DIPAK CHATTERJEE
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REFLECTIONS ON THE PHILOSOPHY OF MATHEMATICS

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REFLECTIONS
ON
THE PHILOSOPHY OF MATHEMATICS

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Dedicated

to

the memory of

Late Benoy Kumar Mukherjee

and

Late Gopal Chandra Chatterjee

who instilled in us an indomitable thirst

for pure knowledge.

FOREWORD

The spectacular growth of scientific knowledge in the last two centuries has called for an introspection as to its necessity and relevance. By this time the nature and character of a subject have also undergone a radical change resolving the often artificial distinction between different subjects and their fields. The questions of survival and values have thus gained importance in the context of various overlapping theories. A review of the old system of thoughts in relation to the new has become a call of the age. This monograph is an interesting evaluation of various philosophical aspects of mathematics in the said direction. The discussion starts with the nature of mathematics in the changed perspectives and winds its path through the philosophical schools, space and time, intuition in mathematics, crises in mathematics right upto the future of mathematics.

The title of this study is 'Philosophy of Mathematics'. The usual meaning of philosophy is the use of reasoning in seeking truth and reality and knowledge of reality, especially of causes and nature of things. This agrees rather well with the explanation given by the authors when they state that the aim of the philosophy of mathematics is to find its relevance to human existence, analysing its basic concepts, its meaning and essence, to put mathematics on a strong foundation. This means that use will be replete with abstract notions as is natural with mathematics, as the authors say "Mathematics is a formal discipline, abstract and symbolic without reference to meaning."

The authors have made an exhaustive and critical study of the works of mathematicians through the ages, starting with the Greek and Vedic period from the 4th century B.C. right to the most recent mathematics of the present age. Throughout this study one is faced with a mind boggling array of mathematicians with short references to their personal contributions to the world of mathematics.

There is an interesting discussion on the contribution of intuition to the acquisition of mathematical knowledge. Intuition is an immediate apprehension by the mind without reasoning, it is an immediate insight, as Descartes puts it "By intuition I understand not the fluctuating testimony of the senses, nor the misleading judgement that proceeds from the blundering construction of imagination, but the conception which an unclouded and attentive mind gives us so readily and distinctly that we are wholly freed from doubt about that which we understand." One might almost say that intuition is a sort of revelation, except that revelation is disclosure of knowledge by a transcendent agency, whereas intuition is disclosure of knowledge by the action of the mind itself unconsciously working on notions acquired in previous experiences.

The authors refer to periodic crises in the world of mathematics, particularly in recent times. Einstein says, "Insofar as the propositions of mathematics give an account of reality, they are not certain and insofar as they are certain, they do not describe reality".

However these crises act as an incentive to further and deeper research into the nature of mathematics. That mathematics is growing is clear from the nearly 50,000 papers published every year.

The authors insist that mathematics is the queen of science for ever. On the other hand they also say that it is a conspicuous fact that mathematics and physics play complimentary roles, presumably on equal terms and not in the position of one subordinate to the other.

Calcutta
12.4.1993

Rev. A. Verstaeten, S.J.

PREFACE

The essence of life is the quest for knowledge and this world is a wonderful place that reveals at each corner of its vast domain unexpected new beauty and grace that is called knowledge. So the kingdom of knowledge is infinite and ever-green with fascinating ideas that surprise any one with a serious mind. Eventually we, while walking down the road of life entered into a new world, namely, the Philosophy of Science whose firmament is studded with innumerable glittering stars. Astounded we looked up and then bathed in divine grace, we resolved to share our joy with others. This monograph is a gentle expression of our joy that we want to share. If any one gets pleasure in reading this, that would be our best reward.

The theme essentially circles round and round one arch-idea—the essence of Mathematics and it is arranged so as to enable the reader roam comfortably and happily over our reflections, except for the last article which is little involved. Our musing on different aspects of the life of Mathematics has manifested itself mainly through philosophical overviews, introspections and conclusions of different chapters. Many may differ from us and that is natural but no realization can be nullified by controversies. Only data can be rectified. Arguments and counter-arguments enrich knowledge but not wisdom. Mistakes are inevitable, more so for persons like us with humble means and ability. All responsibilities in this regard are ours and we beg no excuse for this but will be glad to receive comments and criticisms from all.

Coming back to the contents, we consider it important to mention that a reader after reading the first two chapters can move to any chapter with ease, though an orderly approach will be the best in our consideration. We have tried to be as explicit as possible in discussing the earlier theses on various topics, but at times our limitations might have surfaced.

We appreciate the courage Mr. Mukherjee of the Firma KLM has shown in publishing this monograph and our thanks are due not only to him but also to all who worked hard to make the publication as neat and beautiful as possible.

Thanks are also due to Professor P. M. Gruber, Professor G. Ramharter, Dr. Hartwig Sorger of the Technical University of Vienna who provided us time and inspiration to concentrate on such topics as these during our stay at Vienna. We acknowledge with gratitude our debts to Professor Dilip Kumar Sinha, Dr. Mrinal Kanti Das, Professor Rohinton Kapadia, Late Rev. F. Goreux, Professor Manindra Chandra Chaki, Professor Amarendranath Bhattacharyay and Professor Rama Ranjan Mukherjee who by care and criticism, inspiration and collaboration guided us through this mysterious world of knowledge and to the completion of this monograph.

2nd July, 1992
Calcutta.

Dipak Chatterjee
Suparna Chatterjee

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CHAPTER I

THE PHILOSOPHY OF MATHEMATICS

1.0 Introduction

In the quest for knowledge human intelligence had always asked for the relevance of anything that came to the fore-front, inquired into its essence and meaning and with the passage of time challenged old ideas in the light of the changed horizon of wisdom. New ideas were born often giving rise to new dimensions of life and philosophy. Powerful personalities dominated the world for centuries, then giving way to newer personalities with dazzling new ideas. Thus came Socrates, Plato, Aristotle, Jesus, Hume, Spinoza, Hegel, Mill, Russell, Lakatos in the West and Buddha, Lao Tse, Charvaka, Shankaracharya, Srichaitanya, Sri Ramkrishna, Sri Aurobinda in the East. Each of these greatmen questioned the essence of life on earth, its relation to time and space and its future. Nothing was beyond their critical observation and reflection. Their penetrating ideas are not only gems of wisdom to show the right path to the ordinary people but also sources of an eternal enlightenment to the thirsty souls striving for earthly and spiritual peace and happiness.

Mathematics was born with the dawning of human civilization, if not earlier, as it was needed by all for existence and subsistence as well. In other words, mathematics was equally important as food and shelter. It is the task of the historians to discover when mathematics was born and how, but the philosophers are however inclined not to knowing this but to finding the relevance of mathematics to human existence, to analysing the basic concepts of mathematics, its meaning and essence and to putting mathematics on a strong foundational basis. A lot have been said about this but the philosophers never believe that the last word has been said because all notions are spatio-temporal.

In all centuries this creed of highly talented persons known as philosophers not only added to the vast ocean of knowledge but brought about thorough and often morphological changes of life. They helped people to know their identity and reshape their life style. It is a wellknown fact how Socrates intercepted everybody he met on the road and by asking questions as to their motto and mission of life changed the concept of life in Greece. That the Marxian philosophy had tremendous influence on the twentieth century social and political life is beyond any dispute. Just as the meaning and essence of life had assumed new dimensions so has mathematical truth. Almost all philosophers had sometimes questioned the meaning of the word 'truth' and quite naturally the philosophers of science mused on the word 'mathematical truth'. As the concept of numbers came in the sequel, they questioned the meaning and genesis of these numbers. They challenged even the justifiability of the principles of reasoning that paved the path of modern science including mathematics. Numerous paradoxes, originated right from the Greek period till the nineteenth century rocked the foundation of mathematics, though the success of mathematics in unveiling the mysteries of nature stood irrefutable and immutable. The enormous growth of mathematics till the nineteenth century and the paradoxes necessitated critical analysis not on'y of the notions but also of the methods of mathematics to put it on a sound foundation. Here also approaches varied giving birth to different schools of philosophy. The platonist school that remained almost the solitary authority to explain mathematics broke down to radical schools like the Formalist and the Intuitionist. The philosophers of all ages raised many fundamental questions as to the nature and objective of mathematics, its relevance and relation to science and human existence, its methods and concepts and above all the meaning of mathematics. In this essay we make an overview of some of these fundamental issues only.

1.1 Search for an Identity

What is mathematics?

This is perhaps the toughest question in the whole of human knowledge and surely is as difficult as to answer the question 'who am I ?' The various attempts made by many philosophers and mathematicians have either failed to give a comprehensive description of it or tactfully sidetracked it to give a roundabout answer very much like 'mathematics is what mathematicians do'. One of the major reasons for this difficulty can be attributed to the changing aspect of mathematics for the last five thousand years. In the days of Egyptians and Babylonians, mathematics meant the study of whole numbers and some geometric objects. The Greeks enhanced the scope of mathematics by broadening the frontiers of mathematics to astronomy, mechanics and geography. Thus geometry and arithmetic became much more enriched. So mathematics had a broader implication. Since the abstraction and axiomatization started in the hands of the Greeks, mathematics by now had become more obscure. After the dark period in the history of Europe; when the lights of the Greek wisdom pervaded the European intellectual life and the Renaissance took away the clouds of ignorance, superstition and traditionalism, science had a new birth and mathematics became a driving force in this intellectual upliftment. In multifarious activities, mathematics spread itself and helped uncover the mysteries of nature. During this period of darkness the Chariot of mathematics was drawn by the Arab and Hindu mathematicians who developed the long-neglected algebra to an admirable height. Astronomy too flourished during this period and arithmetic gained momentum by some outstanding discoveries. The 17th, 18th and 19th centuries marked a steady progress of mathematics towards abstraction and formal manipulation. Logic played a key role in this progress. So gradually it became extremely difficult to define mathematics in its entirety. This has been echoed clearly in the attempts of Russell, Poincare, Peirce and Whitehead. Benjamin Peirce said in his famous memoir on "Linear Associative Algebra" "Mathematics is the science which draws necessary conclusions". In his 'Universal Algebra', Whitehead declared, "Mathematics in its widest significance is the development of all types of formal necessary deductive reasoning''. Russell in his characteristic tone uttered. ''Mathematics is the subject in which we do not know what we are talking about or what we say is true''. The logicians of the 20th century went further to prove that all of mathematics can be derived from logic and therefore it is in a sense applied logic. But a careful look into these definitions will, in the light of experience, amply clarify the position of mathematics in the perspective of the above notions. In fact, mathematics is mathematics. It has a distinct status compared to any other subjects including physics and logic. While the hypotheses in physics seek more or less to explore the designs of nature and in this way are somewhat based on empirical observations, mathematical axioms stand for none. Thus the results derived deductively in mathematics can be termed as mathematical truths. Mathematics can be defined as a quest for mathematical truths which can be quite different from empirical truths. So to understand what is mathematics it is more important to know first what are mathematical truths and what the nature of mathematics is.

1.2 Mathematical Truth

Euclid in his 'Elements' has set forth a model on which the whole of mathematics should be built up. Aristotle has bequeathed to us the principles of deducing valid conclusions. These two are the greatest achievements of the Greeks which we have inherited. Since nature appeared to men with its formidably mystic robe, the first objective of human exploration was to expose dreadful nature with its tooth and claw, to understand its mysterious functions and to get over our psychological insecurity about the hands of a controlling God. In this venture, quite naturally and logically, the basis was one's experience and observations.

The model of geometry, built up by Euclid was therefore based on certain axioms which had their genesis rooted in experience and the deductions therefrom were entirely commensurate with the logical principles codified by Aristotle. The theorem obtained thereby were taken as truths. Their overwhelming success in the demystification of nature established the credibility of mathematical truths. But with the advancement of mathematics as non-Euclidean geometries came to explain nature in equal proportion with Euclidean geometry, the 'truth' of any mathematical truth became obscure and the problem of consistency added fuel to this fire of misgiving. So a clearcut distinction between empirical truths and mathematical truths was imperative. Since then mathematical truths are not empirical truths and are not bound to satisfy physical conditions. They stand for logically meaningful hypotheses or theorems derived from such hypotheses on the basis of fundamental logical principles. Eventually the discovery of the non-Euclidean geometries had brought about a complete change in the nature of mathematics. The fact that set theory can be developed with the continuum hypothesis or its negation, with or without the axiom of choice has rendered us to a position where we can not claim any theorem to be a truth. The Gödel's Proof of the inconsistency of any formal mathematical system that encompasses arithmatic has taken away the ground beneath our feet. The challenge of the logical principles, particularly the law of the excluded middle has subdued the status of mathematics to that of a lunatic who deserves no respect from any sane person. So mathematical truths now stand for no truth but merely statements which have been arrived at by some logical principles, right or wrong, and which have not met with exactly opposite assertions as such.

1.3 Nature of Mathematics

As the nature of mathematics is time-dependent, with the passage of time, this nature of mathematics has also changed. What mathematics meant in the Greek period or even before the 17th century is very different from what it means now. But there are certain intrinsic characteristics that have survived through ages. The nature of mathematics will be best understood if we look into these intrinsic properties on one hand and their interactions with other sciences.

From the time of Plato, it had been a dominant question among others whether the numbers and the geometric notions are parts of reality or not. The question of reality did not only pertain to mathematics alone but turned out to be a problem of philosophy in general. For Plato, an important perhaps man's most important intellectual task was to distinguish appearance from reality. He held that there are mind-independent, definite eternal objects which we call 'one' 'two', 'three' etc., the arithmetical forms and there are mind-independent definite eternal objects which we call 'point', 'line', 'circle' etc., the geometrical forms and this world of forms is different from the world of sense perception and this world of forms is apprehended not by the senses but by reason and constitute the subject matter of mathematics. Mathematical propositions are only about mathematical forms. Thus the proposition that $1 + 1 = 2$ and all the other true propositions of arithmetic and of geometry are necessarily true because they describe unchangeable relations between unchangeable objects, namely the arithmetical and geometrical forms. Their necessity is independent of their being apprehended by the discoverers of mathematical truths, independent of any formulation and thus of any rules governing a natural or artificial language. Rejecting Plato's distinction between the world of forms said to be true reality and that of sense experience which is only to be understood as an approximation to the world of Forms, Aristotle observes that the form or essence of any empirical object constitutes a part of it in the same way as does its matter. He pays much more attention than Plato did to the structure of whole theories in mathematics as opposed to isolated propositions, and distinguishes clearly between—

(i) the principles which are common to all sciences,

(ii) the special principles which are taken for granted by the mathematicians engaged in the demonstration of theorems.

(iii) the definitions which do not assume that what is defined exists.

(iv) existential hypotheses which assume that what has been defined exists independently of our thought and perception.

Like Plato and Aristotle, Leibnitz, one of the most distinguished philosopher-mathematicians of the 18th century developed a philosophy of mathematics which reflected his metaphysical vision of great beauty and profundity. His position in logic that every proposition is reducible to subject predicate form is paralleled by his mataphysical doctrine that the world consists of self-contained substances which he called monads and which do not interact. He presented a philosophical thesis concerning the difference between truths of reason and truths of fact and their mutually exclusive and jointly exhaustive character on one hand and the methodological idea of using mechanical calculation in aid of deductive reasoning not only within those disciplines which belong traditionally to mathematics, but also beyond them on the other hand. Truths of reasoning are necessary and their opposite is impossible. Truths of fact are contingent and their opposite is possible. Thus truths of reason are grounded in the principle of contradiction which he takes to cover the principles of identity and of the excluded middle. Not only trivial tautologies but all the axioms, postulates, definitions and theorems of mathematics are truths of reason. In a similar sense a truth of fact is to be regarded as having a subject containing its predicate is far less clear. Indeed, in order to explain the meaning of his assertion that the subject of a truth of fact contains its predicate, Leibnitz had to bring in the notions of God and of infinity. Leibnitz's philosophy does not render much help to Mathematics. According to him that $2+3=5$ is true on the basis of the law of contradiction. The concrete representation in suitable symbols of a complicated deduction is in his words a 'thread of Ariadne' which leads the mind. Leibnitz's programme is first of all to devise a method of so forming and arranging characters and signs that they represent thoughts i.e. they are related to each other as are the corresponding thoughts. In fact what Leibnitz said about the symbolization of deductive reasoning is full of prophetic insights varying from the clear grasp of possible tasks to vague hints. Immanuel Kant (1724-1804), German Philosopher and Scientist, propounded a philosophy which lies in a sense between the rationalist philosophy, represented by Leibnitz and the empiricist philosophy represented by Hume. Differing from both Leibnitz and Hume he observed that though all propositions are divisible into two classes viz. analytic and synthetic, yet synthetic propositions can be further classified as a priori and a posteriori i.e. non-empirical and empirical. According to him, synthetic a posteriori propositions are dependent on sense-perception in that any a posteriori proposition, if it be true, must either describe a possible sense perception or logically imply propositions describing sense perception. Synthetic a priori propositions on the other hand are not dependent on sense perception. He further divided a priori propositions into two classes 'intuitive' and 'discursive'. The intuitive are primarily connected with the structure of perception and perceptual judgement, the discursive with the ordering function of general notions. The so-called principle of causality is an example of a synthetic a priori proposition. According to Kant, all propositions of mathematics belong to the intuitive class of synthetic a priori propositions. Kant's classification reminds us of Plato's and Aristotle's attitude towards mathematics. To Plato, mathematics meant a subject which studies idealized notions of the empirical objects and these objects he called Forms. So his was an idea close to Pure Mathematics. Aristotle on the other hand put equal emphasis on both Forms and the empirical objects and therefore to him mathematics was divisible into two types, pure and applied. Definitely this was implicit in Plato' observation also. But what Aristotle emphasized is that the statements of applied mathematics would approximate to statements of pure mathematics. In fact, Plato considered mathematics not as an idealization, by the mathematicians, of certain aspects of the empirical world but as the description of reality and to Aristotle, both idealization and the empirical observation were reality i.e., $1+1 = 2$ is true as an idealization and so is the fact that one pen and another make two pens.

Kant made this position much more vivid by distinguishing all synthetic propositions as a priori and a posteriori. To him pure mathematics is not analytic, but synthetic a priori and applied mathematics is synthetic a posteriori. The propositions of pure mathematics which by the time of Kant were no more than pure geometry, pure arithmetic and algebra are synthetic because they are about the structure of space and time as revealed by what can be constructed in them and they are a priori because space and time are invariant conditions of any perception of physical objects. Pure mathematics has for its subject matter the structure of space and time free from empirical material. Applied mathematics has for its subject matter the structure of space and time together with material filling it.

What Plato, Aristotle, Leibnitz and Kant viewed regarding the nature of mathematics has permeated in a polished way to recent mathematical doctrines. Pure mathematics though fed by physics and other natural and social sciences has divorced them on the plea that the rich fabric of creations by problems of the real world can be strengthened and illuminated by the recognition of identical mathematical structures in dissimilar situations and their common abstract basis. There is no doubt enough truth in it, but fanatic isolationist abstructionisms are dangerous because the life blood of mathematics comes from reality and if the body does not get blood, it courts death. The nature of mathematics as has been discussed in the earlier paragraphs relates to the intrinsic characters only. To understand it from the relational standpoint it is important to have an overview of the different philosophical movements that took the whole of the first half of the twentieth century. In fact these movements started much earlier but took a serious dimension only after the turn of the century.

