frac{1}{2}\pi\ \mathrm{Im}\tau$

2.35 $\sum_{1}^{\infty} [(-1)^n n^{-1}q^{2n}/(1-q^{2n})] \sin(2nz_1)\sin(2nz_2) = \frac{1}{4}\log\left(\frac{\vartheta_3[(z_1+z_2)/\pi]}{\vartheta_3[(z_1-z_2)/\pi]}\right)$

$|\mathrm{Im} z_1| + |\mathrm{Im} z_2| < \frac{1}{2}\pi\ \mathrm{Im}\tau$

2.36 $\sum_{1}^{\infty} [n^{-1}q^{2n}/(1-q^{2n})] \sin(2nz_1)\sin(2nz_2) = \frac{1}{4}\log\left(\frac{\vartheta_1[(z_1+z_2)/\pi]}{\vartheta_1[(z_1-z_2)/\pi]}\right) - \frac{1}{4}\log\left(\frac{\sin(z_1+z_2)}{\sin(z_1-z_2)}\right)$

$|\mathrm{Im} z_1| + |\mathrm{Im} z_2| < \pi\ \mathrm{Im}\tau$

2.37 $\sum_{1}^{\infty} \frac{(-1)^n n^{-1} q^{2n}}{1-q^{2n}} \sin(2nz_1)\sin(2nz_2) = \frac{1}{4}\log\left(\frac{\vartheta_2[(z_1+z_2)/\pi]}{\vartheta_2[(z_1-z_2)/\pi]}\right) - \frac{1}{4}\log\left(\frac{\cos(z_1+z_2)}{\cos(z_1-z_2)}\right)$

$|\operatorname{Im} z_1| + |\operatorname{Im} z_2| < \pi \operatorname{Im}\tau$

2.38 $\sum_{0}^{\infty} \{[(2n)!]^2/(n!)^4\} 2^{-4n} \sin[(2n+\frac{1}{2})x] = \pi^{-1} K(\sin\frac{1}{2}x), \quad 0<x<\pi$

2.39 $\sum_{0}^{\infty} [(2n)!(2n+1)!/(n!)^3(n+1)!] 2^{-4n} \sin[(2n+\frac{3}{2})x]$

$= (2/\pi)[2E(\sin\frac{1}{2}x) - K(\sin\frac{1}{2}x)], \quad 0<x<\pi$

2.40 $\sum_{0}^{\infty} \{[(2n)!]^2/(n!)^4\} 2^{-4n} \cos[(2n+\frac{1}{2})x] = \pi^{-1} K(\cos\frac{1}{2}x), \quad 0<x<\pi$

2.41 $\sum_{0}^{\infty} [(2n)!(2n+1)!/(n!)^3(n+1)!] 2^{-2n} \cos[(2n+\frac{3}{2})x]$

$= (2/\pi)[K(\cos\frac{1}{2}x) - 2E(\cos\frac{1}{2}x)], \quad 0<x<\pi$

2.42 $\sin(n+1)x + [(n+1)/(2n+3)]\sin(n+3)x + \frac{1\cdot 3(n+1)(n+2)}{2!(2n+3)(2n+5)} \sin(n+5)x$

$+ \frac{1\cdot 3\cdot 5}{3!} \cdot \frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+5)(2n+7)} \sin(n+7)x + \cdots$

$= \pi(2n+1)!(n!)^{-2} 2^{-2-2n} P_n(\cos x), \quad 0<x<\pi; \quad n=0, 1, 2, \ldots$

2.43 $\cos(n+1)x + \frac{n+1}{2n+3}\cos(n+3)x + \frac{1\cdot 3(n+1)(n+2)}{2!(2n+3)(2n+5)} \cos(n+5)x$

$+ \frac{1\cdot 3\cdot 5(n+1)(n+2)(n+3)}{3!(2n+3)(2n+5)(2n+7)} \cos(n+7)x + \cdots = (2n+1)!2^{-2n-1}(n!)^{-2} Q_n(\cos x)$

$0<x<\pi; \quad n=0, 1, 2, \ldots$

2.44 $\sum_{0}^{\infty} \frac{(2r+2n)!(l+r+n)!(l+1+n)!}{n!(r+n)!(2l+2+2n)!} \sin[(2n+l+r+1)x] = \pi 2^{-3-r-2l} (\sin x)^{-r} P_l^r(\cos x)$

$0<x<\pi; \quad r, l=0, 1, 2, \ldots$

2.45 $$\sum_{0}^{\infty} \frac{(2r+2n)!(l+r+n)!(l+1+n)!}{n!(r+n)!(2l+2+2n)!} \cos[(2n+l+r+1)x] = 2^{-2-r-2l}(\sin x)^{-r} Q_l^r(\cos x)$$

$0<x<\pi; \quad r, l = 0, 1, 2, \ldots$

2.46 $$\sum_{0}^{\infty} [(\tfrac{1}{2}+\mu)_n(1+\nu+\mu)_n/n!(\nu+\tfrac{3}{2})_n] \sin[(2n+\nu+\mu+1)x]$$
$$= \pi^{\frac{1}{2}} 2^{-1-\mu}[\Gamma(\tfrac{3}{2}+\nu)/\Gamma(\nu+\mu+1)](\sin x)^{-\mu} P_\nu^\mu(\cos x), \quad 0<x<\pi$$

2.47 $$\sum_{0}^{\infty} [(\tfrac{1}{2}+\mu)_n(1+\nu+\mu)_n/n!(\tfrac{3}{2}+\nu)_n] \cos[(2n+\nu+\mu+1)x]$$
$$= \pi^{-\frac{1}{2}} 2^{-\mu}[\Gamma(\tfrac{3}{2}+\nu)/\Gamma(\nu+\mu+1)](\sin x)^{-\mu} Q_\nu^\mu(\cos x), \quad 0<x<\pi$$

2.48 $$\sum_{1}^{\infty} \log[n/(n+1)] \sin(2n+1)x = \sin x\,\psi(x/\pi) + \tfrac{1}{2}\pi \cos x + (\gamma + \log 2\pi) \sin x, \quad 0<x<\pi$$

2.49 $$\sum_{1}^{\infty} n^{-1} \log n \sin(nx) = \pi \log\Gamma(x/2\pi) + \tfrac{1}{2}\pi \log(\sin\tfrac{1}{2}x)$$
$$-\pi[1-(x/2\pi)]\log\pi - \tfrac{1}{2}\pi[1-(x/\pi)](\gamma+\log 2), \quad 0<x<2\pi$$

2.50 $$\sum_{0}^{\infty} \epsilon_n\{b^2+[n(\pi/a)]^2\}^{-\frac{1}{2}} \sin\{a[b^2+[n(\pi/a)]^2]^{\frac{1}{2}}\} \cos[n(\pi x/a)] = aJ_0[b(a^2-x^2)^{\frac{1}{2}}],$$
$$-a<x<a$$

2.51 $$\cos(\tfrac{1}{2}\pi z) \sum_{1}^{\infty} n^{z-1} \sin(2\pi n\alpha) + \sin(\tfrac{1}{2}\pi z) \sum_{1}^{\infty} n^{z-1} \cos(2\pi n\alpha)$$
$$= \sum_{1}^{\infty} n^{z-1} \sin(2\pi n\alpha + \tfrac{1}{2}\pi z) = \tfrac{1}{2}(2\pi)^{1-z}[\Gamma(1-z)]^{-1}\zeta(z,\alpha), \quad \mathrm{Re}\,z<1, \quad 0<\alpha<1$$

2.52 $$\cos(\tfrac{1}{2}\pi z) \sum_{1}^{\infty} (2n-1)^{z-1} \sin[(2n-1)\alpha\pi] + \sin(\tfrac{1}{2}\pi z) \sum_{1}^{\infty} (2n-1)^{z-1} \cos[(2n-1)\alpha\pi]$$
$$= \sum_{1}^{\infty} (2n-1)^{z-1} \sin[(2n-1)\alpha\pi + \tfrac{1}{2}\pi z]$$
$$= 2^{-z-1}\pi^{1-z}[\Gamma(1-z)]^{-1}[\zeta(z,\tfrac{1}{2}\alpha) - \zeta(z,\tfrac{1}{2}+\tfrac{1}{2}\alpha)], \quad \mathrm{Re}\,z<1, \quad 0<\alpha\leq 1$$

2.53 $$\sum_{0}^{\infty} \epsilon_n \cos(nx)[b^2+(nd)^2]^{-\frac{1}{2}} \sin\{a[b^2+(nd)^2]^{\frac{1}{2}}\} = (\pi/d) \sum_{m_1}^{m_2} J_0(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$

$$m_{\substack{1\\2}} = \mp[(ad\pm x)/2\pi]$$

If $(ad\pm x)/2\pi$ is an integer, then one half of the corresponding term in the sum is taken.

III

FOURIER SERIES WITH HIGHER FUNCTION COEFFICIENTS REPRESENTING ELEMENTARY FUNCTIONS

3.1 $\sum_{0}^{\infty} \{(-1)^n \epsilon_n \cos(2nx)/[\Gamma(1+\nu+n)\Gamma(1+\nu-n)]\} = [2^{2\nu}/\Gamma(1+2\nu)] \sin^{2\nu}x$

$\mathrm{Re}\,\nu > -\tfrac{1}{2}, \quad 0 < x < \pi$

3.2 $\sum_{0}^{\infty} \{(-1)^n \sin[(2n+1)x]/[\Gamma(\nu+\tfrac{3}{2}+n)\Gamma(\nu+\tfrac{1}{2}-n)]\} = [2^{2\nu-1}/\Gamma(1+2\nu)] \sin^{2\nu}x$

$\mathrm{Re}\,\nu > -\tfrac{1}{2}, \quad 0 < x < \pi$

3.3 $\sum_{0}^{\infty} \{\epsilon_n \cos(2nx)/[\Gamma(1+\nu+n)\Gamma(1+\nu-n)]\} = [2^{2\nu}/\Gamma(1+2\nu)] \cos^{2\nu}x$

$\mathrm{Re}\,\nu > -\tfrac{1}{2}, \quad -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi$

3.4 $\sum_{0}^{\infty} \{\cos[(2n+1)x]/[\Gamma(\nu+\tfrac{3}{2}+n)\Gamma(\nu+\tfrac{1}{2}-n)]\} = [2^{2\nu-1}/\Gamma(1+2\nu)] \cos^{2\nu}x$

$\mathrm{Re}\,\nu > -\tfrac{1}{2}, \quad -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi$

3.5 $\sum_{0}^{\infty} (-1)^n \epsilon_n J_{2n}(z) \cos(2nx) = \cos(z \cos x)$

3.6 $\sum_{0}^{\infty} (-1)^n J_{2n+1}(z) \cos[(2n+1)x] = \frac{1}{2} \sin(z \cos x)$

3.7 $\sum_{0}^{\infty} \epsilon_n J_{2n}(z) \cos(2nx) = \cos(z \sin x)$

3.8 $\sum_{0}^{\infty} J_{2n+1}(z) \sin[(2n+1)x] = \frac{1}{2} \sin(z \sin x)$

3.9 $\sum_{0}^{\infty} \epsilon_n J_{2n}(z) \cos(2nt) \cos(2nx) = \cos(z \sin x \cos t) \cos(z \cos x \cos t)$

3.10 $\sum_{0}^{\infty} J_{2n+1}(z) \sin[(2n+1)t] \cos[(2n+1)x] = \frac{1}{2} \cos(z \sin x \cos t) \sin(z \cos x \sin t)$

3.11 $\sum_{0}^{\infty} (-1)^n \epsilon_n J_{2n}(z) \cos(2nt) \cos(2nx) = \cos(z \cos x \cos t) \cos(z \sin x \sin t)$

3.12 $\sum_{0}^{\infty} (-1)^n J_{2n+1}(z) \sin[(2n+1)t] \sin[(2n+1)x] = \frac{1}{2} \cos(z \cos x \cos t) \sin(z \sin x \sin t)$

3.13 $\sum_{0}^{\infty} \epsilon_n I_n(z) \cos(nx) = \exp(z \cos x)$

3.14 $\sum_{0}^{\infty} (-1)^n \epsilon_n I_n(z) \cos(nx) = \exp(-z \cos x)$

3.15 $\sum_{0}^{\infty} \epsilon_n \exp[inm(\pi/2)] J_{nm}(z) \cos(nx) = m^{-1} \sum_{r=r_1}^{r_2} \exp\{iz \cos[(2\pi r + x)/m]\}$

$m = 0, 1, 2, \ldots; \quad r_{\frac{1}{2}} = \mp[(m\pi \pm x)/2\pi]$

3.16 $\sum_{0}^{\infty} \epsilon_n I_{nm}(z) \cos(nx) = m^{-1} \sum_{r_1}^{r_2} \exp\{z \cos[(2\pi r + x)/m]\}$

$m = 0, 1, 2, \ldots; \quad r_{\frac{1}{2}} = \mp[(m\pi \pm x)/2\pi]$

If $(m\pi \pm x)/2\pi$ is an integer or zero, one half of the corresponding term at the right side finite sum of 3.15 and 3.16 has to be taken. Formulas 3.5–3.16 hold also for arbitrary complex values of z, x.

3.17 $$\sum_{1}^{\infty} J_0(nx)\cos(nxt) = -\tfrac{1}{2} + \sum_{l=1}^{m} [x^2-(2\pi l+tx)^2]^{-\frac{1}{2}} + x^{-1}(1-t^2)^{-\frac{1}{2}} + \sum_{l=1}^{k} [x^2-(2\pi l-tx)^2]^{-\frac{1}{2}}$$

3.18 $$\sum_{1}^{\infty} J_0(nx)\sin(nxt) = (2\pi)^{-1}\Big[\sum_{l=1}^{k} l^{-1} - \sum_{l=1}^{m} l^{-1}\Big] + \sum_{l=m+1}^{\infty} \{[(2\pi l+tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\}$$
$$- \sum_{l=k+1}^{\infty} \{[(2\pi l-tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\}$$

3.19 $$\sum_{1}^{\infty} Y_0(nx)\cos(nxt) = -\pi^{-1}[\gamma+\log(x/4\pi)] + (2\pi)^{-1}\Big[\sum_{l=1}^{m} l^{-1} + \sum_{l=1}^{k} l^{-1}\Big]$$
$$- \sum_{l=m+1}^{\infty} \{[(2\pi l+tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\} - \sum_{l=k+1}^{\infty} \{[(2\pi l-tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\}$$

3.20 $$\sum_{1}^{\infty} J_0(nx)\cos(nxt) = -\tfrac{1}{2} + \sum_{l=m+1}^{k} [x^2-(2\pi l-tx)^2]^{-\frac{1}{2}}$$

3.21 $$\sum_{1}^{\infty} J_0(nx)\sin(nxt) = \sum_{l=0}^{m} [(2\pi l-tx)^2-x^2]^{-\frac{1}{2}} + (2\pi)^{-1}\sum_{l=1}^{k} l^{-1}$$
$$+ \sum_{l=1}^{\infty} \{[(2\pi l+tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\} - \sum_{l=k+1}^{\infty} \{[(2\pi l-tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\}$$

3.22 $$\sum_{1}^{\infty} Y_0(nx)\cos(nxt) = -\pi^{-1}[\gamma+\log(x/4\pi)] - \sum_{l=0}^{m} [(2\pi l-tx)^2-x^2]^{-\frac{1}{2}}$$
$$+(2\pi)^{-1}\sum_{l=1}^{k} l^{-1} - \sum_{l=1}^{\infty} \{[(2\pi l+tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\} - \sum_{l=1}^{\infty} \{[(2\pi l-tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\}$$

