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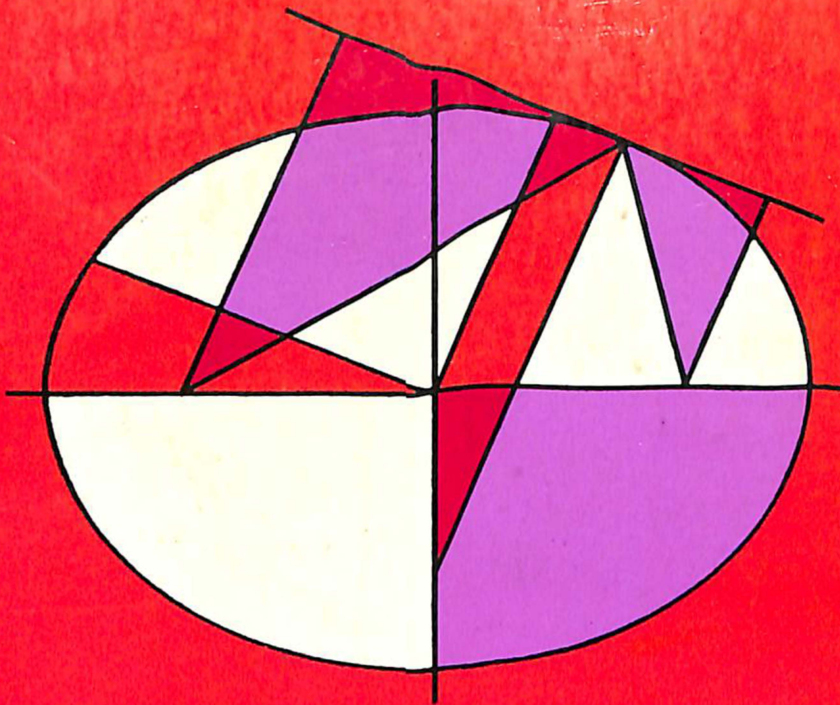

introduction to ELLIPTIC FUNCTIONS with applications

F. Bowman

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INTRODUCTION TO
ELLIPTIC FUNCTIONS
with applications

BY

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DOVER PUBLICATIONS, INC.

NEW YORK

Published in Canada by General Publishing Company, Ltd., 30 Lesmill Road, Don Mills, Toronto, Ontario.
Published in the United Kingdom by Constable and Company, Ltd., 10 Orange Street, London WC 2.

This Dover edition, first published in 1961, is an unabridged and corrected republication of the work originally published by the English Universities Press Ltd. in 1953.

Library of Congress Catalog Card Number: 61-66218

Manufactured in the United States of America
Dover Publications, Inc.
180 Varick Street
New York, N. Y. 10014

PREFACE TO DOVER EDITION

THIS new paperbound edition is identical to the first edition, except for the correction of typographical and other minor errors which have come to my attention. I have also added a few useful reference books to the brief bibliography on page 109.

F. B.

May, 1961

PREFACE TO THE FIRST EDITION

THE purpose of this book is to give a short practical introduction to some applications of elliptic functions; it is confined to the Jacobian functions; the Weierstrassian function is not even mentioned. In the first few chapters, only an elementary knowledge of differentiation and integration is required from the reader; in the later stages he will find it helpful to be familiar with the elements of the theory of functions of a complex variable.

Articles are numbered consecutively, each chapter beginning with §1. References are made in the form " III, §7 ", the Roman numeral indicating the chapter, but the Roman numeral is omitted when reference is made to an article in the current chapter. Equations and figures are numbered and referred to in the same way as articles.

It is a pleasure to record my thanks to former colleagues Dr. S. Verblunsky and the late Mr. W. Hunter, who read part of the manuscript, and to Dr. C. A. Stewart, who read it all. Thanks are also due to Mr. H. Tilsley for drawing the figures.

F. B.

CONTENTS

CHAPTER I

JACOBIAN ELLIPTIC FUNCTIONS

§ 1. Introduction. The method by which the properties of the elliptic functions will be introduced in this book will first be illustrated by showing how the method could be applied to the circular functions. Put

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)}} \cdot \qquad \text{(i)}$$

$$\tfrac{1}{2}\pi = \int_0^1 \frac{dt}{\sqrt{(1-t^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

and suppose x to be real, $-1 \leqslant x \leqslant 1$, and $\sqrt{(1-t^2)} \geqslant 0$.

Equation (i) defines u as an odd function of x which, since the integrand on the R.H.S. is positive, increases steadily from 0 to $\frac{1}{2}\pi$ as x increases from 0 to 1. Inversely,* the same equation defines x as an odd function of u which steadily increases from 0 to 1 as u increases from 0 to $\frac{1}{2}\pi$; let this function be denoted by $\sin u$, so that in place of (i) we can put

$$u = \sin^{-1} x, \quad x = \sin u \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

The function $\cos u$ can then be defined by the equation

$$\cos u = \sqrt{(1-\sin^2 u)} \quad . \quad . \quad . \quad . \qquad \text{(iv)}$$

the square root being positive as long as u is confined to the interval $-\frac{1}{2}\pi < u < \frac{1}{2}\pi$, so that $\cos u$ is an even function.

Since $u = 0$ when $x = 0$, and $u = \frac{1}{2}\pi$ when $x = 1$, we note, in particular, that $\sin 0 = 0$, $\cos 0 = 1$, $\sin \frac{1}{2}\pi = 1$, $\cos \frac{1}{2}\pi = 0$ all follow from these definitions. It is not necessary here that the value of π should be known; for our present purpose, $\frac{1}{2}\pi$ is simply a number defined by (ii), from which its value could be calculated if need be.

From (iv) follows at once the well-known identity

$$\sin^2 u + \cos^2 u = 1 \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

From (i) follows $du/dx = 1/\sqrt{(1-x^2)}$, and hence

$$\frac{d(\sin u)}{du} = \frac{dx}{du} = \sqrt{(1-x^2)} = \sqrt{(1-\sin^2 u)} = \cos u \quad . \qquad \text{(vi)}$$

and further, by differentiating (v), $d(\cos u)/du = -\sin u$.

* Hardy, *A Course of Pure Mathematics*, § 110.

By repeated differentiation and substitution in Maclaurin's series, we could now find the expansions of $\sin u$ and $\cos u$ in ascending powers of u.

We could next prove the addition formulæ

$$\sin(u+v) = \sin u \cos v + \cos u \sin v \quad . \quad . \quad . \quad \text{(vii)}$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v \quad . \quad . \quad . \quad \text{(viii)}$$

assuming u, v, and $u + v$ to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Thus, putting $z = \sin u \cos v + \cos u \sin v$, and differentiating partially, we should see that $\partial z/\partial u = \partial z/\partial v$, from which would follow $z = F(u + v)$ and hence

$$F(u+v) = \sin u \cos v + \cos u \sin v$$

where $F(u + v)$ denotes some function of $u + v$. Putting $v = 0$ gives $F(u) = \sin u$, and hence $F(u + v) = \sin(u + v)$, thus proving (vii). Then (viii) follows from (vii) and (iv).

So far, u has been restricted to the interval $-\frac{1}{2}\pi \leqslant u \leqslant \frac{1}{2}\pi$, and it remains to define $\sin u$ for all other real values of u. One way of doing this is provided by the addition formula (vii), which may be assumed to hold good for all values of u and v. Thus, if we put $u = u_1$ and $v = \frac{1}{2}\pi$ in (vii), we obtain

$$\sin(u_1 + \tfrac{1}{2}\pi) = \cos u_1$$

which, by now putting $u = u_1 + \frac{1}{2}\pi$, may be written

$$\sin u = \cos(u - \tfrac{1}{2}\pi),$$

which may be assumed to define $\sin u$ when $\frac{1}{2}\pi \leqslant u \leqslant \pi$. The interval within which $\sin u$ is defined will thus be extended, and in the same kind of way can be further extended to include all real values of u.

The above method will now be applied to introduce the Jacobian elliptic functions.

§ 2. The Jacobian elliptic functions sn u, cn u, dn u. Put

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)}\sqrt{(1-k^2t^2)}} \quad . \quad . \quad (1)$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)}\sqrt{(1-k^2t^2)}} \qquad (2)$$

and at first suppose that x and k are real, $0 < k < 1$, $-1 \leqslant x \leqslant 1$, and that the square roots $\sqrt{(1-t^2)}$, $\sqrt{(1-k^2t^2)}$ are positive.

Equation (1) defines u as an odd function of x which, since the integrand on the R.H.S. is positive, increases steadily from 0 to K as x increases from 0 to 1. Inversely, the same equation defines x as an odd function of u which increases steadily from 0 to 1 as u increases from 0 to K; this function is denoted by $\operatorname{sn} u$, so that in place of (1) we can put

$$u = \operatorname{sn}^{-1} x, \quad x = \operatorname{sn} u \quad . \quad . \quad . \quad . \quad (3)$$

The functions $\operatorname{cn} u$, $\operatorname{dn} u$ can then be defined by the equations

$$\operatorname{cn} u = \sqrt{(1 - \operatorname{sn}^2 u)}, \quad \operatorname{dn} u = \sqrt{(1 - k^2 \operatorname{sn}^2 u)} \quad . \quad (4)$$

the square roots being positive so long as u is confined to the interval $- K < u < K$, so that $\operatorname{cn} u$ and $\operatorname{dn} u$ are even functions of u.

The function $\operatorname{sn} u$ " is a sort of sine-function, and $\operatorname{cn} u$, $\operatorname{dn} u$ are sorts of cosine-functions " (Cayley), but whereas the circular functions $\sin u$, $\cos u$ are *simply periodic*, the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are *doubly periodic*, as will be seen later. They are also called *elliptic* functions, owing originally to the connection of such functions with the rectification of the ellipse, and they are usually referred to as *Jacobian elliptic functions* because they were much used by Jacobi in his researches on doubly-periodic functions.

§ 3. **The modulus.** Besides being functions of u, the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ depend upon the parameter k, which is called the *modulus*. When it is desirable to put the modulus in evidence, the functions are denoted by $\operatorname{sn}(u, k)$, $\operatorname{cn}(u, k)$, $\operatorname{dn}(u, k)$.

Since $u = 0$ when $x = 0$, and $u = K$ when $x = 1$, we have

$$\operatorname{sn} 0 = 0, \quad \operatorname{cn} 0 = 1, \quad \operatorname{dn} 0 = 1 \quad . \quad . \quad (5)$$

$$\operatorname{sn} K = 1, \quad \operatorname{cn} K = 0, \quad \operatorname{dn} K = k' \quad . \quad . \quad (6)$$

where

$$k' = \sqrt{(1 - k^2)}, \text{ or } k^2 + k'^2 = 1 \quad . \quad . \quad . \quad (7)$$

and k' is called the *complementary modulus*.

§ 4. **Identities.** From (4) easily follow the identities

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$k^2 \operatorname{cn}^2 u + k'^2 = \operatorname{dn}^2 u \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\operatorname{cn}^2 u + k'^2 \operatorname{sn}^2 u = \operatorname{dn}^2 u \quad . \quad . \quad . \quad . \quad . \quad (11)$$

§ 5. **Derivatives.** The derivatives of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are given by

$$d(\operatorname{sn} u)/du = \operatorname{cn} u \operatorname{dn} u \quad . \quad . \quad . \quad (12)$$

$$d(\operatorname{cn} u)/du = - \operatorname{sn} u \operatorname{dn} u \quad . \quad . \quad (13)$$

$$d(\operatorname{dn} u)/du = - k^2 \operatorname{sn} u \operatorname{cn} u \quad . \quad . \quad (14)$$

Proof. From (1),

$$du/dx = (1 - x^2)^{-\frac{1}{2}} (1 - k^2 x^2)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (15)$$

and hence

$$dx/du = (1 - x^2)^{\frac{1}{2}} (1 - k^2 x^2)^{\frac{1}{2}}$$

from which and (4) follows (12), since $x = \operatorname{sn} u$. Then (13) and (14) follow by differentiating (4), or (8) and (9).

§ 6. Expansions in ascending powers of u. The first few terms of the expansions of sn u, cn u, dn u in ascending powers of u are given by

$$\operatorname{sn} u = u - (1 + k^2)u^3/3\,! + (1 + 14k^2 + k^4)u^5/5\,! + \ldots \quad (16)$$

$$\operatorname{cn} u = 1 - u^2/2\,! + (1 + 4k^2)u^4/4\,! - \ldots \quad (17)$$

$$\operatorname{dn} u = 1 - k^2u^2/2\,! + k^2(4 + k^2)u^4/4\,! - \ldots \quad (18)$$

These terms can be obtained by finding the values of the successive derivatives of sn u, cn u, dn u when $u = 0$, and substituting in Maclaurin's series (or otherwise—see Examples I (*a*), 3).

Convergence of these expansions. It may be stated here that, if $k^2 < 1$, the infinite power-series, of which the first few terms have just been given, have a radius of convergence equal to K', where

$$K' = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)}\sqrt{(1 - k'^2t^2)}} \cdot \quad (19)$$

so that K' is the same function of the complementary modulus k' as K is of k. This radius of convergence follows from a general theorem in the theory of functions, to the effect that the circle of convergence of the Maclaurin expansion of an analytic function passes through the singularity nearest to the origin, which in the present case is $u = iK'$, as will be shown when the definitions of sn u, cn u, dn u have been extended to complex values of u (see IV, § 8).

§ 7. Degenerate Cases. The elliptic functions reduce to circular functions when $k = 0$, and to hyperbolic functions when $k = 1$. Thus, when we put $k = 0$ we find, from (1), (2), (4), (7) and (19),

$$k = 0, \quad k' = 1, \quad K = \tfrac{1}{2}\pi, \quad K' = \infty \quad . \quad . \quad (20)$$

$$\operatorname{sn} u = \sin u, \quad \operatorname{cn} u = \cos u, \quad \operatorname{dn} u = 1 \quad (21)$$

and when $k = 1$ we find

$$k = 1, \quad k' = 0, \quad K = \infty, \quad K' = \tfrac{1}{2}\pi \quad (22)$$

$$\operatorname{sn} u = \tanh u, \quad \operatorname{cn} u = \operatorname{dn} u = \operatorname{sech} u \quad . \quad (23)$$

EXAMPLES I (*a*)

1. If $\Delta = 1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v$, show that
$$\Delta = \operatorname{cn}^2 u + \operatorname{sn}^2 u \operatorname{dn}^2 v = \operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u \operatorname{cn}^2 v$$
$$= \operatorname{cn}^2 v + \operatorname{sn}^2 v \operatorname{dn}^2 u = \operatorname{dn}^2 v + k^2 \operatorname{sn}^2 v \operatorname{cn}^2 u$$

2. Show that:

(i) $\dfrac{d}{du}\log(\mathrm{dn}\, u - k\,\mathrm{cn}\, u) = k\,\mathrm{sn}\, u$

(ii) $\dfrac{d^2}{du^2}\mathrm{sn}\, u = -(1+k^2)\,\mathrm{sn}\, u + 2k^2\,\mathrm{sn}^3\, u$

(iii) $\dfrac{d}{du}(\mathrm{sn}\, u\,\mathrm{cn}\, u\,\mathrm{dn}\, u) = 1 - 2(1+k^2)\,\mathrm{sn}^2\, u + 3k^2\,\mathrm{sn}^4\, u$

(iv) $\dfrac{d}{du}\left(\dfrac{\mathrm{cn}\, u\,\mathrm{dn}\, u}{\mathrm{sn}\, u}\right) = -\dfrac{1}{\mathrm{sn}^2\, u} + k^2\,\mathrm{sn}^2\, u$

(v) $\dfrac{d}{du}\left(\dfrac{\mathrm{sn}\, u}{\mathrm{cn}\, u\,\mathrm{dn}\, u}\right) = \dfrac{1}{\mathrm{cn}^2\, u} + \dfrac{1}{\mathrm{dn}^2\, u} - 1$

(vi) $\dfrac{d}{du}\left(\dfrac{\mathrm{sn}^2\, u - \mathrm{sn}^2\, a}{\mathrm{cn}\, u\,\mathrm{dn}\, u}\right) = \mathrm{sn}\, u\left(\dfrac{\mathrm{cn}^2\, a}{\mathrm{cn}^2\, u} + \dfrac{\mathrm{dn}^2\, a}{\mathrm{dn}^2\, u}\right)$

(vii) $\dfrac{d}{du}\left(\dfrac{\mathrm{sn}\, u}{1+\mathrm{cn}\, u}\right) = \dfrac{\mathrm{dn}\, u}{1+\mathrm{cn}\, u}$

(viii) $\dfrac{d}{du}\left(\dfrac{\mathrm{dn}\, u}{1-k\,\mathrm{sn}\, u}\right) = \dfrac{k\,\mathrm{cn}\, u}{1-k\,\mathrm{sn}\, u}$

(ix) $\dfrac{d}{du}\left(\dfrac{\mathrm{cn}\, u\,\mathrm{dn}\, u}{1-\mathrm{sn}\, u}\right) = k^2\,\mathrm{cn}^2\, u + \dfrac{k'^2}{1-\mathrm{sn}\, u}$

(x) $\dfrac{d}{du}\left(\dfrac{\mathrm{sn}\, u\,\mathrm{cn}\, u}{\mathrm{dn}\, u - k'}\right) = \mathrm{cn}^2\, u + \dfrac{k'}{\mathrm{dn}\, u - k'}$

3. Show that the functions $\mathrm{sn}\, x$, $\mathrm{cn}\, x$, $\mathrm{dn}\, x$ satisfy the differential equations

$$\begin{aligned} d^2y/dx^2 &= -(1+k^2)y + 2k^2y^3 \\ d^2y/dx^2 &= -(1-2k^2)y - 2k^2y^3 \\ d^2y/dx^2 &= (2-k^2)y - 2y^3 \end{aligned}$$

respectively. Deduce the first few terms of the Maclaurin expansions of the functions.