1.4 The Philosophical Schools

Even since the creation of logical principles by Aristotle, mathematicians were greatly impressed by its tremendous success in almost all areas of human knowledge. This state of obsession continued till in the late 18th century the fifth postulate of Euclid's 'Elements' raised good deal of confusions. In fact, never before has this parallel postulate given mathematicians enough solace and satisfaction. When a geometry was discovered independently by Lobatchevsky (1793-1856) and Bolyai (1802-1860) which essentially differed from that of Euclid but fitted our physical space quite well, rather equally well as Euclid's geometry, the age-old conviction and faith both in the logical principles and in the absolute supremacy of Euclid's geometry were torn into pieces. In effect this was a crisis of reason. But this crisis did not deter the progress of mathematics. New and revolutionary discoveries poured into the world of mathematics on one hand and on the other a group of intellectuals determined to examine each of the earlier discoveries critically to make mathematics rigorous and well founded. They questioned almost all basic notions of mathematics, its essence and validity, its significance and relevance and tried to give explanations from their own philosophical views. Thus different schools of thought were formed and the first of them is known as the logistic school. They were so impressed by the power of the laws of logic that they contemplated to prove the whole of mathematics as a part of logic. Since, to them the laws of logic were a body of truths, they contended that mathematics must also be true and since truth is consistent, mathematics must also be consistent. Thus what was seminal in the ideas of Leibnitz and what really moved Dedekind got by now an explicit expression in the hands of Gottlob Frege (1848-1925) towards the end of the 19th century. Frege believed that the laws of mathematics are what is called analytic and they say no more than what is implicit in the principles of logic which are a priori truths. Not all of mathematics may be applicable to the physical world but certainly it consists of truths of reason. Frege's attempt was no doubt sincere but lacked clarity and insight which was amply demonstrated in Russell's work. In fact, Russell (1872-1970) in collaboration with Whitehead (1861-1947) worked out the details of the programme. The work starts with the development of logic itself and then axioms of logic are carefully stated from which the theorems are deduced to be used in subsequent reasons. The reason for the culmination of this logistic school can be attributed not only to the discovery of non-Euclidean geometries but to the revolutionary work of Cantor (1845-1918) in developing set theory. It is beyond the scope of this article to discuss how much success was achieved by the logistic school. Nonetheless it is worth mentioning here at least a large portion of mathematics could be shown as derivable from logic. This amply illustrates the situation with regard to the nature of mathematics. When the philosophical texture of mathematics had been experiencing onslaughts of logic, another radical school was in the making. The ingredients that remained buried in the hearts of Descartes (1596-1650), Pascal (1623-62), Kant (1924-1804) and Kronecker (1823-1891) and could never precipitate in totality now became manifest in the writings of Brouwer. What Pascal tried to say in his statement "The heart has its own reasons, which reasons does not know" now became explicit in the Dutch Professor's thesis that Mathematics is a human activity which originates and takes place in the mind and it has no real existence outside human minds. Thus it is independent of the real world and the mind recognizes basic, clear intuitions —not sensuous or empirical but immediate certainties about some concepts of mathematics. To Brouwer, mathematics is synthetic, composed of truths rather than derived implications of logic. He further observed that mathematics is a wholly autonomous, self-sufficient activity and is independent of language. The world of mathematical intuitions is opposed to the world of perceptions. According to him, mathematics is not bound to respect the rules of logic and for this reasons knowing mathematics does not require knowing formal proofs. He insisted that as natural numbers, the operations of addition, multiplication and mathematical induction are intuitively clear, Cantor's infinite sets and Zermelo's axiom of choice were rejected by him. The concepts or objects acceptable to the intuitionists are those that can be constructed i.e. one must give a method of exhibiting the entity or entities in a finite number of steps or a method of calculating them to any desired degree of accuracy. The intuitionists maintain with respect to infinite sets that there is a third state of affairs viz. there may be propositions which are neither provable nor disprovable. Under the intuitionist view, the classical and the logistic constructions of the system of real numbers, the calculus, the modern theory of real functions, the Lebesque integral are not acceptable. In fact intuitionists gave a new dimension to the mataphysics of mathematics. Whether this really made mathematics healthier or not is still a matter of controversy. The third school that left a lasting impression to the world of 20th century mathematicians is popularly known as the formalist school founded by David Hilbert (1862-1943) who wanted to see whatever mathematical truths are discovered, they are of real significance. Thus came the question of consistency of any mathematical system which became poignant in the hearts of the formalists. Though Frege developed logic to encompass much of mathematics, he eventually demonstrated the power of symbols which was immediately taken up by the formalists who realised the tremendous potentiality of such formalism. The thesis of the formalists was to prove that all of mathematics can be put in a formal system to achieve the highest degree of generality. A formal system consists of a finite set of symbols and of a finite number of rules by which these symbols can be combined into formulae or statements. A number of such statements are designated as axioms and by repeated applications of the rules of the system one obtains an evergrowing body of provable statements. A proof of a given statement or formula is a finite sequence of statements that starts with an axiom and ends with the desired statement. The sequence is such that every intermediate statement is either an axiom or is derivable by the rules of the system from statements that precede it. Hilbert and his collaborators in the scheme also developed a well-defined theory of how a proof works and what a proof is, which is often referred to as metamathematics. In this theory he not only emphasised on the classical constructivism like the intuitionists, but also required finitary methods for the demonstration of a proof, which in essence differs from the intuitionists' scheme and consistency and completeness were the most essential requirement of any mathematical system.

The objective of the set theoretic school pioneered by Dedekind (1831-1916) and Cantor (1845-1918) was to reduce all of pure mathematics to set theory. Though the programme of the logistic school was very much the same, the set theorists preferred a direct approach through axioms. The aximatization of set theory was first undertaken by Zermelo in a paper of 1908 and later (1922) improved by Fraenkel (1891-1965). The set theorists were never seriously concerned about the consistency and completeness of the axiomatized set theory.

Before the middle of the nineteenth century there was little conscious effort at synthesis or unification. Of course, Euclid's 'Elements' represent a major synthesis and Descartes' analytic geometry was a major unification of algebra and geometry, but mathematics after Newton were too busy joyfully exploring the new vistas opened by Calculus. Then came a reaction and a trend towards organization developed because the body of mathematics had grown so large that organization became necessary, lest parts of the subject ceased to communicate with one another. Also unrestrained intuition unhampered by the rigid standards imposed by a formal system was beginning to get mathematicians into trouble. Euclid was once the unsurpassed model of rigour. But as mathematicians were exposed to an ever-widening stream of problems, critical senses sharpened and logical senses grew subtle and more refined. So what was once obvious to all now became objects of critical analysis. Some notions which never critically examined now surfaced and drew the attention of many. One such is the notion of 'infinity'. Aristotle is the first of all mathematicians to have given detailed formulation of the problem of mathematical infinity. He analysed the notion in two ways, viz. actual infinity and potential infinity. He distinguishes the possibility of adding a further unit to the last member of any sequence of numbers, say, the sequence of natural numbers from the possibility of conceiving a collection of all natural numbers. According to him, the first implies the notion of potential infinity and the second implies the notion of actual infinity. Though Aristotle advocated the acceptance of potential infinity, for nearly two millenia both were used by mathematicians without sufficient justification. Kant is perhaps the first to foresee the tremendous potentiality of the notion of actual infinity. The contrast between the actual infinity which cannot be constructed but is nevertheless 'needed' and the potential infinity which can be constructed was often emphasized by Kant. But the real distinction was felt deeply and rigorously when the monumental work by G. Cantor (1845-1918) came to light. A conscious examination of the foundation of analysis and inquiry into the meaning of the real number system and of the nature of functions defined on them led to problems that are at the origin of modern set theory and of modern mathematical logic. Cantor's set theory not only gave answers to the question posed previously by Galileo, Giordans Bruno, Bolzano, Weirstrass, Dedekind, Frege and Peano but also opened a new vista and dimension of the existing mathematics. A philosophical school was also enrooted by this discovery which sought to explain many of the mathematico-philosophical phenomena. In fact after Peano's work on axiomatization of arithmetic together with that of Boole's on algebra of sets, the creation of set theory made it imperative to attempt the construction of a system of axioms for the whole of mathematics. The great formalist programme of Hilbert was to erect an axiomatic edifice sufficient for all work in mathematics. Not that there was unanimity as to the admissibility of all axioms. In particular, the axiom of choice was felt by some to be of dubious character and perhaps inadmissible because of the strange, seemingly paradoxical consequences of its applications. In fact, throughout the history of mathematics new objects were constantly being discovered with properties to which the mathematical thinking of the period was unaccustomed even without the use of axioms stating the existence of 'non-constructive' entities. The process of generalization in mathematics has very often started from such 'surprising' discoveries. Their logical consequences, no matter how strange they might have appeared at the moment, had to be accepted and have often formed a basis for new systems. The school of intuitionists headed by Brouwer and for a time by Lebesgue and H. Weyl have attempted to confine mathematics to more constructive or operational system than mere existential ones.

Hilbert's programme implied a faith in the completeness of an all embracing axiomatic system for the whole of mathematics. The work of Bernays, Fraenkel and von Neumann had already laid solid foundations for axiomatic systems of set theory and mathematical logic. There was reason to hope that all meaningful problems in such systems were decidable. Then Gödel (1906-1978) in his revolutionary paper of 1931 proved that in any sufficiently rich system of axioms there will exist statements which though meaningful are undecidable within the system and cannot be made complete by the addition of any finite number of axioms. Another problem that drew the attention of the mathematicians of the twentieth century is known as the consistency problem which requires every mathematical system to be consistent i.e. contradictory theorems are impossible. Gödel's proof of the incompleteness was based on the assumption of the consistency of the system. In fact for healthy mathematics it should be a quite logical requirement that any mathematical system be consistent. But this demand regarding the nature of mathematics met with rejection by the Gödel's result that it is not possible to prove consistency of any system that is broad enough to endorse the system of whole numbers.

Thus the different schools of thought whether they are the classical Platonism or modern Formalism or Logicism or the radical intuitionism have had tremendous influence in reshaping mathematics from the foundational standpoint preserving most of the mathematics that existed till the turn of the century.

1.5 Indian Panorama

Since India is accredited with the genesis of one of the richest philosophies that has survived at least five millenia, it is quite reasonable to think that mathematics was surely within the ambit of its musing. The amazingly rich arithmetic and algebra, the tremendous success in areas of astronomy and incredible vastness of Buddhist logic bear clear evidence to the depth of the Indian philosophy of mathematics. Owing to cryptic aphoristic expressions of the ancient Indian philosophy and non-availability of a written history, no cogent development is noticeable except some stray and scattered writings and what is available to hand is difficult to put into a coherent whole. But fragmentary that may be, it evinces the tendency of the Indian philosophical thoughts towards mathematics. Interestingly, enough materials on the philosophy of science have been bequeathed to us but there is very little on mathematics proper. The argument that has been given by Brojendranath Seal, a great savant of India, for this is interesting and noteworthy. He maintains that Hindu Philosophy on its empirical side was dominated by concepts derived from physiology and philology just as Greek Philosophy was dominated by geometrical concepts and methods. He continues to say that comparative philosophy in its criticism and estimate of Hindu thought must take note of the empirical basis on which the speculative superstructure was raised. Not only about mathematical truths but also about any scientific truth they had clear and distinct position. According to them, the ultimate criterion of truth is found not in mere cognitive part presentation but in the correspondence between the cognitive and the practical activity of the self, which together are supposed to form the circuit of consciousness. Truth, the Buddhists contend, is not self evident, not self-evidence, not the agreement between ideas, nor the agreement of the idea with the reality beyond, if any, for this cannot be attained direct, but the harmony of experience which is implied when the volitional reaction that is prompted by a cognition and that completes the circuit of consciousness meets with fruition. This is the material aspect of truth. The formal aspect is given in a principle which governs all presentations in consciousness and which combines the three moments of Identity, non-Contradiction and Excluded Middle in every individual cognitive operation. As regards inference also, they had a differing view from the Greeks. They maintain that inference is to be drawn not by perception or direct observation but through the instrumentality or medium of a mark, that a thing possesses a certain character. Inference is therefore in Indian Philosophy based on the establishment of an invariable concomitance between the mark and the character inferred. Hindu Inference is therefore neither merely formal nor fully material, but a combined Formal-Material Deductive-Inductive process. It is neither the Aristotelian Syllogism nor Mill's Induction i.e. Material Inductive process, but the real inference which must combine formal validity with the material truth, inductive generalization with deductive particularisation. One of the predominantly materialist Hindu philosophies propounded by Chārvaka had two subschools, one who accepted perception as a valid source of knowledge as well as the reality of natural law, and the other who impugned all kinds of knowledge, immediate as well as mediate and all evidence, Perception as well as Inference. The Chārvakas hold that the principle of causality which the Buddhists assume to be a ground of an induction is itself an induction which amounts to circularity in reasoning that every inference is based on an unconditional invariable concomitance which itself must be inferred, as universal propositions cannot be established by our limited perceptions. Thus there is a regression ad infinitum and that the nexus between cause and effect or between the sign and the thing signified is only a mental step or objective association based on former perception.

The Buddhists however take their stand on the principle of the Uniformity of Nature particularly the uniformity of succession in the relation of cause and effect and the uniformity of coexistence in the relation of genus and species and in all cases of inference based on the Uniformity of nature, the relation is that of inseparableness or non-disjunction between the mark and the character inferred. The Nyaya School of Philosophy, more sophisticated and radical, demolishes the Buddhist contention about identity of essence. The Nyaya writers, being realists, do not impugn the reality of the genus like the nominalists or the nominalistic conceptualists, but they point out that the inseparableness in such cases can only be established by the experience of unbroken uniformity. Uniform agreement in presence with uniform agreement in absence not the mysterious identity of essence irresistibly perceived in any individual case or cases is the only basis for constituting genera and species in natural classification. Some of the later Nyaya writers point out that individuals do not always possess in nature all the characters that go to form the definition of the class to which they are referred.

Thus though not very specific, the Hindu Methodology or the Hindu doctrine of Scientific Method has very clear indication as to how mathematical truths are to be derived, how empirical observation coupled with intuition discover mathematical truths and what is the complete form of valid proof. In fact, a perfect blend of the logico-intuitionist approach is clearly visible, though the obvious trend was towards intuition. Numbers, according to them, are to be taken as names of certain classes and the rest of mathematics is to be developed through experimental ratifications. It is interesting to note that when the Buddhist scholars engaged themselves with the development of logic in its full generality, the Jainas devoted themselves in harnessing the potentiality of mathematics proper, regarding mathematics as an integral part of their religion. A section of their religious literature was named 'Ganitanuyoga'. This could happen possibly because Mahavira, the founder of Jainism, was himself a good mathematician. Among the religious works of Jainism those that are important from the viewpoint of mathematics are Surya Prajnapti, Jamboo Dwipa Prajnapti, Sthananga Sutra, Uttaradhyayana Sutra, Bhagawati Sutra and Anuyoga Dwara Sutra, all written between 500 B.C. and 300 B.C. Of the later commentators Bhadrabahu and Umaswati had exceptional brilliance in dealing with the subject in their original texts—'Bhadrabahavi Samhita' and 'Tattwarthadhigaina Sutra Bhashya' respectively. Whereas in the Buddhist works 'Lalitta Visthara (circa 100 B.C.) numbers upto to the size of 10^{53} are noticeable, in the Jaina scriptures apropos of time and space numbers of the size of 10^{194} have been used. This quite naturally hints at the plausibility of the concept of infinity. In fact, it is no more a matter of conjecture that the Jainas had the concept of infinity—both actual and potential as enunciated by Aristotle and later Kant, Leibnitz and Cantor. Surprisingly, they had even the notion of infinite dimension.

In a nutshell, it can be affirmed that the ancient Hindus had a very rich philosophy of science which flourished in diverse areas including mathematics, but certainly it would not be reasonable to compare its prolificity with the 20th century emanations, because science and technology as such have undergone enormous change by this time. It should neither be belittled in view of the period when it developed, viz. between 1000 B.C. to 500 A.D. The development of the philosophy of mathematics during the period from 500 A.D. to 1500 A.D. is so insignificant that it hardly demands any credit, the reason being the aggressions by Muslim invaders and the predominance of Muslim culture and tradition which patronized anything except the Hindu Philosophy of religion of which the philosophy of mathematics was an integral part. The Buddhist school, the Jaina tradition, the Hindu Nyaya and Nabya Nyaya Schools did not add any new theory during this period except continuing the debate that started long since. This does not certainly mean that the growth of mathematics was hindered during this time due to lack of enough philosophy. The situation was rather opposite. The interaction of Hindu and Muslim mathematics that came from Baghdad enriched Hindu Mathematics, particularly in the sphere of Algebra and Astronomy. The classic works like Brahma Sphuta Siddhanta (630 A.D.) Khandakhandyaka (628 A.D.) by Brahmagupta, Ganita Sarasangraha (850 A.D.) by Mahavira, Trisatika (750 A.D.) by Sridharacharya, Patiganitasara (750 A.D.) by Sridhara, Ganitatilaka by Sripati, Arya Bhateeya (499 A.D.) by Arya Bhatta I and Arya Siddhanta (950 A.D.) by Arya Bhatta II, Siddhanta Siromoni by Bhaskara Charya and Surya Sidhanta stand as glaring evidence to the height of achievements attained during this period.

1.6 Conclusion

It is argued sometimes that Mathematics was born before the philosophy of mathematics came into existence and mathematics did serve mankind to unveil the mysteries of nature, to have control over the nature and to help our civilization progress to an incredible height. There is no doubt that there is sufficient truth in this, but that to be precise— is only a half truth. The fact remains that there had always been a current of philosophy with a current of mathematics and these two currents are so inseparably related that none can go ahead of the other. Their interdependence is so intricate that sometimes we are dazzled by the progress of mathematics and at other time, by the advancement of philosophy. A serious look into the current of historical events in the world of mathematics will make it clear that whenever a new idea came that puzzled the mathematicians, they started looking back, examining the validity of reasoning and the logical basis of the notions. Again when a plausible solution was obtained, they looked forward to create new mathematics. This is how the physiology and anatomy of mathematics are to be looked at. This certainly does not mean all problems of the philosophy of science have been solved or all questions have been answered, but surely the retrospections and the subsequent attempts to resolve the crises have given new insights into mathematics —the insight that have made mathematics the queen of science forever.

CHAPTER II

THE PHILOSOPHICAL SCHOOLS OF MATHEMATICS

2.0 Introduction

Mathematics is a majestic creation of the human reason. Endowed with this faculty man made mathematics not merely for his pragmatic interests but also for the pleasure of creation. It took several thousand years to establish the authority of mathematics with its boundless treasures. The history of development of mathematics is not a history of conquests alone, it is replete with innumerable failures, controversies and crises. The growth of mathematics was not continuous all along or at the same rate. There had been many catastrophic turns and twists. The factors that are responsible for such disasters are basically philosophical—man's never ending queries as to the definition and nature of mathematics, its objectives and scope.