3.23 $$\sum_{1}^{\infty} (-1)^n J_0(nx)\cos(nxt) = -\tfrac{1}{2} + \sum_{l=1}^{m} [x^2-(2l\pi-\pi+tx)^2]^{-\frac{1}{2}} + \sum_{l=1}^{k} [x^2-(2l\pi-\pi-tx)^2]^{-\frac{1}{2}}$$

3.24 $$\sum_{1}^{\infty} (-1)^n J_0(nx)\sin(nxt) = (2\pi)^{-1}\Big[\sum_{l=1}^{k} l^{-1} - \sum_{l=1}^{m} l^{-1}\Big]$$
$$+ \sum_{l=m+1}^{\infty} \{[(2l\pi-\pi+tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\} - \sum_{l=k+1}^{\infty} \{[(2l\pi-\pi-tx)^2-x^2]^{-\frac{1}{2}} - (2\pi l)^{-1}\}$$

3.25 $$\sum_{1}^{\infty}(-1)^n Y_0(nxt)\cos(nxt)=-\pi^{-1}[\gamma+\log(x/4\pi)]+(2\pi)^{-1}[\sum_{l=1}^{m}l^{-1}+\sum_{l=1}^{k}l^{-1}]$$
$$-\sum_{l=m+1}^{\infty}\{[(2l\pi-\pi+tx)^2-x^2]^{-\frac{1}{2}}-(2\pi l)^{-1}\}-\sum_{l=k+1}^{\infty}\{[(2l\pi-\pi-tx)^2-x^2]^{-\frac{1}{2}}-(2\pi l)^{-1}\}$$

3.26 $$\sum_{1}^{\infty}(-1)^n J_0(nx)\cos(nxt)=-\tfrac{1}{2}+\sum_{l=m+1}^{k}[x^2-(2l\pi-\pi-tx)^2]^{-\frac{1}{2}}$$

3.27 $$\sum_{1}^{\infty}(-1)^n J_0(nx)\sin(nxt)=(2\pi)^{-1}\sum_{l=1}^{k}l^{-1}+\sum_{l=1}^{m}[(2l\pi-\pi-tx)^2-x^2]^{-\frac{1}{2}}$$
$$+\sum_{l=1}^{\infty}\{[(2l\pi-\pi+tx)^2-x^2]^{-\frac{1}{2}}-(2\pi l)^{-1}\}-\sum_{l=k+1}^{\infty}\{[(2l\pi-\pi-tx)^2-x^2]^{-\frac{1}{2}}-(2\pi l)^{-1}\}$$

3.28 $$\sum_{1}^{\infty}(-1)^n Y_0(nx)\cos(nxt)$$
$$=-\pi^{-1}[\gamma+\log(x/4\pi)]+(2\pi)^{-1}\sum_{l=1}^{k}l^{-1}-\sum_{l=1}^{m}[(2l\pi-\pi-tx)^2-x^2]^{-\frac{1}{2}}$$
$$-\sum_{l=1}^{\infty}\{[(2l\pi-\pi+tx)^2-x^2]^{-\frac{1}{2}}-(2\pi l)^{-1}\}-\sum_{l=k+1}^{\infty}\{[(2l\pi-\pi-tx)^2-x^2]^{-\frac{1}{2}}-(2\pi l)^{-1}\}$$

3.29 $$\sum_{1}^{\infty}K_0(nx)\cos(nxt)=\tfrac{1}{2}[\gamma+\log(x/4\pi)]+(\pi/2x)(1+t^2)^{-\frac{1}{2}}$$
$$+\tfrac{1}{2}\pi\sum_{l=1}^{\infty}\{[x^2+(2l\pi-tx)^2]^{-\frac{1}{2}}-(2l\pi)^{-1}\}+\tfrac{1}{2}\pi\sum_{l=1}^{\infty}\{[x^2+(2l\pi+tx)^2]^{-\frac{1}{2}}-(2l\pi)^{-1}\}$$

3.30 $$\sum_{1}^{\infty}(-1)^n K_0(nx)\cos(nxt)=\tfrac{1}{2}[\gamma+\log(x/4\pi)]$$
$$+\tfrac{1}{2}\pi\sum_{l=1}^{\infty}\{[x^2+(2l\pi-\pi-tx)^2]^{-\frac{1}{2}}-(2l\pi)^{-1}\}+\tfrac{1}{2}\pi\sum_{l=1}^{\infty}\{[x^2+(2l\pi-\pi+xt)^2]^{-\frac{1}{2}}-(2l\pi)^{-1}\}$$

In 3.11–3.19: $0\le t<1$, $x>0$, $m, k=0, 1, 2, \ldots$

$2\pi m<x(1-t)<2(m+1)\pi$; $2k\pi<x(1+t)<2(k+1)\pi$

In 3.20–3.22: $t>1$, $x>0$, $m, k=0, 1, 2, \ldots$

$2m\pi<x(t-1)<2(m+1)\pi$; $2k\pi<x(t+1)<2(k+1)\pi$

In 3.23–3.25: $0<t\le 1$, $x>0$, $m, k=1, 2, 3, \ldots$

$(2m-1)\pi<x(1-t)<(2m+1)\pi$; $(2k-1)\pi<x(1+t)<(2k+1)\pi$

In 3.26–3.28: $t>1$, $x>0$, $m, k=1, 2, 3, \ldots$

$$(2m-1)\pi<x(t-1)<(2m+1)\pi; \qquad (2k-1)\pi<x(1+t)<(2k+1)\pi$$

3.31 $$\sum_0^\infty \epsilon_n n^{-\nu} J_\nu(n\pi) \cos[n\pi(x/a)] = [2^{1-\nu}/\Gamma(\tfrac{1}{2}+\nu)]\pi^{\nu-\frac{1}{2}} a^{1-2\nu}(a^2-x^2)^{\nu-\frac{1}{2}}$$

$-a<x<a$, $\operatorname{Re}\nu>-\frac{1}{2}$

for $\nu=1$, Fourier series of a circle of radius a

3.32 $$\sum_0^\infty \epsilon_n(-1)^n n^{-\nu} J_\nu(n\pi) \cos[2n\pi(x/a)] = [2^\nu/\Gamma(\tfrac{1}{2}+\nu)]\pi^{\nu-\frac{1}{2}} a^{1-2\nu}(ax-x^2)^{\nu-\frac{1}{2}}$$

$\operatorname{Re}\nu>-\frac{1}{2}$, $-a<x<a$

3.33 $$\sum_1^\infty (-1)^n(2n+1)^{-\nu} J_\nu[(n+\tfrac{1}{2})\pi] \sin[(2n+1)(x/a)] = [a^{1-2\nu}/2\Gamma(\tfrac{1}{2}+\nu)]\pi^{\nu-\frac{1}{2}}(ax-x^2)^{\nu-\frac{1}{2}}$$

$\operatorname{Re}\nu>-\frac{1}{2}$, $0<x<a$

3.34 $$\sum_1^\infty n^{-\nu} J_{\nu+1}(n\pi) \sin[n\pi(x/a)] = [a^{-2\nu}/2\Gamma(\tfrac{1}{2}+\nu)](\tfrac{1}{2}\pi)^{\nu-\frac{1}{2}} x(a^2-x^2)^{\nu-\frac{1}{2}}$$

$\operatorname{Re}\nu>-\frac{1}{2}$, $0<x<a$

3.35 $$\sum_0^\infty (-1)^n J_0\{\tfrac{1}{2}a[b^2+(2n+1)^2(\pi^2/a^2)]^{\frac{1}{2}}\} \sin[(2n+1)\pi(x/a)]$$

$$=(a/2\pi)(ax-x^2)^{-\frac{1}{2}} \cos[b(ax-x^2)^{\frac{1}{2}}], \quad 0<x<a$$

3.36 $$\sum_0^\infty (-1)^n \epsilon_n J_0(\tfrac{1}{2}a\{b^2+[2n(\pi/a)]^2\}^{\frac{1}{2}}) \cos[2n\pi(x/a)]$$

$$=(a/\pi)(ax-x^2)^{-\frac{1}{2}} \cos[b(ax-x^2)^{\frac{1}{2}}], \quad -a<x<a$$

3.37 $$\sum_0^\infty (-1)^n J_0\{\tfrac{1}{2}a[(2n+1)^2(\pi^2/a^2)-b^2]^{\frac{1}{2}}\} \sin[(2n+1)\pi(x/a)]$$

$$=(a/2\pi)(ax-x^2)^{-\frac{1}{2}} \cosh[b(ax-x^2)^{\frac{1}{2}}], \quad 0<x<a$$

3.38 $$\sum_0^\infty (-1)^n \epsilon_n J_0(\tfrac{1}{2}a\{[2n(\pi/a)]^2-b^2\}^{\frac{1}{2}}) \cos[2\pi n(x/a)]$$

$$=(a/\pi)(ax-x^2)^{-\frac{1}{2}} \cosh[b(ax-x^2)^{\frac{1}{2}}], \quad -a<x<a$$

3.39 $\sum_{0}^{\infty} \epsilon_n n^{\frac{1}{2}} J_\nu(\tfrac{1}{2}n\pi) J_{-\nu-\frac{1}{2}}(\tfrac{1}{2}n\pi) \cos[n\pi(x/a)]$

$= \pi^{-2}(2a)^{\frac{3}{2}}(a^2x-x^3)^{-\frac{1}{2}} \cos[(2\nu+\tfrac{1}{2}) \arccos(x/a)], \quad 0<x<a$

3.40 $\sum_{1}^{\infty} n^{\frac{1}{2}} J_{\nu+\frac{1}{2}}(\tfrac{1}{2}n\pi) J_{-\nu}(\tfrac{1}{2}n\pi) \sin[n\pi(x/a)]$

$= \tfrac{1}{2}\pi^{-2}(2a)^{\frac{3}{2}}(a^2x-x^3)^{-\frac{1}{2}} \cos[(2\nu+\tfrac{1}{2}) \arccos(x/a)], \quad 0<x<a$

3.41 $\sum_{0}^{\infty} n^{-\nu} \mathbf{H}_\nu(n\pi) \sin[n\pi(x/a)] = [2^{-\nu}/\Gamma(\tfrac{1}{2}+\nu)] a^{1-2\nu} \pi^{\nu-\frac{1}{2}} (a^2-x^2)^{\nu-\frac{1}{2}}$

$\operatorname{Re}\nu > -\tfrac{1}{2}, \quad 0<x<a$

for $\nu=1$, Fourier series of circle of radius a

3.42 $\sum_{0}^{\infty} \epsilon_n (nd)^{-\nu} J_\nu(and) \cos(nx) = [\pi^{\frac{1}{2}} 2^{1-\nu} a^{-\nu}/d\Gamma(\tfrac{1}{2}+\nu)] \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{\nu-\frac{1}{2}}$

$\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re}\nu > \tfrac{1}{2} \quad \text{if} \quad x=\pm ad$

3.43 $\sum_{0}^{\infty} \epsilon_n \cos(\tfrac{1}{2}and)(nd)^{-\nu} J_\nu(\tfrac{1}{2}and) \cos(nx)$

$= [\pi^{\frac{1}{2}} a^{-\nu}/d\Gamma(\tfrac{1}{2}+\nu)] \sum_{m_1}^{m_2} |(2\pi m+x)/d|^{\nu-\frac{1}{2}} [a-|(2\pi m+x)/d|]^{\nu-\frac{1}{2}}$

$\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re}\nu > \tfrac{1}{2} \quad \text{if} \quad x=\pm ad, \quad \text{or} \quad x=0$

3.44 $\sum_{0}^{\infty} \epsilon_n J_0\{a[b^2+(nd)^2]^{\frac{1}{2}}\} \cos(nx) = (2/d) \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}}$

$\cdot \cos(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}}), \quad x \neq \pm ad$

3.45 $\sum_{0}^{\infty} \epsilon_n J_0\{a[(nd)^2-b^2]^{\frac{1}{2}}\} \cos(nx) = (2/d) \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}}$

$\cdot \cosh(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}}), \quad x \neq \pm ad$

3.46 $\sum_{0}^{\infty} \epsilon_n \cos(\tfrac{1}{2}and) J_0\{\tfrac{1}{2}a[b^2+(nd)^2]^{\frac{1}{2}}\} \cos(nx) = d^{-\frac{1}{2}} \sum_{m_1}^{m_2} |(2\pi m+x)/d|^{-\frac{1}{2}}$

$\cdot [a-|(2\pi m+x)/d|]^{-\frac{1}{2}} \cos\{b|(2\pi m+x)/d|^{\frac{1}{2}}[a-|(2\pi m+x)/d|^{\frac{1}{2}}]^{\frac{1}{2}}\}, \quad x \neq 0, \pm ad$

3.47 $$\sum_{0}^{\infty} \epsilon_n (nd)^{\frac{1}{2}} J_\nu(\tfrac{1}{2}and) J_{-\nu-\frac{1}{2}}(\tfrac{1}{2}and) \cos(nx) = 2(\tfrac{1}{2}\pi nd)^{-\frac{1}{2}} \sum_{m_1}^{m_2} | 2\pi m + x |^{-\frac{1}{2}}$$
$$\cdot \{a^2 - [(2\pi m + x)/d]^2\}^{-\frac{1}{2}} \cos\{(2\nu + \tfrac{1}{2}) \operatorname{arc} \cos[(2\pi m + x)/ad]\}, \quad x \neq 0, \pm ad$$

3.48 $$\sum_{0}^{\infty} \epsilon_n [\mathbf{J}_\nu(a + nd) + \mathbf{J}_{-\nu}(a + nd)] \cos(nx) = 4d^{-1} \cos(\tfrac{1}{2}\pi\nu) \sum_{m_1}^{m_2} \{a^2 - [(2\pi m + x)/d]^2\}^{-\frac{1}{2}}$$
$$\cdot \cos\{\nu \operatorname{arc} \cos[(2\pi m + x)/ad]\}, \quad x \neq \pm ad$$

3.49 $$\sum_{0}^{\infty} \epsilon_n [\mathbf{J}_{and/\pi}(z) + \mathbf{J}_{-and/\pi}(z)] \cos(nx) = -2\pi (ad)^{-1} \sum_{m_1}^{m_2} \sin\{z \sin[(\pi/ad)(2\pi m + x)]\}$$

3.50 $$\sum_{0}^{\infty} \epsilon_n [\cos(and)]^{-1} [\mathbf{J}_{2and/\pi}(z) + \mathbf{J}_{-2and/\pi}(z)] \cos(nx)$$
$$= 2\pi (ad)^{-1} \sum_{m_1}^{m_2} \cos\{z \cos[(\pi/2ad)(2\pi m + x)]\}$$

3.51 $$\sum_{0}^{\infty} \epsilon_n [\cos(and)]^{-1} [E_{2and/\pi}(z) + E_{-2and/\pi}(z)] \cos(nx)$$
$$= -2\pi (ad)^{-1} \sum_{m_1}^{m_2} \sin\{z \cos[(\pi/2ad)(2\pi m + x)]\}$$

3.52 $$\sum_{0}^{\infty} \epsilon_n [E_{and/\pi}(z) + E_{-and/\pi}(z)] \cos(nx) = -2\pi (ad)^{-1} \sum_{m_1}^{m_2} \sin\{z \sin[(\pi/ad)(2\pi m + x)]\}$$

In Formulas 3.42–3.52 $m_1 = -[(ad + x)/2\pi]$; $m_2 = [(ad - x)/2\pi]$. If $(ad \pm x)/2\pi$ is an integer or zero, one half of the corresponding term in the right side sum has to be taken.