[In the first equation, substitute $y = x + a_3x^3 + a_5x^5 + \ldots$, etc.]

4. If $y = \mathrm{sn}^2\, x$, show that $d^2y/dx^2 = 2 - 4(1+k^2)y + 6k^2y^2$.

5. Show that

$$\begin{aligned} \mathrm{sn}\, u &= \frac{\sin\{u\sqrt{(1+k^2)}\}}{\sqrt{(1+k^2)}} + \alpha u^4 \\ \mathrm{cn}\, u &= \cos u + \beta u^3 \\ \mathrm{dn}\, u &= \cos ku + \gamma u^3 \end{aligned}$$

where α, β, γ all tend to zero when $u \to 0$.

6. If $0 < u < K$, show that

$$\frac{1}{\mathrm{cn}\, u} > \frac{u}{\mathrm{sn}\, u} > 1 > \mathrm{dn}\, u > \mathrm{cn}\, u$$

§ 8. The addition formulæ. The addition formulæ for the elliptic functions sn u, cn u, dn u, are [compare § 1, (vii), (viii)]

$$\operatorname{sn}(u+v) = (\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v) \quad (24)$$

$$\operatorname{cn}(u+v) = (\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v) \quad (25)$$

$$\operatorname{dn}(u+v) = (\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v) \quad (26)$$

Brief proofs are given below. For other proofs see, e.g., Whittaker and Watson, *Modern Analysis*, § 22·2.

Proof of (24). For brevity, put $s_1 = \operatorname{sn} u$, $s_2 = \operatorname{sn} v$, $c_1 = \operatorname{cn} u$ $c_2 = \operatorname{cn} v$, $d_1 = \operatorname{dn} u$, $d_2 = \operatorname{dn} v$, and $\Delta = 1 - k^2 s_1^2 s_2^2$. Also put

$$z = (s_1 c_2 d_2 + s_2 c_1 d_1)/\Delta \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Then, by partial differentiation with respect to u by the quotient rule, we get

$$\begin{aligned}
\Delta^2 \partial z/\partial u &= \Delta\{c_1 d_1 c_2 d_2 - s_1 s_2 (d_1^2 + k^2 c_1^2)\} + 2k^2 s_1 c_1 d_1 s_2^2 (s_1 c_2 d_2 + s_2 c_1 d_1) \\
&= c_1 d_1 c_2 d_2 (\Delta + 2k^2 s_1^2 s_2^2) - s_1 s_2 \{\Delta(d_1^2 + k^2 c_1^2) - 2k^2 s_2^2 c_1^2 d_1^2\} \\
&= c_1 d_1 c_2 d_2 (1 + k^2 s_1^2 s_2^2) - s_1 s_2 \{d_1^2(\Delta - k^2 s_2^2 c_1^2) \\
&\qquad + k^2 c_1^2 (\Delta - s_2^2 d_1^2)\} \\
&= c_1 d_1 c_2 d_2 (1 + k^2 s_1^2 s_2^2) - s_1 s_2 (d_1^2 d_2^2 + k^2 c_1^2 c_2^2)
\end{aligned}$$

We thus see that $\partial z/\partial u$ is symmetrical in u and v, and since z itself is symmetrical it follows that $\partial z/\partial v$ will be the same as $\partial z/\partial u$. Hence, z satisfies the partial differential equation $\partial z/\partial u = \partial z/\partial v$. Consequently, $z = f(u + v)$, where $f(u + v)$ denotes some function of $u + v$ and, by (27),

$$f(u + v) = (s_1 c_2 d_2 + s_2 c_1 d_1)/\Delta$$

Putting $v = 0$ gives $f(u) = \operatorname{sn} u$, and hence $f(u + v) = \operatorname{sn}(u + v)$. This proves (24).

Proof of (25). By (24) we have

$$\begin{aligned}
\operatorname{cn}^2(u+v) &= 1 - \operatorname{sn}^2(u+v) \\
&= \{(1 - k^2 s_1^2 s_2^2)^2 - (s_1 c_2 d_2 + s_2 c_1 d_1)^2\}/(1 - k^2 s_1^2 s_2^2)^2
\end{aligned}$$

In the numerator on the R.H.S. we now put

$$(1 - k^2 s_1^2 s_2^2)^2 = (c_1^2 + s_1^2 d_2^2)(c_2^2 + s_2^2 d_1^2)$$

then the numerator reduces to $(c_1 c_2 - s_1 s_2 d_1 d_2)^2$, and on taking the square roots of both sides, and removing the ambiguity of sign by putting $v = 0$, we deduce (25).

Proof of (26). By (24) we have

$$\begin{aligned}
\operatorname{dn}^2(u+v) &= 1 - k^2 \operatorname{sn}^2(u+v) \\
&= \{(1 - k^2 s_1^2 s_2^2)^2 - k^2 (s_1 c_2 d_2 + s_2 c_1 d_1)^2\}/(1 - k^2 s_1^2 s_2^2)^2
\end{aligned}$$

In the numerator on the R.H.S. we now put

$$(1 - k^2 s_1^2 s_2^2)^2 = (d_1^2 + k^2 s_1^2 c_2^2)(d_2^2 + k^2 s_2^2 c_1^2)$$

and after a little reduction we obtain (26).

§ 9. Extension of the definitions of sn u, cn u, dn u to all real values of u. The addition formulæ may now be used to extend the definitions of the elliptic functions to values of u beyond the range $-K \leqslant u \leqslant K$. We shall take it for granted, and the reader may verify, that the functions continue to satisfy the same identities as when the range was restricted.

Putting $v = K$ in (24) and using (6), we obtain

$$\operatorname{sn}(u + K) = \frac{\operatorname{sn} u \operatorname{cn} K \operatorname{dn} K + \operatorname{sn} K \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 K} = \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{dn}^2 u}$$

and hence, and similarly from (25) and (26),

$$\operatorname{sn}(u + K) = \operatorname{cn} u/\operatorname{dn} u \quad . \quad . \quad (28)$$

$$\operatorname{cn}(u + K) = -k' \operatorname{sn} u/\operatorname{dn} u \quad . \quad (29)$$

$$\operatorname{dn}(u + K) = k'/\operatorname{dn} u \quad . \quad . \quad (30)$$

By putting $u = K$, we see that

$$\operatorname{sn} 2K = 0, \quad \operatorname{cn} 2K = -1, \quad \operatorname{dn} 2K = 1 \quad . \quad . \quad (31)$$

Similarly, by putting $v = 2K$ in (24), (25), and (26), and using (31), we have

$$\operatorname{sn}(u + 2K) = -\operatorname{sn} u \quad . \quad . \quad . \quad (32)$$

$$\operatorname{cn}(u + 2K) = -\operatorname{cn} u \quad . \quad . \quad . \quad (33)$$

$$\operatorname{dn}(u + 2K) = \operatorname{dn} u \quad . \quad . \quad . \quad (34)$$

From (32) and (33), by replacing u by $u + 2K$,

$$\operatorname{sn}(u + 4K) = \operatorname{sn} u, \quad \operatorname{cn}(u + 4K) = \operatorname{cn} u \quad . \quad . \quad (35)$$

From (35) we see that sn u and cn u are periodic functions of u, with period $4K$, and from (34) that dn u is periodic with period $2K$.

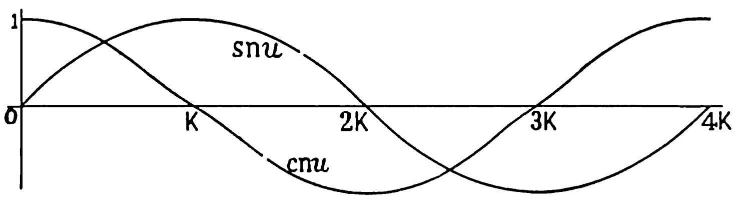

FIG. 1.—Graphs of sn u and cn u ($K = 2$, $k = 0{\cdot}8$ approx.)

§ 10. Graphs. The graphs of the three functions can now be sketched for all real values of u. They are indicated in Fig. 1

and Fig. 2 for $k = 0{\cdot}8$ and positive values of u; for negative values of u, it is to be remembered that sn u is an odd function and that cn u and dn u are even functions. Note that the graphs

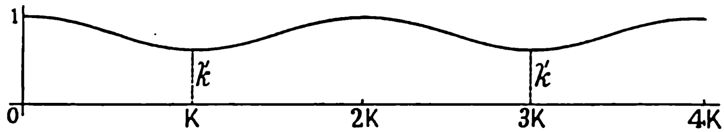

FIG. 2.—Graph of dn u.

of sn u and cn u have not the same shape, i.e., one is not merely a displacement of the other (see Examples I (*b*), 6). For other values of k, the graphs in Jahnke and Emde, *Funktionentafeln*, should be consulted.

EXAMPLES I (*b*)

1. If $s = \text{sn}\, u$, $c = \text{cn}\, u$, $d = \text{dn}\, u$, $S = \text{sn}\, 2u$, $C = \text{cn}\, 2u$, $D = \text{dn}\, 2u$, show that:

$$S = \frac{2scd}{1 - k^2s^4}, \quad C = \frac{1 - 2s^2 + k^2s^4}{1 - k^2s^4}, \quad D = \frac{1 - 2k^2s^2 + k^2s^4}{1 - k^2s^4};$$

$$s^2 = \frac{1 - C}{1 + D}, \quad c^2 = \frac{D + C}{1 + D}, \quad d^2 = \frac{k'^2 + k^2C + D}{1 + D} = \frac{D + C}{1 + C}.$$

[Cf. $\sin 2u = 2 \sin u \cos u$, $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$, etc.]

2. Show that:

$$\text{sn}\, \frac{K}{2} = \frac{1}{\sqrt{(1 + k')}}, \quad \text{cn}\, \frac{K}{2} = \frac{\sqrt{k'}}{\sqrt{(1 + k')}}, \quad \text{dn}\, \frac{K}{2} = \sqrt{k'}.$$

3. Show that, if $\Delta = 1 - k^2s_1^2s_2^2$, $s_1 = \text{sn}\, u$, $s_2 = \text{sn}\, v$, etc., then:

$$\begin{aligned}
\text{sn}\,(u + v) + \text{sn}\,(u - v) &= 2s_1c_2d_2/\Delta \\
\text{sn}\,(u + v) - \text{sn}\,(u - v) &= 2s_2c_1d_1/\Delta \\
\text{cn}\,(u + v) + \text{cn}(u - v) &= 2c_1c_2/\Delta \\
\text{cn}\,(u + v) - \text{cn}\,(u - v) &= -\,2s_1s_2d_1d_2/\Delta \\
\text{dn}\,(u + v) + \text{dn}\,(u - v) &= 2d_1d_2/\Delta \\
\text{dn}\,(u + v) - \text{dn}\,(u - v) &= -\,2k^2s_1s_2c_1c_2/\Delta
\end{aligned}$$

4. Show that:

$$\begin{aligned}
\text{sn}\,(u + v)\, \text{sn}\,(u - v) &= (s_1^2 - s_2^2)/\Delta \\
\text{cn}\,(u + v)\, \text{cn}\,(u - v) &= (1 - s_1^2 - s_2^2 + k^2s_1^2s_2^2)/\Delta \\
\text{dn}\,(u + v)\, \text{dn}\,(u - v) &= (1 - k^2s_1^2 - k^2s_2^2 + k^2s_1^2s_2^2)/\Delta
\end{aligned}$$

5. If C_1, $C_2 = \text{cn}\,(u \pm v)$, and D_1, $D_2 = \text{dn}\,(u \pm v)$, prove that

$$\text{sn}\, u\, \text{sn}\, v = \frac{C_2 - C_1}{D_2 + D_1} = \frac{1}{k^2}\frac{D_2 - D_1}{C_2 + C_1}$$

and hence that

$$1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v = \frac{2(C_1D_2 + C_2D_1)}{(C_1 + C_2)(D_1 + D_2)}$$

$$\operatorname{cn} u \operatorname{cn} v = \frac{C_1D_2 + C_2D_1}{D_1 + D_2}, \qquad \operatorname{dn} u \operatorname{dn} v = \frac{C_1D_2 + C_2D_1}{C_1 + C_2}$$

By putting $u = \alpha + \beta$, $v = \alpha - \beta$, deduce that:

$$\operatorname{sn}(\alpha + \beta) \operatorname{sn}(\alpha - \beta) = \frac{\operatorname{cn} 2\beta - \operatorname{cn} 2\alpha}{\operatorname{dn} 2\beta + \operatorname{dn} 2\alpha}$$

$$\operatorname{cn}(\alpha + \beta) \operatorname{cn}(\alpha - \beta) = \frac{\operatorname{cn} 2\alpha \operatorname{dn} 2\beta + \operatorname{dn} 2\alpha \operatorname{cn} 2\beta}{\operatorname{dn} 2\alpha + \operatorname{dn} 2\beta}$$

$$\operatorname{dn}(\alpha + \beta) \operatorname{dn}(\alpha - \beta) = \frac{\operatorname{cn} 2\alpha \operatorname{dn} 2\beta + \operatorname{dn} 2\alpha \operatorname{cn} 2\beta}{\operatorname{cn} 2\alpha + \operatorname{cn} 2\beta}$$

6. Show that: (i) the graph of $\operatorname{sn} u$ has no point of inflexion, except $u = 0, \pm 2K, \ldots$; (ii) the graph of $\operatorname{cn} u$ has no point of inflexion if $k < 1/\sqrt{2}$, except $u = \pm K, \pm 3K, \ldots$; but has a point of inflexion where $\operatorname{sn} u = 1/k\sqrt{2}$ if $k > 1/\sqrt{2}$; (iii) the graph of $\operatorname{dn} u$ has a point of inflexion where $\operatorname{sn} u = \operatorname{cn} u = 1/\sqrt{2}$.

CHAPTER II

ELLIPTIC INTEGRALS

§ **1.** An integral of the type

$$\int R(x, \sqrt{X})dx \qquad (1)$$

where X is a cubic or a quartic in x, and R denotes a rational function, is called an *elliptic* integral, the name being due originally to the fact that the problem of rectifying an ellipse depends upon an integral of this kind.

§ **2. Legendre's standard forms.** Any elliptic integral can, by suitable linear transformations and reduction formulæ, be expressed as the sum of a finite number of elementary integrals and integrals of the three types:

$$\int \frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-k^2x^2)}} \qquad (2)$$

$$\int \frac{(1-k^2x^2)dx}{\sqrt{(1-x^2)}\sqrt{(1-k^2x^2)}} = \int \frac{\sqrt{(1-k^2x^2)}dx}{\sqrt{(1-x^2)}} \qquad (3)$$

$$\int \frac{dx}{(1+nx^2)\sqrt{(1-x^2)}\sqrt{(1-k^2x^2)}} \qquad (4)$$

which are called elliptic integrals of the first, second, and third kinds, respectively. If the coefficients in X are real, the reduction to these standard types can be effected so that k is real and $0 < k < 1$ (see Chapter IX).

If we put $x = \sin\phi$ and take the lower limit of integration to be zero, these integrals take the forms

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} \qquad (5)$$

$$E(k, \phi) = \int_0^{\phi} \sqrt{(1-k^2\sin^2\phi)}d\phi \qquad (6)$$

$$\Pi(k, n, \phi) = \int_0^{\phi} \frac{d\phi}{(1+n\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}} \qquad (7)$$

If, further, we put $\operatorname{sn} u = x = \sin\phi$, they take the forms:

$$F(k, \phi) = u \qquad (8)$$

$$E(k, \phi) = E(u) = \int_0^{u} \operatorname{dn}^2 u\, du \qquad (9)$$

$$\Pi(k, n, \phi) = \int_0^u \frac{du}{1 + n \operatorname{sn}^2 u} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

§ 3. Complete elliptic integrals. The definite integrals K, E, defined by:

$$K = F(k, \tfrac{1}{2}\pi) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)}\sqrt{(1 - k^2x^2)}} \quad . \quad (11)$$

$$\left.\begin{aligned} E = E(k, \tfrac{1}{2}\pi) &= \int_0^{\frac{1}{2}\pi} \sqrt{(1 - k^2 \sin^2 \phi)}d\phi = \int_0^1 \frac{\sqrt{(1 - k^2x^2)}}{\sqrt{(1 - x^2)}}\,dx \\ \text{or} \qquad E(K) &= \int_0^K \operatorname{dn}^2 u\, du \end{aligned}\right\} \quad . \quad (12)$$

are called *complete* elliptic integrals of the first and second kinds, respectively.

§ 4. Examples. The reduction of (1) to dependence on the standard forms will be deferred. In the present chapter we shall consider a few integrals that can be readily expressed in terms of the standard elliptic integrals of the first and second kinds. If numerical values are required, they can be obtained from tables (see § 9).

We begin with examples of integrals in which the integrands are elliptic functions. Only those integrals of which the integrated forms involve u or $E(u)$ are, strictly speaking, elliptic; thus, the first of the following examples is an elementary integral, as we see from its expression in terms of t.