The genesis of such questions can be traced back to one thousand years before Christ when the Hindus and the Greeks had been musing over what is true mathematics. The discovery of geometry gave a model, but such models were missing for arithmetic. The crises created by the discovery of zero in the Vedic period and negative numbers around 600 A.D. by the Hindus made the wheel of progress rotate back and forth till the 18th century when the discovery of non-Euclidean geometries by Lobatchevsky (1793-1856), Bolyai (1802-1860) and Riemann (1826-1866) actually brought it to a standstill. The centuries old ideas of explaining nature by Euclidean geometry had a shocking rebuff. With the advent of the 19th century the scenes of the mathematical world started changing fast. Mathematicians concerned themselves with the foundational aspects which had been hitherto uncared for. The subsequent discoveries of complex numbers, quarternions by Hamilton (1805-1865) and matrices by Cayley (1821-1995) complicated the situation. In the mean time Rene Descartes (1596-1650) and Immanuel Kant (1724-1804) started questioning the principles of logic that had been the basis of all mathematical judgements and deductions. The strongest blow to the philosophy of mathematics was given by George Cantor (1845-1918) who brought about almost a revolution in mathematics by introducing the concept of transfinite sets. The concept of infinity had been perturbing mathematicians long since, almost from the Greek period and the mathematicians had made very weak attempts to solve and settle the problems of infinity by distinguishing potential infinity from actual infinity. However nothing else than the concept of transfinite sets could shake the foundation of mathematics so severely and forced mathematicians to solve the crisis from different angles. The result was a clear division of the elite of the mathematical world into four major factions, now more popularly known as philosophical schools. Each of these schools had its own philosophy with which it tried to explain all major mathematical events including mathematical concepts. In the International Congress of Mathematics held at Paris in 1900, this split was distinctly noticed. The following onslaughts on the foundation especially with regard to consistency and completeness of mathematics by Russell (1872-1970), Hilbert (1862 - 1943), Gödel (1906 - 1978), Church (1903 - 1919), Löwenheim (1887-1940), Skolem (1887-1963) compelled all leading mathematicians of the 20th century to join one or the other of the camps.

The concerted efforts by groups of mathematicians belonging to different schools engineered new outlook towards mathematics and certainly were not futile. The world experienced sometimes fierce battle between the intellectual giants—the battles that are still not over but not as conspicuous as before.

The most interesting of these schools was certainly the Intuitionist one.

2.1 The Intuitionist School

To intuit does not simply mean to visualize but also that the vision should come from the mind which is illuminated with the light of wisdom and knowledge. To Kant, the mind organizes the perceptions and these organizations are intuitions of space and time. Space and time do not exist objectively but are the contributions of the mind. The experiences of discoveries of deep and penetrating results convinced many discoverers of the tremendous role of intuition in comparison to rational reasoning. So when the centuries old ideas and constructs of mathematics fell flat exposing the emptiness of human faculty of reasoning (e.g. in Euclidean Geometry), many started believing intuition dominates logic in any mathematical discovery.

The celebrated mathematician Descartes who almost solitarily took over the task of rigorous mathematization of the Laws of Mechanics, Philosophy, Logic, Law, Biology, Physics and what not, (one of the most successful of all mathematicians ever born) supported the intuitionist view of mathematical philosophy. But a queer fact is that their view was never conspicuous in any of their writings. Brouwer (1881-1966) can in this sense be designated as the founder of this philosophical school and Descartes and Kant as mere forerunners.

According to Brouwer, mathematics is a human activity which originates and takes place in the mind. It has no existence outside human minds. Thus it is independent of the real world. The mind organizes basic, clear intuitions. These are not sensuous or empirical but immediate certainties about some concepts of mathematics. It is as such clear from the father of this school that proofs are absolutely redundant as long as intuition approves of any result. The questions of consistency and completeness of mathematics are therefore meaningless to the intuitionists. To them mathematics is synthetic and it composes truth rather than derives implications of logic.

Before Brouwer many including Descartes, Pascal, Kant, propounded this intuitionist philosophy, but many more flocked around him in the 20th century which is looked upon as the height of human reason. Mathematicians like Leopold Kronecker (1823-1891), Poincare (1854-1912) and Hermann Weyl who attained spectacular successes in mathematical researches and who earned the distinction of profoundest mathematicians of this century, lost faith in human reason and joined hands with Brouwer in defining the new structure of mathematics and its scope and purpose. They advocate that the heart has its own reasons which reason does not know and it is not to be subdued to the objective principles of logic.

As a dominant philosophy that permeated all of 19th and 20th century mathematics and that lies almost on the opposite pole to that of the intuitionists is the logicist school.

2.2 The Logicist School

Logic is the science of reasoning and is based on certain principles known as laws. These laws were first expounded by the great Greek intellectual Aristotle in his 'Organon' around 300 B.C. He codified three basic principles namely the principle of inductive reasoning, the principle of deductive reasoning and the principle of analogy. When inference is drawn about a whole class of objects on the basis of observation of a part of the class, it is done on the principle of induction. The principle of analogy is the law of concluding about an object by studying a similar object. The principle of deductive reasoning is composed of three fundamental laws viz. the law of excluded middle, law of contradiction and law of syllogism. The law of excluded middle demands that an assertion p is true or its negation $\neg p$ is ture, i.e., symbolically $p V(\neg p)$. The law of contradiction states that an assertion and its negation can not be simultaneously true. The syllogistic reasoning stipulates that there must be two premises to draw a conclusion. That man is mortal and Socrates is a man implies Socrates is mortal is an example of syllogism. These principles of logic had been in vogue for thousands of years and are responsible for the spectacular successes of mathematics in interpreting the designs of nature and the universe. Though they proved their worthiness, mathematicians were never content, as the paradoxes right from the Greek period like the one of Zeno had been discomforting. Many pointed at the insufficiency of the principles of logic to explain every mathematical relation and suggestions for improvement poured in from different corners. Leibnitz (1646-1716) in the 17th century and Gottlob Frege (1848-1925) in the 19th century proposed a revaluation of the logical principles and suggested overhauling which was actually done by Russell (1872-1970) and Whitehead (1861-1947) in the early 20th century. While these men were busy developping the frame of logic they, on the other hand, were consciously or otherwise, contributing to the development of the logistic thesis that all of mathematics is derivable from logic. Infact, in his three outstanding works viz. Concept Writing (1879), Foundations of Mathematics (1884) and two volume Fundamental laws of Mathematics (1893, 1903), Frege proceeded to derive the concepts of Arithmetic and the definitions and laws of numbers from logical premises and he believed that from the laws of numbers it is possible to deduce Algebra, Analysis and Analytic Geometry. Russell had conceived the same programme and ran across Frege's work. In his 'Principles of Mathematics' (1903) he expressed his conviction with exultation "The fact that All Mathematics is Symbolic Logic is one of the greatest discoveries of our age". As regards the most disturbing problem of the 20th century viz. the consistency problem he believed with Frege that if the fundamental laws of mathematics could be derived from Logic then the problem of consistency would be solved as Logic was a body of truths and hence the laws of mathematics would also be truths. This faith he did not leave for the future logicists to prove but he himself and Whitehead developed in details the proof of this identification of logic and mathematics. For this purpose he began with some undefined ideas, proposition, the negation of a proposition, the conjunction and the disjunction of two propositions and propositional functions and six axioms of logic. To resolve the paradoxes that evolved due to the introduction of the notion of a set, it was necessary to enlarge the frame of logic and this was done by Russell and Whitehead by introducing the theory of types. On the basis of this theory they showed that it is possible to avoid the paradoxes but it was discovered soon that the theory has posed again certain problems in connection with the concept of a least upper bound of a bounded set of real numbers. So another subtle axiom viz. the axiom of reducibility was introduced by them and the notion of natural numbers could then by introduced with ease in terms of propositional functions. From natural numbers they wanted to carry the program to rational and irrational numbers, and finally to transfinite numbers when they realized the necessity of involving two more axioms, namely, the axiom of infinity and the axiom of choice. With these axioms Russell and Whitehead tried to found mathematics on logic and undoubtedly attained great success.

But there was no dearth of critics of Russell's logistic program. Objections were voiced by many including supporters of logistic philosophy like Ramsey and the major attacks were directed towards the axiom of reducibility. Hermann Weyl rejected the axiom unequivocally, while Poincare observed that it was not an axiom of logic and the principle of mathematical induction which is proved by this axiom is in effect the axiom itself and hence the thesis fails. Similar objections were raised against the axiom of infinity and many refused to accept this as an axiom of logic, as they could not find the slightest reason for believing its truth. The axiom of choice again engendered maximum controversy and discussions compared to any other axiom. But these three axioms were retained in the frame of logic, as but for these the major of mathematics could not be derived and certainly such a gymnastic was healthy for mathematics.

The Russell-Whitehead program, no doubt, reduced Arithmetic, Algebra and Analysis to logic, but the non-arithmetical parts of mathematics such as Geometry, Topology and Abstract Algebra could not be fitted into the general structure of logic. So a serious philosophical criticism of the entire logistic position was that if the logistic view is correct, all of mathematics is a purely formal logico-deductive Science based on the laws of thought. Then how can this logistic view explain the wide varieties of natural phenomena like Cosmology, Electromagnetic Theory, Geometry of Space etc. Another serious philosophical concern was whether logicism can explain creativity, which is essentially a product of intuition first. The questions of how new ideas could enter mathematics and how mathematics can possibly apply to the physical world if its contents are derivable entirely from logic are not readily answered and were not answered by Russell or Whitehead.

The long period of conversion of mathematics to logic and the subsequent controversies did expose the limits of the pre-Russellian logic and the fruits of the programme were the enormous expansion of the scope of logic with actual success standing far beyond the reach of the logicians. In 'My Philosophical Development' (1959) Russell confessed "The splendid certainty which I had always hoped to find in mathematics was lost in a bewildering maze. It is truly a complicated conceptual labyrinth". This amply depicts the present state of the logistic school fathered by Russell and furthered by Whitehead, Church, Quine and many others.

When the mathematical world was throbbing with two lively philosophies, namely, logicist and intuitionist, another philosophy fashioned by David Hilbert was gaining ground. This was the popular school of the formalists.

2.3 The Formalist School

Since the presentation of his paper by Hilbert in 1900 at the Second International Congress held at Paris, the mathematical world had been busy settling the burning questions of consistency and completeness and resolution of the paradoxes of set theory. What the logicists did in the meantime was not at all satisfactory to Hilbert, as he was convinced that in the long and complicated development of logic, the notion of whole numbers were used not conspicuously but in effect. He also criticized defining sets by their properties as this required the axiom of reducibility to which he had, like many others, strong objections. Like Russell, he accepted infinite sets but the axiom of infinity was to him an axiom of mathematics and not of logic. Unlike the intuitionists he welcomed the proofs of existence. In fact, his differences in many respects with the logical and intuitionist schools compelled him to set up a new school around 1920. He observed that if classical mathematics is to be preserved, the correct approach to mathematics must include concepts and axioms of both logic and mathematics and the most reliable way to treat mathematics is to regard it not as factual knowledge but as a formal discipline, i.e., abstract, symbolic and without reference to meaning and deductions are to be manipulations of symbols according to logical principles. So Hilbert started with all the logical axioms that Russell began with, perhaps a few more, because he was not interested in establishing an axiomatic basis of logic and some axioms of mathematics such as the axiom of mathematical induction. With the sketch of this program, three of Hilbert's most talented students viz. Ackerman (1896-1962), Bernays (1888-1978) and von Neumann (1903-1957) developed an entirely new approach to mathematics during the years from 1920 to 1930. This approach is now know as "Metamathematics". In this they explained in clear terms what is meant by an objective proof, how to prove the consistency of any formal system and how the paradoxes could be resolved. In metamathematics Hilbert proposed that all controversial reasoning like proof of existence by contradiction, transfinite induction, actually infinite sets, unpredicative definitions and the axiom of choice should be avoided and he called his method 'finitary'. With his metamathematical program he again confidently asserted in the International Congress of Mathematicians held in 1928 : "With this new foundation of mathematics, which one can properly call a proof theory I believe I can banish from the world all the foundational problems".

Whether the formalists were hundred percent successful or not is a matter of controversy but as usual they gave room for prolonged debates on the efficacy and acceptability of many of its axioms and notions. Russell objected to the use of the axioms of Arithmetic and the concept of existence. Though some of the principles were close to those of the intuitionists, Brouwer, the founder of the intuitionist school waged wars against the formalists and made a blistering remark "An incorrect theory even if it cannot be rejected by any contradiction that could refute it, is nevertheless incorrect just as a criminal act is nonetheless criminal whether or not any court could prevent it." He alleged that formalistic approach may avoid contradiction and save a sizable portion of classical mathematics but nothing of mathematical value will be obtained this way".

But Hilbert and his fellows never lost heart. By 1930, he proved the consistency of a somewhat artificial system containing only a portion of arithmetic and was exalted therefore. But two startling results of his student Kurt Gödel (1906-1978), a Professor at the Institute of Advanced Study, proved in 1931, fused Hilbert's dream. He proved that the consistency of any mathematical system that is extensive enough to embrace even the arithmetic of whole numbers cannot be established by the logical principles adopted by the foundational schools including the formalist and that if it happens to be consistent then it must be incomplete. Both of Gödel's results thus dealt a death blow to Hilbert's formalist Philosophy.

2.4 The Set Theoretic School

When the intuitionist, logicist and formalist schools were flowering, attempts were there to explore the possibility of rigorizing mathematics in terms of sets and Dedek'nd and Cantor can be taken as the pioneers in this program. The proper systematic approach was, however, made by Ernst Zermelo around 1908 and developed by Fraenkel. In essence this approach is not very different from the logistic, as set theory is incorporated in the logistic approach, but the exponents of the set theoretic approach wished to define sets without the jugglery of logical principles. The axiomatics used by Zermelo and refined by Fraenkel is now-a-days known as the Zermelo-Fraenkel system. The thesis of this school is to reduce pure mathematics to set theory by founding Arithmatic on the basis of a carefully defined axiomatics of set formation. In fact, the axiomatic system laid down by Zermelo and Fraenkel did actually avoid looseness in the earlier work on sets and their properties and thus steered clear of the paradoxes. As regards consistency of this system, the set theorists are least concerned. The program of Zermelo and Fraenkel attracted many mathematicians of the thirties even some from the Formalist and Logicist Schools who served to modify the axioms. In particular, the system developed by von Neumann (1925) and Bernays in 1937 and sharpened by Gödel in 1940 serves an alternative ax'ometic system and avoided still many paradoxes.

Objections were raised against the use of axiom of choice by the set theoretic school and the latter's casual attitude towards the logical principles was severely criticized. According to the critics the axioms are arbitrary, artificial and based on intuition. Why not then start with arithmetic itself, they argued, since the logical principles are presupposed by the set theorists ?

Of all adherents to this philosophy the most sensitive is the group of French mathematicians working under the pseudonym "Nicholas Bourbaki" who believes that logic is the grammar of Mathematics and does not bother with the problem of consistency. These mathematicians who include Deudonne maintain "For twentyfive centuries mathematicians have been correcting their errors, and seeing their science enriched and not impoverished in consequence, and this gives them the right to contemplate the future with equanimity".

Another classical school eclipsed by the modern schools of thought was revamped and revived to synchronize the differing views of the other schools. This is the Platonist school propounded and rejuvenated primarily by Kurt Gödel, the most distinguished contemporary mathematician.

2.5 The Platonist School

To this philosophical school mathematics is a body of truths about abstract structures, existing independently of us and the logical proofs based on the modified logical principles as the formal manipulation of symbols that express those arguments and truths and of nothing else. Natural Numbers are taken for granted independently of us and the fundamental theorem of Arithmetic is taken literally as a matter of fact. When intuitionism, logicism, formalism and set theory deny the semantic content, the platonist school puts much emphasis on the semantic content of mathematical theories popularly known as the model theory. The central problem of this theory is the question of what properties of structures can be expressed in particular languages. Thus according to platonism, the objects which mathematics studies are necessarily abstract. In this sense the platonist school is close to the formalist school and as such considers pure mathematics as real mathematics divorced from other sciences. The crisis facing twentieth century mathematicians with regard to their stand about the set-theoretic universe resulted in the split of set theory into many pieces and paved the path of another split in the large structure of science. The mathematical platonism itself is the result of this split and does not throw away intuition altogether. The models form the building blocks of the platonist school and accepts major parts of the formalist and logicist schools.

2.6 The Popperian School

Another philosophical school that had taken roots in the intellectual world in the post sixties is the Popperian school which is in essence closest to the traditional utilitarian view and is due to Karl Popper, the great philosopher of the recent times. To the Popperians, mathematics is a public activity and it occurs in a social context. Mathematics has been done and is being done for social need. So mathematics can also survive so long it caters to the social benefits. Posing a problem, formulating a definition, proving a theorem are all parts of a large social process, called science. Its main function should also be to facilitate the ongoing social process of doing mathematics. Thus whatever theorems or concepts are of practical value, standing by the social progress and having stood the tests of time with regard to its validity are incorporated in the mathematics of this utilitarian school. One may misunderstand the Popperian philosophy as advocating applied mathematics, but the fact remains that it is not doing so. According to the Popperian school a serious philosophy of mathematics must satisfy the principle of objectivity, i.e., it must not deny objective reality to any aspects of mathematical activity which have practical utility. In this sense applicable mathematics comes within the purview of this school so long mathematics is meant for the benefit of society.

2.7 Conclusion

Though the movement of rigorization of mathematics mothered different philosophical schools, "it did not end with that achievement rather it enlarged the scope of mathematics and logic as well. The long controversies, attacks and counter-attacks did certainly generate tremendous heat in the mathematical atmosphere, but ultimately the huge firmament of mathematics did help that to radiate brightly and by the first half of the 20th century a decent atmosphere prevailed to the relief of the majority of scientists. An important aspect of this philosophical tangle was its great impact on other branches of science, especially physics. Waves of philosophico-mathematical problems bang against the shore of physics which witnessed an almost parallel movement of axiomatization and rigorization. As a result science in general has been enriched and mathematics studded with jewels. The caves and dentures that might cause collapse in certain parts of the huge palace of mathematics have been repaired. Mathematics grew without its philosophy and it will grow for ever till the last breath of human civilization, because "it is as limitless as that space which it finds too narrow for its aspirations, its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer's gaze, it is as incapable of being restricted within assigned boundaries or being reduced to definitions of permanent validity as the consciousness of life, which seems to slumber in each monad, in every atom of matter, in each leaf and bud cell and is for ever ready to burst forth into new forms of vegetable and animal existence"[2] (James Joseph Sylvester).

CHAPTER III

SPACE AND TIME—A PHILOSOPHICAL OVERVIEW

3.0 Introduction

Ever since the dawn of civilization, man has questioned the meaning of space and time. The measurability of time, its irreversibility, its relation with space, has perturbed the intellectuals of all ages. Whether space is ethereal or a perfect vacuum has been the issue of prolonged debate and controvercy untill Einstein's theory of relativity brought about a revolutionary change in the intellectual world. Standing on the peak of 20th century scientific and technological advancements it is amusing to observe what our ancestors thought about space and time.