3.53 $$\sum_{1}^{\infty} (\tfrac{1}{2}nz)^{-\nu} \mathbf{H}_\nu(nz) \sin(nx) = 0, \qquad 0 < z < x$$
$$= [\pi^{\frac{1}{2}}/z\Gamma(\tfrac{1}{2} + \nu)][1 - (x^2/z^2)]^{\nu - \frac{1}{2}}, \quad x < z < \pi$$

3.54 $$\sum_{0}^{\infty} P_n(\cos\vartheta) \sin[(n + \tfrac{1}{2})x] = 0, \qquad x < \vartheta$$
$$= -2^{-\frac{1}{2}} (\cos\vartheta - \cos x)^{-\frac{1}{2}}, \quad 0 < \vartheta < x < \pi$$
$$0 < \vartheta < \pi$$

3.55 $\sum_0^\infty P_n(\cos\vartheta)\cos[(n+\frac{1}{2})x]=2^{-\frac{1}{2}}(\cos x-\cos\vartheta)^{-\frac{1}{2}}, \quad -\vartheta<x<\vartheta<\pi$

$=0, \quad \vartheta<x<2\pi-\vartheta$

$=-2^{-\frac{1}{2}}(\cos x-\cos\vartheta)^{-\frac{1}{2}}, \quad 2\pi-\vartheta<x<2\pi+\vartheta$

$=0, \quad 2\pi+\vartheta<x<4\pi-\vartheta$

$=2^{-\frac{1}{2}}(\cos x-\cos\vartheta)^{-\frac{1}{2}}, \quad 4\pi-\vartheta<x<4\pi+\vartheta$

$0<\vartheta<\pi$

3.56 $\sum_0^\infty \epsilon_n P_{n-\frac{1}{2}}(\cos\vartheta)\cos(nx)=2^{\frac{1}{2}}(\cos x-\cos\vartheta)^{-\frac{1}{2}}, \quad -\vartheta<x<\vartheta<\pi$

$=0, \quad \vartheta<x<2\pi-\vartheta$

$=2^{\frac{1}{2}}(\cos x-\cos\vartheta)^{-\frac{1}{2}}, \quad 2\pi-\vartheta<x<2\pi+\vartheta$

$0<\vartheta<\pi$

3.57 $\sum_0^\infty (-1)^n\epsilon_n P_{n-\frac{1}{2}}(-\cos\vartheta)\cos(nx)=0, \quad 0<x<\vartheta$

$=2^{\frac{1}{2}}(\cos\vartheta-\cos x)^{-\frac{1}{2}}, \quad \vartheta<x<\pi$

3.58 $\sum_0^\infty P_n(\cos\vartheta)\sin(nx)=-\sin(\frac{1}{2}x)(2\cos x-2\cos\vartheta)^{-\frac{1}{2}}, \quad x<\vartheta$

$=-\cos(\frac{1}{2}x)(2\cos\vartheta-2\cos x)^{-\frac{1}{2}}, \quad x>\vartheta$

3.59 $\sum_0^\infty P_n(\cos\vartheta)\cos(nx)=\cos(\frac{1}{2}x)(2\cos x-2\cos\vartheta)^{-\frac{1}{2}}, \quad x<\vartheta$

$=-\sin(\frac{1}{2}x)(2\cos\vartheta-2\cos x)^{-\frac{1}{2}}, \quad x>\vartheta$

3.60 $\sum_0^\infty \epsilon_n P_{-\frac{1}{2}+n\pi/\alpha}(\cos\vartheta)\cos(nx)=0, \quad (\pi/\alpha)\vartheta<x<2\pi-(\pi/\alpha)\vartheta$

$=2^{\frac{1}{2}}\pi^{-1}\alpha\{\cos[(\alpha/\pi)x]-\cos\vartheta\}^{-\frac{1}{2}}, \quad -(\pi/\alpha)\vartheta<x<(\pi/\alpha)\vartheta$

3.61 $\sum_0^\infty (-1)^n Q_n(z)\sin[(n+\frac{1}{2})x]=(2z+2\cos x)^{-\frac{1}{2}}\arctan\{[(1-\cos x)/(z+\cos x)]^{\frac{1}{2}}\}, \quad z>1$

3.62 $\sum_0^\infty Q_n(z)\cos[(n+\frac{1}{2})x]=(2z-2\cos x)^{-\frac{1}{2}}\arctan\{[(1+\cos x)/(z-\cos x)]^{\frac{1}{2}}\}, \quad z>1$

3.63 $\sum_0^\infty \epsilon_n Q_{n-\frac{1}{2}}(\cos\vartheta)\cos(nx)=\pi(2\cos\vartheta-2\cos x)^{-\frac{1}{2}}, \quad x>\vartheta$

$=0, \quad x<\vartheta$

3.64 $\sum_0^\infty \epsilon_n Q_{-\frac{1}{2}+\frac{1}{2}n}(z)\cos(nx) = [2/(z-\cos 2x)]^{\frac{1}{2}}(\frac{1}{2}\pi + \arctan\{[2/(z-\cos 2x)]^{\frac{1}{2}}\cos x\}), \quad z>1$

3.65 $\sum_0^\infty (-1)^n \epsilon_n Q_{-\frac{1}{2}+\frac{1}{2}n}(z)\cos(nx) = [2/(z-\cos 2x)]^{\frac{1}{2}}(\frac{1}{2}\pi - \arctan\{[2/(z-\cos 2x)]^{\frac{1}{2}}\cos x\})$

$z>1$

3.66 $\sum_0^\infty \epsilon_n (-1)^n Q_{n-\frac{1}{2}}(z)\cos(nx) = \pi(z+\cos x)^{-\frac{1}{2}}, \quad z>1$

3.67 $\sum_0^\infty \epsilon_n P^\mu_{-\frac{1}{2}+n(\pi/\vartheta)}(\cos\vartheta)\cos[n\pi(x/a)] = (\frac{1}{2}\pi)^{-\frac{1}{2}}\vartheta[\Gamma(\frac{1}{2}-\mu)]^{-1}(\sin\vartheta)^\mu$

$\cdot\{\cos[(\vartheta/a)x]-\cos\vartheta\}^{-\mu-\frac{1}{2}}, \quad \mathrm{Re}\mu<\frac{1}{2}, \quad 0<\vartheta<\pi, \quad -a<x<a$

3.68 $\sum_0^\infty \epsilon_n P^\mu_{-\frac{1}{2}+n(\pi/\vartheta)}(\cosh\vartheta)\cos[n\pi(x/a)] = (\frac{1}{2}\pi)^{-\frac{1}{2}}\vartheta[\Gamma(\frac{1}{2}-\mu)]^{-1}(\sinh\vartheta)^\mu$

$\cdot\{\cos[(\vartheta/a)x]-\cos\vartheta\}^{-\mu-\frac{1}{2}}, \quad \mathrm{Re}\mu<\frac{1}{2}, \quad \vartheta>0, \quad -a<x<a$

3.69 $\sum_0^\infty \epsilon_n P^\mu_{-\frac{1}{2}+n}(\cos\vartheta)\cos(nx) = [(2\pi)^{\frac{1}{2}}/\Gamma(\frac{1}{2}-\mu)](\sin\vartheta)^\mu(\cos x-\cos\vartheta)^{-\mu-\frac{1}{2}}, \quad 0<x<\vartheta$

$= 0, \quad \vartheta<x<\pi$

$0<\vartheta<\pi, \quad \mathrm{Re}\mu<\frac{1}{2}$

3.70 $\sum_0^\infty \epsilon_n (-1)^n P^\mu_{n-\frac{1}{2}}(-\cos\vartheta)\cos(nx)$

$= [(2\pi)^{\frac{1}{2}}/\Gamma(\frac{1}{2}-\mu)](\sin\vartheta)^\mu(\cos x-\cos\vartheta)^{-\mu-\frac{1}{2}}, \quad \vartheta<x<\pi$

$= 0, \quad 0<x<\vartheta$

$0<\vartheta<\pi, \quad \mathrm{Re}\mu<\frac{1}{2}$

3.71 $\sum_0^\infty P_n{}^\mu(\cos\vartheta)\cos[(n+\frac{1}{2})x] = [(\frac{1}{2}\pi)^{\frac{1}{2}}/\Gamma(\frac{1}{2}-\mu)](\sin\vartheta)^\mu(\cos x-\cos\vartheta)^{-\mu-\frac{1}{2}}, \quad 0<x<\vartheta$

$= 0, \quad \vartheta<x<\pi$

$0<\vartheta<\pi, \quad \mathrm{Re}\mu<\frac{1}{2}$

3.72 $\sum_0^\infty \epsilon_n\{\Gamma[\nu+(a/\pi)nd]\Gamma[\nu-(a/\pi)nd]\}^{-1}\cos(nx)$

$= 2^{2\nu-2}\pi[ad\Gamma(2\nu-1)]^{-1}\sum_{m_1}^{m_2}\{\cos[(\pi/2ad)(2\pi m+x)]\}^{2\nu-2}$

$\mathrm{Re}\nu>\frac{1}{2}, \quad \mathrm{Re}\nu>1 \text{ if } x=\pm ad$

3.73 $\sum_{0}^{\infty} \epsilon_n P^{\mu}_{-\frac{1}{2}+nd}(\cos a)\cos(nx) = [(2\pi)^{\frac{1}{2}}/d\Gamma(\frac{1}{2}-\mu)](\sin a)^{\mu}$

$\cdot \sum_{m_1}^{m_2} \{\cos[(2\pi m+x)/d] - \cos a\}^{-\mu-\frac{1}{2}}, \quad 0<a<\pi, \quad \mathrm{Re}\mu<\frac{1}{2}, \quad \mathrm{Re}\mu<-\frac{1}{2} \text{ if } x=\pm ad$

3.74 $\sum_{0}^{\infty} \epsilon_n P^{\mu}_{-\frac{1}{2}+ind}(\cosh a)\cos(nx) = [(2\pi)^{\frac{1}{2}}/d\Gamma(\frac{1}{2}-\mu)](\sinh a)^{\mu}$

$\cdot \sum_{m_1}^{m_2} \{\cosh a - \cosh[(2\pi m+x)/d]\}^{-\mu-\frac{1}{2}}, \quad \mathrm{Re}\mu<\frac{1}{2}, \quad \mathrm{Re}\mu<-\frac{1}{2} \text{ if } x=\pm ad$

3.75 $\sum_{0}^{\infty} \epsilon_n Q^{\mu}_{n-\frac{1}{2}}(z)\cos(nx) = \exp(i\pi\mu)(\frac{1}{2}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{2}+\mu)(z^2-1)^{\frac{1}{2}\mu}(z-\cos x)^{-\mu-\frac{1}{2}}, \quad z>1, \quad \mathrm{Re}\mu>-\frac{1}{2}$

3.76 $\sum_{0}^{\infty} \epsilon_n Q^{\mu}_{-\frac{1}{2}+nm}(z)\cos(nx) = \exp(i\pi\mu)(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\mu)(z^2-1)^{\frac{1}{2}\mu}$

$\cdot(\pi/m)\sum_{r_1}^{r_2}\{z-\cos[(2\pi r+x)/m]\}^{-\mu-\frac{1}{2}}$

$\mathrm{Re}\mu>-\frac{1}{2}, \quad m=1,2,3,\ldots, \quad r_1=-[(\pi m+x)/2\pi], \quad r_2=[(\pi m-x)/2\pi], \quad z>1$

If $(\pi m\pm x)/2\pi$ is an integer or zero, one half of the corresponding term at the right side has to be taken.

In 3.72–3.74, $m_1=-[(ad+x)/2\pi]$, $m_2=[(ad-x)/2\pi]$. If $(ad\pm x)/2\pi$ is an integer or zero, one half of the corresponding term at the right side has to be taken.

3.77 $\sum_{0}^{\infty}(-1)^n\epsilon_n\Gamma(\nu+2n+1)P_{\nu}^{-2n}(y)\cos(nx) = \Gamma(1+\nu)[y^2+(1-y^2)\cos^2 x]^{-\frac{1}{2}\nu-\frac{1}{2}}$

$\cdot\cos\{(\nu+1)\arctan[(y^{-2}-1)^{\frac{1}{2}}\cos x]\}, \quad 0<y<1, \quad \mathrm{Re}\nu>-1$

3.78 $\sum_{0}^{\infty}(-1)^n\Gamma(\nu+2n+2)P_{\nu}^{-2n-1}(y)\cos[(2n+1)x] = \frac{1}{2}\Gamma(\nu+1)[y^2+(1-y^2)\cos^2 x]^{-\frac{1}{2}\nu-\frac{1}{2}}$

$\cdot\sin\{(\nu+1)\arctan[(y^{-2}-1)^{\frac{1}{2}}\cos x\}, \quad 0<y<1, \quad \mathrm{Re}\nu>-1$

3.79 $\sum_{0}^{\infty}(-1)^n\epsilon_n Q^{\mu}_{n-\frac{1}{2}}(z)[\Gamma(\frac{1}{2}+n+\mu)\Gamma(\frac{1}{2}-n+\mu)]^{-1}\cos(2nx)$

$=[(\frac{1}{2}\pi)^{\frac{1}{2}}/\Gamma(\frac{1}{2}+\mu)](z^2-1)^{-\frac{1}{2}\mu}\exp(i\pi\mu)(1+z-2\cos^2 x)^{\mu-\frac{1}{2}}, \quad z>1$

3.80 $\sum_{0}^{\infty}[\epsilon_n(-i)^n/(n+m)!]P_l^n(\cos\vartheta)\cos(nx) = (l!)^{-1}(\cos\vartheta+i\sin\vartheta\cos x)^l, \quad l=1,2,3,\ldots$

3.81 $$\sum_0^\infty [\epsilon_n(-i)^n/\Gamma(\nu+n+1)]P_\nu^n(\cos\vartheta)=[\Gamma(\nu+1)]^{-1}(\cos\vartheta+i\sin\vartheta\cos x)^\nu,\quad \mathrm{Re}\,\nu>-1$$

3.82 $$\sum_0^\infty [\epsilon_n/\Gamma(\nu+n+1)]P_\nu^n(\cosh z)\cos(nx)=[\Gamma(\nu+1)]^{-1}(\cosh z+\sinh z\cos x)^\nu,\quad \mathrm{Re}\,\nu>-1$$

3.83 $$\begin{aligned}\sum_0^\infty C_n^\nu(\cos\vartheta)\sin(nx)&=-\sin(\nu x)(2\cos x-2\cos\vartheta)^{-\nu}, && x<\vartheta\\ &=-\sin[\nu(x+\pi)](2\cos\vartheta-2\cos x)^{-\nu}, && x>\vartheta\\ &\qquad 0<\vartheta<\pi,\quad \mathrm{Re}\,\nu<1\end{aligned}$$

3.84 $$\begin{aligned}\sum_0^\infty C_n^\nu(\cos\vartheta)\cos(nx)&=\cos(\nu x)(2\cos x-2\cos\vartheta)^{-\nu}, && x<\vartheta\\ &=\cos[\nu(x+\pi)](2\cos\vartheta-2\cos x)^{-\nu}, && x>\vartheta\\ &\qquad 0<x<\vartheta,\quad \mathrm{Re}\,\nu<1\end{aligned}$$

3.85 $$\sum_0^\infty \epsilon_n A_n\cos(nx)=(1-2z\cos x+z^2)^{-\nu}$$

$$A_n=(z^n/n!)[\Gamma(n+\nu)/\Gamma(\nu)]{}_2F_1(\nu,n+\nu;n+1;z^2),\quad |z|<1$$