Ex. 1.
$$\begin{aligned} \int \operatorname{sn} u\, du &= \int \frac{x\, dx}{\sqrt{(1 - x^2)}\sqrt{(1 - k^2x^2)}} && (x = \operatorname{sn} u) \\ &= \int \frac{\frac{1}{2}dt}{\sqrt{(1 - t)}\sqrt{(1 - k^2t)}} && (t = x^2) \\ &= k^{-1} \log \{\sqrt{(1 - k^2t)} - k\sqrt{(1 - t)}\} \\ &= k^{-1} \log (\operatorname{dn} u - k \operatorname{cn} u) \end{aligned}$$

Ex. 2.
$$\int \operatorname{sn}^2 u\, du = \int \frac{1 - \operatorname{dn}^2 u}{k^2}\, du = \frac{u - E(u)}{k^2}$$

Ex. 3. By differentiation we find

$$d(scd)/du = 1 - 2(1 + k^2)s^2 + 3k^2s^4$$

where $s = \operatorname{sn} u$, etc. Hence, by the last example,

$$3k^4 \int \operatorname{sn}^4 u\, du = (2 + k^2)u - 2(1 + k^2)E(u) + k^2 scd.$$

Ex. 4. By differentiation,

$$\frac{d}{du}\left(\frac{sc}{d}\right) = \frac{d^4 - k'^2}{k^2d^2} = \frac{1}{k^2}\left(d^2 - \frac{k'^2}{d^2}\right)$$

and hence, by integration, after multiplication by k^2,

$$k'^2 \int \frac{du}{\operatorname{dn}^2 u} = E(u) - \frac{k^2 sc}{d}$$

The reader may find it worth while to construct a table of the integrals of such differentials as $\operatorname{sn} u\, du$, $\operatorname{cn} u\, du$, $\operatorname{dn} u\, du$; $(1/\operatorname{sn} u)du$, $(1/\operatorname{cn} u)du$, $(1/\operatorname{dn} u)du$; $\operatorname{sn}^2 u\, du$, $\operatorname{cn}^2 u\, du$, $\operatorname{dn}^2 u\, du$; $(1/\operatorname{sn}^2 u)du$, $(1/\operatorname{cn}^2 u)du$, $(1/\operatorname{dn}^2 u)du$; $\operatorname{sn}^4 u\, du$, $\operatorname{cn}^4 u\, du$, $\operatorname{dn}^4 u\, du$; $\operatorname{cn}^2 u \operatorname{dn}^2 u\, du$, $\operatorname{sn}^2 u \operatorname{cn}^2 u\, du$, $\operatorname{sn}^2 u \operatorname{dn}^2 u\, du$; etc.

EXAMPLES II (*a*)

1. Verify the following results:

(i) $\int \operatorname{cn} u \operatorname{dn} u\, du = \operatorname{sn} u$ (ii) $\int \operatorname{sn} u \operatorname{cn} u\, du = -(\operatorname{dn} u)/k^2$

(iii) $\int \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} du = \log \operatorname{sn} u$ (iv) $\int \frac{\operatorname{sn} u\, du}{\operatorname{cn} u \operatorname{dn} u} = \frac{1}{k'^2} \log \frac{\operatorname{dn} u}{\operatorname{cn} u}$

(v) $\int \operatorname{cn} u\, du = k^{-1} \sin^{-1}(k \operatorname{sn} u)$ (vi) $\int \operatorname{dn} u\, du = \sin^{-1}(\operatorname{sn} u)$

(vii) $\int \frac{du}{\operatorname{sn} u} = \log \frac{\operatorname{sn} u}{\operatorname{dn} u + \operatorname{cn} u} = \log \frac{\operatorname{sn} \frac{1}{2}u}{\operatorname{cn} \frac{1}{2}u \operatorname{dn} \frac{1}{2}u}$

(viii) $\int \frac{\operatorname{cn} u\, du}{\operatorname{sn} u} = \log \frac{\operatorname{sn} u}{1 + \operatorname{dn} u} = \log \frac{\operatorname{sn} \frac{1}{2}u \operatorname{cn} \frac{1}{2}u}{\operatorname{dn} \frac{1}{2}u}$

(ix) $\int \frac{du}{\operatorname{cn} u} = \frac{1}{k'} \log \frac{\operatorname{dn} u + k' \operatorname{sn} u}{\operatorname{cn} u}$

(x) $\int \frac{du}{\operatorname{dn} u} = \frac{1}{k'} \tan^{-1} \frac{k' \operatorname{sn} u}{\operatorname{cn} u}$

(xi) $\int \operatorname{sn}^{-1} x\, dx = x \operatorname{sn}^{-1} x - k^{-1} \log \{\sqrt{(1 - k^2x^2)} - k\sqrt{(1 - x^2)}\}$

(xii) $3k^2 \int \operatorname{cn}^2 u \operatorname{dn}^2 u\, du = (1 + k^2)E(u) - (1 - k^2)u + k^2 scd$

(xiii) $\displaystyle\int \frac{du}{\mathrm{sn}^2 u} = u - E(u) - \frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}$

(xiv) $\displaystyle\int \frac{\mathrm{dn}^2 u}{\mathrm{cn}^2 u}\, du = u - E(u) + \frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{\mathrm{cn}\, u}$

(xv) $\displaystyle k'^4 \int \frac{du}{\mathrm{cn}^2 u\, \mathrm{dn}^2 u} = k'^2 u - (1 + k^2)E(u) + \frac{s(d^2 + k^4 c^2)}{cd}$

(xvi) $\displaystyle 3 \int \frac{du}{\mathrm{sn}^4 u} = (2 + k^2)u - 2(1 + k^2)E(u) - 2(1 + k^2)\frac{cd}{s} - \frac{cd}{s^3}$

(xvii) $\displaystyle k'^2 \int \frac{du}{1 + \mathrm{sn}\, u} = k'^2 u - E(u) - \frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{1 + \mathrm{sn}\, u}$

(xviii) $\displaystyle k^2 \int \frac{\mathrm{dn}\, u\, du}{\mathrm{dn}\, u + k'} = u - E(u) + \frac{k^2\, \mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u + k'}$

(xix) $\displaystyle\int_0^K \frac{du}{1 + \mathrm{cn}\, u} = K - E + k'$

(xx) $\displaystyle\int_0^K \frac{\mathrm{cn}\, u\, du}{1 - k\, \mathrm{sn}\, u} = \frac{1 + k - k'}{kk'}$

2. If $s = \mathrm{sn}\, u$, $c = \mathrm{cn}\, u$, $d = \mathrm{dn}\, u$, show that

$$\frac{d}{du}(s^{m+1} cd) = (m + 1)s^m - (m + 2)(1 + k^2)s^{m+2} + (m + 3)k^2 s^{m+4}.$$

Deduce that, if n is an even integer, positive or negative, then the integral of $\mathrm{sn}^n u\, du$ is expressible in terms of u and $E(u)$. [If n is odd, the integral is an elementary integral.]

3. If $V = 1 + n\, \mathrm{sn}^2 u$, show that

$$\frac{d}{du}\left(\frac{scd}{V^{m+1}}\right) = \frac{A}{V^m} + \frac{B}{V^{m-1}} + \frac{C}{V^{m-2}} + \frac{D}{V^{m-3}}$$

where A, B, C, D are constants. Deduce that, if m is a positive integer, then the integral

$$\int \frac{du}{(1 + n\, \mathrm{sn}^2 u)^m}$$

is expressible in terms of u, $E(u)$ and $\Pi(u)$, where

$$\Pi(u) = \int \frac{du}{1 + n\, \mathrm{sn}^2 u} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

4. Prove that any integral of the type

$$\int R\{x, \sqrt{(1 - x^2)}, \sqrt{(1 - k^2x^2)}\}dx,$$

where R denotes a rational function, can be expressed in terms of elementary integrals and the three elliptic integrals u, $E(u)$, $\Pi(u)$.

Proof. An integral of the given type can be expressed in the form

$$\int \frac{a' + b'\xi + c'\eta + d'\xi\eta}{a + b\xi + c\eta + d\xi\eta}\, dx$$

where $\xi = \sqrt{(1 - x^2)}$, $\eta = \sqrt{(1 - k^2x^2)}$, and $a, a', \ldots$ are polynomials in x. Now multiply numerator and denominator of the integrand by $a - b\xi - c\eta + d\xi\eta$; then the integral takes the form

$$\int \frac{A' + B'\xi + C'\eta + D'\xi\eta}{A + D\xi\eta}\, dx$$

where $A, A', \ldots$ are polynomials in x. If we now multiply numerator and denominator of the integrand by $A - D\xi\eta$, then the integral can be expressed as the sum of four integrals in the form

$$\int R_1 dx + \int R_2 \xi dx + \int R_3 \eta dx + \int R_4 \xi\eta dx,$$

where R_1, R_2, R_3, R_4 are rational functions of x. The first three of these are elementary integrals. The fourth can be written in the form

$$\int \frac{P' + xQ'}{P + xQ} \frac{dx}{\xi\eta}$$

where P, P', Q, Q' are polynomials in x^2, and when we now multiply numerator and denominator of the integrand by $P - xQ$, the integral can be expressed as the sum of two integrals of the form

$$\int \frac{R(x^2)dx}{\sqrt{(1 - x^2)}\sqrt{(1 - k^2x^2)}} + \int \frac{S(x^2)d(x^2)}{\sqrt{(1 - x^2)}\sqrt{(1 - k^2x^2)}}$$

where $R(x^2)$ and $S(x^2)$ are rational functions of x^2. The second of these is an elementary integral. To simplify the first, we begin by putting $R(x^2)$ into partial fractions, thus,

$$R(x^2) = \Sigma a x^{2p} + \Sigma \frac{b}{(1 + nx^2)^q}$$

where a, b, n are constants, p and q are positive integers, and only the most general typical terms are indicated. Hence, putting $x = \operatorname{sn}(u, k)$, we find

$$\int \frac{R(x^2)dx}{\sqrt{(1 - x^2)}\sqrt{(1 - k^2x^2)}} = \Sigma a \int \operatorname{sn}^{2p} u\, du + \Sigma b \int \frac{du}{(1 + n \operatorname{sn}^2 u)^q}$$

and it follows from examples 2 and 3 that the original integral is expressible in terms of u, $E(u)$, $\Pi(u)$.

§ 5. Variation of the complete elliptic integrals when k varies. From (11) and (12), when $k = 0$ and $k = 1$ we have

$$k = 0, \quad K = \tfrac{1}{2}\pi, \quad E = \tfrac{1}{2}\pi \qquad (14)$$

$$k = 1, \quad K = +\infty \quad E = 1 \qquad (15)$$

Now, if we differentiate (11) and (12) with respect to k, and then put $x = \operatorname{sn} u$, we get

$$\frac{dK}{dk} = \int_0^1 \frac{kx^2\,dx}{(1-x^2)^{\frac{1}{2}}(1-k^2x^2)^{\frac{3}{2}}} = k\int_0^K \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}\,du \quad . \quad . \quad . \quad (16)$$

$$\frac{dE}{dk} = -\int_0^1 \frac{kx^2\,dx}{(1-x^2)^{\frac{1}{2}}(1-k^2x^2)^{\frac{1}{2}}} = -k\int_0^K \operatorname{sn}^2 u\,du \quad . \quad (17)$$

It follows that $dK/dk > 0$ and $dE/dk < 0$ when $0 < k < 1$, and hence that K increases steadily from $\frac{1}{2}\pi$ to $+\infty$, while E decreases steadily from $\frac{1}{2}\pi$ to 1, when k increases from 0 to 1.

The integrals in (16) and (17) can be evaluated. Making use of Examples 4 and 2 in § 4, we find

$$\frac{dK}{dk} = \frac{E - k'^2K}{kk'^2}, \quad \frac{dE}{dk} = -\frac{K-E}{k} \quad . \quad . \quad (18)$$

§ 6. Approximations to K and K' when k is small. When k is small we have

$$K = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = \int_0^{\frac{1}{2}\pi} (1 + \tfrac{1}{2}k^2\sin^2\phi + \ . \ . \ . \ )d\phi$$

and hence

$$K = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}k^2 + \ . \ . \ . \ ) \quad . \quad . \quad . \quad (19)$$

It will be proved that an approximation to K' is given by

$$K' = \log_e(4/k) + \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where $\alpha \to 0$ when $k \to 0$.

Proof. Write K' in the form

$$K' = \int_0^{\frac{1}{2}\pi} \frac{1 - k'\sin\phi + k'\sin\phi}{\sqrt{(1-k'^2\sin^2\phi)}}\,d\phi = A + B$$

where

$$A = \int_0^{\frac{1}{2}\pi} \left(\frac{1-k'\sin\phi}{1+k'\sin\phi}\right)^{\frac{1}{2}} d\phi, \quad B = \int_0^{\frac{1}{2}\pi} \frac{k'\sin\phi\,d\phi}{(k^2 + k'^2\cos^2\phi)^{\frac{1}{2}}}$$

The integrand of A is a continuous function of k' and ϕ $(0 \leqslant k' \leqslant 1,\ 0 \leqslant \phi \leqslant \frac{1}{2}\pi)$, and hence the integral A itself is a continuous function of k' $(0 \leqslant k' \leqslant 1)$, and so we can put

$$A = \int_0^{\frac{1}{2}\pi} \left(\frac{1-\sin\phi}{1+\sin\phi}\right)^{\frac{1}{2}} d\phi + \alpha_1$$

where $\alpha_1 \to 0$ when $k' \to 1$ or $k \to 0$. Putting $\phi = \frac{1}{2}\pi - \psi$, we then find

$$A = \int_0^{\frac{1}{2}\pi} \tan\tfrac{1}{2}\psi\,d\psi + \alpha_1 = \log 2 + \alpha_1$$

Also we find

$$B = \log \frac{k' + 1}{k} = \log \frac{2}{k} + a_2$$

where $a_2 \rightarrow 0$ when $k \rightarrow 0$. Hence

$$K = A + B = \log 2 + a_1 + \log (2/k) + a_2 = \log (4/k) + a$$

where $a = a_1 + a_2 \rightarrow 0$ when $k \rightarrow 0$.

COR. 1. It follows from (19) and (20) that, if k is small,

$$\frac{K'}{K} \doteqdot \frac{2}{\pi} \log_e \frac{4}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

COR. 2. From (21), if k is small.

$$k^2 \doteqdot 16e^{-\pi K'/K} \quad . \quad . \quad . \quad . \quad (22)$$

Note. By solving in series the differential equation satisfied by K and K' (see Examples II (*b*), 8), it can be shown that, when k is small, a better approximation to K' than (20) is given by

$$K' \doteqdot \left(\log_e \frac{4}{k}\right)\left(1 + \frac{k^2}{4}\right) - \frac{k^2}{4} \quad . \quad . \quad . \quad (23)$$

and then from (19) and (23) we have

$$\frac{K'}{K} \doteqdot \frac{2}{\pi}\left(\log_e \frac{4}{k} - \frac{k^2}{4}\right) \quad . \quad . \quad . \quad . \quad (24)$$

Putting $k = 1/5$, we have $\log_e (4/k) = \log_e 20 \doteqdot 3$, and $\frac{1}{4}k^2 = 0{\cdot}01$. Consequently, when $k = 1/5$ the error in the approximation (21) is about 1 in 300. We infer that for most practical purposes, (21) may be used as an approximation to K'/K when $k < 1/5 \doteqdot \sin 12°$, and that (22) may be used as an approximation to k^2 when $K'/K > 2$.

§ 7. The addition formula for the integral $E(u)$ is

$$E(u + v) = E(u) + E(v) - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} (u + v) \quad . \quad (25)$$

In particular, putting $v = K$, we find, by I, (28),

$$E(u + K) = E(u) + E - k^2 \operatorname{sn} u \operatorname{cn} u / \operatorname{dn} u \quad . \quad (26)$$

When $u = K$, this gives $E(2K) = 2E$. Then (25) gives

$$E(u + 2K) = E(u) + 2E \quad . \quad . \quad . \quad . \quad (27)$$

Proof of (25). From the last two identities in Examples I (*b*), (3), we have, by multiplication,

$$\operatorname{dn}^2 (x + y) - \operatorname{dn}^2 (x - y) = - \frac{4k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{sn} y \operatorname{cn} y \operatorname{dn} y}{(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y)^2}$$

and hence, by integration with respect to y,

$$E(x + y) + E(x - y) = C - \frac{2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x}{\operatorname{sn}^2 x(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y)}$$

where C may depend on x, but not on y. Putting $y = x$, we get

$$E(2x) = C - \frac{2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x}{\operatorname{sn}^2 x(1 - k^2 \operatorname{sn}^4 x)}$$

and by subtraction

$$E(x + y) + E(x - y) - E(2x) = \frac{2k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^4 x} \cdot \frac{\operatorname{sn}^2 x - \operatorname{sn}^2 y}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}$$

and hence, by Examples I (*b*), (1) and (4),

$$E(x + y) + E(x - y) - E(2x) = k^2 \operatorname{sn} 2x \operatorname{sn} (x + y) \operatorname{sn} (x - y)$$

from which follows (25) when we put $u = x + y$, $v = x - y$.

§ 8. **Jacobi's elliptic integral of the second kind** (Jacobi's Zeta-function) is denoted by $Z(u)$ and is defined by

$$Z(u) = \int_0^u \left(\operatorname{dn}^2 u - \frac{E}{K}\right) du = E(u) - \frac{E}{K} u \,. \quad (28)$$

Its properties are simpler than those of $E(u)$ in some respects; for instance, $Z(u)$ is simply periodic, with period $2K$, as will be seen below.