Just as the Greeks had queer ideas about space and time, the Indians too had peculiar perceptions about them, expressed in general through maxims requiring elaboration or interpretation. Not all of them can be taken as truths in the light of the latest discoveries nor can they be described as untrue at face value. To have a clear-grasp of the ideas and their philosophical impacts let us look at the Indian and Western views on space and time.

3.1 Western View on Space

In the west the earliest consideration of these concepts is seen in the Greek period by the Pythagorians who identified empty space with air. For special metaphysical reasons Permenides and Melisus denied the existance of truly empty space.

Democritus like the atomists distinguished the atoms and the void which separated them. Lucretius who conceived space as an infinite material entity like a container, distinguished space as a pure void. In 'Timaeus', Plato expressed his idea of space as a receptacle and of the matter in this receptacle as itself mere empty space limited by geometrical surfaces. In thrashing out the question "whether an empty space is a thing or not a thing", Aristotle defined space in terms of place and arrived at a dichotomy—space as a stuff and space as a system of relations between bodies.

In the sixteenth century Descartes held that the essence of matter is extension and so space and stuff are identical, for if the essence of matter is to be extended, then any volume of space must be a portion of matter and there can be no such thing as vacuum. From this view point, Descartes could not explain how one material object is distinguishable from another. Though almost contemporary, Leibnitz held a relational theory of space whereby space is in no sense a stuff but is merely a system of relations in which indivisible substances (which he called monads) stand to one another. We remark here that the issue between the two theories viz. the absolute theory of Descartes and the relational theory of Leibnitz is yet to be decisively settled in the light of the spacetime relation accorded by the theory of relativity. Immanuel Kant, in his 'Prolegomena', provided a curious argument supporting the absolute theory of space initially but in his 'Critique of Pure Reason' argued against both a naive absolute theory and a relational theory and held that space is something merely subjective or phenomenal. Newton, one of the greatest of all scientists ever born, held an absolute theory of space and time and dispensed with any metaphysical view. In fact he had in his mind the development of the Newtonian mechanics that was to revolutionize the existing scientific knowledge. To be more specific, what is important in Newtonian mechanics is not the notion of absolute space but that of an inertial system. An inertial system is one in which there are no accelerations of the heavenly bodies except those which can be accounted for by the mutual gravitational attractions of these bodies. Relative to an inertial system of axes a body, not acted on by a force, moves with uniform speed in a straight line.

However if S is an inertial system and another such system S′ moving with uniform velocity relative to S, then S′ is also an inertial system. So Newton must have taken only one such system at rest in absolute space. Newtonian mechanics is invariant with respect to uniform velocities. In contrast accelerations are absolute, and since the particles that constitute a rotating rigid body accelerate towards the centre of rotation, a system that rotates relative to an inertial system is not itself inertial. Therefore in view of the rotation of the earth round the sun and rotation of the sun round the centre of the galaxy to which it belongs it is fallacious to start with an inertial frame. In 1883 the Austrian physicist-philosopher Ernst Mach published his classic scientific polemic 'The Science of Mechanics' in which he maintained that an intertial frame of reference was not to be defined as one at rest or in motion with respect to absolute space but was to be determined by the general distribution of matter in the universe. This is known as Mach's principle and till date the controversy over the acceptability of this principle is not over. What was seminal in Mach's demonstration of space became conspicuous in Einstein's theory of relativity which sought to unify space with time as a unitary space-time. So before coming to this culmination of the idea of space, let us look back on time.

3.2 Western View on Time

Just as space perturbed the philosophers of all ages, time also struck them as mysterious. Interesting enough are their explanations as to its distinct nature and essence. Some of them felt that it was incapable of rational discursive treatment and it was able to be grasped only by intuition. But others propounded new theses regarding the direction, continuity and infiniteness of time.

The earliest reference to the 'Nature of time' in the European literature is seen in St. Augustine's 'Confessions'. Here he asks 'What is time ?' and then proceeds to answer this question. This is anything but a definition for it leads to greater confusions. In explaining the word 'time' he brings in cognate temporal words like 'before', 'after, 'past' and 'future' and then finally in an attempt to explain how time should be measured he develops an analogy between temporal measurement and spatial measurement. While the same measuring stick can be used to measure the same length (say of a table) in two different lengths at different positions, two temporal measurements, say, walking a mile on two days implies two different movements of the watch hand. This leads Augustine to a puzzle and it seems that has been influenced by the thought that the present is real although the past and future are not. The above puzzle emanates apparently from the cofusion of the flow of time, popularly known as 'the myth of passage'. To many of us time is a stream that flows past us. But this again leads us to another controversy. If time is a flow then this would be a motion with respect to hypertime. A motion in space is with respect to time but motion in time cannot be so with respect to time and has to be with respect to something else, hypertime. Then in a similar argument, hypertime is a motion with respect to hyper-hypertime—an endless hypothecation. Another way to look at time is to consider the changeability of events from future to past. We experience events as approaching from future whereupon they are momentarily caught in the spotlight of the present and then recede into the past i.e. events are happening to continuants i.e. to things that change or stay the same. But can we intelligently talk of a change itself as changing or not changing ?

The philosophers of the 17th century and afterwards seem to have been drifted by the metaphysics of time and they have, whenever convenient, mixed up 'duration' with 'time'. In his Essay (Bk. II, Ch. 14, Sec. 1) Locke says that 'duration is fleeting extension'. Bergson makes 'duration' central in his philosophy and says that physical time is something spatialized and intellectualized whereas the real thing with which we are acquainted in intuition is duration. Unlike physical time, duration is the experienced change itself —the directly intuited non-spatial stream of consciousness in which past, present and future flow into one another. This subjective notion of duration lacks clarity as it is not understandable why duration is to be intiutively and not intellectually grasped. His observation that the past survives in the present by memory is also questionable as he confuses the memory of the past event with the past event.

It is interesting to note here that while almost none of the philosophers of science questioned the reality of time. McTaggart in his 1908 article in 'Mind' tried to prove through a fallacious argument the unreality of time; none the less his argument provides an excellent case study with which to elucidate—the relations between tensed and tenseless language.

The earliest reference to the problem of continuity of space and time is seen in the works of Zeno of Elea in Greece of the fifth century B.C. Zeno posed a number of problems, now known as paradoxes which he could not solve or explain. The more popular of these is the paradox of Achilles and the tortoise. Achilles chases a fleeing tortoise at a certain distance. When Achilles reaches the position of the tortoise, the tortoise is farther away at another position. So to catch the tortoise, Achilles traverses infinitely many segments. Then how does he catch the tortoise ? A moment's thought will make it clear that the expalnation rather solution of the problem requires understanding of the concepts of dimension and measure, which were non-existent during the time of Zeno. The consideration of Zeno's other paradoxes like Race course Paradox, Stadium Paradox, Flying Arrow paradox etc. amply illustrate the fact that space and time are not discrete but continuous. In fact, the set theory developed by Cantor and subsequent development of the concepts of dimension has been able to solve the paradoxes of Zeno, providing thereby the continuity of space and time. It would not be out of place to mention another paradox by Kant on space, time and causality, now known as Kant's antinomy. There are two antithetical arguments in this antinomy. The first argument states that the world had a beginning in time whereas the second argument shows with equal plausibility that the world had no beginning. A critical scrutiny of Kant's argument will make it clear that Kant's definition of infinity, besides being objectionably psychologistic is inapplicable to uncountable infinite sets and the antinomy can be fully explained if the thesis of unidirectional flow of time is rejected.

So the temporal directionality of the universe or at least our present cosmic era of the universe appear to be a deep-lying cosmological fact to be understood with care and caution. Since the laws of classical dynamics, electromagnetism (i.e. Maxwell's equation) and quantum mechanics (i.e. Schrödinger's equation) are expressed by time-symmetrical differential equations, it is to be generally accepted that symmetry is the essence of the laws of physics. This may sound quite astonishing in view of the multitude of common experiences. When a stone falls on the ground, we never see the energy in the ground causing the stone move upwards. Milk mixes with coffgee, but we never see a milky coffee separated into black coffee surmounted by a layer of milk. These examples certainly do not ratify the presence of symmetry in nature. But the modern physicists contend that the asymmetry observed in these examples and the like are only a partial truth and the reverse phenomena are observable if possibly several billions of years one can wait to see. The entire thesis has been nicely described by the concept of 'Entropy' which means a measure of disorder. To understand this one can look at the example of shuffling a pack of cards. If a pack of cards is arranged according to some order and then the pack is shuffled many times, obviously the entropy will gradually increase and a stage will come when it will reach a maximum, and then the entropy will start decreasing untill eventually it comes back to its original order. There is a celebrated mathematical theorem credited to the great French mathematical physicist Jules Henri Poincare, a contemporary of Einstein who made significant contributions to relativity theory as well. Poincare's theorem says that eventually every state in a system will repeat itself infinitely often. But the period of repetition of the state i.e. 'Poincare's cycles' may be longer than the present age of the universe which is approximately fifteen billion years. From cosmology it is known that the universe is expanding with the distant galaxies moving away from one another at speeds approximating the velocity of light. If one traces this expansion backward, it appears to have originated in a gigantic cosmic explosion—'The Big Bang' some fifteen billion years ago. At its creation, the universe was at a low ebb in its entropy and since then the entropy has been increasing constantly. Einstein's theory suggests that either the universe will eventually die a slow but inexorable 'heat death', temperature of the entire cosmos arriving at absolute zero or the entropy will reach a maximum when the universe will start contracting until in about seventyfive billion years it recontracts to its original state of a ball of superdense matter and then there will be a new explosion conforming the theory of Poincare's cycles. In the era in which the universe was getting less disordered, time would seem to run in the opposite direction to that in which it seems to run to us in deed, there would be an infinite sequence of cosmic eras with time running in opposite ways in alternate eras. Thus, in a sufficiently large view there would be temporal symmetry in this universe. It is therefore not correct to talk about the direction of time but of the temporal asymmetry of the universe.

As regards absolute and relational theories of space and time we have seen earlier that Newton held to an absolute theory whereas his contemporary Leibnitz held that space and time are sets of relations between things which are in space and time. Definitely Newton had reasons to identify time as a unidirectional flow and state "Absolute, true and mathematical time, of itself and from its own nature, flows equably without relation to anything external". But this is unnecessary and misleading. Leibnitz, on the otherhand, believed that time is simply a relation by which we order our experience, time does not exist in the absence of things. In an empty universe, Leibnitz maintained, there would be neither space nor time since these concepts refer to relations among material objects. What Leibnitz propounded has now become an acceptable thesis by the discovery of the theory of relativity. The special theory of relativity has made it impossible to consider time as something absolute; rather it stands neutrally between absolute and relational theories. In fact, the great discovery of the theory of relativity had done away with the separate ideas of space and time and invoked a unitary space time, stated in the words of Hermann Minkowski "henceforth space by itself and time by itself are doomed to fade away into mere shadows and only a kind of union of the two will preserve an independent reality".

3.3 Relativistic Analysis

In the history of science, Einstein's theory of relativity revolutionized the whole idea of science. The classical mechanics founded by the legendary Newton that worked as the basis of physics for several centuries was challenged as to its validity for high velocities. In deed, Newtonian mechanics and Maxwell's electromagnetic theory do not fit very well together. In 1905 Einstein modified Newtonian mechanics keeping electromagnetic theory in tact and established that new mechanics becomes practically important when high velocities are involved. In this theory which is popularly known as special theory of relativity—special because it deals with only a special category of motions, Einstein proved not only that time and space are inseparable but also time and space are functions of velocity. He explained how under the effect of velocities, a rod can be contracted and time can be dilated. Technically these are known as Lorentz contraction and time dilation. He identified time with simultaneity of events and not as a flow from the past to future. Minkowski explained Lorentz contraction, time dilation and the relativity of simultaneity in Einstein's theory in a purely geometrical way, which has since been the accepted way of developing and understanding relativity theory. In additions to the three axes of the physical space he took time as the fourth axis and showed that his geometry is non-Euclidean, more specifically hyperbolic in time like directions and Euclidean in space like directions. This space time world picture enables to give a very simple account of change and motion. In 1916, Einstein presented a generalization of his 1905 theory, now known as the General Theory of Relativity and considered as the most aesthetically beautiful scientific theory ever invented, in which he showed how not only space but also time is affected by gravitating matter. In empty space a light ray propagates in a straight line but if the same light ray is acted on by gravitating matter, it is bent out of that line. Thus if a giant triangle is constructed near gravitating matter out of light rays, the triangle will not obey the celebrated theorem of Euclidean Geometry that states that the sum of the interior angles of a triangle is exactly 180 degrees. This establishes the fact that the Minkowski's Geometry is non-Euclidean and the Geometry differs from place to place in the universe depending on the distribution of mass. Without such objects, in a sense, space and time disappear. This has nicely been put by Einstein, "There is no such thing as an empty space i.e. a space without a field. Space-time does not claim existence on its own, but only as a structural quality of the field". Thus space and time in the absence of things is a purely metaphysical concept about which in Mach's phrase, we know 'ought'. The relativity of time has been nicely proved by Einstein in his 1905 paper "Zur Elektrodynamik bewegter Körper". He first explains how time is assigned to a given event as "We have to take account that all our judgements in which time plays a part are always judgements of simultaneous events. If, for instance, I say, "that train arrives here at 7 O'clock", I mean something like this. "The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events". Then he warns that the recording of simultaneity is not so easy. If the event is in close proximity to my watch there is nothing very complicated about recording, but if the event takes place on the Moon, say, then one had to take resort to some signalling, preferably of light, since light travels in straight lines with a velocity of about 1,86,000 miles a second in vacuum. Hence, if an atomic clock is used which measures minute fractions of seconds, simultaneity of the events to the observer does not imply simultaneity of occurrence since light must have taken more time to reach the observer's eye from the moon than from the clock. In deed, for most practical purposes we can assume that this light propagation is instantaneous, it is only when the speeds of objects become comparable to that of light then the impact of the theory of relativity becomes dramatic. If the distant object is in motion, simultaneity has no longer a universal meaning. Two events that we judge to be simultaneous will not be judged simultaneous by a moving observer. This situation was nicely explained by Einstein by an example of a moving train. Suppose an observer stands at the midpoint between two places where two lightning bolts strike simultaneously. The actual simultaneity of the events can be judged by the simultaneous arrival of the light at his eye ; but if the observer watches the events from a moving train whose front end is at the point where one of the bolt strikes down and whose rear end is near where the other bolt hits and the observer watches from a car in the middle of the train, a moment's thought will convince us that the light from the bolts will not reach the observer's eye simultaneously. This phenomenon is the relativity of simultaneity. Since the time of an event is determined by the simultaneity of two events and since simultaneity is relative, then time itself must be relative.

3.4 Indian Explorations

It is quite reasonable to think that an ancient civilization like India's which is so rich in its philosophy would have mused over the different aspects of time and space. The scriptures available till date amply corroborates this supposition. Innumerable references to space and time are noticeable in all works right from the Vedas to the Puranas and Tantras. But what is distinguishable from the Greek references is its means and mode of reference. Most of the references are cloaked in aphorisms and mysticism and the understanding of the deeper significance and nature of space and time requires unveiling of the mystic mythologies and careful interpretation of the abstract symbols of truth hidden in their multifarious folds. In deed, under the abstract symbols of the fables and fairy tales of the mythologies lie buried deep observations about nature which surprisingly conform to the theories of modern physics and mathematics. It is really astounding how the Risis (Scholars) of ancient India discovered these deep theories without modern instruments and this quite reasonably lead us to believe that knowledge is acquired not by perceptions alone but by 'intuitions' also. These Risis had a vision of the whole universe attained through observation and meditation and to perpetuate their knowledge took resort to symbols and mythologies so that the ordinary people learn and communicate to generations but the intelligent people dicipher and discover the mysteries of nature.

The citations of these multitude of observations and their interpretations are not our objective; many European and Indian savants have already done this job. We shall make some passing references only to a few of them to evince that Indian scholars too, like their Greek counterparts, meditated on space and time. It would not be fair to say only this much, as has been proved by Wallace, Max Müller, Capra, Suzuki, they propounded sophisticated theories like the Relativity theory much before the modern physicists proved them. This has been nicely echoed in the words of a great scientist of the 20th Century, W. Heisenberg who says, "The great scientific contribution in theoretical physics that has come from Japan since the last war may be an indication of certain relationship between philosophical ideas in the tradition of the Far East and the philosophical substance of quantum theory". A remark of Professor J. R. Oppenheimer, the mastermind behind the preparation of the first atom bomb is also worth mentioning here. He says, "The general notions about human understanding ... which are illustrated by discoveries in atomic physics are not in the nature of things wholly unfamiliar, wholly unheard of or new. Even in our own culture they have a history and in Buddhist and Hindu thought a more considerable and central place. What we shall find in an exemplification, an encouragement and a refinement of old wisdom".

As mentioned earlier, in the Vedas Desa (Space) and Kāla (Time) have been mentioned many times and statements about their nature are also quite frequent. But deciphering the codes one can easily observe deep and penetrating theories as has been observed by Wallace in one case. In Rig Veda 10.72.4, the following has been stated.

"From Aditi, Daksa was born and from Daksa Aditi was born".

If Aditi is interpreted as 'unbounded space' and Daksa as 'creative intelligence, the apparent contradiction boils down to the following philosophical overtones :

A creative intelligence assuredly dwells and functions within the womb of unbounded space, yet unbounded space is nevertheless the product of a creative intelligence.

As regards Aditi, Max Müller writes, "Aditi, an ancient god, is in reality the earliest name invented to express the infinite, ... the visible infinite, ... the endless expanse beyond the earth, beyond the clouds, beyond the sky". The Vedas were written around 1500 B.C. So the concepts of space and time can be supposed to have been dealt with as early as 1500 B.C. or earlier. The Buddhist philosophy which is more down to earth than the Hindu philosophy and which flourished between 450 B.C. and 100 A.D. is seen to have given due consideration on these concepts. The Avatamsaka Sutra on which the Avatamsaka school of Mahayana Buddhism is based gives a vivid description of how the world is experienced in the state of enlightenment. In the words of D.T. Suzuki—

"The significance of the Avatamsaka and its philosophy is unintelligible unless we once experience a state of complete dissolution where there is no more distinction between mind and body, subject and object. We look around and perceive that every object is related to every other object not only spatially but temporally. As a fact of pure experience, there is no space without time, no time without space ; they are interpenetrating".

In another sequence he writes,

"In this spiritual world there are no time divisions such as the past, present and future ; for they have contracted themselves into a single moment of the present where life quivers in its true sense. The past and the future are both rolled up in this present moment of illumination and this present moment is not something standing still with all its contents for it ceaselessly moves on".

Is it not a sure testimony to the acquisition of relativistic ideas of space and time by the Indian sages in the light of the modern relativistic physics ? Is it difficult to draw an analogy if we recall Louis De Broglie's following remark ?

"In space-time, everything which for each of us constitutes the past, the present and the future is given en block. Each observer, as his time passes, discovers, so to speak new slices of spacetime which appear to him as successive aspects of the material world, though in reality the ensemble of events constituting space-time exist prior to his knowledge of them".