IV

FOURIER SERIES WITH HIGHER FUNCTION COEFFICIENTS REPRESENTING HIGHER FUNCTIONS

4.1 $\sum_{0}^{\infty} \epsilon_n[\Gamma(\frac{1}{4}+\frac{1}{2}n)/\Gamma(\frac{3}{4}+\frac{1}{2}n)]^2 \cos(nx) = 4K(\cos\frac{1}{2}x), \quad -\pi<x<\pi$

4.2 $\sum_{0}^{\infty} \epsilon_n(-1)^n[\Gamma(\frac{1}{4}+\frac{1}{2}n)/\Gamma(\frac{3}{4}+\frac{1}{2}n)]^2 \cos(nx) = 4K(\sin\frac{1}{2}x), \quad 0<x<2\pi$

4.3 $\sum_{0}^{\infty} \epsilon_n(-1)^n[\Gamma(\frac{1}{4}+\frac{1}{2}n)\Gamma(\frac{1}{4}-\frac{1}{2}n)/\Gamma(\frac{3}{4}+\frac{3}{2}n)\Gamma(\frac{3}{4}-\frac{3}{2}n)] \cos(nx) = 4K(\cos\frac{1}{2}x), \quad 0<x<\pi$

4.4 $\sum_{0}^{\infty} \epsilon_n[\Gamma(\frac{1}{4}+\frac{1}{2}n)\Gamma(\frac{1}{4}-\frac{1}{2}n)/\Gamma(\frac{3}{4}+\frac{1}{2}n)\Gamma(\frac{3}{4}-\frac{1}{2}n)] \cos(nx) = 4K(\sin\frac{1}{2}x), \quad 0<x<2\pi$

4.5 $\sum_{1}^{\infty} (-1)^n n^{-1}[\sin(n\pi s)\ \mathrm{Ci}(n\pi s) - \cos(n\pi s)\ \mathrm{si}(n\pi s)] \sin(nx)$
$= -\frac{1}{4}\{\psi[\frac{1}{2}+\frac{1}{2}s+(x/2\pi)] - \psi[\frac{1}{2}+\frac{1}{2}s-(x/2\pi)]\}, \quad -\pi\le x\le\pi, \quad s>0$

4.6 $$\sum_0^\infty (-1)^n\{\cos[(n+\tfrac{1}{2})\pi s]\,\mathrm{Ci}[(n+\tfrac{1}{2})\pi s]+\sin[(n+\tfrac{1}{2})\pi s]\,\mathrm{si}[(n+\tfrac{1}{2})\pi s]\}\,\sin[(n+\tfrac{1}{2})x]$$
$$=-\tfrac{1}{8}\{\psi[\tfrac{1}{4}+\tfrac{1}{4}s+(x/4\pi)]-\psi[\tfrac{1}{4}+\tfrac{1}{4}s-(x/4\pi)]-\psi[\tfrac{3}{4}+\tfrac{1}{4}s+(x/4\pi)]$$
$$+\psi[\tfrac{3}{4}+\tfrac{1}{4}s-(x/4\pi)]\},\quad s>0,\quad -\pi\le x\le\pi$$

4.7 $$\sum_0^\infty (-1)^n\{\sin[(n+\tfrac{1}{2})\pi s]\,\mathrm{Ci}[(n+\tfrac{1}{2})\pi s]-\cos[(n+\tfrac{1}{2})\pi s]\,\mathrm{si}[(n+\tfrac{1}{2})\pi s]\}\,\cos[(n+\tfrac{1}{2})x]$$
$$=\tfrac{1}{8}\{\psi[\tfrac{3}{4}+\tfrac{1}{4}s+(x/4\pi)]+\psi[\tfrac{3}{4}+\tfrac{1}{4}s-(x/4\pi)]-\psi[\tfrac{1}{4}+\tfrac{1}{4}s+(x/4\pi)]$$
$$-\psi[\tfrac{1}{4}+\tfrac{1}{4}s-(x/4\pi)]\},\quad -\pi\le x\le\pi,\quad s>0$$

4.8 $$\sum_0^\infty \epsilon_n J_n(z)\,\cos(nx)=\cos(z\,\sin x)+z\,\cos x\int_0^1\cos(tz\,\sin x)J_0(zt)\,dt$$

4.9 $$\sum_1^\infty J_n(z)\,\sin(nx)=\tfrac{1}{2}\sin(2\,\sin x)-\tfrac{1}{2}z\,\cos x\int_0^1\sin(tz\,\sin x)J_0(zt)\,dt$$

4.10 $$\sum_0^\infty(-1)^n J_{2n+\frac{1}{2}}(z)\,\cos[(2n+\tfrac{1}{2})x]=2^{-\frac{1}{2}}[\cos(z\,\cos x)C(2z\,\cos^2\tfrac{1}{2}x)+\sin(z\,\cos x)S(2z\,\cos^2\tfrac{1}{2}x)]$$

4.11 $$\sum_0^\infty(-1)^n J_{2n+\frac{3}{2}}(z)\,\cos[(2n+\tfrac{3}{2})x]=2^{-\frac{1}{2}}[\sin(z\,\cos x)C(2z\,\cos^2\tfrac{1}{2}x)-\cos(z\,\cos x)S(2z\,\cos^2\tfrac{1}{2}x)]$$

4.12 $$\sum_1^\infty I_n(z)\,\sin(nx)=\tfrac{1}{2}z\,\sin x\int_0^1\exp(-zt\,\cos x)\,I_0(zt)\,dt$$

4.13 $$\sum_0^\infty \epsilon_n J_0^2(\tfrac{1}{2}n\pi)\,\cos[n\pi(x/a)]=4\pi^{-2}K'(x/a),\quad 0<x<a$$

4.14 $$\sum_0^\infty \epsilon_n J_\nu(\tfrac{1}{2}n\pi)J_{-\nu}(\tfrac{1}{2}n\pi)\,\cos[n\pi(x/a)]=2\pi^{-1}P_{\nu-\frac{1}{2}}[2(x^2/a^2)-1],\quad -a<x<a$$

4.15 $$\sum_0^\infty \epsilon_n\{b^2+[n(\pi/a)]^2\}^{-\frac{1}{2}\nu}J_\nu(a\{b^2+[n(\pi/a)]^2\}^{\frac{1}{2}})\,\cos[n\pi(x/a)]$$
$$=(2/\pi a)^{\frac{1}{2}}(ab)^{\frac{1}{2}-\nu}(a^2-x^2)^{\frac{1}{2}\nu-\frac{1}{4}}J_{\nu-\frac{1}{2}}[b(a^2-x^2)^{\frac{1}{2}}],\quad \mathrm{Re}\,\nu>-\tfrac{1}{2},\quad -a<x<a$$

4.16 $$\sum_0^\infty \epsilon_n J_\nu(\tfrac{1}{2}a\{[b^2+n^2(\pi^2/a^2)]^{\frac{1}{2}}+n(\pi/a)\})J_\nu(\tfrac{1}{2}a\{[b^2+n^2(\pi^2/a^2)]^{\frac{1}{2}}-n(\pi/a)\})\,\cos[n\pi(x/a)]$$
$$=2a\pi^{-1}(a^2-x^2)^{-\frac{1}{2}}J_{2\nu}[b(a^2-x^2)^{\frac{1}{2}}],\quad \mathrm{Re}\,\nu>-\tfrac{1}{2},\quad -a<x<a$$

4.17 $$\sum_{0}^{\infty} (-1)^n \epsilon_n J_\nu(\tfrac{1}{4}a\{[b^2+4n^2(\pi^2/a^2)]^{\frac{1}{2}}+2n(\pi/a)\}) \cdot J_\nu(\tfrac{1}{4}a\{[b^2+4n^2(\pi^2/a^2)]^{\frac{1}{2}}-2n(\pi/a)\}) \cos[2n\pi(x/a)]$$
$$= a\pi^{-1}(ax-x^2)^{-\frac{1}{2}} J_{2\nu}[b(a^2-x^2)^{\frac{1}{2}}], \quad \mathrm{Re}\nu > -\tfrac{1}{2}, \quad -a<x<a$$

4.18 $$\sum_{0}^{\infty} (-1)^n J_\nu(\tfrac{1}{4}a\{[b^2+(2n+1)^2(\pi^2/a^2)]^{\frac{1}{2}}+(2n+1)(\pi/a)\}) \cdot J_\nu(\tfrac{1}{4}a\{[b^2+(2n+1)^2(\pi^2/a^2)]^{\frac{1}{2}}-(2n+1)(\pi/a)\}) \sin[(2n+1)\pi(x/a)]$$
$$= \tfrac{1}{2}a\pi^{-1}(ax-x^2)^{-\frac{1}{2}} J_{2\nu}[b(a^2-x^2)], \quad \mathrm{Re}\nu > -\tfrac{1}{2}, \quad 0<x<a$$

4.19 $$\sum_{0}^{\infty} \epsilon_n I_{\frac{1}{2}n}(z) \cos(nx) = \exp(z\cos 2x)\{1+\mathrm{Erf}[(2z)^{\frac{1}{2}}\cos x]\}$$
$$= \exp(z\cos 2x)\{2-\mathrm{Erfc}[(2z)^{\frac{1}{2}}\cos x]\}$$

4.20 $$\sum_{0}^{\infty} (-1)^n \epsilon_n I_{\frac{1}{2}n}(z) \cos(nx) = \exp(z\cos 2x)\ \mathrm{Erfc}[(2z)^{\frac{1}{2}}\cos x]$$
$$= \exp(z\cos 2x)\{1-\mathrm{Erf}[(2z)^{\frac{1}{2}}\cos x]\}$$

4.21 $$\sum_{0}^{\infty} I_{n+\frac{1}{2}}(z) \cos[(2n+1)x] = \tfrac{1}{2}\exp(z\cos 2x)\ \mathrm{Erf}[(2z)^{\frac{1}{2}}\cos x]$$

4.22 $$\sum_{0}^{\infty} (-1)^n I_{n+\frac{1}{2}}(z) \sin[(2n+1)x] = \tfrac{1}{2}\exp(-z\cos 2x)\ \mathrm{Erf}[(2z)^{\frac{1}{2}}\sin x]$$

4.23 $$\sum_{0}^{\infty} (-1)^n I_{n+\frac{1}{2}}(z) \cos[(2n+1)x] = -\tfrac{1}{2}i\exp(-z\cos 2x)\ \mathrm{Erf}[i(2z)^{\frac{1}{2}}\cos x]$$

4.24 $$\sum_{0}^{\infty} I_{n+\frac{1}{2}}(z) \sin[(2n+1)x] = -\tfrac{1}{2}i\exp(z\cos 2x)\ \mathrm{Erf}[i(2z)^{\frac{1}{2}}\sin x]$$

4.25 $$\sum_{0}^{\infty} \epsilon_n J_{n-\frac{1}{2}}(z) J_{n+\frac{1}{2}}(z) \cos(2nx)$$
$$= 2(2\pi z\sin x)^{-\frac{1}{2}}[\cos(2z\sin x)\,C(2z\sin x)+\sin(2z\sin x)\,S(2z\sin x)]$$

4.26 $$\sum_{0}^{\infty} (-1)^n \epsilon_n J_{n-\frac{1}{2}}(z) J_{n+\frac{1}{2}}(z) \cos(2nx)$$
$$= 2(2\pi z\cos x)^{-\frac{1}{2}}[\cos(2z\cos x)\,C(2z\cos x)+\sin(2z\cos x)\,S(2z\cos x)]$$

4.27 $$\sum_{0}^{\infty} (-1)^n J_{n+\frac{1}{4}}(z) J_{n+\frac{3}{4}}(z) \cos[(2n+1)x]$$
$$= (2\pi z \cos x)^{-\frac{1}{2}}[\sin(2z \cos x)\, C(2z \cos x) - \cos(2z \cos x)\, S(2z \cos x)]$$

4.28 $$\sum_{0}^{\infty} J_{n+\frac{1}{4}}(z) J_{n+\frac{3}{4}}(z) \sin[(2n+1)x]$$
$$= (2\pi z \sin x)^{-\frac{1}{2}}[\sin(2z \sin x)\, C(2z \sin x) - \cos(2z \sin x)\, S(2z \sin x)]$$

4.29 $$\sum_{0}^{\infty} \epsilon_n J_n{}^2(z) \cos(nx) = J_0(2z \sin\tfrac{1}{2}x)$$

4.30 $$\sum_{0}^{\infty} \epsilon_n J_n(z)\, Y_n(z) \cos(nx) = Y_0(2z\, |\sin\tfrac{1}{2}x\,|)$$

4.31 $$\sum_{0}^{\infty} \epsilon_n (-1)^n [I_n(z)]^2 \cos(nx) = I_0(2z \sin\tfrac{1}{2}x)$$

4.32 $$\sum_{0}^{\infty} \epsilon_n I_n(z)\, K_n(z) \cos(nx) = K_0(2z\, |\sin\tfrac{1}{2}x\,|)$$

4.33 $$\sum_{0}^{\infty} J_n(z) J_{n+1}(z) \sin[(2n+1)x] = \tfrac{1}{2}J_1(2z \sin x)$$

4.34 $$\sum_{0}^{\infty} (-1)^n J_n(z) J_{n+1}(z) \cos[(2n+1)x] = \tfrac{1}{2}J_1(2z \cos x)$$

4.35 $$\sum_{0}^{\infty} (-1)^n [J_{n+\frac{1}{2}}(z)]^2 \cos[(2n+1)x] = \tfrac{1}{2}\mathbf{H}_0(2z \cos x)$$

4.36 $$\sum_{0}^{\infty} [J_{n+\frac{1}{2}}(z)]^2 \sin[(2n+1)x] = -\tfrac{1}{2}\mathbf{H}_0(2z \sin x)$$

4.37 $$\sum_{0}^{\infty} (-1)^{n+1} J_{n+\frac{1}{2}}(z)\, Y_{n+\frac{1}{2}}(z) \cos[(2n+1)x] = \tfrac{1}{2}J_0(2z \cos x)$$

4.38 $$\sum_{0}^{\infty} J_{n+\frac{1}{2}}(z)\, Y_{n+\frac{1}{2}}(z) \sin[(2n+1)x] = -\tfrac{1}{2}J_0(2z \sin x)$$

4.39 $$\sum_0^\infty \epsilon_n J_n(z_1) J_n(z_2) \cos(nx) = J_0[(z_1^2+z_2^2-2z_1z_2\cos x)^{\frac{1}{2}}]$$

4.40 $$\sum_0^\infty \epsilon_n J_n(z_1) Y_n(z_2) \cos(nx) = Y_0[(z_1^2+z_2^2-2z_1z_2\cos x)^{\frac{1}{2}}], \quad z_2>z_1$$

4.41 $$\sum_0^\infty (-1)^n \epsilon_n I_n(z_1) I_n(z_2) \cos(nx) = I_0[(z_1^2+z_2^2-2z_1z_2\cos x)^{\frac{1}{2}}]$$

4.42 $$\sum_0^\infty \epsilon_n I_n(z_1) K_n(z_2) \cos(nx) = K_0[(z_1^2+z_2^2-2z_1z_2\cos x)^{\frac{1}{2}}], \quad z_2>z_1$$

4.43 $$\sum_0^\infty (-1)^n \epsilon_n I_{nm}(z_1) I_{nm}(z_2) \cos(nx) = m^{-1}\sum_{r_1}^{r_2} I_0(\{z_1^2+z_2^2-2z_1z_2\cos[(2\pi r+x)/m]\}^{\frac{1}{2}})$$

4.44 $$\sum_0^\infty \epsilon_n J_{nm}(z_1) J_{nm}(z_2) \cos(nx) = m^{-1}\sum_{r_1}^{r_2} J_0(\{z_1^2+z_2^2-2z_1z_2\cos[(2\pi r+x)/m]\}^{\frac{1}{2}})$$

4.45 $$\sum_0^\infty \epsilon_n J_{nm}(z_1) Y_{nm}(z_2) \cos(nx) = m^{-1}\sum_{r_1}^{r_2} Y_0(\{z_1^2+z_2^2-2z_1z_2\cos[(2\pi r+x)/m]\}^{\frac{1}{2}})$$

4.46 $$\sum_0^\infty \epsilon_n I_{nm}(z_1) K_{nm}(z_2) \cos(nx) = m^{-1}\sum_{r_1}^{r_2} K_0(\{z_1^2+z_2^2-2z_1z_2\cos[(2\pi r+x)/m]\}^{\frac{1}{2}})$$

In the last four formulas $m=1, 2, 3, \ldots$, $r_1=-[(m\pi+x)/2\pi]$, $r_2=[(m\pi-x)/2\pi]$. If $(m\pi\pm x)/2\pi$ is an integer or zero, one half of the corresponding term in the sum at the right side has to be taken.