From (25) and (28) it follows that the addition formula for $Z(u)$ is the same as that for $E(u)$, viz.,

$$Z(u + v) = Z(u) + Z(v) - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} (u + v) \quad . \quad (29)$$

In particular,

$$Z(u + K) = Z(u) - k^2 \operatorname{sn} u \operatorname{cn} u/\operatorname{dn} u \quad (30)$$

$$Z(u + 2K) = Z(u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

From (31) we see that $Z(u)$ is periodic, with period $2K$.

From the definition (28), we have

$$Z(0) = 0, \quad Z(K) = 0, \quad Z(-u) = -Z(u) \quad . \quad (32)$$

$$\frac{dZ(u)}{du} = \operatorname{dn}^2 u - \frac{E}{K}, \quad \frac{d^2Z(u)}{du^2} = -2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \quad (33)$$

and it follows that, between $u = 0$ and $u = K$, the graph of $Z(u)$ has a maximum point where $\operatorname{dn}^2 u = E/K$ (see on, Examples

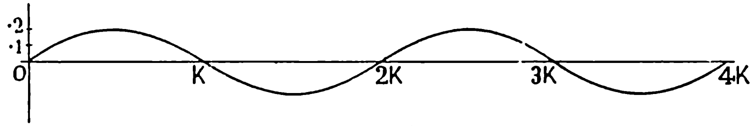

FIG. 1.—Graph of $Z(u)$ for $k = 0{\cdot}8$ approx.

II (*b*), 4) and no point of inflexion. The graph is indicated in Fig. 1 for $k = 0{\cdot}8$. For other values of k, see the figures in Jahnke and Emde, *Funktionentafeln*.

§ 9. **Tables.** Tables of values of the functions sn u, cn u, dn u, $Z(u)$, and of the complete elliptic integrals K, K', E, E' have been calculated by Milne-Thomson. The tables of sn u, cn u, dn u are five-figure tables, those of K, K', E, E' are seven-figure tables, and they were published together under the title *Die Elliptischen Funktionen von Jacobi*, by J. Springer, Berlin, 1931. The tables of $Z(u)$ are seven-figure tables, under the title " The Zeta-function of Jacobi ", reprinted from the *Proceedings of the Royal Society of Edinburgh*, 1932. Milne-Thomson's tables have also recently (1950) been published in New York by Dover Publications, Inc.

Tables of values of the integrals

$$u = F(k, \phi) = \int_0^\phi \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}}$$

$$E(u) = E(k, \phi) = \int_0^\phi \sqrt{(1 - k^2 \sin^2 \phi)} d\phi$$

and of the complete integrals K, E were originally compiled by Legendre; they are included in many books of tables: e.g., in Dale, *Five-figure Mathematical Tables* (Arnold), as functions of θ and ϕ, where $k = \sin \theta$; or in Jahnke and Emde's *Funktionentafeln*, as functions of α and ϕ, where $k = \sin \alpha$.

§ 10. **Given u and k, to calculate sn u, cn u, dn u, $E(u)$ from Legendre's tables.** In the table of values of $F(k, \phi)$ we find the column headed $\sin^{-1} k$. In this column, in the body of the table, we find u, $= F(k, \phi)$. Opposite this value of u, in the left-hand column, we then read the value of ϕ. Then we have

$$\text{sn}\,(u, k) = \sin \phi, \qquad \text{cn}\,(u, k) = \cos \phi,$$
$$\text{dn}\,(u, k) = \sqrt{(1 - k^2 \sin^2 \phi)}, \qquad E(u, k) = E(k, \phi),$$

the values of sn (u, k) and cn (u, k) being read from the tables of sines and cosines, that of dn (u, k) being calculated from the values of k and $\sin \phi$, and that of $E(u, k)$ being read from the table of values of $E(k, \phi)$.

EXAMPLES II (*b*)

1. From (11) and (12), by expanding the integrands in ascending powers of $k^2 \sin^2 \phi$, show that

$$K = \frac{\pi}{2}\left\{1 + (\tfrac{1}{2})^2 k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 k^4 + \ldots\right\} = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ k^2)$$

$$E = \frac{\pi}{2}\left\{1 - \frac{1}{2^2} k^2 - \frac{1^2\,.\,3}{2^2\,.\,4^2} k^4 - \ldots\right\} = \frac{\pi}{2} F(\tfrac{1}{2}, -\tfrac{1}{2};\ 1;\ k^2)$$

[Here F denotes the hypergeometric function.]

2. Show that

$$\frac{dK'}{dk} = -\frac{E' - k^2K'}{kk'^2}, \quad \frac{dE'}{dk} = \frac{k(K' - E')}{k'^2}$$

3. Verify Legendre's formula

$$KE' + K'E - KK' = \tfrac{1}{2}\pi$$

[First show that the derivative of the L.H.S. with respect to k vanishes.] Deduce that, when $k = 1/\sqrt{2}$, then

$$E = \tfrac{1}{2}K + \tfrac{1}{4}\pi/K$$

4. From (18) show that $E/K > k'^2$, and deduce that the equation $\operatorname{dn}^2 u = E/K$ has a root between 0 and K (see § 8).

5. If u is small, show that

$$E(u) = u - \tfrac{1}{3}k^2u^3 \ldots, \quad Z(u) = \frac{K - E}{K}u - \tfrac{1}{3}k^2u^3 \ldots$$

6. Show that the average values of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, between $u = 0$ and $u = K$, are respectively

$$\frac{1}{kK}\log\frac{1+k}{k'}, \quad \frac{\sin^{-1} k}{kK}, \quad \frac{\pi}{2K}$$

Examine the limits of these average values when $k \to 0$ and when $k \to 1$.

7. By using the addition formulæ, show that

$$E(\tfrac{1}{2}K) = \tfrac{1}{2}E + \tfrac{1}{2}(1 - k'), \quad Z(\tfrac{1}{2}K) = \tfrac{1}{2}(1 - k')$$

8. Show that $y = K$ and $y = K'$ are solutions of the equation

$$\frac{d}{dk}\left(kk'^2\frac{dy}{dk}\right) = ky$$

By solving this equation in series by Frobenius's method, and using the known behaviour of K' when k is small (§ 6), prove that

$$K' = \frac{2K}{\pi}\log\frac{4}{k} - 2\left\{\frac{1^2}{2^2}\frac{1}{1\,.\,2}k^2 + \frac{1^2\,.\,3^2}{2^2\,.\,4^2}\left(\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4}\right)k^4 + \ldots\right\}$$

CHAPTER III

APPLICATIONS. ARGUMENT REAL

§ 1. Perimeter of ellipse. The co-ordinates of any point on the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may be written $x = a \sin t$, $y = b \cos t$, where t is the complement of the eccentric angle. Hence, if s is the perimeter of the ellipse, and if $b^2 = a^2(1 - e^2)$, then

$$s = 4 \int_0^{\frac{1}{2}\pi} \sqrt{(a^2 \cos^2 t + b^2 \sin^2 t)} dt = 4a \int_0^{\frac{1}{2}\pi} \sqrt{(1 - e^2 \sin^2 t)} dt$$

and therefore

$$s = 4aE(e) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where e is the eccentricity. For example, taking the values of the complete elliptic integral E, $= E(e)$, from tables, we find, approximately,

$$\begin{array}{lll} e = 0, & E = \frac{1}{2}\pi, & s = 2\pi a = 6{\cdot}28a, \\ e = \frac{1}{2}, & E = 1{\cdot}467, & s = 5{\cdot}87a, \\ e = \frac{1}{2}\sqrt{3}, & E = 1{\cdot}211, & s = 4{\cdot}84a, \\ e = 1, & E = 1, & s = 4a. \end{array}$$

§ 2. Parametric equations of the ellipse in terms of sn *u*, cn *u*. One way of writing the co-ordinates of any point on the ellipse (1) in parametric form is

$$x = a \operatorname{sn} u, \quad y = b \operatorname{cn} u \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the modulus k may have any value. In particular, we may put $k = e =$ the eccentricity, $e' = k' = \sqrt{(1 - e^2)}$.

Then we find, in the usual notation of the ellipse (Fig. 1),

$$k = e, \quad k' = e', \quad b = ae', \quad l = ae'^2, \quad x = a \operatorname{sn} u, \quad y = b \operatorname{cn} u.$$

$$CG = ae^2 \operatorname{sn} u, \quad Cg = ae^2 \operatorname{cn} u/e', \quad gG = ae^2 \operatorname{dn} u/e'.$$

$$PG = ae' \operatorname{dn} u, \quad Pg = a \operatorname{dn} u/e'.$$

$$CK = ae'/\operatorname{dn} u, \quad CD = a \operatorname{dn} u, \quad \rho = a \operatorname{dn}^3 u/e'.$$

$$r = SP = a(1 - e \operatorname{sn} u), \qquad r' = S'P = a(1 + e \operatorname{sn} u).$$

$$p = SZ = b(1 - e \operatorname{sn} u)/\operatorname{dn} u, \quad p' = S'Z' = b(1 + e \operatorname{sn} u)/\operatorname{dn} u.$$

$$\cos GSP = (e - \operatorname{sn} u)/(1 - e \operatorname{sn} u),$$

$$\sin GSP = e' \operatorname{cn} u/(1 - e \operatorname{sn} u),$$

$$\cos SPG = e'/\operatorname{dn} u, \qquad \sin SPG = e \operatorname{cn} u/\operatorname{dn} u,$$

$$\cos PGS = e' \operatorname{sn} u/\operatorname{dn} u, \qquad \sin PGS = \operatorname{cn} u/\operatorname{dn} u.$$

Further, if s is the length of the arc from B to P, we have

$$\frac{s}{a} = \int_0^u \sqrt{(\mathrm{cn}^2 u \,\mathrm{dn}^2 u + k'^2 \mathrm{sn}^2 u \,\mathrm{dn}^2 u)}\, du$$

$$= \int_0^u \mathrm{dn}\, u \sqrt{(\mathrm{cn}^2 u + k'^2 \mathrm{sn}^2 u)}du = \int_0^u \mathrm{dn}^2 u \, du = E(u) \quad (4)$$

Ex. Deduce geometrical properties of the ellipse by eliminating u between pairs of the above equations.

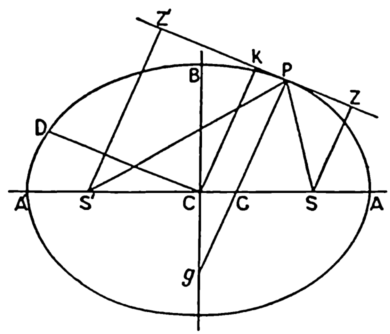

FIG. 1.

§ 3. Fagnano's Theorem. If $u + v = K$ we have, from II (25),

$$E(u) + E(v) - E = k^2 \mathrm{sn}\, u \,\mathrm{sn}\, v$$

This result has the following geometrical meaning (Fagnano's Theorem), as may be readily verified from the formulæ in § 2:

If $P(x, y)$, $P'(x', y')$ are two points on the ellipse (1) whose eccentric angles ϕ, ϕ' are such that $\tan\phi \tan\phi' = b/a$, then

$$\text{arc } BP + \text{arc } BP' - \text{arc } BA = e^2 xx'/a \quad . \quad . \quad (5)$$

When P and P' coincide, the point F in which they coincide is called Fagnano's point. For this point * it may be verified that

$$\text{arc } BF - \text{arc } AF = a - b \quad . \quad . \quad . \quad . \quad (6)$$

§ 4. The pendulum. Suppose a light rod of length l, suspended from one end and having a " bob " attached to the other end, to be moving freely under gravity in a vertical plane. At time t, let v be the speed of the bob and θ the angle between the rod and the downward vertical. At $t = 0$, let $v = v_0$ and $\theta = 0$. Then the equation of energy is

$$\tfrac{1}{2}(v_0^2 - v^2) = gl(1 - \cos\theta) \quad . \quad . \quad . \quad . \quad (7)$$

* For other properties of Fagnano's point, see, e.g., Greenhill, *Applications of Elliptic Functions*, p. 182.

which may be written

$$v^2 = v_0^2 - 4gl \sin^2 \tfrac{1}{2}\theta \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

or, when we put $v = ld\theta/dt$ and $\omega^2 = g/l$, and divide by l^2,

$$\left(\frac{d\theta}{dt}\right)^2 = 4\omega^2\left(\frac{v_0^2}{4gl} - \sin^2 \tfrac{1}{2}\theta\right) \quad . \qquad (9)$$

There are now three cases to be considered:

1°. $v_0^2 < 4gl$. *The rod oscillates.*

Put $k^2 = v_0^2/4gl < 1$. Let α be the angular amplitude of the oscillations, so that $v = 0$ when $\theta = \alpha$. Then, putting $v = 0$, $\theta = \alpha$ in (8), we have

$$k^2 = v_0^2/4gl = \sin^2 \tfrac{1}{2}\alpha \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Since $d\theta/dt$ changes sign when $\theta = \pm\,\alpha$, we have, from (9),

$$\omega dt = \pm\, \tfrac{1}{2}d\theta/\sqrt{(k^2 - \sin^2 \tfrac{1}{2}\theta)} \quad . \quad . \quad . \qquad (11)$$

To reduce this to the standard form, make the substitution

$$\sin \tfrac{1}{2}\theta = k \sin \phi = \sin \tfrac{1}{2}\alpha \sin \phi \quad . \quad . \quad . \qquad (12)$$

and let ϕ oscillate between $\pm\, \tfrac{1}{2}\pi$ while θ oscillates between $\pm\,\alpha$; then, after a little reduction, we find

$$\omega dt = \pm\, d\phi/\sqrt{(1 - k^2 \sin^2 \phi)} \qquad (13)$$

from which follows $\sin \phi = \text{sn}\,(\omega t, k)$ and hence

$$\sin \tfrac{1}{2}\theta = k\, \text{sn}\, \omega t \qquad \cos \tfrac{1}{2}\theta = \text{dn}\, \omega t$$
$$d\theta/dt = 2k\omega\, \text{cn}\, \omega t \qquad \theta = 2 \sin^{-1} (k\, \text{sn}\, \omega t) . \qquad (14)$$

These equations represent a periodic motion, of which the complete period T is given by $\omega T = 4K$, or

$$T = 4K\sqrt{(l/g)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

the modulus of K being k, $= \sin \tfrac{1}{2}\alpha$. When $\alpha = 0$, $k = 0$, $K = \tfrac{1}{2}\pi$ and so (15) gives the usual formula for the period of small oscillations. For other values of α, taking values of K from the tables, we find approximately, putting $T_0 = 2\pi\sqrt{(l/g)}$:

α	k	K	$T/T_0 = 2K/\pi$
30°	$\sin 15° = (\sqrt{3} - 1)/(2\sqrt{2})$	1·598	1·02
60°	$\sin 30° = \tfrac{1}{2}$	1·686	1·07
90°	$\sin 45° = 1/\sqrt{2}$	1·854	1·18
120°	$\sin 60° = \tfrac{1}{2}\sqrt{3}$	2·156	1·37
150°	$\sin 75° = (\sqrt{3} + 1)/(2\sqrt{2})$	2·768	1·76
180°	$\sin 90° = 1$	∞	∞

2°. $v_0^2 > 4gl$. *The rod makes complete revolutions,* and $d\theta/dt$ always has the same sign, which we take to be positive. Note that θ is not periodic.

Put $k^2 = 4gl/v_0^2 < 1$. Then, from (9),

$$\omega dt/k = \tfrac{1}{2}d\theta/\sqrt{(1 - k^2 \sin^2 \tfrac{1}{2}\theta)} \quad . \quad (16)$$

Since $\theta = 0$ when $t = 0$, it follows that

$$\sin \tfrac{1}{2}\theta = \operatorname{sn}(\omega t/k) \qquad \cos \tfrac{1}{2}\theta = \operatorname{cn}(\omega t/k)$$

$$\frac{d\theta}{dt} = \frac{2\omega}{k}\operatorname{dn}\frac{\omega t}{k} \qquad \theta = 2\sin^{-1}\left(\operatorname{sn}\frac{2gt}{v_0}\right) \quad (17)$$

This represents a periodic motion, of which the period T is given by $\omega T/k = 2K$, or

$$T = 2kK/\omega = 2kK\sqrt{(l/g)} \quad . \quad . \quad . \quad (18)$$

3°. $v_0^2 = 4gl$. Putting $k = 1$ in (14) and using I (23), we find

$$\sin \tfrac{1}{2}\theta = \tanh \omega t, \quad \cos \tfrac{1}{2}\theta = \operatorname{sech} \omega t, \quad d\theta/dt = 2\omega \operatorname{sech} \omega t \ . \quad (19)$$

The rod never quite reaches the upward vertical position.

§ 5. Rectification of the curve $y/b = \operatorname{sn}(x/a, k)$ in a special case. The curve

$$y/b = \operatorname{sn}(x/a, k) \ . \quad . \quad . \quad . \quad . \quad (20)$$

is rectifiable in terms of elliptic integrals in the special case in which k is determined by the equation

$$b/a = \tan \psi_0 = 2k/k'^2 \quad . \quad . \quad . \quad . \quad (21)$$

where $\tan \psi_0$ is the gradient at $x = 0$. (This special case has a bearing on § 6 below.)