Not in the Vedas of the Hindus and the Buddhist Scriptures alone, in the Puranas too, space and time appeared cloaked in mysticism. The Hindu trinity of gods, Brahma, Vishnu and Siva if interpreted in the right perspective, demonstrates through their functions and interrelations deep theories of this cosmic world. One will miss the whole flavour if one considers them as the forces of creation, preservation and destruction. One has to recount the various highlights in the life of these deities to appreciate the symbology in which this basic truth was enshrined. Of the three deities, Siva is the only deity perpetually on the move and hence is to be understood as the Energy of motion. He wears as a sacred thread a serpent which represents energy inherent in spiral form of motion. Often he is drawn with a third eye in the middle of his forehead—an eye that could see both backward and forward in time. Hidden in this symbolism is the concept of Siva as the creator of time. Just as Siva is drawn as the energy of motion, Vishnu symbolizes space. In every action of Vishnu we find his deep need to preserve our material reality. As space, Vishnu is drawn with Brahman growing out of his navel but in another episode he is said to have measured the entire universe in three strides. This is just a symbolic way of saying that he has the space like qualities of the universe and the emergence of Brahma from Vishnu's navel is a symbolic way of showing that the energy of expression manifested itself in space. As regards the nature of space, we thus get from the Vishnu Puran.

"The world was produced from Vishnu, it exists in him; he is the cause of it, continuance and cessation, he is the world." The Padma Puran also describes Vishnu (Space) as "without beginning or end" i.e. capable of infinite expansion. In the Mahabharat too, written circa 400 B.C., Vishnu's abode is described as studded with jewels (i.e. galaxies), the pillars and the facade of which are embedded with precious stones. Vishnu himself is presented as dark complexioned since space beyond our atmosphere is black. The episode of Vishnu's appearance to his wife Lakshmi as 'Siva' symbolizes the unification of space and time.

Going back to the Vedas, one who is conscious of temporal asymmetry will be pleased to note the following from the Maitrayana.

"In truth, there are two forms of Brahman, time and not time. That is to say, that which existed before the Sun is not time (i.e. negative time) and that which began to be with the Sun is time, is the devisible".

In Brihadaranyaka 3.8.11, space is characterised as "In truth, O. Gargi, this (Brahman) imperishable one sees but is not seen, hears but is not heard, comprehends but is not comprehended, knows but is not known. In truth, O. Gargi, in this imperishable one is space inwoven and interwoven".

Regarding the Buddhist experience of spacetime, Lama (1973) writes.

"If we speak of the space experience in meditation we are dealing with an entirely different dimension. In this space experience the temporal sequence is converted into a simultaneous co-existence, the side by side existence of things and this does not remain static but becomes a living continuum in which time and space are integrated".

In the eye of Swami Vivekananda (1863-1902) a comparatively recent philosopher, "Time, space and causation are like the glass through which the absolute is seen. In the absolute there is neither time, space nor causation".

3.5 Philosophical Overview

From time immemorial man has been concerned with space and time, but a rigorous consideration of the problem is noticed in the Vedic period circa 2500 B.C. in India and in Greece around 1000 B.C. From this early stage of civilization, the natural scientists like the early atomists including Democritus, Epicurus identified space with vacuum, which they considered absolute, always and everywhere the same and motionless with time running always at the same space. There had been always dissidents but the main current of thoughts remained unaltered till the 20th century when modern physics discarded the old conceptions of space as an empty receptacle of bodies and of time as something uniform for the boundless universe. Einstein's theory of relativity played the most crucial role in bringing about radical morphological change of the ideas of space and time. The main conclusion of Einstein's theory is precisely the establishment of the fact that time and space do not exist by themselves in isolation from matters but are part of a universal interrelation in which they lose their independence and emerge as aspects of a single and diverse whole. The general theory of relativity has proved that the elapse of time and extent of bodies depend upon the velocity of these bodies and that the structure of the four dimensional space-time continuum changes according to the distribution of masses of substances and the field of gravitation to which they give rise. Indeed, the discovery of non-Euclidean geometry by Labatchevsky, Riemann, Bolyai refuted Kant's teaching on time and space as forms of sense perception outside the range of experience. Thus the theory of relativity settles the question of relationship between time and space and matter that concerned the philosophers ever since the beginning. The long standing thesis of the idealist philosophers like Berkeley, Hume, Mach in which they sought to establish that space and time are forms of individual consciousness or a priori forms of sense contemplation (Kant) or as categories of the absolute spirit (Hegel) thus got rejected by the material philosophy backed by the physics. Materialism stresses the objectivity of space and time which according to the materialists are inseparable from matter, this being a manifestation of their universality. Space expresses the distribution of simultaneously existing objects while time expresses the sequence of existence of phenomena which replace one another. Thus time is irreversible i.e. every material process develops only in one direction from the past to the future. According to the materialists, motion is the essence of space and time and matter and motion, space and time are inseparable. The Big Bang theory of the origin of this universe has modified further the concept of irreversibility of time when judged in the most general context of cosmic eras and has urged to consider temporal asymmetry in place of irreversibility. Thus a thorough change has taken place in the philosophical views about space and time due to the revolutionary discovery of the theory of relativity by Einstein. Before concluding, we feel tempted and privileged to say that certainly Einstein is to be accorded with the credit of a formal theory that unified space with time but the same observations were made long before by the Indian sages. Let the following comments of Capra conclude this article in a placid mood, "In my opinion, the time minded intuition of Eastern mysticism is one of the main reasons why its views of nature seem to correspond, in general, much better to modern scientific views than those of most Greek philosophers. Greek natural philosophy was, on the whole, essentially static largely based on geometrical considerations. It was, one could say, extremely 'non-relativistic' and its strong influence on Western thoughts may well be one of the reasons why we have such great conceptual difficulties with relativistic models in modern physics. The Eastern philosophies and thus their intuition often comes very close to the views of nature implied by our modern relativistic theories".

CHAPTER IV

INTUITION IN MATHEMATICS

4.0 Introduction

The history of human civilization is a history of numerous intellectual upheavals, popularly known as 'movements'. This is perhaps more true about and better applicable to the history of literature and art, particularly after the Renaissance in Italy. This is equally true, if not more, about Philosophy. No sooner one philosophical movement had taken root in the minds of people than another started throwing new light in our life. Though rationality had althrough played a predominantly uncontroversial role in shaping our life following the traditions of the Greek, anti-rational movements did wage wars against the rationalist traditions and exercised for decades together a dominant role in the intellectual forefront of our civilization. But as time is the best judge, the anti-rational movements had to quit the arena leaving room to the better for human existence and sustenance. Thus came the empiricism, the naturalism, the romanticism, the existentialism, the impressionism and many other philosophies of art and literature that charmed and fascinated the intellectual world for years. Some of them lasted several years and many died a premature death to occupy a meagre space in the history of civilization. But despite the fact that liberation from superstitions and religious oppressions is the product of a revolution waged by reason and humanity, the twentieth century has witnessed an upsurge of anti-rationalism. Even among the professed philosophers, the high priests of the sanctuary of reasons, faith in rational or demonstrative science is noticed to be on the wane in the interests of practical idealism, vitalism, humanism, intuitionism and other forms of avowed anti-intellectualism. A striking instance of this is William Jame's attack, in his 'Pluralistic Universe', on the whole enterprise of intellectual logic in favour of Bergsonian intuitionism and Fechner's mythologic speculations about the earthspirit. In the philosophy of science too, Brouwer, Heyting and Dummett have revived faith in intuition in respect of mathematical comprehensions and discoveries. Romanticism on the other hand denies the existing order in literature to affirm what may be non-existent in the real world, and finds sanction in the realm of fancy and mysticism. In deed almost all of the popular philosophies of the day, those which emanate from James, Croce, Bergson, Nietzsche, Spengler show a clear tendency towards deviation from rational forms and ideas seeking refuge in some forms of anti-intellectual pluralistic universe with the growing contempt for the ideal of humanity that was professed in the days of the rationalistic enlightenment by illuminati like Voltaire, Lessing, Diderot, Kant, Condorcet, Paine and Goethe. This current anti-rationalism strives to soften its opposition to rigorous logical procedure by representing modern science as empirical rather than rational. But it is surprising how the vanguards of the present anti-rational movement forget that the science which has unveiled the mysteries of nature and liberated mankind from the gloomy world of superstitions and savagery is the product of reason, that the great scientific achievements of Copernicus, Kepler, Galileo, Descartes, Newton and Leibnitz were thoroughly mathematical and rationalistic and not based on mystic vision, intuition or higher non-rational illumination. In deed, in science rationalism opposes the traditionally authoritative view of the world as well as popular credulity in the strange, the marvellous and the magical. But this does not mean that intuition had no role to play in the advancement of science and civilization. Certainly it had. Now to see in what way it permeated the intellectual life, it is necessary to first know what intuition really means.

4.1 Meaning of Intuition

In the broadest sense the term 'intuition' means 'immediate apprehension' where apprehension covers such states as sensation, knowledge and mystical rapport and 'immediate' has as many senses as there are kinds of mediation like the absence of inference, the absence of causes, the absence of justification, the absence of concept, the absence of symbols or the absence of thought processes. Given these mediations, there can be as many rational connotations as one likes, but there are only four generally accepted principal meanings in vogue, namely,

1. The faculty that provides us unjustified true belief not backed by inference i.e. non-inferential flashes like hunches.

2. The faculty that provides us immediate knowledge of the truth of a proposition i.e. non-inferential knowledge of the truth of a proposition.

3. The faculty that provides us immediate knowledge of a concept without the ability to define the concept.

4. The faculty that provides us non-propositional knowledge of an entity like sense perception or mystic realisation or insensible illumination.

If carefully studied, Bergson's inexpressible intuition of duration, Fichte's intuition of the transcendental ego and Shankaracharya's mystic intuition of God will be found to conform to the fourth category of meaning. There had been variety of opinions among the philosophers as to how intuition comes or whether intuition is the only modus operendi in the acquisition of knowledge, which have been grouped as linguistic theory, faculty theory, behaviourist theory etc. Since our intention is not to explore these theories but to see how intuition works both in acquisition of mathematical knowledge and in discovery of mathematical theorems, we would concentrate on only that type of intuition that pertains to mathematics proper. In deed, it would be a totally wrong approach if we study the role of intuition with a very wrong idea about intuition itself. To have a fairly clear idea about what mathematicians mean by intuition let us recall what Descartes, a great philosopher-mathematician said. He stated, "By intuition I understand, not the fluctuating testimony of the senses, or the misleading judgement that proceeds from the blundering constructions of imagination, but the conception which an unclouded and attentive mind gives us so readily and distinctly that we are wholly freed from doubt about that which we understand". This idea is not surely uncontroversial because many would argue—that ready acceptance of the truth without doubt has created so many crises in mathematics, has dislodged mathematicians for decades if not centuries from the correct rational paths. Yet Descartes' notion of intuition is what every mathematician experiences. As a matter of fact, it is an essential psychological process which every mathematician must undergo in order to learn mathematics and to discover mathematics. In this regard, the more experienced he becomes, the more reliable becomes his intuition. That is, mathematical intuition' is a psychological quality stemming out from an experience-based faculty and which contains one's attitude towards a mathematical situation which one has never faced before. The common experience is this that a mathematician is seen to have no intuition or little intuition regarding a branch of mathematics in which he has never worked.

4.2 Intuition in Mathematics Learning

While most mathematicians are least concerned in expressing how mathematics is learnt, there are a few who have expounded in very clear terms how the concepts of mathematics, their interrelations and properties are understood by a human being. The question as to the understanding of mathematics requires special attention because as a subject mathematics is quite distinct in the world of learning. Therefore it is important to see what mathematicians and philosophers of mathematics have said. In this context, the following remark by Immanual Kant, the greatest of the philosophers of the modern world who started his career as a mathematician is worthmentioning. "Thus all human cognition begins with intuition, proceeds from there to conceptions and ends with ideas''. This is applicable not to learning of mathematics alone but to acquisition of any knowledge as such. In deed, he has categorically emphasized the role of intuition in the acquisition of any knowledge. He points out in his ''Citique of Pure Reason'', ''Our knowledge comes from two basic sources in mind, of which the first is the faculty of receiving sensations, the second ability to recognize an object by these perceptions. Through the first an object is given to us, through the second this object is thought in relation to these perceptions as a simple determination of the mind. Thus intuition and concepts constitute the elements of all our knowledge''.

Thus Kant talks about passive role of intuition which plays first and then the active role of thought which plays next to form new ideas. He also admits of two ingredients of intuition viz., a posteriori part which forms the content of intuition and a priori part which forms the form of intuition independent of all experiences. He proceeds further to say that we possess two such pure intuitional forms, namely, space and time, where space is, according to him, the intuitional form of our external sense by means of which we picture things as outside ourselves and time is the intuitional form of our inner sense by means of which the mind observes itself or its inner state. These intuitional forms of space and time constitute the a priori frame into which we fit all physical events experienced by us and every such event has a precisely determined place in space and time. Before we pass on to the observations made by others in respect of intuitive acquisition of knowledge, we simply like to mention that Kant's views have been strongly criticized by many. Particularly the physical aspects of his thesis about the a priori nature of space and time, because its dependence on the Newtonian concepts of mechanics has turned out to be fallacious in the light of the relativity theory of Einstein. The psychological aspect, i.e., the intuitional aspect has been vehemently encountered by Russell who forcefully denied the role of intuition in the acquisition of knowledge and set out to prove Kant's antithesis that even arithmetic belong exclusively to the domains of the intellect and of logic and certainly not of pure intuition.

Now let us look at the Greek who provided us the models of modern science and are considered to be the founders of the rationalist thesis. A careful study of the views of Socrates, Plato and Aristotle will convince anybody that they were basically proponents of the rationalist thesis in every sphere of human knowledge possibly without any exception, none the less at times they could not deny the elements of intuition in some kinds of scientific activities. Particularly, for Aristotle, the greatest of the Greek intellectuals, intuitive reasoning was imperative in the understanding of the axioms as self-evident truths which were to be followed by discursive reasoning for understanding the relationship between the concepts.

Rene Descartes the greatest of the mathematicians of the 17th century and staunch rationalist voted essentially in favour of Aristotle. He distinguishes intuitive knowledge as particular and general and goes forward to say that our knowledge of the particular has often been referred to as sensory intuition and of the very general as non-sensory intuition. Thus, according to Descartes also, the presence in our mind of the original starting points of knowledge is due to the peculiar faculty of our mind and is therefore inexplicable.

Thus we see that in Cartesianism this inexplicability was woven into the fabric of a metaphysical dualism, according to which no mental event can be caused by any sequence of physical events and in which the only mental relation that can bring about a coming-to-know is the relation of being inferred. Thus, according to Descartes sensory intuitions are not really cases of knowledge at all but are merely physical ones. This, in effect, implies that sense perception is in principle nonessential to attaining knowledge, although it is mysteriously necessary in practice. The rationalist

Locke criticized this paradoxical position of Descartes but subscribed to the rationalist contexts of Cartesian thesis.

Leibnitz's position in respect of this is much the same as the Greeks. Though a pioneer and an uncompromising Vanguard of logicism and rationality in general, he accepted the Greek models of scientific pursuits which base on the acceptance of a number of axioms purely intuitively and then the application of deductive logic for proofs of the results.

Bertrand Russell (1980), a stalwart logician and rationalist of the 20th century, waged wars against any kind of intuition refuting all arguments in favour of that and in persuasive language established that intuitive acquisition of knowledge is a vague term and that knowledge is acquired only by a rational process once the language of mathematics is known by sense perception.

Most of the Indian philosophers possibly because of their religious background have in some way or other subscribed to the intuitionist thesis of acquisition of knowledge. Radhakrishnan remarks, "while all varieties of cognitive experience result in a knowledge of the real, it is produced in three ways, which are sense perception, discursive reasoning and intuitive apprehension". It seems that Radhakrishnan has tried to propound a compromising thesis supporting the role of intuition as a way of acquiring knowledge just as the discursive reasoning is another distinct way. It cannot be denied that intuition certainly plays an important role in the assimilation of preliminary mathematical knowledge, but experience suggests that it is not a must. Much of mathematics can be grasped purely on the basis of the language and deductive logic; just as mathematical logic can be grasped without an intuitive aid. Thus intuition is merely an aid, not an ingredient of our mental faculty.

4.3 Intuition in Mathematical Discoveries

The intuitionists have advocated not only the presence of an element of intuition in the acquisition of knowledge but also the active role of intuition in every mathematical discovery. The physicist mathematician Pascal has put great emphasis on intuition in derivation of mathematical results. According to him, "the heart has its own reasons which reason does not know". He maintains that "reason is the slow and tortuous method by which those who do not know the truth discover it".

Jaques Hadamard in the "Psychology of invention in the Mathematical Field" investigated the question of how mathematicians think and discover. He came out with very interesting findings in this regard. He concluded that in the creative process practically all mathematicians avoid the use of precise language ; they use vague images, visual or tactile. The great scientist Albert Einstein subscribes to the same view. In a letter to a friend he once remarked "The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined. The above-mentioned elements are in my case, visual and some of mascular type. Conventional words or other signs have to be sought for laboriously only in a secondary state."

The greatest exponent of intuitionism is however L. E. J. Brouwer, a Dutch professor of mathematics who on his own developed an intuitionist school of philosophy. He maintains that mathematics is a human activity which originates and takes place in the mind and therefore it is independent of the real world; the mind recognizes basic, clear intuitions which are not sensuous or empirical but immediate certainties about some concepts of mathematics. He further said. "Mathematics arises when the subject of twoness, which results, from the passage of time, is abstracted from all special occurrences. The remaining empty form of the common content of all these twonesses becomes the original intuition of mathematics and repeated unlimitedly creates new mathematical subjects''. To Brouwer, mathematics is synthetic and is composed of truths rather than derived implications of logic. He held that ''in this constructive process, bound by the obligation to notice with reflection, refinement and cultivation of thinking which theses are acceptable to the intuition, self-evident to the mind and which are not, the only possible foundation for mathematics is to be looked for''. In deed, Brouwer believed that intuition determines the soundness and acceptability of ideas, not experience or logic. He did not recognize any a priori obligatory logical principles to deduce conclusions from axioms. Herman Weyl, one of the greatest mathematician of the 20th century, in his 'Philosophy of Mathematics and Natural Science' has held a supporting view also. He says that the Principia (by Russell and Whitehead) based mathematics not on logic alone, but on a sort of logician's paradise, a universe endowed with an ultimate furniture of rather complex structure.... Would any realistically-minded man dare say he belives in this transcendental world ?... This complex structure taxes the strength of our faith hardly less than the doctrines of the early Fathers of the Church or of the Scholastic philosophers of the Middle Ages''. In deed, in Principia Mathematica, Russell and Whitehead attacked the intuitionist thesis of the foundation of mathematics rejecting the claim of Brouwer and his followers. In the logistic programme contained in Principa, they proved systematically how a large portion of classical mathematics can be developed purely on deductive logic without resort to the intuitive appeal of the concepts even. The famous Bourbaki school of France remarked, ''The memory of the intuitionist school will without doubt be destined to survive only as a historical curiosity.'' This proves how much critical it was in respect of the intuitionist thesis. Felix Klein, Moritz Pasch, David Hilbert, Alonzo Quine, Paul Bernay and many others of the 20th century had shown their least faith in intuition and very strongly advocated the logicist thesis or the rationalist thesis. The rationalists in general criticize the intuitionist citing examples of the Greek and their marvellous achievement in unveiling the mysteries of nature. They question, if intuition is the principal force in discovering mathematical truths, then why not the poets or novelists or historians with high degree of intuitive faculty have not been able to discover sophisticated mathematical theorems, given relevant mathematical concepts ?