4.47 $$\sum_0^\infty J_{n+\frac{1}{2}}(az) J_{n+\frac{1}{2}}(bz) \cos[(n+\tfrac{1}{2})x] = \pi^{-1}\int_{t_1}^{t_2} [t^2-(a^2+b^2-2ab\cos x)]^{-\frac{1}{2}} \sin(tz)\, dt$$

$$t_1=(a^2+b^2-2ab\cos x)^{\frac{1}{2}}, \quad t_2=a+b$$

4.48 $$\sum_0^\infty (-1)^{n+1} J_{n+\frac{1}{2}}(az) Y_{n+\frac{1}{2}}(bz) \cos[(n+\tfrac{1}{2})x]$$

$$=\pi^{-1}\int_{t_1}^{t_2} [(a^2+b^2+2ab\cos x)-t^2]^{-\frac{1}{2}} \cos(tz)\, dt$$

$$t_1=b-a, \quad t_2=(a^2+b^2+2ab\cos x)^{\frac{1}{2}}; \quad b>a$$

4.49 $$\sum_{0}^{\infty} (-1)^n \epsilon_n J_{2n}(z) K_{2n}(z) \cos(2nx) = \tfrac{1}{2}\{K_0[z(2i\cos x)^{\frac{1}{2}}] + K_0[z(-2i\cos x)^{\frac{1}{2}}]\}$$

4.50 $$\sum_{0}^{\infty} (-1)^n J_{2n+1}(z) K_{2n+1}(z) \cos[(2n+1)x] = \tfrac{1}{4} i\{K_0[z(2i\cos x)^{\frac{1}{2}}] - K_0[z(-2i\cos x)^{-\frac{1}{2}}]\}$$

4.51 $$\sum_{0}^{\infty} \epsilon_n J_{2n}(z) K_{2n}(z) \cos(2nx) = \tfrac{1}{2}\{K_0[z(2i\sin x)^{\frac{1}{2}}] + K_0[z(-2i\sin x)^{\frac{1}{2}}]\}$$

4.52 $$\sum_{0}^{\infty} J_{2n+1}(z) K_{2n+1}(z) \sin[(2n+1)x] = \tfrac{1}{4} i\{K_0[z(2i\sin x)^{\frac{1}{2}}] - K_0[z(-2i\sin x)^{\frac{1}{2}}]\}$$

4.53 $$\sum_{0}^{\infty} (-1)^n I_{n+\frac{1}{2}}(z) K_{n+\frac{1}{2}}(z) \cos[(2n+1)x] = \tfrac{1}{4}\pi[I_0(2z\cos x) - L_0(2z\cos x)]$$

4.54 $$\sum_{0}^{\infty} I_{n+\frac{1}{2}}(x) K_{n+\frac{1}{2}}(x) \sin[(2n+1)x] = \tfrac{1}{4}\pi[I_0(2z\sin x) - L_0(2z\sin x)]$$

4.55 $$\sum_{1}^{\infty} J_n(z) J_{n+2m}(z) \cos[2x(n+m)] = \tfrac{1}{2}(-1)^m J_{2m}(2z\sin x) - \tfrac{1}{2}(-1)^m$$
$$\cdot \sum_{k=0}^{m} \epsilon_k (-1)^k J_{m-k}(z) J_{m+k}(z) \cos(2kx), \quad m = 0, 1, 2, \ldots$$

4.56 $$\sum_{1}^{\infty} (-1)^n J_n(z) J_{n+2m}(z) \cos[2x(n+m)]$$
$$= \tfrac{1}{2} J_{2m}(2z\cos x) - \tfrac{1}{2} \sum_{k=0}^{m} \epsilon_k J_{m-k}(z) J_{m+k}(z) \cos(2kx), \quad m = 0, 1, 2, \ldots$$

4.57 $$\sum_{0}^{\infty} (-1)^n J_n(z) J_{n+2m+1}(z) \cos[x(2n+2m+1)]$$
$$= \tfrac{1}{2} J_{2m+1}(2z\cos x) - \sum_{k=0}^{m-1} J_{m-k}(z) J_{m+k+1}(z) \cos[(2k+1)x]$$
$$m = 0, 1, 2, \ldots; \quad \sum_{k=0}^{m-1} (\ ) = 0 \quad \text{if} \quad m = 0$$

4.58 $\sum_{0}^{\infty} J_n(z)J_{n+2m+1}(z)\sin[x(2n+2m+1)]$

$= \frac{1}{2}(-1)^m J_{2m+1}(2z\sin x) - (-1)^m \sum_{k=0}^{m-1} (-1)^k J_{m-k}(z)J_{m+k+1}(z)\sin[(2k+1)x]$

$m=0,1,2,\ldots;\quad \sum_{k=0}^{m-1}(\ )=0 \quad \text{if} \quad m=0$

4.59 $\sum_{0}^{\infty} (-1)^n \epsilon_n J_{n-\nu}(z)J_{n+\nu}(z)\cos(2nx) = [2\cos(\pi\nu)]^{-1}[\mathbf{J}_{2\nu}(2z\cos x)+\mathbf{J}_{-2\nu}(2z\cos x)]$

$= [2\sin(\pi\nu)]^{-1}[E_{2\nu}(2z\cos x) - E_{-2\nu}(2z\cos x)]$

4.60 $\sum_{0}^{\infty} (-1)^n J_{n+\frac{1}{2}+\nu}(z)J_{n+\frac{1}{2}-\nu}(z)\cos[(2n+1)x]$

$= [4\sin(\pi\nu)]^{-1}[\mathbf{J}_{2\nu}(2z\cos x) - \mathbf{J}_{-2\nu}(2z\cos x)]$

$= -[4\cos(\pi\nu)]^{-1}[E_{2\nu}(2z\cos x) + E_{-2\nu}(2z\cos x)]$

4.61 $\sum_{0}^{\infty} (-1)^n \epsilon_n [J_{n+\frac{1}{2}\nu}(z)Y_{n-\frac{1}{2}\nu}(z) + J_{n-\frac{1}{2}\nu}(z)Y_{n+\frac{1}{2}\nu}(z)]\cos(2nx)$

$= 2\sin(\frac{1}{2}\pi\nu)J_\nu(2z\cos x) + 2\cos(\frac{1}{2}\pi\nu)Y_\nu(2z\cos x), \quad \text{Re}\nu > -1$

4.62 $\sum_{0}^{\infty} (-1)^{n+1}[J_{n+\frac{1}{2}+\frac{1}{2}\nu}(z)Y_{n+\frac{1}{2}-\frac{1}{2}\nu}(z) + J_{n+\frac{1}{2}-\frac{1}{2}\nu}(z)Y_{n+\frac{1}{2}+\frac{1}{2}\nu}(z)]\cos[(2n+1)x]$

$= \cos(\frac{1}{2}\pi\nu)J_\nu(2z\cos x) - \sin(\frac{1}{2}\pi\nu)Y_\nu(2z\cos x), \quad \text{Re}\nu > -1$

4.63 $\sum_{0}^{\infty} (-1)^n \epsilon_n J_{\nu-n}(z)J_{\nu+n}(z)\cos(2nx) = J_{2\nu}(2z\,|\sin x|), \quad \text{Re}\nu > -\frac{1}{2}$

4.64 $\sum_{0}^{\infty} \epsilon_n J_{\nu-n}(z)J_{\nu+n}(z)\cos(2nx) = J_{2\nu}(2z\cos x), \quad \text{Re}\nu > -\frac{1}{2}$

4.65 $\sum_{0}^{\infty} (-1)^n J_{\nu-n-\frac{1}{2}}(z)J_{\nu+n+\frac{1}{2}}(z)\sin[(2n+1)x] = \frac{1}{2}J_{2\nu}(2z\,|\sin x|), \quad \text{Re}\nu > -\frac{1}{2}$

4.66 $\sum_{0}^{\infty} J_{\nu-n-\frac{1}{2}}(z)J_{\nu+n+\frac{1}{2}}(z)\cos[(2n+1)x] = \frac{1}{2}J_{2\nu}(2z\cos x), \quad \text{Re}\nu > -\frac{1}{2}$

4.67 $\sum_{0}^{\infty} (-1)^n \epsilon_n I_{\nu-n}(z)I_{\nu+n}(z)\cos(2nx) = I_{2\nu}(2z\,|\sin x|), \quad \text{Re}\nu > -\frac{1}{2}$

4.68 $\sum_0^\infty \epsilon_n I_{\nu-n}(z) I_{\nu+n}(z) \cos(2nx) = I_{2\nu}(2z \cos x), \quad \operatorname{Re}\nu > -\frac{1}{2}$

4.69 $\sum_0^\infty (-1)^n I_{\nu-n-\frac{1}{2}}(z) I_{\nu+n+\frac{1}{2}}(z) \sin[(2n+1)x] = \frac{1}{2} I_{2\nu}(2z \sin x), \quad \operatorname{Re}\nu > -\frac{1}{2}$

4.70 $\sum_0^\infty I_{\nu-n-\frac{1}{2}}(z) I_{\nu+n+\frac{1}{2}}(z) \cos[(2n+1)z] = \frac{1}{2} I_{2\nu}(2z \cos x), \quad \operatorname{Re}\nu > -\frac{1}{2}$

4.71 $\sum_0^\infty (-1)^n \epsilon_n I_{n-\frac{1}{2}\nu}(z) K_{n+\frac{1}{2}\nu}(z) \cos(2nx)$

$= -[\pi/\sin(\pi\nu)][\frac{1}{2}\mathbf{J}_{-\nu}(2iz \cos x) + \frac{1}{2}\mathbf{J}_{-\nu}(-2iz \cos x) - \cos(\frac{1}{2}\pi\nu) I_{-\nu}(2z \cos x)], \quad \operatorname{Re}\nu < 1$

4.72 $\sum_0^\infty (-1)^n I_{n+\frac{1}{2}+\frac{1}{2}\nu}(z) K_{n+\frac{1}{2}-\frac{1}{2}\nu}(z) \cos[(2n+1)x] = [\pi/2 \sin(\pi\nu)][\frac{1}{2} i \mathbf{J}_\nu(2iz \cos x)$

$-\frac{1}{2} i \mathbf{J}_\nu(-2iz \cos x) + \sin(\frac{1}{2}\pi\nu) I_\nu(2z \cos x)], \quad \operatorname{Re}\nu > -1$

4.73 $\sum_0^\infty (-1)^n \epsilon_n [I_{n+\nu}(z) K_{n-\nu}(z) + I_{n-\nu}(z) K_{n+\nu}(z)] \cos(2nx) = 2 \cos(\pi\nu) K_{2\nu}(2z \cos x)$

$\operatorname{Re}\nu > -\frac{1}{2}$

4.74 $\sum_0^\infty (-1)^n [I_{n+\frac{1}{2}-\nu}(z) K_{n+\frac{1}{2}+\nu}(z) - I_{n+\frac{1}{2}+\nu}(z) K_{n+\frac{1}{2}-\nu}(z)] \cos[(2n+1)x]$

$= 2 \sin(\pi\nu) K_{2\nu}(2z \cos x), \quad \operatorname{Re}\nu > -\frac{1}{2}$

4.75 $\sum_0^\infty \epsilon_n (nd)^{-\nu} J_{\nu+2l}(nda) \cos(nx)$

$= [(-1)^l (\frac{1}{2}a)^{-\nu} (2l)!/d\Gamma(2l+2\nu)] \sum_{m_1}^{m_2} \{a^2 - [(2\pi m + x)/d]^2\}^{\nu-\frac{1}{2}} C_{2l}^\nu [(2\pi m + x)/ad]$

$l = 0, 1, 2, \ldots; \quad \operatorname{Re}\nu > -\frac{1}{2}; \quad \operatorname{Re}\nu > \frac{1}{2} \quad \text{if} \quad x = \pm ad$

4.76 $\sum_0^\infty \epsilon_n [J_0(\frac{1}{2}and)]^2 \cos(nx) = (4/\pi ad) \sum_{m_1}^{m_2} K'[|(2\pi m + x)/ad|]$

4.77 $\sum_0^\infty \epsilon_n J_\nu(\frac{1}{2}and) J_{-\nu}(\frac{1}{2}and) \cos(nx) = (2/ad) \sum_{m_1}^{m_2} P_{\nu-\frac{1}{2}}[(2/a^2d^2)(2\pi m + x)^2 - 1]$

4.78 $$\sum_{0}^{\infty} \epsilon_n [b^2+(nd)^2]^{-\frac{1}{2}\nu} J_\nu\{a[b^2+(nd)^2]^{\frac{1}{2}}\} \cos(nx)$$
$$= (2\pi b^{\frac{1}{2}}/d)(ab)^{-\nu} \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > \frac{1}{2}$ if $x = \pm ad$

4.79 $$\sum_{0}^{\infty} \epsilon_n T_{2l}\{nd/[b^2+(nd)^2]^{\frac{1}{2}}\} J_{2l}\{a[b^2+(nd)^2]^{\frac{1}{2}}\} \cos(nx)$$
$$= (-1)^l (2/d) \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} \cos(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$$\cdot T_{2l}[(2\pi m+x)/ad], \quad l = 0, 1, 2, \ldots; \quad x \neq ad$$

4.80 $$\sum_{0}^{\infty} \epsilon_n T_{2l}\{b[b^2+(nd)^2]^{-\frac{1}{2}}\} J_{2l}\{a[b^2+(nd)^2]^{\frac{1}{2}}\} \cos(nx)$$
$$= (-1)^l (2/d) \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} \cos(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$$\cdot T_{2l}(\{1-[(2\pi m+x)/ad]^2\}^{\frac{1}{2}}), \quad l = 0, 1, 2, \ldots; \quad x \neq ad$$

4.81 $$\sum_{0}^{\infty} \epsilon_n T_{2l+1}\{b[b^2+(nd)^2]^{-\frac{1}{2}}\} J_{2l+1}\{a[b^2+(nd)^2]^{\frac{1}{2}}\} \cos(nx)$$
$$= (-1)^l (2/d) \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} \sin(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$$\cdot T_{2l+1}(\{1-[(2\pi m+x)/ad]^2\}^{\frac{1}{2}}), \quad l = 0, 1, 2, \ldots$$

4.82 $$\sum_{0}^{\infty} \epsilon_n [b^2+(nd)^2]^{-\frac{1}{2}} U_{2l}\{nd[b^2+(nd)^2]^{-\frac{1}{2}}\} J_{2l+1}\{a[b^2+(nd)^2]^{\frac{1}{2}}\} \cos(nx)$$
$$= (-1)^l (2abd)^{-1} \sum_{m_1}^{m_2} \sin(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}}) U_{2l}[(2\pi m+x)/ad], \quad l = 0, 1, 2, \ldots$$