For we then find, putting $x = au$, $y = b \operatorname{sn} u$,

$$(ds/du)^2 = (dx/du)^2 + (dy/du)^2 = a^2 + b^2 \operatorname{cn}^2 u \operatorname{dn}^2 u$$

and hence, after substituting for b from (21),

$$k'^2 ds/du = a(2 \operatorname{dn}^2 u - k'^2)$$

from which, by integration, if $s = 0$ at $x = 0$,

$$k'^2 s/a = 2E(u) - k'^2 u$$

which may be written

$$k'^2(s + x) = 2aE(x/a) \quad . \quad . \quad . \quad . \quad (22)$$

§ 6. The skipping rope. The ends of a uniform flexible rope, of length $2l$ and mass m per unit length, are fixed to two points at a distance $2c$ apart, and the rope revolves in relative equilibrium, with constant angular velocity ω, about the line joining its ends; to find the shape of the rope, gravity being ignored.

It will be shown that the shape of the rope is a curve of the type discussed in § 5.

Let A, B be the ends of the rope, A the origin, AB the axis of x (Fig. 2). Let P be any point of the rope, ψ the angle which the

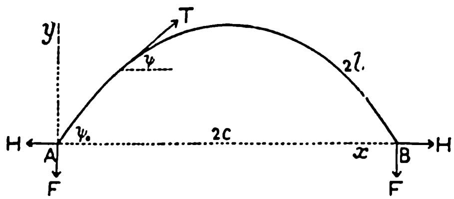

FIG. 2.

tangent at P makes with AB, and T the tension at P. Let $s = \text{arc } AP$, and consider the motion of an element of rope at P of length ds and mass mds. Since the components of T are $T \cos \psi$ and $T \sin \psi$ parallel and perpendicular to AB, respectively, therefore the equations of motion of the element are

$$d(T \cos \psi) = 0, \quad d(T \sin \psi) = -\, mds\,.\,y\omega^2 \quad . \quad (23)$$

From the first of these two equations follows

$$T \cos \psi = \text{constant} = H \quad . \quad . \quad . \quad (24)$$

where H is the component along AB of the reaction at A, and hence the second equation can be written

$$\frac{d}{dx}(H \tan \psi) = \frac{d}{dx}\left(H \frac{dy}{dx}\right) = H \frac{d^2y}{dx^2} = -\, m\omega^2 y \frac{ds}{dx}$$

from which follows

$$p \frac{dp}{dy} = -\frac{4}{h^2} y \sqrt{(1 + p^2)}\ . \quad . \quad (25)$$

where $p = dy/dx$ and h is given by

$$m\omega^2/H = 4/h^2 \quad . \quad . \quad . \quad . \quad . \quad (26)$$

By integrating (25), with $y = b$, $p = 0$ at the middle of the rope, we now find

$$h^2\sqrt{(1 + p^2)} = h^2 + 2(b^2 - y^2)$$

which, after squaring, gives

$$h^4p^2 = 4(b^2 - y^2)(h^2 + b^2 - y^2)$$

and hence

$$p^2 = \left(\frac{dy}{dx}\right)^2 = \frac{4b^2(h^2 + b^2)}{h^4}\left(1 - \frac{y^2}{b^2}\right)\left(1 - \frac{b^2}{h^2 + b^2}\frac{y^2}{b^2}\right)$$

Taking square roots, and putting

$$y/b = \eta, \quad 1/a = 2\sqrt{(h^2 + b^2)}/h^2, \quad k = b/\sqrt{(h^2 + b^2)} \quad . \quad (27)$$

we now find, after separating the variables,

$$\frac{d\eta}{\sqrt{(1-\eta^2)}\sqrt{(1-k^2\eta^2)}} = \frac{dx}{a}$$

and hence, since $y = 0$ when $x = 0$,

$$\eta = y/b = \operatorname{sn}(x/a, k) \quad (28)$$

This is of the same form as (20), since $b/a = 2k/k'^2$, as may be verified. The length s of the arc, measured from A, is given by (22). Putting $x = c$, $c/a = K$, $s = l$, we find

$$(l + c)/2c = E/Kk'^2 \quad . \quad . \quad . \quad . \quad . \quad (29)$$

an equation which determines k in terms of the ratio l/c. Then a can be found from $c/a = K$, equations (27) give b and h, and equation (26) gives H. Further, if F is the component of the reaction at A, perpendicular to the line AB, then F is given by $F/H = \tan \psi_0 = 2k/k'^2$.

§ 7. **Area of the surface of an ellipsoid.** If p is the perpendicular from the centre of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \quad . \quad . \quad . \quad . \quad (30)$$

on the tangent plane at the point (x, y, z), and if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ denote the direction cosines of the normal at this point, it is known from geometry that

$$\cos\alpha = px/a^2, \quad \cos\beta = py/b^2, \quad \cos\gamma = pz/c^2 \quad . \quad (31)$$

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\,. \quad (32)$$

It follows that the points at which the normals make a constant angle γ with the axis of z lie on the cone

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\cos^2\gamma = \frac{z^2}{c^4} \quad . \quad (33)$$

and hence also, by eliminating z between (30) and (33), that they lie on the elliptic cylinder

$$\left(\frac{\cos^2\gamma}{a^2} + \frac{\sin^2\gamma}{c^2}\right)\frac{x^2}{a^2} + \left(\frac{\cos^2\gamma}{b^2} + \frac{\sin^2\gamma}{c^2}\right)\frac{y^2}{b^2} = \frac{\sin^2\gamma}{c^2} \quad . \quad (34)$$

Let A be the area of the cross-section of this cylinder, and S the area of the surface of the ellipsoid intercepted by the cylinder where $z > 0$. Then

$$dS = dA \sec\gamma \quad . \quad . \quad . \quad . \quad (35)$$

Now from (34) we find

$$A = \frac{\pi a^2 b^2 \sin^2 \gamma}{\sqrt{(c^2 \cos^2 \gamma + a^2 \sin^2 \gamma)}\sqrt{(c^2 \cos^2 \gamma + b^2 \sin^2 \gamma)}}$$
$$= \frac{\pi ab \sin^2 \gamma}{\sqrt{(1 - e_1^2 \cos^2 \gamma)}\sqrt{(1 - e_2^2 \cos^2 \gamma)}} \quad . \quad . \quad (36)$$

where

$e_1^2 = (a^2 - c^2)/a^2$, $e_2^2 = (b^2 - c^2)/b^2$, $e_1^2 > e_2^2$ if $a^2 > b^2 > c^2$; and hence if we put

$$t = e_1 \cos \gamma, \quad k^2 = e_2^2/e_1^2 \quad . \quad (37)$$

then

$$A = \frac{\pi ab}{e_1^2} \frac{e_1^2 - t^2}{\sqrt{(1 - t^2)}\sqrt{(1 - k^2 t^2)}}$$

If we put further

$$t = \text{sn}\,(u, k), \quad e_1 = \text{sn}\,(\theta, k) \quad . \quad . \quad (38)$$

we have

$$\sec \gamma = \frac{\text{sn}\,\theta}{\text{sn}\,u}, \quad A = \frac{\pi ab}{\text{sn}^2 \theta} \frac{\text{sn}^2 \theta - \text{sn}^2 u}{\text{cn}\,u\,\text{dn}\,u}$$

and after differentiation and simplification

$$\frac{\text{sn}\,\theta}{\pi ab} dA \sec \gamma = -\left(\frac{\text{dn}^2 \theta}{\text{dn}^2 u} + \frac{\text{cn}^2 \theta}{\text{cn}^2 u}\right) du$$

Now, when γ varies from 0 to $\frac{1}{2}\pi$, then t varies from e_1 to 0, and u from θ to 0; hence, by (35), if S now denotes the whole area of the surface of the ellipsoid,

$$\frac{\text{sn}\,\theta}{\pi ab} \frac{S}{2} = \int_0^\theta \left(\frac{\text{dn}^2 \theta}{\text{dn}^2 u} + \frac{\text{cn}^2 \theta}{\text{cn}^2 u}\right) du \quad . \quad (39)$$

After integrating and simplifying, we find that the result can be put in the form

$$S = 2\pi c^2 + \frac{2\pi b}{\sqrt{(a^2 - c^2)}} \{(a^2 - c^2)E(\theta) + c^2\theta\} \quad . \quad (40)$$

EXAMPLES III

1. If s is the length of the arc of the curve $y/b = \sin (x/a)$, measured from $x = 0$, show that

$$s = \sqrt{(a^2 + b^2)}E(k, \phi),$$

where $\phi = x/a$, $k^2 = b^2/(a^2 + b^2)$.

2. Show that the length s of the arc of the trochoid

$$x = a\theta + c \sin \theta, \quad y = a - c \cos \theta,$$

measured from the vertex $\theta = 0$, is given by

$$s = 2(a + c)E(u)$$

where

$$\sin \tfrac{1}{2}\theta = \operatorname{sn} u, \quad k^2 = 4ac/(a + c)^2.$$

3. Show that the length s of the arc of the limaçon

$$r = a + c \cos \theta$$

is given by the same formula as that in Example 2.

4. Show that the length s of the arc of the epitrochoid

$$x = (a + b) \cos \theta + c \cos (a + b)\theta/a$$
$$y = (a + b) \sin \theta + c \sin (a + b)\theta/a$$

measured from the vertex $\theta = 0$, is given by

$$bs = 2(a + b)(a + c)E(u)$$

where

$$\sin (b\theta/2a) = \operatorname{sn} u, \quad k^2 = 4ac/(a + c)^2.$$

5. Show that one way of expressing parametrically the co-ordinates of a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is

$$x = a \operatorname{dn} u/\operatorname{cn} u, \quad y = bk' \operatorname{sn} u/\operatorname{cn} u.$$

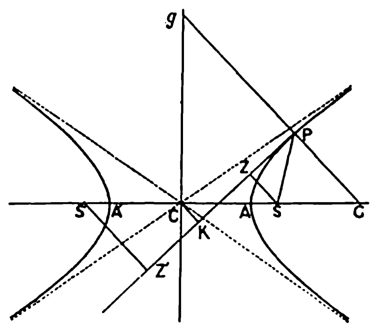

Fig. 3.

Put $\epsilon = 1/e < 1$, $\epsilon' = \sqrt{(1 - \epsilon^2)}$. Then, taking $k = \epsilon$, show that in the usual notation (Fig. 3), measuring s from $u = 0$,

$$k = \epsilon = \frac{1}{e}, \quad b = \frac{a\epsilon'}{\epsilon}, \quad l = \frac{a\epsilon'^2}{\epsilon^2}, \quad x = a\frac{\operatorname{dn} u}{\operatorname{cn} u}, \quad y = b\epsilon'\frac{\operatorname{sn} u}{\operatorname{cn} u},$$

$$\epsilon s/a = \epsilon'^2 u - E(u) + \operatorname{sn} u \operatorname{dn} u/\operatorname{cn} u;$$

$CG = a \operatorname{dn} u/(\epsilon^2 \operatorname{cn} u)$, $\quad Cg = a \operatorname{sn} u/(\epsilon \operatorname{cn} u)$, $\quad gG = a/(\epsilon^2 \operatorname{cn} u)$;
$PG = a\epsilon'^2/(\epsilon^2 \operatorname{cn} u)$, $\quad Pg = a/\operatorname{cn} u$,
$CK = a \operatorname{cn} u$, $\quad \rho = a\epsilon'^2/(\epsilon^2 \operatorname{cn}^3 u)$.
$r = SP = a(\operatorname{dn} u - \epsilon \operatorname{cn} u)/(\epsilon \operatorname{cn} u)$,
$r' = S'P = a(\operatorname{dn} u + \epsilon \operatorname{cn} u)/(\epsilon \operatorname{cn} u)$,
$p = SZ = b(\operatorname{dn} u - \epsilon \operatorname{cn} u)/\epsilon'$, $\quad p' = S'Z' = b(\operatorname{dn} u + \epsilon \operatorname{cn} u)/\epsilon'$.

$$\cos GSP = \frac{\epsilon \operatorname{dn} u - \operatorname{cn} u}{\operatorname{dn} u - \epsilon \operatorname{cn} u}, \qquad \sin GSP = \frac{\epsilon'^2 \operatorname{sn} u}{\operatorname{dn} u - \epsilon \operatorname{cn} u},$$

$$\cos SPG = \operatorname{cn} u, \qquad \sin SPG = \operatorname{sn} u,$$

$$\cos PGS = \operatorname{dn} u. \qquad \sin PGS = \epsilon \operatorname{sn} u,$$

6. The axes of two circular cylinders, of radii a and b $(a > b)$, intersect at right angles. Show that their common volume is

$$\tfrac{8}{3} a\{(a^2 + b^2)E - (a^2 - b^2)K\}, \quad (k = b/a).$$

7. Show that the volume common to the two elliptic cylinders

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \quad \frac{y^2}{b^2} + \frac{z^2}{c'^2} = 1, \quad (c < c'),$$

is
$$(8ab/3c)\{(c'^2 + c^2)E - (c'^2 - c^2)K\}, \quad (k = c/c').$$

8. A heavy bead is projected from the lowest point of a smooth circular wire, fixed in a vertical plane, so that it describes one-third of the circumference before coming to rest. Prove that during the first half of the time it describes one-quarter of the circumference.

9. Show that the length of a pendulum which beats seconds when swinging through an angle $2a$ is given by $l = g/(4K^2)$, $(k = \sin \tfrac{1}{2}a)$.

10. A pendulum swinging through an angle 2α makes N beats a day. If α is increased by $d\alpha$, show that the pendulum will lose $\tfrac{1}{8}N \sin \alpha \, d\alpha$ beats a day, approximately.

A pendulum beats seconds when swinging through an angle of 6°. If the angle of swing is increased to 8°, show that the pendulum will lose 10 beats a day approximately.

11. Show that the gravitational potential of a uniform circular disc, of radius a and mass σ per unit area, at a point on the disc at a distance c from the centre is $4\sigma a E(c/a)$.

12. Let (R, θ, ϕ) be the usual spherical polar co-ordinates on a sphere of radius R, and let $\rho = R \sin \theta$. If ds is the element of arc of any curve drawn on the sphere, show that

$$(ds)^2 = (\rho d\phi)^2 + (R d\rho)^2/(R^2 - \rho^2).$$

Hence show that, for the curve (Seiffert's spherical spiral) defined by the equation $R\phi = ks$, if s is measured from the pole of the sphere, then $\rho = R \operatorname{sn}(s/R)$, $z = R \operatorname{cn}(s/R)$, and $\operatorname{dn}(s/R)$ is the cosine of the angle at which the curve cuts the meridian. $(0 < k < 1.)$

13. A particle of unit mass oscillates on the axis of x so that its position at time t is given by (i) $x = a \operatorname{sn} \omega t$, (ii) $x = a \operatorname{sn}^2 \omega t$. Show that the force f which causes the oscillation is given by:

(i) $x = a \operatorname{sn} \omega t, \quad f = -\omega^2\{(1 + k^2)x - 2k^2x^3/a^2\}$,

(ii) $x = a \operatorname{sn}^2 \omega t, \quad f = 2\omega^2\{a - 2(1 + k^2)x + 3k^2x^2/a\}$.

14. **Application to spherical trigonometry.** Let T be a spherical triangle having three acute sides. The polar triangle T' has three obtuse angles. Every spherical triangle, or one of its colunar triangles,* belongs to type T or T'.

* Greenhill, *Elliptic Functions*, p. 133.

Prove that a spherical triangle with three acute sides has at least two acute angles, and that in such a triangle we may put

$$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C = k < 1.$$

Show also that, if A and B are acute, and if we put (modulus k)

$$\sin A = \operatorname{sn} u, \qquad \sin B = \operatorname{sn} v$$

then

$$\sin p = k \operatorname{sn} u \operatorname{sn} v$$

where p is the perpendicular arc from C on AB, and

$$\cos a = \operatorname{dn} u, \quad \cos b = \operatorname{dn} v, \quad \cos c = \operatorname{dn}(u+v),$$
$$\cos A = \operatorname{cn} u, \quad \cos B = \operatorname{cn} v, \quad \cos C = -\operatorname{cn}(u+v).$$

15. Euler's equations of motion * for a rigid body moving with one point fixed, under no external forces, are

$$A\frac{dp}{dt} = (B-C)qr, \quad B\frac{dq}{dt} = (C-A)rp, \quad C\frac{dr}{dt} = (A-B)pq.$$

Verify that these equations are satisfied by

$$p = \alpha \operatorname{cn} \lambda t, \quad q = -\beta \operatorname{sn} \lambda t, \quad r = \gamma \operatorname{dn} \lambda t$$

where α, β, γ, λ are given by

$$\frac{A\alpha^2}{B-C} = \frac{B\beta^2}{A-C} = \frac{Ck^2\gamma^2}{A-B} = \frac{\alpha\beta\gamma}{\lambda} = \Omega^2$$

the modulus k, and Ω, being arbitrary.

Verify also that p, q, r satisfy the two equations

$$Ap^2 + Bq^2 + Cr^2 = \text{const.}$$

and

$$A^2p^2 + B^2q^2 + C^2r^2 = \text{const.}$$

Show further that Euler's equations have a solution of the form $p = a \operatorname{dn} \mu t$, $q = b \operatorname{sn} \mu t$, $r = c \operatorname{cn} \mu t$.

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* See, e.g., Lamb, *Higher Mechanics*, § 52.