In the light of the above discussions, the justification of Wilder (1967), that the role of mathematical intuition in the evolution of mathematical concepts—our collective intuition of basic concepts has grown by a series of discoveries of faulty features in the current concepts, with ultimate replacement by new concepts which not only clear up the faults, but lead to feverish activity on the new foundation with consequent creation of much good mathematics is too weak a proposition to be acceptable.

Most of the philosophers and scientists of ancient India are no doubt pro-intuitionism, yet they did not want to sail on without a proper rational justification. The Hindus and the Buddhists alike developed the science of logic not for logic's sake but to give a rational finish of what they contemplated.

We conclude this section by recalling one of Descartes' memorable comments about intuition.

"Let us now declare the means whereby our understanding can rise to knowledge without fear of error. There are two such means intuition and deduction. By intuition I mean not the varying testimony of the senses, nor the misleading judgement of imagination naturally extravagant, but the conception of an attentive mind so distinct and so clear that no doubt remains to it with regard to that which it comprehends, or what amounts to the same thing, the self-evidencing conception of a sound and attentive mind, a conception which spring from the light of reason alone, and is more certain, because more simple, than deduction itself, although as we have noted above the human mind cannot err in deducing either. Thus everyone can see by intuition that he exists, that he thinks, that a triangle is bounded by only three lines, a sphere by a single surface and so on.

It may perhaps be asked why to intuition we add that other mode of knowing by deduction, that is to say, the process which, from something of which we have certain knowledge, draws consequences which necessarily follow therefrom. But we are obliged to admit this second step; for there are a great many things which, without being evident of themselves, nevertheless bear the marks of certainty if only they are deduced from true and incontestable principles by a continuous and uninterrupted movement of thought, with distinct intuition of each thing; just as we know that the last link of a long chain holds to the first, although we cannot take in with one glance of the eye the intermediate links, provided that, after having run over them in succession, we can recall them all, each as being joined to its fellows, from the first to the last. Thus we distinguish intuition from deduction, inasmuch as in the later case there is conceived a certain progress or succession while it is not so in the former—whence it follows that primary propositions, derived immediately from principles, may be said to be known, according to the way we view them, now by intuition, now by deduction, although the principles themselves can be known only by intuition, the remote consequences only by deduction''.

It is clear from the above passage, Descartes' view was tuned to the Greek model which in essence accepts the role of intuition to the extent of self-evident axioms. Whatever glorification of intuition Descartes might have done, that intuition sometimes leads to deep crisis has been proved beyond doubt by the discovery of non-Euclidean Geometry and Peano's spacefilling curve. We now give some examples where intuition leads to crisis.

4.4 The Crisis of Intuition

In different ages there were proponents of intuition who glorified the role of intuition not only in mathematics but also in science as such. There were persons who sought to establish the supremacy of intuitionism over formalism and logicism. Even the Greeks who are considered as the founder-fathers of modern science and who evinced in clear terms the strength of logic and rationality spoke of limited but indubitable role of intuition. In deed, the Euclid's geometry stands as a glaring example and a model of the height of logicism and also of limited intuition as far as the acceptance of the axioms as truths is concerned. Many a mathematician and philosophers of the 20th century even thronged to this school of intuition, who include L.E.J. Brouwer (1913-1914); Hermann Weyl, Michael Dummett (1964), Arend Heyting (1964). But some recent discoveries have proved beyond doubt that intuition can lead us astray. Many of the propositions which had been accounted true by intuition has been repeatedly proved false by logic. Even Euclidean geometry which till 19th century lend credibility and greatest support to intuition has succumbed to a serious intuitional crisis by the discovery of non-Euclidean geometries. The discoveries of Gauss, Lobatchevski, Bolyai and Riemann evinced that non-Euclidean geometries demonstrate nature equally effectively as Euclidean geometry though intuitively the fifth postulate of Euclid which is the starting point of non-Euclidean geometries was unacceptable. But perhaps the greatest blow was given by Karl Weirstrass (1815-1897) in the year 1861. Till this time it was a well-accepted truth that every continuous function is derivable at every point. Some even went further to prove it, for example, Andri Marie Ampere (1775-1836), J.L.F. Bertrand (1822-1900), Lacroix and almost all text book authors of the 19th century and many more accepted that like Fourier, Cauchy, Galois, Legendre, Gauss who were the leading mathematicians of their times. One of the reasons for the existence of such an inexcusable error may be attributed to the absence of a rigorous definition of function but the main reason is conspicuously the dictate of intuition which derived the conclusion by studying the graphs of a number of functions commonly known. Weirstrass invented the curve by an intricate and arduous calculation and the character of this curve entirely eludes intuition; indeed only thought or logical analysis can pursue this strange curve to its final form. Thus, had we relied on intuition in this instance, we would have remained in error till another serious crisis had jolted us.

The example of Weirstrass concerns differential calculus. There are similar examples in other areas too, which lead to intuitional crisis. The second example we cite now is from integral calculus. The question facing us is as follows : If two curves in a plane start from a common origin and continuously have the same slope must they coincide in their entire course ? Intuition suggests that the two curves must coincide in their entire course, yet logical analysis shows that this is not necessarily so. The intuitive answer is true of course for ordinary curves but there exist rather complicated curves for which it is not true.

We now give a third example drawn from geometry which happens to be the foothold of intuition. It is well known that a curve is an one-dimensional geometric object. In the year 1890 the Italian mathematician Giuseppe Peano (1858-1952) proved much to the surprise of the intuitionists that a two dimensional object can be produced by such an one-dimensional object, more precisely, it is possible to imagine a point moving in such a way that in a finite time it will pass through every point of a square and yet no one would consider the entire area of a square as simply a curve. This space-filling curve baffles intuitive apprehension and can only be understood by logical analysis. This apparently paradoxical situation can be explained and properly understood if the notion of a curve is precisely defined and the concepts of connectivity and dimension are brought in.

There are many other examples which convince ourselves of the undependability of intuition even as regards very elementary geometrical questions like the Brouwer's three-country corners problem (1910) and Sierpinski's branch point problem (1915). In all these problems we are faced with intuitional crisis which is overcome only by strict logical reasoning. Thus from the gamut of problems we see that intuition is deceptive and an unreliable guide. In fact, this point the formalists and logicists had been harping on voicing absolute dependence on logic and critical examination of all results and concepts of classical mathematics.

A Synthesis : There is hardly any doubt that the rationalist movement after the Greek model launched by the 19th Century philosophers and scientists have brought about a morphological change of the human civilization ; but yet the psychologists inspite of its spectacular advances in exploring the functions of mind, have achieved little success in harnessing the total faculty of mind which is still clouded in mysteries. Clarivoyance and premonition are no more disputable claims but facts of life. So to whatever height reason may soar with flying colours, intuition as such is an accepted truth, more so in mathematical discoveries. One must grant that logic plays an important part, but who can deny the role of intuition in guiding one's thought to a meaningful proposition. In fact it is almost a truism that without intuition there is no creativity in mathematics. Wilder depicts the role concisely in the following words : "The younger man has not only come into the particular field without having to clutter up his brain with concepts and methods which served their purpose and are now discarded but using new concepts and methods he has built up an individual intuition which forms a platform from which he can regard his field of research with an eye undimmed by the recollection of earlier and faulty intuitions." Thus the role of intuition is to provide the "educated guess" which may prove to be true or false, but in either case, progress cannot be made without it and even a false guess may lead to progress. Mathematics undoubtedly rests upon certain intuitions that may be the product of our sense organs, brains and the external world. The essential idea or method is always grasped intuitively long before any rational argument for the conclusion is devised. In deed, the intuitions of great-men are sounder than the deductive demonstrations of mediocrities. The French mathematician Jacques Hadamard, referring intuition as the master and logic as the servant, observes that logic restricts unbridled intuition. He maintains, "The object of mathematical rigor has been only to sanction and legitimize the conquests of intuition". That legitimization too, in the opinion of E. H. Moore, a foremost American Mathematician, is a function of the epoch, that is, its civilization. The same concern has been expressed by Poincaré, "There are no solved problems, there are only problems more or less solved". The faith in reason is shaken due to the disasters of mathematics that built up its kingdom on the strength of reason. The noted philosopher of science Karl Popper in his "The logic of scientific discovery" remarks that mathematical reasoning is never verifiable but only falsifiable and mathematical theorems are not guaranteed in any way. Thus if intuition is deceptive and need to be abandoned in the pursuit of knowledge, can we absolutely depend on the rationality when the crises of reason is no more a folklore but reality so far as the foundation of mathematics is concerned ?

CHAPTER V

THE CRISES IN MATHEMATICS

5.0 Introduction

In an age of tremendous scientific and technological advancement when men have put their footsteps on the moon, when the appearance of the Halley's comet is predicted with minutest precision or when the star wars are not dreams but a reality, it is unbelievable that present mathematics is shaken and shattered by a crisis and that its credibility is at stake. But as truth is stranger than fiction, the fact remains that there is a crisis and the paradise where mathematicians reigned for five thousand years is lost. In fact not just one but a compendium of crises have taken away the ground on which the majesty of mathematics developed, and because the present one concerns the foundation, it is one of the deepest crises that mathematics has ever faced. Worse than that there is no slightest sign of hope flickering in the firmament of the mathematical world. The cry that tore the heart of Hermann Weyl, one of the deepest and certainly the most erudite of modern mathematicians, is now rumbling in the hearts of thousands of mathematicians. "We are less certain than ever about the ultimate foundations of mathematics and logic. Like everybody and everything in the world today we have our "crisis"." In fact, we had several such in the past two millenia, which were resolved partly or fully yet we have another of much deeper significance and gravity. To understand the crises it is most essential to have a knowledge of the structure of mathematics and its evolution.

5.1 Crises of Concepts

The earliest form of mathematics initiated by the Hindus, Egyptians and Babylonians can be termed as a naive play with some whole numbers and geometric figures. Apropos of physical needs, these concepts came into existence and as such never seized the attention of the intellectuals. Thus the subject status it never gained. The Hindus were little advanced at this time because they had already defined very large numbers and zero. They had mastered peculiar calculations with fractions and strange numbers. In a nutshell, Hindu mathematics though developed to satisfy religious and social needs as evidenced in the Vedic scriptures was more sophisticated compared to other civilizations in Egypt and Babylon in the period 3000 to 2000 B.C.

When the Greek civilization spread in length and breadth from the island of Sicily to Egypt circumscribing the Mediterranean coast, the mathematics of numbers and geometric figures caught the attention of the best Greek intellectuals, who in return developed systematically and analytically the whole of mathematics in the light of rigorous reasoning, unlike the Hindus whose contribution was more of an intuitive and empirical nature and least based on reasoning. It is still a matter of imagination and conjecture how the Hindus got such splendid results simply by intuition as the slightest reasoning can hardly be traced there. However, the Greeks, on the contrary, developed the laws of reasoning on one hand and on the other through systematic application of the laws developed geometry to an incredible maturity. The greatest contribution of the Greeks is thus the formation of these principles of logic and was codified in 'Organon' by the unparallel Greek genius Aristotle, a student of Plato and the teacher of Alexander, the great. These principles discovered and codified around 300 B.C. still form the basic principles of reasoning on which almost the whole of mathematics has been developed, if not the whole of science. In fact, there are many types of reasoning,—for example, induction, reasoning by analogy and deduction. Induction means generalization of an inference from a part to the whole. Analogy establishes truths of one from an analogous one and deduction is made by several ways. The first way is known as syllogistic reasoning, which establishes the truth of a statement from two premises. The second is the law of contradiction which assures that a proposition cannot be both true and false and the third one is the law of excluded middle which asserts that a proposition must be either true or false. In most areas of science a hypothesis is made through an induction but that is proved or disproved by deductive reasoning.

There are three characteristic features of the Greek mathematics. The first is the discovery of the principles of logic, the second is the discovery of methodology of describing mathematics in a systematic manner, namely the axiomatic method based on deduction and finally the attitude of the Greeks, at large, towards nature.

The Greek intellectuals adopted a totally new attitude. They dared to look nature in the face. Their intellectual leaders, if not the people at large, rejected traditional doctrines, supernatural forces, superstitions, dogma and other trammels on thought and tried to understand the multifarious, mysterious and complex operations of nature. They applied mathematics to dispel the mystery, mysticism and seeming chaos in the working of nature and establish the existence of irrefutable and immutable laws there. This is echoed in the words of Anaxagoras, a Greek mathematician cum philosopher "Reason rules the world".

As regards methodology, the 'Elements' by Euclid containing the works of many including Apollonius and Pythagoras stands as an examplary testimonial and embodies how by deductive reasoning, Geometry is developed starting from some undefined terms and a host of axioms i.e. assertions about the undefined terms which are to be taken for granted. Since 300 B.C. this has become a model on which the rest of mathematics and even science have been developed. At present almost all sciences including social sciences have been constructed on this model of Euclid's geometry. In fact, the Euclidean geometry starts with some definitions and ten axioms of which the most controversial is the parallel axiom or the fifth postulate in particular. These axioms were based on experience and seemed to fit nature appropriately and as such were accepted as truths.

In another aspect the Greeks can demand special credit as they are the first to apply deductive reasoning to political systems, ethics, justice, education and numerous other concerns of man. Here too they achieved great success just as they had in geometry.

Though the Greeks achieved spectacular successes in geometry and astronomy, they were not as successful in developing arithmetic as the Hindus were. The first crisis therefore was faced by the Greeks as they never bowed down to mere intuition or experience. This was concerning irrational numbers. In geometric constructions of right angle triangles they discovered some quantities like $\sqrt{2}$, $\sqrt{3}$ etc. which they did not accept as numbers. The rampant attempts by them to explain how these can be fit into the system of numbers met with failures as no other kinds of numbers except whole numbers were known to them. These new numbers perturbed them greatly and ultimately they overcame the crisis by ostracizing them. They however tried to give a geometric treatment of such quantities which proved anything but their usefulness.

The period from 300 B.C. to 1300 A.D. in the history of European Mathematics is dipped in darkness. The masterminds behind the construction of this majesty of mathematics were busy saving the fruits of their toil for one thousand years from the angry Romans who waged wars against the pagan Greeks, and destroyed hundreds of thousands of parchments containing invaluable treasures of the human mind. In 47 B.C. the Romans set fire to the Egyptian ships in the harbour of Alexandria, the fire spread and burned the library—the most extensive of ancient libraries—Such destructive acts were repeated under the direction of the Roman emperors, Constantine, Theodosius and his two sons Honorius and Arcadius. The next wave of attacks on the Greek Civilization came from the Moslem hordes in A.D. 640 who captured Egypt causing migration of the majority of Greek scholars to Constantinople. The atmosphere there was still inconducive for any academic pursuits. The scholars were busy preserving somehow the remnants of their researches. A positive effect of the mohammedan rule was that the Arabs came to know the vast treasures of the Greek intellectual world. The Arabian scholars being charmed by the dazzling glow of the Greek intellect resolved to master the art and science of mathematics. But the crisis created around 600 A.D. by the Hindus by discovering negative numbers remained unsolved, though they were prone to using such numbers in greater proportions than the Greeks. Even as late as 1694, the European mathematicians including Blaise Pascal (1623-1662), Nicolas Chuquet, Antoine Arnould were bewildered by this crisis as negativity of numbers was entirely new to them and they refused to accept them as numbers and called them "absurd numbers". But reality, need and convenience forced them to finally accept them. This is how this crisis was avoided without sufficient justification like the crisis of rational numbers.

Though the Greeks ostracized irrational numbers, calculations with irrationals were carried freely in Europe in keeping with the Hindu tradition, but the problem of whether irrationals were really numbers troubled mathematicians even in the 17th century. John Napier (1550-1617) made frequent use of irrationals even in his new creation 'logarithm' for positive numbers one of the most valuable Renaissance products. In the 16th century sincere attempts were made by many to identify irrational numbers. Jerome Cardan (1501-1576), Michael Stifel (1486-1567); Luca Pacioli (1445-1514) and John Wallis (1616-1703) not only accepted irrationals as numbers but tried to justify the operations with irrationals, though any logical foundation for such numbers was lacking. Without having overcome their difficulties with irrationals and negative numbers, the Europeans added to their burdens another by blundering into what we now call complex numbers. These numbers they arrived at from two sides, one in attempting to solve equations like $x^2+1 = 0$ or $x^3=1$ and the other in trying to extend the definition of logarithm to negative numbers. Cardan in his attempt to solve an equation $x(10-x) = 40$ obtained the roots as $5+\sqrt{-15}$ and $5-\sqrt{-15}$ which he called "Useless Sophistic quantities." These sophistic quantities induced clouds of confusion in the world of mathematics and were not received with pleasure. This situation has been vividly narrated by Leibnitz in the following lines. "The Divine Spirit found a sublime outlet in that wonder of analysis that portend of the ideal world, that amphibian between being and not being which we call the imaginary root of negative unity." The quandaries about the meaning and use of complex numbers deepened with the introduction of logarithm of negative numbers and logarithm of complex numbers and over the meaningfulnss of such operations, Leibnitz, Euler and John Bernoulli debated the matter hotly through letters and papers. Finally this crisis was resolved by d'Alembert and Euler, who published two papers in 1947 and in 1951 in wihch they justified the operations with complex numbers. Subsequently Gauss, Hamilton and Argand established the true significance of complex numbers by associating a geometric interpretation and meaning of complex numbers. Thus, though geometry was deductively organized by 300 B.C. and the few defects detected later could be readily corrected, arithmetic and algebra had no logical foundation whatsoever. The whole numbers were taken as God's gift and accepted unquestioningly. The basis for adapting the properties of whole numbers and fractions had certainly been an experience. As new types of numbers were added to the number system, the rules of operations already accepted for the natural numbers and fractions on an empirical basis were applied without logical justification. Fortunately, the use of these new numbers and of Algebra in general in scientific work produced excellent agreement with observations and experiments. As the needs of science prevailed over logical scruples, doubts about the soundness of algebra were cast aside. However, the most potent impetus was inflicted on mathematics in the 17th century. Calculus, the subtlest discipline in all of mathematics was born in an atmosphere of non-existent logical foundations of arithmetic and algebra. The two fundamental concepts namely the derivative and the definite integral appeared in the works of many, the most famous of whom were Kepler (1571-1630), Descartes (1596-1650), Cavalieri (1598-1647), Fermat (1601-1665), Pascal (1623-1662), Gregory (1638-1675), Huygens (1629-1695), Wallis (1616-1703), Newton (1642-1727) and Leibnitz (1646-1716). Each of these men formulated his own approach to the problem of defining and computing the derivative and definite integral. Of the above approaches only two demand special mention namely those by Newton and Leibnitz. Newton explained the concept of derivative purely as a rate measure, more precisely as velocity of a particle, but Leibnitz followed a more abstract approach using a concept known as infinitesimal. The use of such a quantity raised a storm of protest all over the mathematical world probably because Leibnitz himself failed to justify the logic of his approach properly. Not being able fully to clarify concepts and justify operations, both men relied upon the fecundity of their methods and the coherence of their results and pushed ahead with vigour but without rigour. The 17th century ended with calculus as well as arithmetic and algebra in a muddled state and the 18th century mathematicians vastly extended the arena of calculus by deriving new subjects from it namely infinite series, ordinary and partial differential equations, differential geometry, the calculus of variations and the theory of functions of a complex variable which are collectively referred to as Analysis. Needless to mention that all these poured fuel to the burning flame of the rigorization problem. The concept of the infinitesimal or more specifically infinity stood in the way of intuitive understanding of the whole working.