4.83 $$\sum_{0}^{\infty} \epsilon_n J_\nu(\tfrac{1}{2}a\{[b^2+(nd)^2]^{\frac{1}{2}}+nd\}) J_\nu(\tfrac{1}{2}a\{[b^2+(nd)^2]^{\frac{1}{2}}-nd\}) \cos(nx)$$
$$= (2/d) \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} J_{2\nu}(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > -\frac{1}{2}$ if $x = \pm ad$

4.84 $$\sum_{0}^{\infty} \epsilon_n \cos(\tfrac{1}{2}and) J_\nu(\tfrac{1}{4}a\{[b^2+(nd)^2]^{\frac{1}{2}}+nd\}) J_\nu(\tfrac{1}{4}a\{[b^2+(nd)^2]^{\frac{1}{2}}-nd\}) \cos(nx)$$
$$= d^{-\frac{1}{2}} \sum_{m_1}^{m_2} |2\pi m+x|^{-\frac{1}{2}} [a-|(2\pi m+x)/d|]^{-\frac{1}{2}} J_{2\nu}\{b|(2\pi m+x)/d|^{\frac{1}{2}}[a-|(2\pi m+x)/d|^{\frac{1}{2}}]\}$$
$\mathrm{Re}\nu > -\frac{1}{2}$; $\mathrm{Re}\nu > \frac{1}{2}$ if $x=0$ or $x=\pm ad$

4.85 $$\sum_{0}^{\infty} \epsilon_n J_{\nu-(and/\pi)}(y) J_{\nu+(and/\pi)}(y) \cos(nx) = \pi(ad)^{-1} \sum_{m_1}^{m_2} J_{2\nu}\{2y \cos[(\pi/2ad)(2\pi m+x)]\}$$
$\mathrm{Re}\nu > -\frac{1}{2}$; $\mathrm{Re}\nu > 0$ if $x=\pm ad$

4.86 $$\sum_{0}^{\infty} \epsilon_n \cos(\tfrac{1}{2}and) J_{\nu-(and/2\pi)}(y) J_{\nu+(and/2\pi)}(y) \cos(nx)$$
$$= \pi(ad)^{-1} \sum_{m_1}^{m_2} J_{2\nu}\{2y \sin[(\pi/ad)(2\pi m+x)]\}$$
$\mathrm{Re}\nu > -\frac{1}{2}$; $\mathrm{Re}\nu > 0$ if $x=0$ or $x=\pm ad$

4.87 $$\sum_{0}^{\infty} \epsilon_n I_{\nu-(and/\pi)}(y) I_{\nu+(and/\pi)}(y) \cos(nx) = \pi(ad)^{-1} \sum_{m_1}^{m_2} I_{2\nu}\{2y \cos[(\pi/2ad)(2\pi m+x)]\}$$
$\mathrm{Re}\nu > -\frac{1}{2}$; $\mathrm{Re}\nu > 0$ if $x=\pm ad$

4.88 $$\sum_{0}^{\infty} \epsilon_n \cos(\tfrac{1}{2}and) I_{\nu-(and/\pi)}(y) I_{\nu+(and/\pi)}(y) \cos(nx)$$
$$= \pi(ad)^{-1} \sum_{m_1}^{m_2} I_{2\nu}\{2y \sin[(\pi/ad)(2\pi m+x)]\},$$ $\mathrm{Re}\nu > -\frac{1}{2}$; $\mathrm{Re}\nu > 0$ if $x=0$ or $\pm ad$

4.89 $$\sum_{0}^{\infty} \epsilon_n (nd)^{-\nu-1} \mathbf{H}_\nu(and) \cos(nx)$$
$$= 2(2\pi/a)^{\frac{1}{2}} d^{-1} \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}\nu+\frac{1}{4}} P_{\nu-\frac{1}{2}}^{-\nu-\frac{1}{2}}[|(2\pi m+x)/ad|]$$
$\mathrm{Re}\nu > -\frac{3}{2}$; $\mathrm{Re}\nu > -\frac{1}{2}$ if $x=\pm ad$

4.90 $$\sum_{0}^{\infty} \epsilon_n (nd)^{-\mu-1} s_{\mu,\nu}(and) \cos(nx)$$
$$= 2^{\frac{3}{2}+\mu}(\pi/a)^{\frac{1}{2}} d^{-1} [(\mu+1)^2-\nu^2]^{-1} \Gamma[\tfrac{1}{2}(3+\mu+\nu)] \Gamma[\tfrac{1}{2}(3+\mu-\nu)]$$
$$\cdot \sum_{m_1}^{m_2} \{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}\mu+\frac{1}{4}} P_{\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}[|(2\pi m+x)/ad|]$$
$\mathrm{Re}\mu > -\frac{3}{2}$; $\mathrm{Re}\mu > -\frac{1}{2}$ if $x=\pm ad$

4.91 $\sum_{0}^{\infty} \epsilon_n P_\nu^{and/\pi}(y) P_\nu^{-and/\pi}(y) \cos(nx)$

$$= (\pi/ad) \sum_{m_1}^{m_2} P_\nu\{1-2(1-y^2)\cos^2[(\pi/2ad)(2\pi m+x)]\}, \quad 0<y<1, \quad x\neq\pm ad$$

4.92 $\sum_{0}^{\infty} \epsilon_n P_\nu^{and/\pi}(z) P_\nu^{-and/\pi}(z) \cos(nx)$

$$= (\pi/ad) \sum_{m_1}^{m_2} P_\nu\{1+2(z^2-1)\cos^2[(\pi/2ad)(2\pi m+x)]\}, \quad z>1, \quad x\neq\pm ad$$

4.93 $\sum_{0}^{\infty} \epsilon_n \cos(yn) Q_{-\frac{1}{2}+(and/\pi)}(z) Q_{-\frac{1}{2}-(and/\pi)}(z) \cos(nx)$

$$= (\pi^2/ad) \sum_{m_1}^{m_2} \{y^2-\sin^2[(\pi/2ad)(2\pi m+x)]\}^{-\frac{1}{2}}$$

$$\cdot K(\cos[(\pi/2ad)(2\pi m+x)]\{y^2-\sin^2[(2\pi^2 m+\pi x)/2ad]\}^{-\frac{1}{2}}), \quad z>1$$

4.94 $\sum_{0}^{\infty} \epsilon_n [P_{-\frac{1}{2}+\frac{1}{2}ind}(\cosh a)]^2 \cos(nx) = (4/\pi d \sinh a) \sum_{m_1}^{m_2} K'\left(\left|\frac{\sinh[(2\pi m+x)/d]}{\sinh a}\right|\right), \quad a>0$

4.95 $\sum_{0}^{\infty} \epsilon_n P^{\mu}_{-\frac{1}{2}+\frac{1}{2}ind}(\cosh a) P^{-\mu}_{-\frac{1}{2}+\frac{1}{2}ind}(\cosh a) \cos(nx)$

$$= \frac{2}{d \sinh a} \sum_{m_1}^{m_2} P_{\mu-\frac{1}{2}}\left[2\left(\frac{\sinh(2\pi m+x/d)}{\sinh a}\right)^2-1\right], \quad a>0, \quad x\neq\pm ad$$

4.96 $\sum_{0}^{\infty} \epsilon_n D_{\nu+(2and/\pi)}(y) D_{\nu-(2and/\pi)}(y) \cos(nx)$

$$= 2^{-\nu}(\tfrac{1}{2}\pi)^{\frac{1}{2}} \exp(\tfrac{1}{4}y^2)(ad)^{-1} \sum_{m_1}^{m_2} \{\cos[(\pi/2ad)(2\pi m+x)]\}^{-\nu-1}$$

$$\cdot \exp\{-\tfrac{1}{4}y^2 \sec[(\pi/2ad)(2\pi m+x)]\} D_{2\nu+1}(y\{1+\sec[(\pi/2ad)(2\pi m+x)]\}^{\frac{1}{2}})$$

In 4.75–4.96, $m_1 = -[(ad+x)/2\pi]$; $m_2 = [(ad-x)/2\pi]$. If $(ad\pm x)/2\pi$ is an integer or zero, one half of the corresponding term in the right side sum has to be taken.

4.97 $\sum_{0}^{\infty} (-1)^n \epsilon_n \Gamma(\tfrac{1}{2}+2n) D_{-2n-\frac{1}{2}}[(\tfrac{1}{2}ia^2)^{\frac{1}{2}}] D_{-2n-\frac{1}{2}}[(-\tfrac{1}{2}ia^2)^{\frac{1}{2}}] \cos(2nx)$

$$= \pi(\cos x)^{-\frac{1}{2}}\{\cos^2(a^2/4\cos x)[\tfrac{1}{2}-S(a^2/4\cos x)]-\sin(a^2/4\cos x)[\tfrac{1}{2}-C(a^2/4\cos x)]\}$$

4.98 $\sum_0^\infty (-1)^n\Gamma(\frac{3}{2}+2n)D_{-2n-\frac{3}{2}}[(\frac{1}{2}ia^2)^{\frac{1}{2}}]D_{-2n-\frac{3}{2}}[(-\frac{1}{2}ia^2)^{\frac{1}{2}}]\cos[(2n+1)x]$

$$=\tfrac{1}{2}\pi(\cos x)^{-\frac{1}{2}}\{\cos(a^2/4\cos x)[\tfrac{1}{2}-C(a^2/4\cos x)]+\sin(a^2/4\cos x)[\tfrac{1}{2}-S(a^2/4\cos x)]\}$$

4.99 $\sum_0^\infty (-1)^n\epsilon_nJ_{2n}\{z[(a^2+1)^{\frac{1}{2}}-a]^{\frac{1}{2}}\}K_{2n}\{z[(a^2+1)^{\frac{1}{2}}+a]^{\frac{1}{2}}\}\cos(2nx)$

$$=\tfrac{1}{2}K_0[z(a+i\cos x)^{\frac{1}{2}}]+\tfrac{1}{2}K_0[z(a-i\cos x)^{\frac{1}{2}}]$$

4.100 $\sum_0^\infty (-1)^nJ_{2n+1}\{z[(a^2+1)^{\frac{1}{2}}-a]^{\frac{1}{2}}\}K_{2n+1}\{z[(a^2+1)^{\frac{1}{2}}+a]^{\frac{1}{2}}\}\cos[(2n+1)x]$

$$=\tfrac{1}{4}iK_0[z(a+i\cos x)^{\frac{1}{2}}]-\tfrac{1}{4}iK_0[z(a-i\cos x)^{\frac{1}{2}}]$$

4.101 $\sum_0^\infty (-1)^n\epsilon_n[P_{n-\frac{1}{2}}(y)]^2\cos(2nx)=(2/\pi)(1-y^2)^{-\frac{1}{2}}K'[\cos x(1-y^2)^{-\frac{1}{2}}],\quad \cos x<(1-y^2)^{\frac{1}{2}}$

$$=0,\qquad \cos x>(1-y^2)^{\frac{1}{2}}$$

$$0<y<1$$

4.102 $\sum_0^\infty (-1)^n[P_n(y)]^2\cos[(2n+1)x]=\pi^{-1}(1-y^2)^{-\frac{1}{2}}K[(1-y^2)^{-\frac{1}{2}}\cos x],\quad \cos x<(1-y^2)^{\frac{1}{2}}$

$$=(\pi\cos x)^{-1}K[(1-y^2)^{-\frac{1}{2}}\sec x],\qquad \cos x>(1-y^2)^{\frac{1}{2}}$$

$$0<y<1$$

4.103 $\sum_0^\infty (-1)^n\epsilon_n[\Gamma(n-\mu+\frac{1}{2})/\Gamma(n+\mu+\frac{1}{2})][P^\mu_{n-\frac{1}{2}}(y)]^2\cos(2nx)$

$$=(2/\pi)(1-y^2)^{-\frac{1}{2}}Q_{-\mu-\frac{1}{2}}\{1-[2\cos^2x/(1-y^2)]\},\qquad 0<\cos x<(1-y^2)^{\frac{1}{2}}$$

$$=(2/\pi)(1-y^2)^{-\frac{1}{2}}\sin(\pi\mu)Q_{-\mu-\frac{1}{2}}\{[2\cos^2x/(1-y^2)]-1\},\quad (1-y^2)^{\frac{1}{2}}<\cos x<1$$

$$0<y<1$$

4.104 $\sum_0^\infty (-1)^n\epsilon_nP^\mu_{n-\frac{1}{2}}(y)P^{-\mu}_{n-\frac{1}{2}}(y)\cos(2nx)$

$$=(1-y^2)^{-\frac{1}{2}}P_{\mu-\frac{1}{2}}\{[2\cos^2x/(1-y^2)]-1\},\quad 0<\cos x<(1-y^2)^{\frac{1}{2}}$$

4.105 $\sum_0^\infty (-1)^n\epsilon_n[\Gamma(\frac{1}{2}-\mu+n)/\Gamma(\frac{1}{2}+\mu+n)]P^\mu_{n-\frac{1}{2}}(z)Q^\mu_{n-\frac{1}{2}}(z)\cos(2nx)$

$$=(z^2-1)^{-\frac{1}{2}}e^{i\pi\mu}Q_{-\mu-\frac{1}{2}}\{1+[2\cos^2x/(z^2-1)]\},\quad z>1$$

4.106 $\sum_0^\infty (-1)^n\epsilon_nP_{n-\frac{1}{2}}(z)Q_{n-\frac{1}{2}}(z)\cos(2nx)=(z^2-\sin^2x)^{-\frac{1}{2}}K[(z^2-1)^{\frac{1}{2}}(z^2-\sin^2x)^{-\frac{1}{2}}],\quad z>1$

4.107 $\sum_{0}^{\infty} (-1)^n \epsilon_n [Q_{n-\frac{1}{2}}(z)]^2 \cos(2nx) = \pi(z^2-\sin^2 x)^{-\frac{1}{2}} K[\cos x (z^2-\sin^2 x)^{-\frac{1}{2}}], \quad z>1$

4.108 $\sum (-1)^n P_n(z) Q_n(z) \cos[(2n+1)x] = \frac{1}{2}(z^2-\sin^2 x)^{-\frac{1}{2}} K[\cos x (z^2-\sin^2 x)^{-\frac{1}{2}}], \quad z>1$

4.109 $\sum_{0}^{\infty} \epsilon_n P_\nu{}^n(y) P_\nu{}^{-n}(y) \cos[n\pi(x/a)] = P_\nu\{1-2(1-y^2)\cos^2[\pi(x/a)]\}, \quad 0<y<1$

4.110 $\sum_{0}^{\infty} \epsilon_n P_\nu{}^n(z) P_\nu{}^{-n}(z) \cos[n\pi(x/a)] = P_\nu\{1+2(z^2-1)\cos^2[\pi(x/a)]\}, \quad z>1$

4.111 $\sum_{0}^{\infty} (-1)^n \epsilon_n Q_{-\frac{1}{2}+n}(z) Q_{-\frac{1}{2}-n}(z) \cos[n\pi(x/a)]$

$= \pi[z^2-\sin^2(\pi x/2a)]^{-\frac{1}{2}} K\{\cos(\pi x/2a)[z^2-\sin^2(\pi x/2a)]^{-\frac{1}{2}}\}, \quad z>1$

4.112 $\sum_{0}^{\infty} (-1)^n [\Gamma(n-\mu+1)/\Gamma(n+\mu+1)][P_n{}^\mu(y)]^2 \cos[(2n+1)x]$