CHAPTER IV

ARGUMENT COMPLEX

§ 1. Introduction. As in the first chapter, we begin with an introductory article. We suppose that $\sin u$ has been defined as in I, § 1, and that the function $\sinh v$ has been defined in a similar way by means of the integral

$$v = \int_0^y \frac{dt}{\sqrt{(1+t^2)}} = \sinh^{-1} y, \quad y = \sinh v \quad . \quad . \quad \text{(i)}$$

where $-\infty < y < \infty$, $-\infty < v < \infty$. Now consider the integral

$$w = \int_0^{iy} \frac{dt}{\sqrt{(1-t^2)}} \quad . \quad . \quad \text{(ii)}$$

the integration being carried out along the imaginary axis from $t = 0$ to $t = iy$. Putting $t = i\eta$, $dt = id\eta$, we get, by (i),

$$w = \int_0^y \frac{id\eta}{\sqrt{(1+\eta^2)}} = i \sinh^{-1} y \quad . \quad . \quad . \quad \text{(iii)}$$

Thus w is a pure imaginary; so we may put $w = iv$; then (ii) and (iii) give

$$iv = \int_0^{iy} \frac{dt}{\sqrt{(1-t^2)}}, \quad v = \sinh^{-1} y, \quad y = \sinh v \quad . \quad \text{(iv)}$$

If by comparison with I, § 1, we suppose that the first of these can be written $iv = \sin^{-1}(iy)$ and hence $\sin iv = iy$, thus defining $\sin iv$, we arrive at the well-known formula

$$\sin iv = i \sinh v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

§ 2. In order to obtain the corresponding formula for sn (iv, k), we consider the integral

$$w = \int_0^{iy} \frac{dt}{\sqrt{(1-t^2)}\sqrt{(1-k^2t^2)}} \quad . \quad (1)$$

the integration being carried out along the imaginary axis from $t = 0$ to $t = iy$. Putting $t = i\eta$, $dt = id\eta$, we get

$$w = \int_0^y \frac{id\eta}{\sqrt{(1+\eta^2)}\sqrt{(1+k^2\eta^2)}} \quad . \quad . \quad (2)$$

Thus w is a pure imaginary; so we may put $w = iv$; then (1) and (2) give, respectively,

$$iv = \int_0^{iy} \frac{dt}{\sqrt{(1 - t^2)}\sqrt{(1 - k^2t^2)}} \quad . \quad . \quad (3)$$

$$v = \int_0^{y} \frac{d\eta}{\sqrt{(1 + \eta^2)}\sqrt{(1 + k^2\eta^2)}} \quad . \quad . \quad (4)$$

We may suppose (3) to define sn iv, writing, by comparison with I, (1),

$$iv = \text{sn}^{-1}\,(iy), \quad \text{sn}\, iv = iy \quad . \quad . \quad . \quad . \quad (5)$$

If in (4) we now put $\eta = \tan \psi$, $y = \tan \phi$, we find

$$v = \int_0^{\phi} \frac{d\psi}{\sqrt{(1 - k'^2 \sin^2 \psi)}} \quad . \quad . \quad . \quad (6)$$

and hence

$$\sin \phi = \text{sn}\,(v, k'), \quad \cos \phi = \text{cn}\,(v, k')$$
$$y = \tan \phi = \sin \phi / \cos \phi = \text{sn}\,(v, k')/\text{cn}\,(v, k')$$

so that (5) gives

$$\text{sn}\,(iv, k) = i\,\text{sn}\,(v, k')/\text{cn}\,(v, k') \quad . \quad . \quad . \quad (7)$$

Note that as y varies from $-\infty$ to $+\infty$, the angle ϕ varies from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ and v varies from $-K'$ to $+K'$.

§ 3. **The elliptic functions. Argument purely imaginary.** We have just seen that sn (iv, k) is given by formula (7). The functions cn (iv, k) and dn (iv, k) can be expressed by similar formulæ. The three formulæ are

$$\text{sn}\,(iv, k) = i\,\text{sn}\,(v, k')/\text{cn}\,(v, k') \quad . \quad (8)$$

$$\text{cn}\,(iv, k) = 1/\text{cn}\,(v, k') \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{dn}\,(iv, k) = \text{dn}\,(v, k')/\text{cn}\,(v, k') \quad . \quad . \quad . \quad (10)$$

They express elliptic functions in which the argument iv is a pure imaginary in terms of functions in which the argument v is real. Together they are usually known as *Jacobi's imaginary transformation*. Formulæ (9) and (10) are obtained by substituting from (8) in

$$\text{cn}\,(iv, k) = \sqrt{\{1 - \text{sn}^2\,(iv, k)\}}$$

and

$$\text{dn}\,(iv, k) = \sqrt{\{1 - k^2\,\text{sn}^2\,(iv, k)\}}$$

remembering that cn $0 = +1$ and dn $0 = +1$.

Although at the end of § 2 it was pointed out that

$$-K' \leqslant v \leqslant K',$$

we may assume now that v can have any real value, since sn (v, k'), cn (v, k'), dn (v, k') have previously been defined for all real values of v (see I, § 9).

§ 4. **Argument complex.** In order to define sn $(u + iv)$, cn $(u + iv)$, and dn $(u + iv)$, we shall assume that the addition formulæ continue to hold good when the arguments are complex. Accordingly, replacing v by iv in I, (24), etc., and using (8), (9), (10), we find the following formulæ, in which the real and imaginary parts of sn $(u + iv)$, . . . are separated. For brevity, we put $s = \text{sn}\,(u, k)$, $s_1 = \text{sn}\,(v, k')$, . . . Note that the complementary modulus k' goes with the suffix 1:

$$\text{sn}\,(u + iv) = (sd_1 + icds_1c_1)/(1 - d^2s_1^2) \quad . \quad . \quad (11)$$

$$\text{cn}\,(u + iv) = (cc_1 - isds_1d_1)/(1 - d^2s_1^2) \quad . \quad . \quad (12)$$

$$\text{dn}\,(u + iv) = (dc_1d_1 - ik^2scs_1)/(1 - d^2s_1^2) \quad . \quad (13)$$

We deduce the following particular cases:

Putting $u = K$ in (11), etc., we find

$$\text{sn}\,(K + iv) = 1/\text{dn}\,(v, k') \quad . \quad . \quad . \quad . \quad (14)$$

$$\text{cn}\,(K + iv) = -\,ik'\,\text{sn}\,(v, k')/\text{dn}\,(v, k') \quad (15)$$

$$\text{dn}\,(K + iv) = k'\,\text{cn}\,(v, k')/\text{dn}\,(v, k') \quad . \quad (16)$$

Putting $v = K'$ in (11), etc., we find

$$\text{sn}\,(u + iK') = 1/(k\,\text{sn}\,u) \quad . \quad . \quad . \quad . \quad (17)$$

$$\text{cn}\,(u + iK') = -\,i\,\text{dn}\,u/(k\,\text{sn}\,u) \quad . \quad . \quad (18)$$

$$\text{dn}\,(u + iK') = -\,i\,\text{cn}\,u/\text{sn}\,u \quad . \quad . \quad . \quad . \quad (19)$$

Replacing u by $u + iK'$ in (17), etc., we find

$$\text{sn}\,(u + 2iK') = \text{sn}\,u \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\text{cn}\,(u + 2iK') = -\,\text{cn}\,u \quad . \quad . \quad . \quad . \quad (21)$$

$$\text{dn}\,(u + 2iK') = -\,\text{dn}\,u \quad . \quad . \quad . \quad . \quad (22)$$

Replacing u by $u + K$ in (17), etc., we find

$$\text{sn}\,(u + K + iK') = \text{dn}\,u/(k\,\text{cn}\,u) \quad . \quad . \quad . \quad (23)$$

$$\text{cn}\,(u + K + iK') = -\,ik'/(k\,\text{cn}\,u) \quad . \quad . \quad . \quad (24)$$

$$\text{dn}\,(u + K + iK') = ik'\,\text{sn}\,u/\text{cn}\,u \quad . \quad . \quad . \quad (25)$$

Replacing u by $u + 2K$ in (20), etc., we find

$$\text{sn}\,(u + 2K + 2iK') = -\,\text{sn}\,u \quad . \quad . \quad . \quad (26)$$

$$\text{cn}\,(u + 2K + 2iK') = \text{cn}\,u \quad . \quad . \quad . \quad . \quad (27)$$

$$\text{dn}\,(u + 2K + 2iK') = -\,\text{dn}\,u \quad . \quad . \quad . \quad (28)$$

After replacing u by $u + K$ and v by $v + K'$ in (11), etc., and a little reduction, we find

$$\operatorname{sn}(u + iv + K + iK') = k^{-1}(cdd_1 + ik'^2ss_1c_1)/(1 - s^2d_1^2) \quad (29)$$

$$\operatorname{cn}(u + iv + K + iK') = k^{-1}k'(sds_1d_1 - icc_1)/(1 - s^2d_1^2) \quad (30)$$

$$\operatorname{dn}(u + iv + K + iK') = -k'(ds_1 - iscc_1d_1)/(1 - s^2d_1^2) \quad (31)$$

These formulæ, being consequences of the addition formulæ, hold good for real or complex values of u or v.

§ 5. **Doubly periodic functions.** A single-valued function $f(w)$ is said to have a period Ω if, for all values of w,

$$f(w + \Omega) = f(w)$$

A period Ω is said to be a *primitive* period if no submultiple of Ω is a period.

If a single-valued function has two periods Ω, Ω', the ratio of which is not real, the function is said to be *doubly periodic*.

A pair of primitive periods of a doubly periodic function is not unique (e.g., $4K$, $2iK$, is one pair of primitive periods of the function $\operatorname{sn} w$; another pair is $4K$, $4K + 2iK'$).

If Ω, Ω' is a pair of primitive periods for the function $f(w)$, then every point of the form $w + m\Omega + m'\Omega'$, where m, m' are any two integers, is said to be *congruent* to w and to every other point of the same form. At all congruent points $f(w)$ has the same value.

A parallelogram with its vertices at four points of the form w, $w + \Omega$, $w + \Omega'$, $w + \Omega + \Omega'$ is called a *period-parallelogram.* The whole w-plane can be divided into a network of period-parallelograms by means of two systems of parallel lines. It is sufficient to study the properties of a doubly periodic function inside one of its period-parallelograms, because every point outside is congruent to some inside point.

§ 6. **Periods of sn w, cn w, dn w.** From (20), (22) and (27) and from I (34) and (35) it follows that pairs of periods for these functions are, respectively,

$$\operatorname{sn} w: \quad 4K,\ 2iK' \quad . \quad . \quad (32)$$

$$\operatorname{cn} w: \quad 4K,\ 2K + 2iK' \quad (33)$$

$$\operatorname{dn} w: \quad 2K,\ 4iK' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

§ 7. **Behaviour of sn w, cn w, dn w near special points.** Using the Maclaurin expansions of the functions sn u, cn u, dn u along with formulæ (17) to (25) and I, (28) to (34), we find that the behaviour of these functions near the points $w = 0$, $w = 2K$, $w = \pm K$, $w = K \pm iK'$, $w = iK'$, $w = 2K + iK'$, $w = -iK'$

can be represented in the forms given in the table below, in which $P, Q, R \ldots$ denote power-series in u^2:

w	sn w	cn w	dn w
u	$u + Pu^3$	$1 + Qu^2$	$1 + Ru^2$
$2K + u$	$-u - Pu^3$	$-1 - Qu^2$	$1 + Ru^2$
$K + u$	$1 + P_1u^2$	$-k' + Q_1u^3$	$k' + R_1u^2$
$-K + u$	$-1 - P_1u^2$	$k'u - Q_1u^3$	$k' + R_1u^2$
$K + iK' + u$	$k^{-1} + P_2u^2$	$-ik'k^{-1} + Q_2u^2$	$ik'u + R_2u^3$
$K - iK' + u$	$k^{-1} + P_2u^2$	$ik'k^{-1} - Q_2u^2$	$-ik'u - R_2u^3$
$iK' + u$	$k^{-1}u^{-1} + P_3u$	$-ik^{-1}u^{-1} + Q_3u$	$-iu^{-1} + R_3u$
$2K + iK' + u$	$-k^{-1}u^{-1} - P_3u$	$ik^{-1}u^{-1} - Q_3u$	$-iu^{-1} + R_3u$
$-iK' + u$	$k^{-1}u^{-1} + P_3u$	$ik^{-1}u^{-1} - Q_3u$	$iu^{-1} - R_3u$

§ 8. **Zeros and poles of sn w, cn w, dn w.** Each of the functions sn w, cn w, dn w has two simple zeros and two simple poles in a period-parallelogram. Thus, the entries in the above table show that the functions have simple zeros and poles at the points given in the next table. The functions have no other zeros or poles, except at congruent points (see the corollaries in § 9). Note that sn $w = 0$ at the point $w = 0$, that cn $w = 0$ at the point $w = K$, that dn $w = 0$ at the point $w = K + iK'$, and that all become infinite at the point $w = iK'$. These four points are the corners in order of a rectangle.

	Periods	Simple zeros at	Simple poles at
sn w .	$4K, 2iK'$	$0, 2K$	$iK', 2K + iK'$
cn w .	$4K, 2K + 2iK'$	$K, -K$	$iK', 2K + iK'$
dn w .	$2K, 4iK'$	$K + iK', K - iK'$	$iK', -iK'$

§ 9. **Expressions for $|\text{sn}\, w|$, $|\text{cn}\, w|$, $|\text{dn}\, w|$,** where $w = u + iv$. From (11), and by replacing v by $-v$ on both sides of (11), we have

$$\text{sn}\,(u + iv) = x + iy, \quad \text{sn}\,(u - iv) = x - iy$$

and therefore

$$\text{sn}\,(u + iv)\,\text{sn}\,(u - iv) = x^2 + y^2$$

where

$$x = sd_1/(1 - d^2s_1^2), \quad y = cds_1c_1/(1 - d^2s_1^2).$$

Now, if u and v are real and $0 < k < 1$, then x and y are real and $|x + iy|^2 = (x + iy)(x - iy)$. Hence, in this case,

$$|\text{sn}\, w|^2 = |\text{sn}\,(u + iv)|^2 = \text{sn}\,(u + iv)\,\text{sn}\,(u - iv) = \text{sn}\, w\, \text{sn}\, \bar{w} \quad (35)$$

and similarly

$$|\text{cn}\, w|^2 = \text{cn}\, w\, \text{cn}\, \bar{w}, \quad |\text{dn}\, w|^2 = \text{dn}\, w\, \text{dn}\, \bar{w} \quad . \quad (36)$$

It will now be proved that, under the same conditions,

$$|\text{sn}\,(u + iv)|^2 = \frac{1 - c^2 c_1^2}{1 - d^2 s_1^2} = \frac{1 - CC_1}{D_1 + DC_1} \quad (37)$$

$$|\text{cn}(u + iv)|^2 = \frac{1 - s^2 d_1^2}{1 - d^2 s_1^2} = \frac{D + CD_1}{D_1 + DC_1} \quad (38)$$

$$|\text{dn}\,(u + iv)|^2 = \frac{d_1^2 - k^2 s^2}{1 - d^2 s_1^2} = \frac{D + CD_1}{1 + CC_1} \quad (39)$$

where $s = \text{sn}\,(u, k)$, $s_1 = \text{sn}\,(v, k')$, $C = \text{cn}\,(2u, k)$, $C_1 = \text{cn}\,(2v, k')$. . . .

Proof. From Examples I (*b*) (4), we have

$$\text{sn}\,(u + iv)\,\text{sn}\,(u - iv) = (\text{sn}^2 u - \text{sn}^2 iv)/(1 - k^2\,\text{sn}^2 u\,\text{sn}^2 iv)$$

which reduces to the first form given in (37), with the help of (8).

Again, from Examples I (*b*) (5), we have

$$\text{sn}\,(u + iv)\,\text{sn}\,(u - iv) = (\text{cn}\, 2iv - \text{cn}\, 2u)/(\text{dn}\, 2iv + \text{dn}\, 2u)$$

which reduces to the second form in (37).

The proofs of (38) and (39) are similar.

Cor. 1. The only zeros of sn w, cn w, dn w are given by

$$\text{sn}\, w = 0, \quad w = 2mK + 2niK' \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\text{cn}\, w = 0, \quad w = (2m + 1)K + 2niK' \quad . \quad . \quad . \quad . \quad (41)$$

$$\text{dn}\, w = 0, \quad w = (2m + 1)K + (2n + 1)iK' \quad . \quad . \quad (42)$$

where m, n denote any integers.

Proof. From (37), since u and v are real, $|\text{sn}\,(u + iv)|$ can vanish only at points where $c^2 = 1$, $c_1^2 = 1$, that is, where $u = 2mK$, $v = 2nK'$, which proves (40). The proofs of (41), (42) are similar.

Cor. 2. The only zeros of the derivatives of sn w, cn w, dn w are given respectively by

$$\text{cn}\, w\, \text{dn}\, w = 0, \quad w = (2m + 1)K + niK' \quad . \quad . \quad . \quad (43)$$

$$\text{sn}\, w\, \text{dn}\, w = 0, \quad w = mK + \left[\, 2n + \tfrac{1}{2}\{1 - (-)^m\}\right] iK' \quad (44)$$

$$\text{sn}\, w\, \text{cn}\, w = 0, \quad w = mK + 2niK' \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where m, n denote any integers. This follows from Cor. 1.

Cor. 3. The only poles of the functions sn w, cn w, dn w are given by

$$w = 2mK + (2n + 1)iK' \quad . \quad . \quad . \quad . \quad (46)$$

where m, n denote any integers.

Proof. From (37) etc., $|\text{sn}\, w|$, $|\text{cn}\, w|$, $|\text{dn}\, w|$ can only become infinite, for real values of u and v, at points where $d^2 = 1$, $s_1^2 = 1$, i.e., where $u = 2mK$, $v = (2n + 1)K'$.