As Newton, Leibnitz, Euler, d'Alembert, Lagrange and many others of this century struggled with the strange problem of infinity and employed them in analysis, they perpetrated all sorts of blunders, made false proofs and drew incorrect conclusions; they even put forward arguments which we are now able to call ludicrous. To overcome this crisis and to build the foundations of the calculus, Lagrange

(1736-1813) came forward and while he himself was trying for a solution, as the Director of the Berlin Academy of Sciences also he proposed in 1784 that a prize be awarded in 1786 for the best solution to the problem of the infinite in mathematics. best solution to the problem of the infinite in mathematics. But the best solution was not obtained and the state of the foundations was worse at the end of the 18th century though its successes in representing and predicting physical phenomena were beyond all expectation superlative.

The history of the medieval Europe is marked by a fast and steady growth of knowledge in almost all directions. Enlightened minds fused with Greek ideals applied reason to all human activities including the search of a mathematical design of the universe and discovered in succession hundreds of thousands of truths about nature, psychology, philosophy and society at large. Nicolaus Copernicus (1473-1543), Johannes Kepler (1571-1630), Galileo Galilee (1564-1642) overhauled the already existing misconceptions about the heavenly bodies and René Descartes (1596-1650) set about erecting a new philosophy of science. His contribution to mathematics did not offer new truths but rather a powerful methodology which we now call analytic geometry. In the history of science if any century is to be ascribed to any person, it is the 17th century which is illuminated by the shining genius of Sir Isaac Newton (1642-1727) the greatest of all scientists ever born. His scintillating intelligence was multidirectional and superseded all his predecessors. The mechanistic theory that was fertilized by Aristotle and laid by Descartes surrendered to Newtonian mechanics. The most powerful, almost omnipresent, Calculus was discovered. Gottfried Wilhelm Leibnitz (1646-1716) with much more versatility came as an aide to endow this century with an exceptional brilliance. His contributions to mathematics, science, history, logic, law, diplomacy and theology were remarkable. Quite independently, using an original idea of infinitesimal he developed Calculus though he was not as clear as to this new idea as one needs be. The 18th century accounted for many more discoveries of great significance but there was no serious crisis as such.

5.2 Crises of Intuition

The 19th century dawned quite auspiciously, attained immense progress but ended with several severe crises that tore the hearts of hundreds of thousands. Karl Friedrich Gauss (1777-1855) and Cauchy (1789-1857) the two leaders in the mathematical world and Immanuel Kant (1724-1804) the great Philosopher not only influenced the entire intellectual world by their brilliant ideas but also challenged the foundation of the existing beliefs and scientific truths. The contributions of Wilhelm Weber (1804-1891), Nicolai Ivanovich Lobatchevsky (1793-1856), Johann Bolyai (1802-1860), Georg Bernhard Riemann (1826-1866), Karl Gustav Jacob Jacobi (1804-1851), Casper Wessel (1745-1818), Jean-Robert Argand (1786-1822), William R. Hamilton (1805-1865), Arthur Cayley (1821-1895), Hermann Gunther (1809-1877), Felix Klein (1849-1925) were no less significant. Almost each of these mathematicians made a mark in the history of mathematics.

The first half of the nineteenth century heralded with a remarkable success in unveiling the mysteries of nature. Both Gauss and Cauchy though first rate mathematicians were equally efficient physicists. In the words of James Clerk Maxwell, the founder of electromagnetic theory, Gauss's studies of magnetism reconstituted the whole science, the instruments used, the methods of observation and the calculation of results. Gauss worked on Arithmetic, Algebra, Geometry, Astronomy, Optics, Magnetism and several other disciplines and achieved remarkable success. Cauchy laid the foundation of the theory of functions of a complex variable and left an indelible impression of his superb mathematical genius through as many as seven hundred papers, second only to Euler in number, in different areas of science. But in the midst of such splendid discoveries, who could imagine that a severe crisis was looming large in the horizon of mathematics ?

From the time of the Greeks, though Euclid's geometry was accepted with exhileration as a model of scientific approach, doubts were being heaped as to the acceptability of the axioms. For centuries together, the axioms of Euclid were taken as truths as they conformed the truths of the physical space and except one these did not stand in our way to normal intuition. The one that really caused trouble is the fifth postulate, also known as the parallel postulate which stipulated a geometric object of infinite length. Certainly experience did not vouch for the behaviour of infinite straight lines whereas axioms were supposed to be self-evident truths about the physical world. So all round attempts were made to rebuild Euclid's geometry with the replacement of the parallel axiom by a seemingly more self-evident statement or to prove the parallel axiom by the other nine axioms. The attempts in particular by John Playfair (1748-1819), Georg S. K'ügel (1739-1812), Abraham G. Kästner (1719-1800) were far reaching in that they proved the independence of the axiom, but Gauss's work in this direction can be termed revolutionary. What he visualized and deduced unprecisely was rigorously established by Lobatchevsky and Bolyai who proved that the variations of the fifth postulate give birth to new geometries, called non-Euclidean geometries and they do fit into the physical space as accurately as the Euclidean geometry. This stimulated another novel creation—a new geometry, that gave the mathematical world further inducement to believe that the geometry of physical space could be non-Euclidean. Infact Riemann's 1854 paper published in 1868 covinced many mathematicians that a non-Euclidean geometry could be the geometry of physical space and that we could no longer be sure which geometry is true. The mere fact that there could be alternative geometries was in itself a shock. But the greater shock was that one could no longer be sure which geometry was true or whether any one of them was at all true, because it was clear by this time that the axioms framed were based on our limited experience and could never be taken for granted as truths. So the supporters of Plato's view that God geometrizes were at a loss and having no reinforced argument to support their view had to surrender. After the discovery of non-Euclidean geometries, it was really difficult to accept that the axioms and theorems of mathematics in general were not necessary truths about the physical world. This formidable crisis moved a physicist like Einstein so much that he expressed his despair in 1921 "Insofar as the propositions of mathematics give an account of reality they are not certain and insofar as they are certain they do not describe reality". The reverberations of this disaster reached almost all areas of our culture of the 19th century, shattering the confidence man had gathered by spectacular achievements in all branches of science till the beginning of the 19th century. That there is no truth in mathematics destroyed the evidence that man can acquire truths with the help of mathematics. The discovery of quarternions by Hamilton in the mean time deepened the crisis that suggests that conceptual order is not a veridical account of the perceptual. The pride of human reason suffered a fall which brought down with it the temple of truth.

But history suggests that man cannot live happily in an unpleasant darkness; he strives to turn darkness into light. So a critical movement started in the second decade of the century, because the founders of the movement realized that for over two thousand years mathematicians had wandered in a wilderness of intuitions, plausible arguments, inductive reasoning and formal manipulation of symbolic expressions. As two thousand years of reliable usage lent assurance to what logic had failed to demonstrate, the founders of the movement chose calculus as their starting point and not geometry, for calculus, the fount of analysis had already incorporated in its body loose proofs, paradoxes and even contradictions and many results without pragmatic sanctions.

As Cauchy (1789-1857) at that time was at the centre of the mathematical world, he had to shoulder this enormous task. He decided to found calculus on the limit concept built on the logic of numbers. Certainly he was not cent-percent successful and he also lost sight of rigor in many of his arguments, but much credit is to be given to him for this venture and the success he attained. The chief credit however should go to another master, Karl Weierstrass (1815-1897) who freed analysis from all dependence upon motion, intuitive understanding and geometric notions.

5.3 Crises of Reason

When the rigorization was going on in full swing, another metamorphic event was under way. The science of logic was founded by Aristotle in his Organon around 300 B.C. but the crises exposed its limitedness of scope. Leibnitz proposed a universal logic to enhance the utility of mathematics. He founded symbolic logic and tried to develop a symbolic calculus of reasoning but met with little success. So until the 19th century, Aristotelian logic held sway. George Boole (1815-1864) was instrumental in using logic to rigorize algebra and developed an entirely new kind of algebra of logic, now popular as Boolean algebra. Following Boole, DeMorgan provided another innovation. The logic of relation that was lacking so far was then propounded by Charles Sanders Peirce (1839-1914) and systematized by Giuseppe Peano and Ernest Schröder (1841-1902). The final step in mathematical logic was taken by Gottlob Frege (1848-1925). With these new instruments of logic the structure of mathematics was given soundness but this was almost gratuitous as no new theorem of arithmetic, algebra, or Euclidean geometry evolved and the theorems of analysis had only to be more carefully formulated. Thus though mathematics lost its grounding in reality, the axiomatic activity of the 19th century resolved a severe crisis in its history.

But the goddess of fortune smiled perhaps at the satisfaction of the 19th century mathematicians and storm clouds were gathering over the firmament of the mathematical world. In the International Congress held at Paris in 1900. David Hilbert presented a list of twenty three problems whose solution he regarded as most important for the advancement and even existence of mathematics. The first two problems are worth mentioning as they stand at the root of all controversies that swept away foundation built thus far. George Cantor (1845-1918) introduced transfinite numbers called cardinal numbers to represent the number of objects in infinite sets. Hilbert's first problem was to prove that there is no cardinal number between that of the set of natural numbers and that of the set of real numbers and also to prove that the set of real numbers is well ordered. The second problem raised the question of consistency of the Arithmetic system. This was pertinent as Hilbert had already proved that the Euclidean Geometry is consistent if the Arithmetic system is consistent. There was enough ground prepared by Cantor for the presentation of the above problems. The rigorization of analysis had to take into account the distinction between infinite series that converge and those that diverge. The difference between 'actual infinite' and 'potential infinite, propounded by Aristotle had to be carefully analysed since these had caused lot of confusion and challenged our intuition as is evident by a remark of Descartes "The infinite is recognizable but not comprehensible". In fact, the concept of infinity did thwart comprehension for centuries together. When in 1873, Cantor introduced infinite sets as existing totalities, as entities but he set about distinguishing them, the world of mathematics was in a whirlpool. Just as we have number symbols 1, 2, 3, 10 etc., to denote the number of objects in a finite collection, so Cantor decided to use symbols for the number of objects in infinite sets. He used the notation N_0* to denote the number of objetcts of a set whose objects can be put into one-to-one correspondence with the whole numbers. He used the notation N_1* or c to denote the number of objects in the set of all real numbers, because he proved this set of real numbers is larger than the set of whole numbers. Cantor was further able to show that for any given set there is always a larger one and this he demonstrated by taking the set of all subsets of the original set, which he called Power set and denoted as P(X) or 2^x. Considering the set of whole

*N_0 = $\aleph_0$ (aleph null), *N_1 = $\aleph_1$ (aleph one).

numbers, he thus obtained the power set whose cardinal number i.e. the number of objects he denoted by 2^{N_0} and proved that $_2N_0$* = N_1*. Contor's proof of the fact that there were larger and larger transfinite sets deepened the crisis of logic and intuition. The logician Russell also was initially stunned and he added his famous paradoxes to the problems that took away the sleep of many. Cantor introduced ordinal numbers also for sets and used the notion ω to denate the ordinal number of the set of whole numbers. In fact, an ordinal number is the cardinal number of a well ordered set i.e. a set every subset of which has a first element. Cantor introduced a hierarchy of transfinite ordinal numbers, like ω, $\omega + 1$, $\omega + 2$, ..., ω . ω , ..., ω^n ..., ω^ω and beyond. After creating the theory of transfinite ordinal numbers, Cantor believed that the set of ordinal numbers can be ordered in some suitable fashion; but a theorem that the ordinal number of the set of all ordinals upto and including α , say, is larger than α shook his convictions as the fact that the set of all ordinals should have a larger ordinal than the largest in the set leads to contradiction. With regard to the axiomatization a bundle of paradoxes flooded the mathematical arena this time. Aristotle's 'Liar paradox', Russell's 'Barber paradox', G. G. Berry's 'Word paradox' and many others by Kurt Gödel (1906-1978), Kurt Grelling (1886-1941), Leonard Nelson (1882-1927), Jules Richard (1862-1956), Frank Plumton Ramsey (1903-1930) and Burali Forti (1861-1931) set the stage in a bewildering condition. A number of attempts were made to resolve the paradoxes and as many of them could be resolved, the fallacies were discovered. Russell pointed at the unpredictiveness of the definitions which had been in use for long and were at the root of confusions. So set theory called for carefully worded axioms. Ernst Zermelo (1871 - 1953) and Fraenkel (1891 - 1965) offered a scheme, Russell-Whitehead offered another, while von Neumann (1903-1957) offered a third and in the effort to build solid foundation for mathematics, establishing consistency cer-

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*N_0 = $\aleph_0$ (aleph null), N_1 = $\aleph_1$ (aleph one).

tainly became the most demanding problem of the early 20th century. The earlier proofs were critically examined in the light of the new axiomatics. An innocent looking assertion, now known as Axiom of choice which had been in use in many earlier works struck the mathematicians. This assertion is that given any collection of sets, one can select one object from each set and thus form a new set. Cantor had used the axiom of choice in many situations including the proof of the theorem that an infinite set contains a subset with cardinal number N_0*. Peano first called attention to the axiom of choice in 1890 when he wrote that one cannot apply an infinite number of times an arbitrary law that selects a member of a class from each of many classes. The nub of the criticism was that unless a definite law specified which object is to be chosen from each set, no real choice can be made and so the new set cannot be really formed. So attempts were made by many to replace the axiom by a less controversial one but success was a far cry. The key issue with respect to the axiom of choice was what mathematics means by existence. Martin's Axiom, Countable Axiom of choice, Finite Axiom of choice came as substitutes but none proved equal to the Axiom of Choice.

Though the problems posed by Hilbert put mathematics into an uncomfortable situation, the state of the foundations of mathematics was tolerable till 1930. Two problems however continued to trouble the mathematical conscience, namely the problem of consistency and the problem of completeness. This second problem amounts to whether a reasonable conjecture can be proved or disproved. With regard to these two problems and the other vital questions like infinity and existence mathematicians were divided into different schools of thought, now known as the intuitionist school headed by Brouwer (1881-1966), the Logicist school headed by Russell and the Formalist school headed by Hilbert. Each of these schools tried to settle the problems within the philosophical framework. Russell had abandoned his belief in the

*N_0 = $\aleph_0$ (aleph null),

truth of the logical axioms used in the logistic approach and had confessed tthe artificiality of his axiom of reducibility.

His theory of types avoided the known paradoxes and the completeness problem was outside his purview. The intuitionists were indifferent to the problem of consistency, since they were convinced that the intuitions accepted by the human mind were eo ipso consistent and formal proofs were unnecessary and irrelevent and as to completeness they believed that human intuition was powerful enough to decide the truth or falsity of any meaningful proposition.

The formalists did not like to leave these two major problems unchallenged. In 1920 Hilbert returned to these problems. In his metamathematics he outlined the approach to a proof of consistency and expressed his confidence in the possibility of a solution of both the problems. But in 1931, Gödel published a paper entitled "On Formally Undecidable Propositions of Principia Mathematica and Related Systems", containing two startling results regarding consistency and completeness. He demonstrated first that the consistency of any mathematical system that is extensive enough to embrace the arithmetic of whole numbers cannot be proved by the logical principles of the different philosophical schools, and secondly that if any formal theory adequate to embrace the theory of whole numbers is consistent, then it must be incomplete, which means the price of consistency is incompleteness. This was a death blow to comprehensive axiomatization, because mathematicians, formalists in particular, had expected that any valid statement could certainly be established within the framework of some axiomatic system. Thus Gödel's formidable results exposed the hollowness of mathematics. Since consistency could not be proved, mathematicians risked talking nonsense because anyday a contradiction could be found. Gödel's incompleteness theorem is to an extent a denial of the law of excluded middle, a logical principle applied so long with remarkable success. The fact that no contradiction has been found in the axiomatic set theory with the continuum hypothesis and also in the theory with the negative of the continuum hypothesis added credence to this futility of the law of excluded middle. That theorems like Fermat's which assert that there is no integral solution of $x^n + y^n = z^n$ when $n > 2$ and like Goldbach's which assert that every even positive integer is the sum of two primes have baffled the endeavours of hundreds of mathematicians for centuries together supports Gödel's incompleteness theorem. A result of Alonzo Church (1903), a professor at Princeton University deepened this crisis. Developing a notion of function he showed that in general no decision procedure or an algorithm can be found to establish provability or disprovability of any meaningful statement. Thus Euclid's parallel postulate is not decidable on the basis of other Euclidean axioms. In view of the further result by Paul Cohen (1934), a professor of mathematics at Stanford University that the axiom of choice and the continuum hypothesis are independent of the other Zermelo-Fraenkel Axioms if the latter are consistent, it became obvious then that there are many mathematics. The troubles multiplied when Leopold Löwenheim (1878-1940) and Thoralf Skolem (1887-1963) disclosed new flaws in the structure of mathematics. Löwenheim and Skolem proved that a set of axioms permits many more essentially different interpretations than the one intended—a result that denies categoricalness in a more radical way. The efforts to eliminate possible contradictions and establish the consistency of the mathematical structures thus failed and there was no longer any agreement on whether to accept the axiomatic approach or if so, with which axioms or the non-axiomatic intuitionist approach. No school has now the right to claim that it represents mathematics.

Thus present mathematics is a collection of results—the results which may be true or false because their derivations are based on faulty logic and what are taken as rigorous proofs now are anything but secular. There is no unified theory of proof or any solid foundation. The proofs are parochial in the sense they obey only one set of axioms which are not acceptable to many. The question of the consistency problem has doomed the credibility of mathematics because what we prove now to be correct may turn out false tomorrow. What a stumbing disaster !

Still mathematics is growing. Nearly 50,000 papers are being published every year in different journals of the world. There are several reasons for this growth amidst this quandary. The one that accounts for by and large is that most contemporary mathematicians are unaware of this crisis, and many who are aware prefer taking an indifferent attitude towards this because publication of more and more papers is their prime interest for survival and prestige. They prefer to be suckled in a creed outworn. Then, is there no authority to urge restraint on the ground that foundational issues remain to be resolved ? The fact is there is none. The editors of journals could refuse papers but they are peers having the same position as mathematicians at large. Hence papers that maintain some semblance of rigor, the rigor of 1900, are being published. A few mathematicians have expressed their deep concern about foundational issues among whom, are Emile Borel, René Baire, Henri Lebesgue and Hermann Weyl. Despite this deplorable stage of affairs, those who advocate the growth of matter whether it is an advancement or a digression hold this view that mathematics is the sustenance of science and therefore it must develop. This tune is echoed in the voices of many amongst whom are Karl Popper and André Weil, the greatest philosophers of science of modern times. We conclude this article by a remark of Weil :

> "For us, whose shoulders sag under the weight of the heritage of Greek thought and who walk in the paths traced out by the heroes of the Renaissance, a civilization without mathematics is unthinkable. Like the parallel postulate, the postulate that mathematics will survive has been stripped of its "evidence", but while the former is no longer necessary, we would not be able to get on without the latter".