$= \frac{1}{2}(1-y^2)^{-\frac{1}{2}} P_{-\mu-\frac{1}{2}}\{1-[2\cos^2 x/(1-y^2)]\}, \qquad \cos x < (1-y^2)^{\frac{1}{2}}$

$= \pi^{-1}(1-y^2)^{-\frac{1}{2}} Q_{-\mu-\frac{1}{2}}\{[2\cos^2 x/(1-y^2)]-1\}\cos(\pi\mu), \quad \cos x > (1-y^2)^{\frac{1}{2}}$

$\operatorname{Re}\mu < 1$

4.113 $\sum_{0}^{\infty} (-1)^n \epsilon_n [\Gamma(\frac{1}{2}+n-\mu)/\Gamma(\frac{1}{2}+n+\mu)][Q^\mu_{n-\frac{1}{2}}(z)]^2 \cos(2nx)$

$= \frac{1}{2}\pi^2(z^2-1)^{-\frac{1}{2}} \sec(\pi\mu) P_{-\mu-\frac{1}{2}}\{1+[2\cos^2 x/(z^2-1)]\} \exp(i2\pi\mu), \quad z>1$

4.114 $\sum_{0}^{\infty} (-1)^n \epsilon_n P_\nu{}^n(\cos\vartheta_1) P_\nu{}^{-n}(\cos\vartheta_2) \cos(nx)$

$= \sum_{0}^{\infty} \epsilon_n [\Gamma(\nu-n+1)/\Gamma(\nu+n+1)] P_\nu{}^n(\cos\vartheta_1) P_\nu{}^n(\cos\vartheta_2) \cos(nx)$

$= P_\nu(\cos\vartheta_1 \cos\vartheta_2 + \sin\vartheta_1 \sin\vartheta_2 \cos x), \quad 0 \leq \vartheta_{\substack{1\\2}} < \pi, \quad \vartheta_1+\vartheta_2<\pi$

4.115 $\sum_{0}^{\infty} (-1)^n \epsilon_n P_\nu{}^{-n}(\cos\vartheta_1) Q_\nu{}^n(\cos\vartheta_2) \cos(nx)$

$= \sum_{0}^{\infty} \epsilon_n [\Gamma(\nu-n+1)/\Gamma(\nu+n+1)] P_\nu{}^n(\cos\vartheta_1) Q_\nu{}^n(\cos\vartheta_2) \cos(nx)$

$= Q_\nu(\cos\vartheta_1 \cos\vartheta_2 + \sin\vartheta_1 \sin\vartheta_2 \cos x), \quad 0<\vartheta_2<\frac{1}{2}\pi, \quad 0<\vartheta_1<\pi, \quad 0<\vartheta_1+\vartheta_2<\pi$

4.116 $\sum_{0}^{\infty} (-1)^n \epsilon_n P_\nu^n(z_1) P_\nu^n(z_2) [\Gamma(\nu-n+1)/\Gamma(\nu+n+1)] \cos(nx)$

$$= \sum_{0}^{\infty} (-1)^n \epsilon_n P_\nu^n(z_1) P_\nu^{-n}(z_2) \cos(nx)$$

$$P_\nu[z_1 z_2 - (z_1^2-1)^{\frac{1}{2}}(z_2^2-1)^{\frac{1}{2}} \cos x], \quad \mathrm{Re}\, z_{1,2} > 0, \quad |\arg(z_{1,2}-1)| < \pi$$

4.117 $\sum_{0}^{\infty} (-1)^n \epsilon_n Q_\nu^n(z_1) P_\nu^{-n}(z_2) \cos(nx)$

$$= \sum_{0}^{\infty} (-1)^n \epsilon_n [\Gamma(\nu-n+1)/\Gamma(\nu+n+1)] Q_\nu^n(z_1) P_\nu^n(z_2) \cos(nx)$$

$$= Q_\nu[z_1 z_2 - (z_1^2-1)^{\frac{1}{2}}(z_2^2-1)^{\frac{1}{2}} \cos x], \quad z_{1,2} \text{ real}, \quad 1 < z_2 < z_1, \quad \nu \neq -1, -2, -3, \ldots$$

4.118 $\sum_{0}^{\infty} P_\nu^{n+\frac{1}{2}}(z) P_\nu^{-n-\frac{1}{2}}(z) \cos[(2n+1)x] = \frac{1}{2} P_\nu[z^2 + (z^2-1)\cos(2x)], \quad z > 1$

4.119 $\sum_{0}^{\infty} P_\nu^{n+\frac{1}{2}}(y) P_\nu^{-n-\frac{1}{2}}(y) \cos[(2n+1)x] = \frac{1}{2} P_\nu[y^2 - (1-y^2)\cos(2x)], \quad 0 < y < 1$

4.120 $\sum_{0}^{\infty} \epsilon_n [\Gamma(1+\nu-nm)/\Gamma(1+\nu+nm)][P_\nu^{nm}(y)]^2 \cos(nx)$

$$= m^{-1} \sum_{r_1}^{r_2} P_\nu\{y^2 + (1-y^2)\cos[(2\pi r+x)/m]\}, \quad 0 < y < 1$$

4.121 $\sum_{0}^{\infty} (-1)^{nm} \epsilon_n [\Gamma(1+\nu-nm)/\Gamma(1+\nu+nm)][P_\nu^{nm}(z)]^2 \cos(nx)$

$$= m^{-1} \sum_{r_1}^{r_2} P_\nu\{z^2 - (z^2-1)\cos[(2\pi r+x)/m]\}, \quad z > 1$$

4.122 $\sum_{0}^{\infty} (-1)^{nm} \epsilon_n [\Gamma(1+\nu-nm)/\Gamma(1+\nu+nm)] P_\nu^{nm}(z) Q_\nu^{nm}(z) \cos(nx)$

$$= m^{-1} \sum_{r_1}^{r_2} Q_\nu\{z^2 - (z^2-1)\cos[(2\pi r+x)/m]\}, \quad z > 1$$

In the last three formulas $m = 1, 2, 3, \ldots$, $r_1 = -[(m\pi+x)/2\pi]$, $r_2 = [(m\pi-x)/2\pi]$. If $(m\pi \pm x)/2\pi$ is an integer or zero, one half of the corresponding term in the sum at the right side has to be taken.

V

EXPONENTIAL FOURIER AND FOURIER–BESSEL SERIES

In formulas 5.1–5.33 the properties m_1, m_2, and $H(m)$ on the right-hand sides are

$$m_1=-[(ad+x)/2\pi],\qquad m_2=[(ad-x)/2\pi];\qquad H(m)=\exp[-iy(2\pi m+x)/d]$$

If $(ad\pm x)/2\pi$ is an integer or zero, one half of the corresponding term in the sum has to be taken.

5.1 $$\sum_{-\infty}^{\infty}\exp(inx)[b^2+(y+nd)^2]^{-\frac{1}{2}}\sin\{a[b^2+(y+nd)^2]^{\frac{1}{2}}\}$$
$$=(\pi/d)\sum_{m_1}^{m_2}H(m)J_0(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$

5.2 $$\sum_{-\infty}^{\infty}\exp(inx)\cos[\tfrac{1}{2}a(y+nd)]\{\Gamma[\nu+(ay/2\pi)+(and/2\pi)]\Gamma[\nu-(ay/2\pi)-(and/2\pi)]\}^{-1}$$
$$=\pi 2^{2\nu-2}[ad\Gamma(2\nu-1)]^{-1}\sum_{m_1}^{m_2}H(m)\,|\sin[(\pi/ad)(2\pi m+x)]|^{2\nu-2}$$

$\mathrm{Re}\nu>\frac{1}{2}$; $\mathrm{Re}\nu>1$ if $x=\pm ad$ or $x=0$

5.3 $$\sum_{-\infty}^{\infty} \exp(inx)\{\Gamma[\nu+(a/\pi)y+(a/\pi)nd]\Gamma[\nu-(a/\pi)y-(a/\pi)nd]\}^{-1}$$
$$=2^{2\nu-2}\pi[ad\Gamma(2\nu-1)]^{-1}\sum_{m_1}^{m_2} H(m)\{\cos[(\pi/2ad)(2\pi m+x)]\}^{2\nu-2}$$
$\mathrm{Re}\nu>\frac{1}{2}$; $\mathrm{Re}\nu>1$ if $x=\pm ad$

5.4 $$\sum_{-\infty}^{\infty} \exp(inx)P^{\mu}_{-\frac{1}{2}+y+nd}(\cos a)$$
$$=[(2\pi)^{\frac{1}{2}}/d\Gamma(\tfrac{1}{2}-\mu)](\sin a)^{\mu}\sum_{m_1}^{m_2} H(m)\{\cos[(2\pi m+x)/d]-\cos a\}^{-\mu-\frac{1}{2}}$$
$0<a<\pi$, $\mathrm{Re}\mu<\frac{1}{2}$; $\mathrm{Re}\mu<-\frac{1}{2}$ if $x=\pm ad$

5.5 $$\sum_{-\infty}^{\infty} \exp(inx)P_{-\frac{1}{2}+iy+ind}(\cosh a)$$
$$=[(2\pi)^{\frac{1}{2}}/d\Gamma(\tfrac{1}{2}-\mu)](\sinh a)^{\mu}\sum_{m_1}^{m_2} H(m)\{\cosh a-\cosh[(2\pi m+x)/d]\}^{-\mu-\frac{1}{2}}$$
$\mathrm{Re}\mu<\frac{1}{2}$; $\mathrm{Re}\mu<-\frac{1}{2}$ if $x=\pm ad$

5.6 $$\sum_{-\infty}^{\infty} \exp(inx)P_{\nu}^{a(y+nd)/\pi}(b)P_{\nu}^{-a(y+nd)/\pi}(b)$$
$$=(\pi/ad)\sum_{m_1}^{m_2} H(m)P_{\nu}\{1-2(1-b^2)\cos^2[(\pi/2ad)(2\pi m+x)]\},\quad 0<b<1,\quad x\neq\pm ad$$

5.7 $$\sum_{-\infty}^{\infty} \exp(inx)P_{\nu}^{a(y+nd)/\pi}(b)P_{\nu}^{-a(y+nd)/\pi}(b)$$
$$=(\pi/ad)\sum_{m_1}^{m_2} H(m)P_{\nu}\{1+2(b^2-1)\cos^2[(\pi/2ad)(2\pi m+x)]\},\quad b>1,\quad x\neq\pm ad$$

5.8 $$\sum_{-\infty}^{\infty} \exp(inx)\cos(an)Q_{-\frac{1}{2}+a(y+nd)/\pi}(b)Q_{-\frac{1}{2}-a(y+nd)/\pi}(b)$$
$$=(\pi^2/ad)\sum_{m_1}^{m_2} H(m)\{b^2-\sin^2[(\pi/2ad)(2\pi m+x)]\}^{-\frac{1}{2}}$$
$$\cdot K(\cos[(\pi/2ad)(2\pi m+x)]\{b^2-\sin^2[(2\pi^2 m+\pi x)/2ad]\}^{-\frac{1}{2}}),\quad b>1$$

5.9 $$\sum_{-\infty}^{\infty} \exp(inx)P^{\mu}_{-\frac{1}{2}+\frac{1}{2}iy+\frac{1}{2}ind}(\cosh a)P^{-\mu}_{-\frac{1}{2}+\frac{1}{2}iy+\frac{1}{2}ind}(\cosh a)$$
$$=(2/d\sinh a)\sum_{m_1}^{m_2} H(m)P_{\mu-\frac{1}{2}}\left(2\frac{\sinh^2[(2\pi m+x)/d]}{\sinh^2 a}-1\right),\quad x\neq 0,\quad x\neq\pm ad$$

5.10 $$\sum_{-\infty}^{\infty} \exp(inx)\,(y+nd)^{-\nu} J_\nu[a(y+nd)]$$
$$= [\pi^{\frac{1}{2}} 2^{1-\nu} a^{-\nu}/d\Gamma(\tfrac{1}{2}+\nu)] \sum_{m_1}^{m_2} H(m)\,\{a^2-[(2\pi m+x)/d]^2\}^{\nu-\frac{1}{2}}$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > \frac{1}{2}$ if $x=\pm ad$

5.11 $$\sum_{-\infty}^{\infty} \exp(inx)\,(y+nd)^{-\nu} J_{\nu+2l}[a(y+nd)]$$
$$= [(-1)^l(\tfrac{1}{2}a)^{-\nu}\Gamma(\nu)\,(2l)!/d\Gamma(2l+2\nu)] \sum_{m_1}^{m_2} H(m)\,\{a^2-[(2\pi m+x)/d]^2\}^{\nu-\frac{1}{2}}$$
$$\cdot C_{2l}{}^{\nu}[(2\pi m+x)/ad]$$, $l=0, 1, 2, \ldots$; $\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > \frac{1}{2}$ if $x=\pm ad$

5.12 $$\sum_{-\infty}^{\infty} \exp(inx)\,\cos[\tfrac{1}{2}a(y+nd)] J_\nu[\tfrac{1}{2}a(y+nd)](y+nd)^{-\nu}$$
$$= [\pi^{\frac{1}{2}} a^{-\nu}/d\Gamma(\tfrac{1}{2}+\nu)] \sum_{m_1}^{m_2} H(m)\,|(2\pi m+x)/d|^{\nu-\frac{1}{2}}[a-|(2\pi m+x)/d|]^{\nu-\frac{1}{2}}$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > \frac{1}{2}$ if $x=0$ and $x=\pm ad$

5.13 $$\sum_{-\infty}^{\infty} \exp(inx) J_\nu[\tfrac{1}{2}a(y+nd)] J_{-\nu}[\tfrac{1}{2}a(y+nd)]$$
$$= 2(ad)^{-1} \sum_{m_1}^{m_2} H(m)\,P_{\nu-\frac{1}{2}}[2(ad)^{-1}(2\pi m+x)^2-1]$$

5.14 $$\sum_{-\infty}^{\infty} \exp(inx)\,(y+nd)^{\frac{1}{2}} J_\nu[\tfrac{1}{2}a(y+nd)] J_{-\nu-\frac{1}{2}}[\tfrac{1}{2}a(y+nd)]$$
$$= 2(\tfrac{1}{2}\pi d)^{-\frac{1}{2}} \sum_{m_1}^{m_2} H(m)\,|2\pi m+x|^{-\frac{1}{2}}\{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}}$$
$$\cdot \cos\{(2\nu+\tfrac{1}{2})\arccos[(2\pi m+x)/ad]\}$$, $x\neq 0$, $\pm ad$

5.15 $$\sum_{-\infty}^{\infty} \exp(inx)\,\cos[\tfrac{1}{2}a(y+nd)] J_0\{\tfrac{1}{2}a[(y+nd)^2+b^2]^{\frac{1}{2}}\}$$
$$= d^{-\frac{1}{2}} \sum_{m_1}^{m_2} H(m)\,|2\pi m+x|^{-\frac{1}{2}}[a-|(2\pi m+x)/d|]^{-\frac{1}{2}}$$
$$\cdot \cos\{b[(2\pi m+x)/d]^{\frac{1}{2}}[a-|(2\pi m+x)/d|^{\frac{1}{2}}]^{\frac{1}{2}}\}$$, $x\neq 0$, $\pm ad$

5.16 $$\sum_{-\infty}^{\infty} \exp(inx) J_0\{a[(y+nd)^2+b^2]^{\frac{1}{2}}\}$$
$$= (2/d) \sum_{m_1}^{m_2} H(m)\,\{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} \cos(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$, $x\neq\pm ad$

5.17 $$\sum_{-\infty}^{\infty} \exp(inx) J_0\{a[(y+nd)^2-b^2]^{\frac{1}{2}}\}$$
$$=(2/d)\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} \cosh(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}}), \quad x\neq\pm ad$$