COR. 4. The only periods of $\operatorname{sn} w$, $\operatorname{cn} w$, $\operatorname{dn} w$ are of the forms given by

$$\text{for } \operatorname{sn} w: \quad 4mK + 2niK' \qquad (47)$$

$$\text{for } \operatorname{cn} w: \quad 2(2m+1)K + 2niK' \qquad (48)$$

$$\text{for } \operatorname{dn} w: \quad 2mK + 4niK' \qquad (49)$$

Proof of (47). Let Ω be a period of $\operatorname{sn} w$. Then

$$\operatorname{sn}(w + \Omega) \equiv \operatorname{sn} w \qquad (50)$$

is an identity in w. Hence, by the addition theorem,

$$sc_0d_0 + s_0cd \equiv s(1 - k^2s^2s_0^2)$$

or

$$k^2s_0^2s^3 + (c_0d_0 - 1)s \equiv -s_0cd$$

where $s = \operatorname{sn} w$, $s_0 = \operatorname{sn} \Omega$, . . . Squaring both sides and rearranging in descending powers of s^2, we get

$$k^4s_0^4s^6 + k^2s_0^2(2c_0d_0 - 3)s^4 + \{(c_0d_0 - 1)^2 + (1 + k^2)s_0^2\}s^2 - s_0^2 \equiv 0$$

This being an identity in s, the coefficients of all the powers of s must vanish, and therefore $s_0 = 0$, $c_0d_0 - 1 = 0$; that is,

$$\operatorname{sn} \Omega = 0, \quad \operatorname{cn} \Omega \operatorname{dn} \Omega = 1$$

From the first of these equations and (40) follows $\Omega = 2lK + 2niK'$, where l and n are integers, and then from the second l must be even, so that Ω must be of the form $4mK + 2niK'$.

The proofs of (48), (49) can be constructed similarly.

EXAMPLES IV

1. Replace u by $u + iv$ in (23) and deduce (29).

2. Prove that

$$\frac{1}{\operatorname{sn}^2(iv, k)} + \frac{1}{\operatorname{sn}^2(v, k')} = 1$$

3. Obtain the following values of $\operatorname{sn} \frac{1}{2}iK'$, etc.:

$$\operatorname{sn} \tfrac{1}{2}iK' = i/\sqrt{k}, \quad \operatorname{cn} \tfrac{1}{2}iK' = \sqrt{(1+k)}/\sqrt{k}, \quad \operatorname{dn} \tfrac{1}{2}iK' = \sqrt{(1+k)}$$

4. Obtain the following values of $\operatorname{sn} \frac{1}{2}(K + iK')$, etc.:

$$\operatorname{sn} \tfrac{1}{2}(K + iK') = \{\sqrt{(1+k)} + i\sqrt{(1-k)}\}/\sqrt{(2k)}$$
$$\operatorname{cn} \tfrac{1}{2}(K + iK') = (1 - i)\sqrt{k'}/\sqrt{(2k)}$$
$$\operatorname{dn} \tfrac{1}{2}(K + iK') = \sqrt{(kk')}\{\sqrt{(1+k')} - i\sqrt{(1-k')}\}/\sqrt{(2k)}$$

5. Prove that, if u and v are real,

(i) $|\operatorname{sn}(u + \frac{1}{2}iK')| = 1/\sqrt{k} = \text{const.}$
(ii) $|\operatorname{dn}(\frac{1}{2}K + iv)| = \sqrt{k'} = \text{const.}$
(iii) $|\operatorname{sn}(\frac{1}{2}K + iv)/\operatorname{cn}(\frac{1}{2}K + iv)| = 1/\sqrt{k'} = \text{const.}$
(iv) $|\operatorname{dn}(u + \frac{1}{2}iK')/\operatorname{cn}(u + \frac{1}{2}iK')| = \sqrt{k} = \text{const.}$

6. Prove that, if $w = u + iv$, $s = \operatorname{sn}(u, k)$, $s_1 = \operatorname{sn}(v, k')$, . . . then

(i) $|\operatorname{sn} w \pm 1| = (1 \pm sd_1)/\sqrt{(1 - d^2s_1^2)}$
(ii) $|k \operatorname{sn} w \pm 1| = (d_1 \pm ks)/\sqrt{(1 - d^2s_1^2)}$
(iii) $|\operatorname{cn} w \pm 1| = (1 \pm cc_1)/\sqrt{(1 - d^2s_1^2)}$

(iv) $|k \operatorname{cn} w \pm ik'| = (dd_1 \mp kk'ss_1)/\sqrt{(1 - d^2 s_1^2)}$
(v) $|\operatorname{dn} w \pm 1| = (d_1 \pm dc_1)/\sqrt{(1 - d^2 s_1^2)}$
(vi) $|\operatorname{dn} w \pm k'| = (dd_1 \pm k'c_1)/\sqrt{(1 - d^2 s_1^2)}$

7. Deduce from Example 5 that, if $z = \operatorname{sn} w \operatorname{dn} w/\operatorname{cn} w$, then $|z| = 1$ when w lies on the rectangle $u = \pm \frac{1}{2}K$, $v = \pm \frac{1}{2}K'$.

8. By taking the integral

$$\int e^{\pi i u/K} \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} du$$

round a rectangle with vertices at $\pm K$, $\pm K + 2iK'$, suitably indented, prove that

$$\int_{-K}^{K} e^{\pi i u/K} \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} du = \pi i \tanh \frac{\pi K'}{2K}$$

CHAPTER V

CONFORMAL REPRESENTATION

§ **1.** We shall assume that, if $f(w)$ is an analytic function of w, with a finite non-zero derivative $f'(w)$ at every point of a region D bounded by a simple closed curve C in the w-plane, then the relation $z = f(w)$ " represents " or " maps " D conformally upon a corresponding region Δ bounded by a closed curve Γ in the z-plane.

The region Δ may overlap itself, but will not do so if D is small enough. A region D which has the property that the corresponding region Δ consists of the entire z-plane, without overlapping, is called a *fundamental region* of the w-plane for the function $f(w)$.

§ **2. Conformal representation by the equation $z = \text{sn}\, w$.** Let $z = x + iy$, $w = u + iv$. It will be shown that the rectangle

$$-K < u < K, \quad -K' < v < K' \tag{1}$$

is a fundamental region for the function sn w.

For, by IV, (11), we have

$$x + iy = (sd_1 + icds_1c_1)/(1 - d^2s_1^2) \quad . \tag{2}$$

and hence

$$x = sd_1/(1 - d^2s_1^2), \quad y = cds_1c_1/(1 - d^2s_1^2) \quad . \tag{3}$$

It follows that, if w lies within the rectangle (1), then x has the same sign as u, and y the same sign as v, so that the point (x, y) will lie in the same quadrant as the point (u, v), and to the four points $(\pm u, \pm v)$ will correspond the four points $(\pm x, \pm y)$. It will therefore suffice to consider the rectangle $0 < u < K$, $0 < v < K'$.

§ **3.** Suppose, then, that the point w describes the closed contour indicated in Fig. 1, which approximates to a rectangle $OABC$, the corners A, B, C being avoided by means of small quadrants of circles. These corners are avoided because at A and B the derivative dz/dw, $=$ cn w dn w, vanishes, while at C the function z, $=$ sn w, has a simple pole (IV, § 8). At every point within the region bounded by this contour, sn w is analytic and has a non-zero derivative, by IV, § 9, Cor. 2.

When w describes the contour, starting at O, the consequent variations in u, v, x, y are shown in the table:

	u	v	x	y
O to A . .	0 to K	0	0 to 1	0
A to B . .	K	0 to K'	1 to $1/k$	0
B to C . .	K to 0	K'	$1/k$ to $+\infty$	0
C to O . .	0	K' to 0	0	$+i\infty$ to 0

We can deduce the behaviour of z when w is in the neighbourhood of any of the points A, B, C in Fig. 1 from the table in IV, § 7. Thus,

(i) Near A, put $w = K + w_1$, $z = 1 + z_1$. Then $z_1 \doteqdot a_1 w_1^2$, where a_1 is a constant, the first term in the series P_1 of IV, § 7;

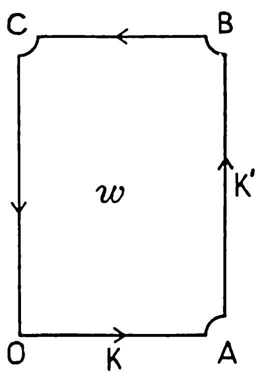

FIG. 1.

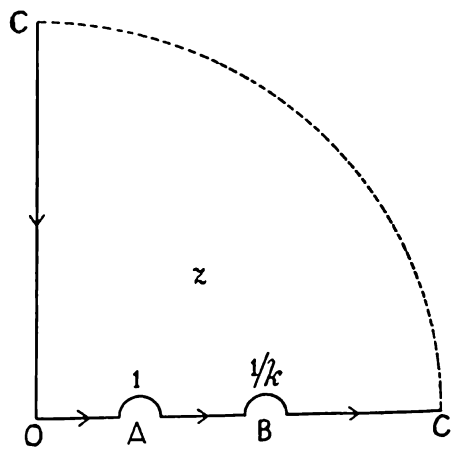

FIG. 2.

consequently, z describes a small semicircle round $z = 1$ while w describes the small quadrant round $w = K$, in the same sense.

(ii) Near B, put $w = K + iK' + w_2$, $z = k^{-1} + z_2$. Then $z_2 \doteqdot a_2 w_2^2$, where a_2 is a constant, the first term in the series P_2 of IV, § 7; consequently, z again describes a small semicircle when w describes the small quadrant round B, in the same sense.

(iii) Near C, put $w = iK' + w_3$. Then $z \doteqdot k^{-1} w_3^{-1}$; consequently, z describes an infinite circular quadrant when w describes the infinitesimal quadrant round C, in the reverse sense.

It should be kept in mind that the representation is conformal at every point on the closed contour, as the points A, B, C do not lie upon it. For this reason, when, for example, the direction of motion in the w-plane turns 90° to the left at the beginning and end of the small quadrantal arc at A, B, or C, the same change takes place in the direction of motion in the z-plane at the corresponding points.

The contour described in the z-plane is indicated in Fig. 2. In Figs. 1 and 2 corresponding points are marked with the same letter. We see that the rectangle $0 < u < K$, $0 < v < K'$ is mapped upon the whole of the positive quadrant $x > 0, y > 0$. The representation is conformal at every pair of corresponding points except $w = K$, $z = 1$ and $w = K + iK'$, $z = 1/k$.

The representation at the pair of points $w = iK'$, $z = \infty$ is defined to be conformal in the sense that, if we put $z = 1/\zeta$, then $\zeta \doteqdot kw_3$, so that the representation would be conformal at the pair of points $w = iK'$, $\zeta = 0$.

It follows, by § 2, that the rectangle $-K < u < K$, $-K' < v < K'$ is mapped upon the entire z-plane.

§ 4. The curves that correspond to the lines u = const., v = const. The lines $v = \pm \frac{1}{2}K'$, $-K < u < K$, transform into the upper and lower halves of the circle $|z| = 1/\sqrt{k}$, by Examples IV, 5, (i). The curves which are the transforms of the other lines of the systems u = const., v = const., may be discussed as follows:

Let A, A', B, B' be the points $z = 1$, $z = -1$, $z = 1/k$, $z = -1/k$ on the real axis of the z-plane, and let Z be any other

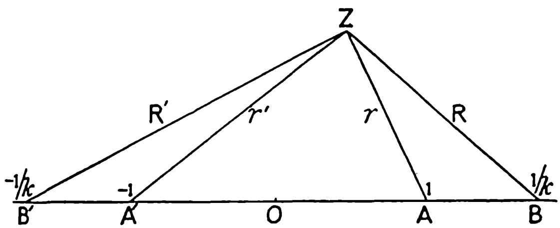

FIG. 3.

point (Fig. 3). Put $r = AZ$, $r' = A'Z$, $R = BZ$, $R' = B'Z$. Then, by Examples IV, 6, (i), (ii), we have

$$r = AZ = |\,\text{sn}\, w - 1| = (1 - sd_1)/\sqrt{(1 - d^2 s_1^2)} \quad . \tag{4}$$

$$r' = A'Z = |\text{sn}\, w + 1| = (1 + sd_1)/\sqrt{(1 - d^2 s_1^2)} \tag{5}$$

$$R = BZ = |\text{sn}\, w - 1/k| = (d_1 - ks)/k\sqrt{(1 - d^2 s_1^2)} \tag{6}$$

$$R' = B'Z = |\text{sn}\, w + 1/k) = (d_1 + ks)/k\sqrt{(1 - d^2 s_1^2)} \tag{7}$$

and hence

$$\frac{r' + r}{1} = \frac{r' - r}{sd_1} = \frac{R' + R}{d_1/k} = \frac{R' - R}{s} = \frac{2}{\sqrt{(1 - d^2 s_1^2)}} \quad . \tag{8}$$

Since $s = \text{sn}\,(u, k)$ and $d_1 = \text{dn}\,(v, k')$, the equations of the curves that correspond to the lines $u = \text{const.}$, $v = \text{const.}$, can be expressed in various ways, e.g., the curves $u = \text{const.}$ can be expressed in either of the forms

$$R' - R = (r' + r)\,\text{sn}\,(u, k) \quad . \quad . \quad (9)$$

$$r' - r = (R' + R)k\,\text{sn}\,(u, k) \quad . \quad . \quad (10)$$

and the curves $v = \text{const.}$ in either of the forms

$$r' - r = (R' - R)\,\text{dn}\,(v, k') \quad . \quad . \quad . \quad (11)$$

$$R' + R = (r' + r)k^{-1}\,\text{dn}\,(v, k') \quad . \quad . \quad . \quad (12)$$

These curves are bicircular quartics,* with foci at A, A', B, B'. Typical curves are indicated in Fig. 4.

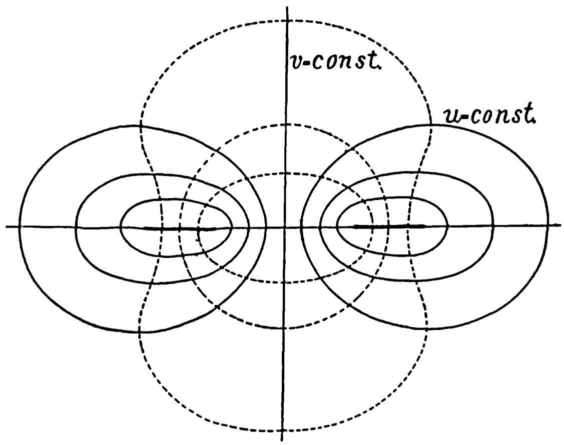

FIG. 4.

When $k \to 0$ the curves degenerate into the confocal ellipses and hyperbolas belonging to the relation $z = \sin w$. When $k \to 1$ they degenerate into the two systems of coaxial circles belonging to the relation $z = \tanh w$.

Ex. Express the equations of the curves $u = \text{const.}$ (i) in terms of r, r' and u, (ii) in terms of R, R' and u.

§ 5. Conformal representation by $z = \text{sn}^2 w$. The equation $z = \text{sn}^2 w$ can be replaced by the two equations $z = z_1^2$, $z_1 = \text{sn}\,w$.

* Salmon, *Higher Plane Curves* (1879), p. 126; Basset, *An Elementary Treatise on Cubic and Quartic Curves* (1901), p. 133; Hilton, *Plane Algebraic Curves*, p. 304.

Hence, in order to obtain the representation of the w-plane on the z-plane by means of $z = \operatorname{sn}^2 w$, we can first represent the w-plane upon an intermediate z_1-plane by means of $z_1 = \operatorname{sn} w$, making use of §§ 2, 3, and then represent the z_1-plane on the z-plane by means of $z = z_1{}^2$. We thus see that the rectangle $OABC$ will be mapped on the upper half of the z-plane (Figs. 5 and 6).

To examine the curves that correspond to the straight lines $u =$ const., $v =$ const., let O, A, B be the points $z = 0$, $z = 1$, $z = 1/k^2$ and Z any other point in the z-plane, and put $r = OZ$, $r' = AZ$, $r'' = BZ$. Then from IV, (37), (38), (39), we have

$$r = |\operatorname{sn}^2 w| = (1 - CC_1)/(D_1 + DC_1) \quad . \quad . \quad . \quad . \quad (13)$$

$$r' = |\operatorname{sn}^2 w - 1| = |\operatorname{cn}^2 w| = (D + CD_1)/(D_1 + DC_1) \quad . \quad (14)$$

$$r'' = |\operatorname{sn}^2 w - 1/k^2| = |\operatorname{dn}^2 w/k^2| = (D + CD_1)/k^2(1 + CC_1) \quad (15)$$

From (13) and (14) follow $r' - Dr = C$ and $D_1 r + C_1 r' = 1$, showing that the curves that correspond to the lines $u =$ const. are the Cartesian ovals

$$r' - Dr = C \quad . \qquad (16)$$

one of which, $u = \frac{1}{2}K$, reduces to the circle

$$r'/r = k', \quad \text{or} \quad |z - 1/k^2| = k'/k^2 \quad . \quad . \quad (17)$$

and that the curves that correspond to the lines $v =$ const. are the Cartesian ovals

$$D_1 r + C_1 r' = 1 \quad . \quad . \quad (18)$$

one of which, $v = \frac{1}{2}K'$, reduces to the circle

$$r = 1/k, \quad \text{or} \quad |z| = 1/k \quad . \quad . \quad . \quad (19)$$

§ 6. Conformal representations by $z = \operatorname{cn} w$, $z = \operatorname{cn}^2 w$, $z = \operatorname{dn} w$, $z = \operatorname{dn}^2 w$. These may be derived independently, or they may be deduced from the case of $z = \operatorname{sn}^2 w$ as follows:

1°. $z = \operatorname{cn}^2 w$. Put $z = 1 - z_1$, $z_1 = \operatorname{sn}^2 w$. See Fig. 8.