CHAPTER VI

THE FUTURE OF MATHEMATICS

6.0 Introduction

Crowned with innumerable glorious achievements mathematics has a special position in our culture—a position that is unique and invulnerable. She has been worshipped by all and sundry as the 'queen of science'. The adventures that began nearly three thousand years back by the Greeks and the Hindus met with memorable successes that shaped our civilization delivering us from the gloomy world of superstitions and ignorance to the new universe of light and wisdom. But every success is the culmination of several failures. The exalted position now held by mathematics has been attained not by a series of successes alone. Just as darkness reigns behind light, failures paved the path of successes. It would be unjust to call them failures, there had been several crises —Crises of concepts, crises of intuition and crises of reasons— more precisely intellectual crises that altered radically mathematicians' attitude towards their work. The discovery of non-Euclidean geometries by Gauss, Bolyai and Riemann compelled mathematicians to forget that mathematics is a body of truths about nature—a faith nourished by all for more than two millenia. The creation of quarternions and matrices forced mathematicians to belive that mathematics of ordinary numbers is not even a priori applicable. Like the well known three-body problem of mechanics and Goldbach's hypothesis of arithmatic, innumerable problems of very candid appearance mock at human intelligence even to day. The great discoveries of Gödel have lead us to an apparently insurmountable ditch, destroying our faith and credibility in mathematics. Standing in this gloomy world of mathematics one may naturally doubt whether the death of mathematics is imminent. If not, how can one believe that in the teeth of this severe intellectual crisis, mathematics has the necessary potential to survive ? This worry of the present generation is not altogether meaningless. Because the loss of truth is a tragedy, for truths are man's dearest possessions and the loss of even one is a cause of grief. In a broader view, our culture is veritably dependent on and intimately connected with a normal growth of mathematics, because the growth of mathematics entails the growth of science and science is the mind of civilization, art is but the heart. Marshall stone, a great mathematician of this century suggests in this regard—"We shall not resolve its tensions untill we accept science as an integral and all pervasive part of our culture, not only at the material level but also in the tangent spheres of the intellectual life and of education. So suddenly has this crisis developed and so far do its effects extend that we are forced to recognize in it the symptoms characteristic of a major mutation in human culture". This optimistic tone is not the solitary one to keep our spirit alive in mathematics; Andre Weil, one of the leading mathematician of our time has also sung the songs of a hopeful future. He remarks in his famous article "The Future of Mathematics". "Like the parallel postulate, the postulate that mathematics will survive has been stripped of its evidence". Professor Weil or Professor Stone was not prophets of science or mathematics in particular. They certainly did not see the dawn of a revolution in mathematics through clairvoyance, but backed by careful observations only they enchanted us with the music of a new epoch. Let us look back in the history of science for evidence of their phrophecies.

6.1 Survey of the Past

The past history of mathematics is replete with innumerable failures and several debacles that seemingly stranded the progress of mathematics. The Greek had many such setbacks. The Ionian Philosophers of the 6th century B.C. gave rational explanations of the nature and functioning of the universe. Almost every one of the famous philosophers of this period like Thales, Anaximander, Anaximenes, Heraditus, Anaxagoras took a distinct substance as the constituent of the universe. Thales, for example, declared that everything is made of water and tried to explain all natural phenomena accordingly. But meticulous studies later on exposed the truth to a great shock of the Greeks ; yet the expedition in the domain of science never stopped. Prompted by the theological tenets and taboos, the Greeks also propagated that the universe has been designed by the god and natural phenomena are only whims of the creator. Even Plato (427-347 B.C.) who dominated Greek thought in the momentous 4th century B.C. and long after believed that God eternally geometrizes. Much of the works of Archimedes (287-212 B.C.), Menelaus (circa A.D. 98), Ptolemy (d. A.D. 168) turned out erroneous in later researches. The same thing happened with almost all scientists of every age but the mere fact of discovering errors in any scientific investigation never stood in the way to progress of mathematics; rather the critical analysis and attempts to rectification of these errors ended with new discoveries, more useful techniques and greater farsight.

Not the indomitable venturesomeness of human nature alone, but the socio-political conditions too had althrough played a vital role in the growth of mathematics. When the beams of intellectual light kindled by the pioneers of Renaissance in Europe charged thousands of hearts, they stood up together to fight against the long cultivated superstitions and anti-rational prejudices that had eclipsed human civilization for a long period and ungrudgingly took service of the queen of science. This enormously boosted up the spirit of mathematics and as the primary objective of science is human welfare, mathematicians of the Renaissance period imbued with new ideals started contributing liberally to science and mathematics in particular.

The history of mathematics till the 17th century is marked by several crises, mostly with regard to the notions. The negative numbers, rational and irrational numbers, infinity infinitesimals posed serious problems to mathematicians. Galileo, Copernicus, Kepler, Descartes, Leibnitz, Newton and many others challenged the traditional concepts about nature and numbers. The catholic doctrine of God's creating the universe and the Greek doctrine of mathematical designs of nature were thrown aside; but the rejection of the old incorrect ideas did not check the sponteneity of growth. The effect was rather beneficial; a spurt of fresh researches was noticed. The ideas and techniques devised by Descartes, Newton and Leibnitz provided new areas of research. The foundational questions started pouring in to threw away the discrepencies, imprecision and irregularities in mathematics on a solid foundational basis.

The 18th century flourished with a steady growth of mathematical results in the hands of several luminaries like Euler, Lagrange, Laplace, Fourier and Gauss. But soon the foundational questions drifted mathematics to one of its worst debacles. Euclid's fifth postulate whose validity and relevance had already been an issue of great concern inevitably led mathematics to an uncomfortable situation. The centuries old perception of describing nature by Euclid's geometry tore into pieces when the non-Euclidean geometry was discovered by Gauss, Lobatchevsky and Bolyai. In deed, from Galileo's time onward, scientists recognised that the fundamental principles of science, as opposed to mathematics, must come from experience, although for at least two centuries they believed that the principles they did find were imbedded in the design of nature. But by the end of 18th century they realised that scientific theories are not truths. With the loss of truth, man lost his intellectual centre, his form of reference and the established authority for all thought. The pride of human reason suffered a fall which brought down with it the temple of truth. But did this debacle put an end to mathematical discoveries ? The history proves the other truth. The needs of science prevailed over logical scruples. Doubts about the soundness of algebra, arithmatic and geometry were cast aside to make way to new concepts and discoveries. The fertile complex numbers, matrices, quanternions, divergent series came with flying colours and swept the 18th century and the first half of the 19th century.

The crises of concepts and the crises of reality that had shaken the faith and infallibility of mathematics now being studied introspectively by the philosophers of science paved the path for serious logical tests.

With this objective, the traditional logic hitherto applied to all branches of science was improvised by Peano, Boole, Frege and Russell. The foundation of mathematics so long uncared for drew the attention of all mathematicians of this generation. With the revolutionary idea of set by Cantor and the subsequent paradoxes, the growth of mathematics though it took a new turn never held sway. Series of valuable discoveries poured in enriching the vast treasure of mathematics. Great mathematicians like Dedekind, Weirstrass, Riemann, Hilbert, Schröder and Poincare worked ceaselessly to broaden the majesty of mathematics. Hilbert alone provided hundreds of outstanding problems the solutions of which could give not only new insight but new dimensions to mathematics as well.

6.2 Survey of the Present

Though the present century witnessed the worst ever foundational crisis, it is this century in which mathematics has revolutionized the whole concept of science. No doubt Gödel's results have marred the credibility of mathematics but not its vitality. Defying the crises mathematics has flourished at an incredible pace, permeating in almost all spheres of human activity. In the words of Marshall stone, a vanguard of modern mathematical culture, "Our conception of the nature of mathematics has been revolutionized, our technical knowledge of the subject vastly enlarged and our dependence upon it for scientific and technological progress enormously increased. In deed, it is becoming clearer and clearer every day that mathematics has to be regarded as the corner-stone of all scientific thinking and hence of the intricately articulated technological society we are busily engaged in building. We can forsee a time in the not-very-distant future when a complete identification of sciences, logic and mathematics will be achieved".

Thus the merging of mathematics with science itself, in the opinion of Stone, ensures a bright future of mathematics since science means survival of the civilization. Andre weil too is highly optimistic about the future of mathematics and does not consider the foundational crisis that has stamped the 20th century with the worst blemish as an omen of an ill fate but as a nucleus storing unbounded energy for future growth. In his famous article "The Future of Mathematics", he goes on to say, "In recent times mathematics has demonstrated its vitality by passing through one of these periods of growing pains, to which it has been accustomed for a long time, and which are designated by the strange name of foundation crisis. It has come through it, not only without damage but with great gain". Referring to a caustic remark of Hilbert that a branch of science is full of life as long as it offers an abundance of problems and a lack of problems is a sign of death, he proceeds in the same article to cite some of the outstanding problems that had remained hitherto unsolved and that suggest great unification of mathematics. In deed, many such lists are available today prepared mostly by the vanguards of modern mathematics like Hilbert, Ulam, Bessaga, Nagy, Bourbaki, Erdos, which have not only kept mathematics lively but also suggest extension of the frontiers of science in general.

In what follows, we make a survey of the present state of mathematical investigations in various fields of research. We do not intend to give as many major problems as are present there, but only some illustrative examples to evince the liveliness of that part of mathematics. To start with we enter the domain of algebra first but ask the reader to remember that different disciplines of mathematics are so interwoven now that it is difficult to assert the major field to which a problem belongs.

Mathematicians have studied intensively the structure of continuous groups. Recent work has successfully clarified many of the topological characteristics of such groups, especially in the case where the group operation, in addition to being continuous, possesses differentiability properties. These are called Lie groups. The representation of such groups as groups of linear transformations of the n-space has enabled important physical interpretations in quantum theory and particle physics. Application of these ideas to the classification of atom spectra has shown the unexpected powers of mathematical foresight. But still there are plenty of unsolved problems. The Riemann hypothesis which could not be proved by function theoretic method now raises hope in the light of new discoveries and seems to be closely connected with the conjecture of Artin on the L-functions and hence requires simultaneous study of all the cyclotomic extensions of a given number field. In this context, Gaussian arithmetic was centred around the law of quadratic reciprocity. Though much has been studied about this law and the like but practically nothing is known as to the deeper lying symmetries behind this. The automorphisms induced in the class groups by the automorphisms of the field, the properties of the norm residues in non-cyclic cases, its direct product and direct sum replacing the base field by extensions of indefinitely increasing degree are questions which are yet to be answered to study the Riemann hypothesis in its complete generality. In brief, practically very little is known about non-abelian extensions and once the first decisive step has been taken, a vast domain will be opened for future work.

The Kronecker's work on generalization of class fields by means of analytic function in the case of imaginary quadratic fields has made a satisfactory progress, but the general problem considered by Hilbert is far from the desired success. The guesses as to the applicability of automorphic functions of Siegel or endomorphism of abelian varieties are yet to be tested.

Hermite's approach to the systematic study of discontinuous groups of arithmatical nature by means of continuous groups in which they can be embedded, of the symmetric Riemannian spaces associated with these groups, of the differential and topological properties of their fundamental domains and of the automorphic functions which belong to them is yet to yield desired results. The works of Siegel, Minkowski, Fermat, Lagrange and Gauss, that establishes a connection between numbers and forms, in particular the genera of quadratic forms, have opened new vistas of research. The analogy between the results of Siegel in arithmetic and Cauchy's theorem for the Riemann surface of an algebraic curve raises the following question. Is it possible to formulate a general principle which would allow us to obtain all results of this character at one stroke ? It is now quite clear through the work of Hecke, that the above work will have a long lasting bearance upon the class of quadratic forms and the theory of theta functions and modular functions. This domain is still full of mystery and shows clear signs of fascinating research potentiality.

In deed, the recent works of Bergman, Hodge, de Rham, Ahlfors, and Chern have proved beyond doubt an intimate connection between analytic number theory, function theory and Topology which has given rise to series of new problems. Thus having received a fresh stimulus from the recent developments in topology and differential geometry, algebraic geometry has exposed multitude of unexplored areas for future research. The theory of surfaces developed mainly by the Italian school calls for a general theory of algebraic varieties. In deed, algebraic geometry over fields other than the complex field like p-adic fields, fields of algebraic numbers deserve to be studied in greater details.

In geometry the work of Cartan is good enough to supply problems for several generations of mathematicians. His general theory of systems in involution is yet to be completed and stands as a perenial source of mathematical investigations. His work on "infinite Lie groups" was just a beginning of a long way and requires sustained efforts on the part of the researchers. The modern techniques from the topological theory of fibre spaces, de Rham's theory and homology theory seem to take the enterpreneurs a long way.

The mathematical treatment of continuous distribution of mass or of fields in physics requires partial differential equations and here functions of several variables play the major role. Problems in hydrodynamics, elasticity, thermodynamics, heat could be stated and solved by partial differential equations. The development of electromagnetic theory by Maxwell also call for differential equations for its description and analysis. Though the theory of partial differential equations has been developed much to satisfy our requirement, yet many more investigations atleast regarding the elliptic and hyperbolic equations are to be carried on. Another path shown by Bergman and his pupils with regard to the study of transformation of differential equations by means of integral or of integro-differential operators demand exploration. This seems to be the germ of entirely new developments in the classification of systems of partial differential equations which falls possibly outside the framework of classical methods. The Banach space theory seems to be inadequate in this regard and calls for more general spaces.

The recent input of physics into ordinary differential equations, particularly by Vander Pal and Hopf has given rise to a new set of problems for further studies.

Information theory shaped by Shannon and his successors forms an elegant and coherent part of mathematics. Intially it sarted with a finite set of events and the associated probabilities. But the recent development in terms of infinite set of events and the subsequent introduction of the idea of entropy from physics into information theory has not only greatly stirred the whole theory but also proved extremely valuable in abstract mathematical questions that prima facie seemed to have nothing to do with notions of probability. The application of computers has tremendously influenced Coding Theory in particular and has sparked off new hopes of further researches.

In Game theory, much has been done in the very simple case when all the strategies of the opponent are known. A completely new type of problems in which coalitions of players are taken instead of players themselves was initially studied by J. von Neumann and his students but now requires much more attention of the researchers.

Ergodic theory concerns dynamical systems of n mass points. If the motion in a dynamical system is described by linear equations; the situation is much simple due to availability of many mathematical tools, but the situation with non-linear representation of motion is much more complicated and demands sustained efforts of the researchers for a useful solution. Some problems of purely mathematical interest are there and the tools of topology and modern algebra have been proved to be useful in this exploration. Non-standard analysis is of recent origin and is a fertile area of research to take analysis further for a few years to come.

In the history of science, it is a conspicuous fact that mathematics and physics played complementary roles and just as much of mathematics received the basic stimulus from physics so did physics get new interpretations, new insights and even new intellectual dimensions. The very essence of theoretical physics lies in its mathematical formulation and the development of a large part of mathematics was stimulated and determined by problems posed by the behaviour of matter. The great success thus attained in physics prompted mathematicians, scientists of pure sciences and social scientists to try mathematization of various aspects of the relative sciences. Overwhelming success has been achieved in the 20th century particularly in Bio-sciences and Economics. In the recent past we have witnessed a rapidly increasing knowledge of primary or elementary biological phenomena which are ripe for mathematization. Much has been done in this direction. Studies like Volterra's on changes in the number of individuals of living species that feed on each other were mathematically beyond their implications in biology. Volterra, using a system of related total non-linear differential equations, has been able to show farflung effectiveness of mathematization. Such work has stimulated study of non-linear problems afresh in pure mathematics and seems to lead mathematical biology quite far in the near future. Similarly Mendel's genetic laws set in mathematical language gave rise to a number of combinatorial problems. In bio-chemistry too mathematics is playing an influential role not only to describe the behaviour of mixtures but also to consider the thermodynamics and quantum theoretical bases of such processes. In economics mathematics has demonstrated beyond doubt its inevitability in the demonstration and better understanding of a variety of economic behaviours and phenomena.

The theory of Fuzzy sets is yet to attain its major status, none the less it is all set for an astonishing influence on social as well as pure sciences.

In biology again, important beginnings have been achieved in understanding operating schemes of the living cells. The exact mechanics, logic and combinatorics of this process are not yet fully understood. New logical schemes that have been proved mathematically sound seem to describe new pattern somewhat different from those used in the formal apparatus of mathematics. A prospective research area is in the offing which will take enough time of our future for a complete understanding.

Recent discoveries in mathematical logic by Gödel as to the incompleteness and consistency of any axiomatic system have posed serious scientific as well as philosophical problems regarding the universe. Since mathematics provides models of physics, time is mature now in the light of the results of Gödel to reconsider the models critically as it is now an accepted fact that no finite system of axioms can be considered as definite or ultimate. Should the universe actually contain an infinity of distinct points in the form of stars or photons or elementary particles of matter statements concerning such assemblies will certainly exist that are undecidable in terms of any finite number of laws and rules stated in advance. In deed, this philosophico-mathematical problems seems to have profound mathematical as well philosophical import in the days to come.

It is a common experience in the history of mathematics whenever mathematics expanded enormously, the search for a pattern aimed at unification had been desperately attempted resulting in a new insight into the subject. For obvious reasons this would be quite expectedly a direction of research in the coming decades.

6.3 Conclusion

The 20th century witnessed on one side splendid mathematical discoveries, marvellous achievements in mathematizations of remotest subjects which could not be dreamt of even a decade back and on the other side severest debacle ever seen in the history of mathematics. The very nature of mathematics, the science of logic which forms the very life blood of mathematics and the most fundamental notions are faced with gruesome challenges. A prominent mathematician like R. L. Wilder comments that proof, absolute rigor and their ilk are will-o-the wisps, ideal concepts with no natural habitat in the mathematical world. Mathematicians are divided into different camps as to what is true mathematics and what is not. Yet even a cursory glance at the history of mathematics will convince anybody that this challenge to mathematics is not new. In deed, it is an intrinsic characteristic of any science that old ideas are challenged to give way to new ideas, old customs are changed. Mathematics in particular faced several such crises in the past but it possess enough vigour, vitality and strength to get over such crises. Days are not far away when mathematicians will find a solution to the present foundational crisis, however grave it is. But history evinces that even in the teeth of such devastating foundational crisis, mathematics had a steady growth—a growth envisaged by plenty of physical problems and an attempt for unity of mathematics in diversity. In the words of Courant and Robbins, "Fortunately, creative minds forget dogmatic philosophical beliefs whenever adherence to them would impede constructive achivement''.

Professor Andre Weil's apprehension of the contingent growth of mathematics is but a father's worry for his son. History ratifies the thesis that mathematics will never die neither for any foundational crisis nor for dearth of problems. Thank God that he made physics the closest friend of mathematics who never refrained from supplying food for mathematics. Thank God Physics has still numerous unsolved problems which will take many more years for solution even with the strongest backing of mathematics. Thank God for his blessing to mankind to discover electronic computers which will further mathematical researches at least for another century.

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