5.18 $$\sum_{-\infty}^{\infty} \exp(inx)[b^2+(y+nd)^2]^{-\frac{1}{2}\nu} J_\nu\{a[b^2+(y+nd)^2]^{\frac{1}{2}}\}$$
$$=[(2\pi b)^{\frac{1}{2}}/d](ab)^{-\nu}\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$\mathrm{Re}\,\nu>-\frac{1}{2}$; $\mathrm{Re}\,\nu>\frac{1}{2}$ if $x=\pm ad$

5.19 $$\sum_{-\infty}^{\infty} \exp(inx) T_{2l}\{(y+nd)/[b^2+(y+nd)^2]^{\frac{1}{2}}\} J_{2l}\{a[b^2+(y+nd)^2]^{\frac{1}{2}}\}$$
$$=(-1)^l(2/d)\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}}$$
$$\cdot\cos(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}}) T_{2l}[(2\pi m+x)/ad], \quad l=0,1,2,\ldots, \quad x\neq ad$$

5.20 $$\sum_{-\infty}^{\infty} \exp(inx) T_{2l}\{b[b^2+(y+nd)^2]^{-\frac{1}{2}}\} J_{2l}\{a[b^2+(y+nd)^2]^{\frac{1}{2}}\}$$
$$=(-1)^l(2/d)\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} \cos(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$$\cdot T_{2l}(\{1-[(2\pi m+x)/ad]^2\}^{\frac{1}{2}}), \quad l=0,1,2,\ldots, \quad x\neq ad$$

5.21 $$\sum_{-\infty}^{\infty} \exp(inx) T_{2l+1}\{b[b^2+(y+nd)^2]^{-\frac{1}{2}}\} J_{2l+1}\{a[b^2+(y+nd)^2]^{\frac{1}{2}}\}$$
$$=(-1)^l(2/d)\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi+x)/d]^2\}^{-\frac{1}{2}} \sin(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$$\cdot T_{2l+1}(\{1-[(2\pi m+x)/ad]^2\}^{\frac{1}{2}}), \quad l=0,1,2,\ldots$$

5.22 $$\sum_{-\infty}^{\infty} \exp(inx)[b^2+(y+nd)^2]^{-\frac{1}{2}} U_{2l}\{(y+nd)/[b^2+(y+nd)^2]^{\frac{1}{2}}\} J_{2l+1}\{a[b^2+(y+nd)^2]^{\frac{1}{2}}\}$$
$$=(-1)^l 2(abd)^{-1}\sum_{m_1}^{m_2} H(m) \sin(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$$\cdot U_{2l}[(2\pi m+x)/ad], \quad l=0,1,2,\ldots$$

5.23 $$\sum_{-\infty}^{\infty} \exp(inx) J_\nu(\tfrac{1}{2}a\{[b^2+(y+nd)^2]^{\frac{1}{2}}+y+nd\}) J_\nu(\tfrac{1}{2}a\{[b^2+(y+nd)^2]^{\frac{1}{2}}-y-nd\})$$
$$= (2/d) \sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]^2\}^{-\frac{1}{2}} J_{2\nu}(b\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}})$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > \frac{1}{2}$ if $x = \pm ad$

5.24 $$\sum_{-\infty}^{\infty} \exp(inx) \cos[\tfrac{1}{2}a(y+nd)] J_\nu(\tfrac{1}{4}a\{[b^2+(y+nd)^2]^{\frac{1}{2}}+y+nd\})$$
$$\cdot J_\nu(\tfrac{1}{4}a\{[b^2+(y+nd)^2]^{\frac{1}{2}}-y-nd\})$$
$$= d^{-\frac{1}{2}} \sum_{m_1}^{m_2} H(m)\, |2\pi m+x|^{-\frac{1}{2}} [a-|(2\pi m+x)/d|]^{-\frac{1}{2}}$$
$$\cdot J_{2\nu}\{b\,|(2\pi m+x)/d|^{\frac{1}{2}} [a-|(2\pi m+x)/d|]^{\frac{1}{2}}\}$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > \frac{1}{2}$ if $x = 0$ or $\pm ad$

5.25 $$\sum_{-\infty}^{\infty} \exp(inx) J_{\nu-[a(y+nd)/\pi]}(b) J_{\nu+[a(y+nd)/\pi]}(b)$$
$$= (\pi/ad) \sum_{m_1}^{m_2} H(m) J_{2\nu}\{2b\cos[(\pi/2ad)(2\pi m+x)]\},$$ $\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > 0$ if $x = \pm ad$

5.26 $$\sum_{-\infty}^{\infty} \exp(inx) \cos[\tfrac{1}{2}a(y+nd)] J_{\nu-[a(y+nd)/2\pi]}(b) J_{\nu+[a(y+nd)/\pi]}(b)$$
$$= (\pi/ad) \sum_{m_1}^{m_2} H(m) J_{2\nu}\{2b\sin[(\pi/ad)(2\pi m+x)]\}$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > 0$ if $x = 0$ or $\pm ad$

5.27 $$\sum_{-\infty}^{\infty} \exp(inx) I_{\nu-[a(y+nd)/\pi]}(b) I_{\nu+[a(y+nd)/\pi]}(b)$$
$$= (\pi/ad) \sum_{m_1}^{m_2} H(m) I_{2\nu}\{2b\cos[(\pi/2ad)(2\pi m+x)]\},$$ $\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > 0$ if $x = \pm ad$

5.28 $$\sum_{-\infty}^{\infty} \exp(inx) \cos[\tfrac{1}{2}a(y+nd)] I_{\nu-[a(y+nd)/2\pi]}(b) I_{\nu+[a(y+nd)/2\pi]}(b)$$
$$= (\pi/ad) \sum_{m_1}^{m_2} H(m) I_{2\nu}\{2b\sin[(\pi/ad)(2\pi m+x)]\}$$
$\mathrm{Re}\,\nu > -\frac{1}{2}$; $\mathrm{Re}\,\nu > 0$ if $x = 0$ or $\pm ad$

5.29 $\sum_{-\infty}^{\infty} \exp(inx)\{\mathbf{J}_\nu[a(y+nd)]+\mathbf{J}_{-\nu}[a(y+nd)]\}$

$$= (4/d)\cos(\tfrac{1}{2}\pi\nu)\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]\}^{-\frac{1}{2}}\cos\{\nu\arccos[(2\pi m+x)/ad]\}$$

$x \neq \pm ad$

5.30 $\sum_{-\infty}^{\infty} \exp(inx)[\mathbf{J}_{a(y+nd)/\pi}(b)+\mathbf{J}_{-a(y+nd)/\pi}(b)]$

$$= (2\pi/ad)\sum_{m_1}^{m_2} H(m)\cos\{b\sin[(\pi/ad)(2\pi m+x)]\}$$

5.31 $\sum_{-\infty}^{\infty} \exp(inx)[E_{a(y+nd)/\pi}(b)+E_{-a(y+nd)/\pi}(b)]$

$$= (-2\pi/ad)\sum_{m_1}^{m_2} H(m)\sin\{b\sin[(\pi/ad)(2\pi m+x)]\}$$

$\sum_{-\infty}^{\infty} \exp(inx)[\cos(ay+and)]^{-1}[\mathbf{J}_{2a(y+nd)/\pi}(b)+\mathbf{J}_{-2a(y+nd)/\pi}(b)]$

$$= (2\pi/ad)\sum_{m_1}^{m_2} H(m)\cos\{b\cos[(\pi/2ad)(2\pi m+x)]\}$$

$\sum_{-\infty}^{\infty} \exp(inx)[\cos(ay+and)]^{-1}[E_{2a(y+nd)/\pi}(b)+E_{-2a(y+nd)/\pi}(b)]$

$$= (-2\pi/ad)\sum_{m_1}^{m_2} H(m)\sin\{b\cos[(\pi/2ad)(2\pi m+x)]\}$$

5.32 $\sum_{-\infty}^{\infty} \exp(inx)(y+nd)^{-1-\nu}\mathbf{H}_\nu[a(y+nd)]$

$$= (2/d)(2\pi/a)^{\frac{1}{2}}\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}\nu+\frac{1}{4}}P_{\nu-\frac{1}{2}}^{-\nu-\frac{1}{2}}[|(2\pi m+x)/ad|]$$

$\mathrm{Re}\,\nu > -\frac{3}{2};\quad \mathrm{Re}\,\nu > -\frac{1}{2}\quad \text{if}\quad x=\pm ad$

5.33 $\sum_{-\infty}^{\infty} \exp(inx)(y+nd)^{-\mu-1}s_{\mu,\nu}[a(y+nd)]$

$$= (2^{\frac{3}{2}+\mu}\pi^{\frac{1}{2}}/a^{\frac{1}{2}}d)\,\Gamma[\tfrac{1}{2}(\mu+\nu+3)]\,\Gamma[\tfrac{1}{2}(\mu-\nu+3)][(\mu+1)^2-\nu^2]^{-1}$$

$$\cdot\sum_{m_1}^{m_2} H(m)\{a^2-[(2\pi m+x)/d]^2\}^{\frac{1}{2}\mu+\frac{1}{4}}P_{\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}[|(2\pi m+x)/ad|]$$

$\mathrm{Re}\,\mu > -\frac{3}{2};\quad \mathrm{Re}\,\mu > -\frac{1}{2}\quad \text{if}\quad x=\pm ad$

5.34 $\sum_{-\infty}^{\infty} J_{\mu-n}(a)J_{\nu+n}(a)\exp(inx)=\exp[i(\mu-\nu)\tfrac{1}{2}x]J_{\nu+\mu}(2a\cos\tfrac{1}{2}x)$

$-\pi\leq x\leq\pi,\quad \mathrm{Re}(\nu+\mu)>0$

5.35 $\sum_{-\infty}^{\infty} J_{\nu-n}(z)J_n(z)\exp(inx)=\exp(i\nu\tfrac{1}{2}x)J_\nu(2z\cos\tfrac{1}{2}x)$

5.36 $\sum_{-\infty}^{\infty} \exp(inx)J_n(a)J_{\nu+n}(b)=[(b-ae^{-ix})/(b-ae^{ix})]^{\frac{1}{2}\nu}J_\nu[(a^2+b^2-2ab\cos x)^{\frac{1}{2}}]$

5.37 $\sum_{-\infty}^{\infty} (-1)^n I_n(a)I_{\nu+n}(b)\exp(inx)=[(b-ae^{-ix})/(b-ae^{ix})]^{\frac{1}{2}\nu}I_\nu[(a^2+b^2-2ab\cos x)^{\frac{1}{2}}]$

5.38 $\sum_{-\infty}^{\infty} \exp(inx)Y_{\nu+n}(b)J_n(a)=[(b-ae^{-ix})/(b-ae^{ix})]^{\frac{1}{2}\nu}Y_\nu[(a^2+b^2-2ab\cos x)^{\frac{1}{2}}]$

5.39 $\sum_{-\infty}^{\infty} I_n(a)K_{\nu+n}(b)\exp(inx)=[(b-ae^{-ix})/(b-ae^{ix})]^{\frac{1}{2}\nu}K_\nu[(a^2+b^2-2ab\cos x)^{\frac{1}{2}}]$

5.40 $\sum_{-\infty}^{\infty} a^{n-\mu}J_{\mu-n}(a)b^{-\nu-n}J_{\nu+n}(b)\exp(inx)=\exp[i(\mu-\nu)\tfrac{1}{2}x][2\cos\tfrac{1}{2}x/(a^2e^{\frac{1}{2}ix}+b^2e^{-\frac{1}{2}ix})]^{\frac{1}{2}\nu+\frac{1}{2}\mu}$

$\cdot J_{\mu+\nu}\{[2\cos\tfrac{1}{2}x(a^2e^{\frac{1}{2}ix}+b^2e^{-\frac{1}{2}ix})]^{\frac{1}{2}}\},\quad -\pi\leq x\leq\pi,\quad \mathrm{Re}(\nu+\mu)>0$

5.41 $\sum_{1}^{\infty} [\tau_{\nu,n}J_{\nu+1}(\tau_{\nu,n})]^{-1}J_\nu(\tau_{\nu,n}x)=\tfrac{1}{2}x^\nu,\quad 0\leq x<1$

5.42 $\sum_{1}^{\infty} [\tau_{0,n}J_1(\tau_{0,n})]^{-2}J_0(X\tau_{0,n})J_0(\tau_{0,n}x)=-\tfrac{1}{2}\log X,\quad 0\leq x\leq X\leq 1$

5.43 $\sum_{1}^{\infty} \tau_{0,n}^{-\mu-1}[J_1(\tau_{0,n})]^{-2}J_{\mu+1}(\tau_{0,n})J_0(\tau_{0,n}x)=[2^{-\mu-1}/\Gamma(1+\mu)](1-x^2)^\mu$

$0<x<1,\quad \mathrm{Re}\mu>-1$

5.44 $\sum_{1}^{\infty} \tau_{\nu,n}^{-\mu-1}[J_{\nu+1}(\tau_{\nu,n})]^{-2}J_{\nu+\mu+1}(\tau_{\nu,n})J_\nu(\tau_{\nu,n}x)=[2^{-\mu-1}/\Gamma(1+\mu)]x^\nu(1-x^2)^\mu$

$0<x<1,\quad \mathrm{Re}\nu>-1$

5.45 $\sum_{1}^{\infty} \tau_{\nu,n}(\tau_{\nu,n}^2 - z^2)^{-1}[J_\nu(\tau_{\nu,n}x)/J_{\nu+1}(\tau_{\nu,n})] = \frac{1}{2}[J_\nu(xz)/J_\nu(z)], \quad 0 \leq x < 1$

5.46 $\sum_{1}^{\infty} (\tau_{\nu,n}^2 - z^2)^{-1}\tau_{\nu,n}^{-1}[J_\nu(\tau_{\nu,n}x)/J_{\nu+1}(\tau_{\nu,n})] = \frac{1}{2}z^{-2}\{[J_\nu(xz)/J_\nu(z)] - x^\nu\}, \quad 0 \leq x < 1$

5.47 $\sum_{1}^{\infty} \tau_{\nu,n}(\tau_{\nu,n}^2 + z^2)^{-1}[J_\nu(\tau_{\nu,n}x)/J_{\nu+1}(\tau_{\nu,n})] = \frac{1}{2}[I_\nu(xz)/I_\nu(z)], \quad 0 \leq x < 1$

5.48 $\sum_{1}^{\infty} (z^2 - \tau_{\nu,n}^2)^{-1}[J_{\nu+1}(\tau_{\nu,n})]^{-2}J_\nu(\tau_{\nu,n}X)J_\nu(\tau_{\nu,n}x)$

$= [\pi J_\nu(xz)/4J_\nu(z)][J_\nu(x)Y_\nu(Xz) - J_\nu(Xz)Y_\nu(z)]$

$= [\pi i J_\nu(xz)/4J_\nu(z)][J_\nu(z)H_\nu^{(2)}(Xz) - J_\nu(Xz)H_\nu^{(2)}(z)]$

5.49 $\sum_{1}^{\infty} (z^2 + \tau_{\nu,n}^2)[J_{\nu+1}^2(\tau_{\nu,n})]^{-2}J_\nu(\tau_{\nu,n}X)J_\nu(\tau_{\nu,n}x)$

$= [I_\nu(xz)/2I_\nu(z)][I_\nu(z)K_\nu(Xz) - K_\nu(z)I_\nu(Xz)]$

In the last two formulas $0 \leq x \leq X \leq 1$. If $0 \leq X \leq x \leq 1$, interchange x and X at the right sides.

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