There are two circles:

$$|z + k'^2/k^2| = k'/k^2 \quad (u = \tfrac{1}{2}K) \quad . \quad (20)$$

$$|z - 1| = 1/k \quad (v = \tfrac{1}{2}K') \quad . \quad . \quad (21)$$

2°. $z = \operatorname{cn} w$. Put $z = \sqrt{z_1}$, $z_1 = \operatorname{cn}^2 w$. See Fig. 7.

3°. $z = \operatorname{dn}^2 w$. Put $z = 1 - k^2 z_1$, $z_1 = \operatorname{sn}^2 w$. See Fig. 10.

There are two circles:

$$|z| = k' \quad (u = \tfrac{1}{2}K) \quad . \quad . \quad . \quad (22)$$

$$|z - 1| = k \quad (v = \tfrac{1}{2}K') \quad . \quad . \quad . \quad (23)$$

4°. $z = \operatorname{dn} w$. Put $z = \sqrt{z_1}$, $z_1 = \operatorname{dn}^2 w$. See Fig. 9.

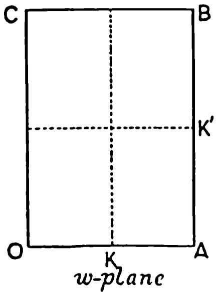

FIG. 5.

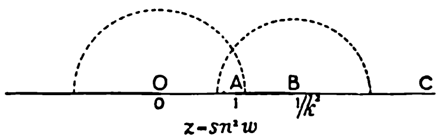

FIG. 6.

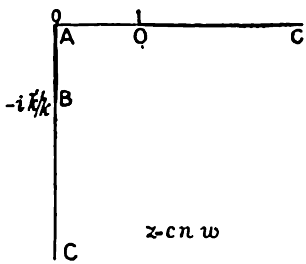

FIG. 7.

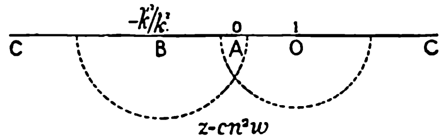

FIG. 8.

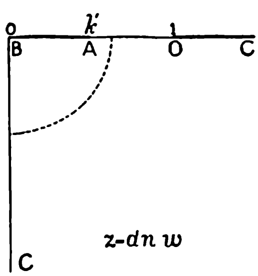

FIG. 9.

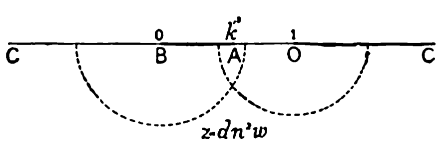

FIG. 10.

There is one circle:

$$|z| = \sqrt{k'} \quad (u = \tfrac{1}{2}K) \quad . \quad . \quad . \quad . \quad (24)$$

§ 7. Conformal representation of a period-parallelogram by $z = \text{sn}\, w$. Returning to $z = \text{sn}\, w$, consider the representation of the period-parallelogram $-K < u < 3K$, $-K' < v < K'$. Let this be divided by the line $u = K$ into two rectangles R and R' in which $-K < u < K$ and $K < u < 3K$, respectively.

The two points $w = K - w_1$ and $w = K + w_1$ lie one in R and the other in R', being the images of each other in the point $w = K$. Since $\text{sn}\,(K + w_1) = \text{sn}\,(K - w_1)$, the function $\text{sn}\, w$ has the same value at each of these two points. Consequently, as the rectangle R is mapped on the entire z-plane (§ 3), the same must be true of the rectangle R', so that the map of the period-parallelogram covers the z-plane twice.

COR. For any given value of z, say z_0, the inverse function $w_1 = \text{sn}^{-1} z$, has two values in a period-parallelogram. When this is defined as above, one value lies in R, the other in R'. The one that lies in R is called the *principal value* of $\text{sn}^{-1} z_0$; let it be denoted by w_0. Then the one that lies in R' will be $2K - w_0$, the image of w_0 in the point $w = K$. Any other value of $\text{sn}^{-1} z_0$ will be congruent to either w_0 or $2K - w_0$.

Ex. 1. Solve the equation $\text{sn}\, w = \text{sn}\, a$.

There are the two non-congruent roots $w = a$, $w = 2K - a$. The general solution consists of these two roots together with all the values of w congruent to either of them.

Ex. 2. Show that $x = \text{sn}\, \tfrac{1}{3}K$ is a root of the equation

$$k^2x^4 - 2k^2x^3 + 2x - 1 = 0$$

and find the other three roots.

If $u = \tfrac{1}{3}K$, then $2u = K - u$ and so

$$\text{sn}\, 2u = \text{sn}\,(K - u) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

the general solution of which, by the first example, is given by

$$2u = K - u + 4mK + 2niK' \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

$$2u = K + u + 4m'K + 2n'iK' \quad . \quad . \quad . \quad \text{(iii)}$$

Now (i) may be written

$$2scd/(1 - k^2s^4) = c/d$$

and hence

$$c(k^2s^4 - 2k^2s^3 + 2s - 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

where $s = \text{sn}\, u$, $c = \text{cn}\, u$. The values of u given by (iii) satisfy the equation $c = 0$. Those given by (ii) satisfy

$$k^2s^4 - 2k^2s^3 + 2s - 1 = 0$$

which is the same as the given equation if s is replaced by x. But from (ii) we find

$$u = \tfrac{1}{3}K, \quad u = \tfrac{1}{3}(K \pm 2iK'), \quad u = 3K + \tfrac{2}{3}iK'$$

and values congruent to these. Hence the four roots of the given equation are given by

$$x = \operatorname{sn} \tfrac{1}{3}K, \quad x = \operatorname{sn} \tfrac{1}{3}(K \pm 2iK'), \quad x = \operatorname{sn}(3K + \tfrac{2}{3}iK').$$

The first and last are real, the other two a conjugate pair.

EXAMPLES V

1. Show that the functions $\operatorname{cn} w$, $\operatorname{dn} w$ take every value once in the rectangle $-K < u < K$, $-K' < v < K'$.

2. Show that $\operatorname{sn}^2 w$ is doubly-periodic, with periods $2K$, $2iK'$; that $\operatorname{sn}^2 w$ has a double zero at $w = 0$, a double pole at $w = iK'$, and that its derivative vanishes at $w = 0$, $w = K$, and $w = K + iK'$.

Show that the relation $z = \operatorname{sn}^2 w$ maps a period-parallelogram on the entire z-plane twice over, and hence that $\operatorname{sn}^2 w$ takes every value twice in a period-parallelogram.

3. Show that, when $k \to 0$, the curves $u =$ const., $v =$ const. of the transformation $z = \operatorname{sn}^2 w$ degenerate into the confocal conics with foci at $z = 0$, $z = 1$, belonging to the transformation $z = \sin^2 w$.

4. Discuss the curves $u =$ const., $v =$ const. in the transformations $z = \operatorname{cn} w$, $z = \operatorname{dn} w$, $z = \operatorname{cn}^2 w$, $z = \operatorname{dn}^2 w$.

5. Show that the four roots of the equation

$$k'^2(1 - 2x) = k^2x^3(2 - x)$$

are $\operatorname{cn} \tfrac{2}{3}K$, $\operatorname{cn} \tfrac{2}{3}iK'$, $\operatorname{cn}(\tfrac{4}{3}K \pm \tfrac{2}{3}iK')$, of which the first two are real.

6. If $c = \operatorname{cn} \tfrac{1}{3}K$, show that $Z(\tfrac{1}{3}K) = k^2c^3/3k'$.

7. If $s = \operatorname{sn} \tfrac{1}{3}K$, $c = \operatorname{cn} \tfrac{1}{3}K$, $d = \operatorname{dn} \tfrac{1}{3}K$,
$S = \operatorname{sn} \tfrac{2}{3}K$, $C = \operatorname{cn} \tfrac{2}{3}K$, $D = \operatorname{dn} \tfrac{2}{3}K$,

show that

$$k^2 = \frac{2s - 1}{s^3(2 - s)}, \quad k'^2 = \frac{(1 - s)^3(1 + s)}{s^3(2 - s)}, \quad d = \frac{k's}{1 - s},$$

$$S^2 = c^2/d^2 = s(2 - s), \quad C = 1 - s, \quad D = (1 - s)/s.$$

8. Show that $\operatorname{sn} \tfrac{1}{3}K$ increases steadily from $\tfrac{1}{2}$ to 1 as k increases from 0 to 1.

9. If $f(x) = k^2x^4 - 2k^2x^3 + 2x - 1$, $(0 < k < 1)$, verify that the equation $f(x) = 0$ has one real root between 0 and 1, and one between $-1/k$ and -1. Also show that the curve $y = f(x)$ has points of inflexion at $x = 0$ and $x = 1$.

CHAPTER VI

CONFORMAL REPRESENTATION (*cont.*)

§ **1.** In the last chapter was considered the representation of the w-plane on the z-plane by means of the transformation $z = \operatorname{sn} w$. It is more useful, however, in many applications to consider the inverse transformation

$$w = \int_0^z \frac{dt}{\sqrt{(1 - t^2)}\sqrt{(1 - k^2t^2)}} \quad . \quad . \quad . \quad (1)$$

as representing the upper half of the z-plane upon a rectangle in the w-plane. As such, it is an example of the Schwarz–Christoffel transformation, the chief properties of which will now be recalled.*

§ **2.** It is first necessary to define $(z - a)^\alpha$ and $(a - z)^\alpha$ so that they will be single-valued functions of position in the upper half of the z-plane, a and α being any real numbers. We therefore put

$$z - a = re^{i\theta}, \text{ where } r = |z - a|, \; 0 \leqslant \theta \leqslant \pi \quad . \quad (2)$$

and define $(z - a)^\alpha$ and $(a - z)^\alpha$ respectively by

$$(z - a)^\alpha = r^\alpha e^{i\alpha\theta} \quad . \quad . \quad (3)$$

and

$$(a - z)^\alpha = (z - a)^\alpha e^{-i\alpha\pi} = r^\alpha e^{-i\alpha(\pi - \theta)} \quad . \quad . \quad (4)$$

Note that, as a consequence, when the point z moving along the real axis, passes from right to left of the point a, avoiding the point itself, e.g., by way of a small semicircle drawn in the upper half-plane, as in Fig. 1, then

$$(z - a)^\alpha \text{ changes into } (a - z)^\alpha e^{i\alpha\pi} \quad . \quad . \quad . \quad (5)$$

and that, when the point z passes the point a from left to right, then

$$(a - z)^\alpha \text{ changes into } (z - a)^\alpha e^{-i\alpha\pi} \quad . \quad . \quad (6)$$

§ **3.** Now let $a, b, \ldots \alpha, \beta, \ldots$ be any real numbers, and let $(z - a)^\alpha, (z - b)^\beta, \ldots$ be defined, as above, so as to be single-valued in the upper half of the z-plane. Then the function $f(z)$ defined by

$$f(z) = (z - a)^\alpha (z - b)^\beta \ldots (z - l)^\lambda \quad . \quad . \quad (7)$$

* For a fuller account, see, e.g., Copson, *Functions of a Complex Variable*, p. 193.

will be single-valued and analytic at all points of the upper half plane, except the points $a, b, \ldots l$. Suppose

$$-R < a < b < \ldots < l < R \quad . \quad . \quad . \quad (8)$$

and let semicircles of small radius ρ be drawn with the points $a, b, \ldots$ as centres, separating these points from the upper half-plane, and a semicircle of large radius R, centre $z = 0$, separating the point $z = \infty$ from the upper half-plane (the large semicircle is not shown completed in Fig. 2). Let these semicircles be joined by the intervening portions of the real axis, to form a simple closed contour C enclosing a simply connected region S.

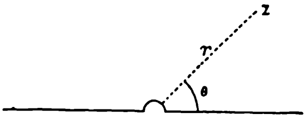

FIG. 1.

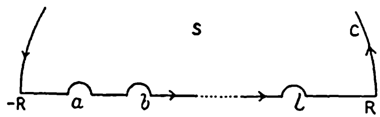

FIG. 2.

The Schwarz–Christoffel transformation may be expressed by

$$w = \int_{z_0}^{z} \frac{dt}{(a-t)^{\alpha}(b-t)^{\beta} \ldots (l-t)^{\lambda}} \qquad (9)$$

where z_0 is a fixed lower limit. Then w is a function of the upper limit z such that

$$dw = (a-z)^{-\alpha}(b-z)^{-\beta} \ldots (l-z)^{-\lambda}\, dz \quad . \quad (10)$$

Since the integrand in (9) is single-valued and analytic in the simply-connected region S, the same will be true of w, by a fundamental theorem in the theory of functions of a complex variable.* Consequently, when z describes any closed curve in S, then w will describe a corresponding closed curve in the w-plane.

§ 4. Suppose, for convenience, that $z_0 = -R$, and let the point z describe the contour C in the positive sense, beginning and ending at the point $z = z_0 = -R$. Put

$$\sigma = \alpha + \beta + \ldots + \lambda \quad . \quad . \quad . \quad (11)$$

and consider the case in which $\alpha < 1$, $\beta < 1$, $\ldots \lambda < 1$, and $\sigma > 1$. This is the simplest case because the integral in (9) will then converge at the points $a, b, \ldots$ when $\rho \to 0$, and at ∞ when $R \to \infty$, and so the closed curve described by w will lie entirely in the finite part of the w-plane.

Moreover, this curve will be a rectilinear polygon in the limit when $\rho \to 0$ and $R \to \infty$. For w will start at $w = 0$ when $z = z_0$, by (9), and dw will be real and positive as long as $z < a$,

* E.g., Copson, § 4.4.

by (10), so that while z moves from z_0 towards the first small semicircle, w will describe a piece of the real axis in the w-plane.

When z leaves the real axis, passes round the semicircle at a, and joins the real axis again, the factor $(a - z)^{-\alpha}$ in dw will change into $(z - a)^{-\alpha}e^{i\alpha\pi}$, by (6), and for this reason $am(dw)$ will increase by $\alpha\pi$; at the same time $am(dz)$ will gain $\frac{1}{2}\pi$ as z joins the semicircle, will lose π as z moves round it, and will gain another $\frac{1}{2}\pi$ as z leaves it; consequently, the net gain in $am(dw)$ will be $\alpha\pi + \frac{1}{2}\pi - \pi + \frac{1}{2}\pi = \alpha\pi$.

While z moves from a to b, we shall then have

$$dw = (z - a)^{-\alpha}e^{i\alpha\pi} \,.\, (b - z)^{-\beta} \;.\;.\;.\; (l - z)^{-\lambda}\, dz$$

and therefore $am(dw) = \alpha\pi =$ constant, so that w will describe another piece of a straight line, inclined at $\alpha\pi$ to the first piece.

Similarly, while z moves from b to c, we shall have

$$dw = (z - a)^{-\alpha}e^{i\alpha\pi} \,.\, (z - b)^{-\beta}e^{i\beta\pi} \;.\;.\;.\; (l - z)^{-\lambda}\, dz$$

and hence $am(dw) = (\alpha + \beta)\pi =$ constant, so that w will again describe a piece of a straight line, inclined at $\beta\pi$ to the second piece, and so on.

On the last part of the real axis of z, between l and R, we shall have $am(dw) = (\alpha + \beta + \ldots + \lambda)\pi = \sigma\pi$. Finally, when z leaves the real axis, describes the large semicircle and joins the real axis again, thus passing from right to left of the points $a, b, \ldots l$, the factors $(z - a)^{-\alpha}e^{i\alpha\pi}$, $(z - b)^{-\beta}e^{i\beta\pi}, \ldots$ will all return to their starting values, and for this reason $am(dw)$ will lose $\sigma\pi$; while $am(dz)$ will gain $\frac{1}{2}\pi + \pi + \frac{1}{2}\pi$, and for this reason $am(dw)$ will gain 2π; thus verifying, as is otherwise obvious, that the net increment in $am(dw)$ after the completion of the circuit will be 2π.

Rule. As a practical rule, we note that, as z passes the points $a, b, \ldots l, \infty$, the direction of motion of the point w turns through angles $\alpha\pi, \beta\pi, \ldots \lambda\pi, (2 - \sigma)\pi$ to the left.

Here it is to be understood that, for example, a turn through an angle $\alpha\pi$ to the left means a turn through an angle $|\alpha\pi|$ to the right if $\alpha < 0$. Further, we note that, if $\sigma = 2$, the directions of motion of the point w just before and just after z has described the large semicircle will be the same.

§ 5. Representation of a half-plane upon a given polygon. A polygon which encloses a simply connected area will be called here an *ordinary* polygon. When z describes its real axis, the polygon described by w may or may not be an ordinary polygon; for example, two or more of its sides may intersect, in which case it will not be ordinary.

If the polygon described by w is ordinary, the upper half of the z-plane will be conformally represented upon the polygon by the transformation (9). The exterior angles of the polygon will be $\alpha\pi$, $\beta\pi$, . . . $(2 - \sigma)\pi$, and since each must be less than two right angles, the conditions $|\alpha| < 1$, $|\beta| < 1$, . . . must be satisfied.

Conversely, if any ordinary polygon is given, as in Fig. 3, it can be proved that the upper half z-plane can be represented upon it by a transformation of the same kind as (9). The steps required in the proof would be as follows:

Let the given polygon be situated anywhere in the w-plane. Let a similar polygon be drawn having one vertex O at the origin, and one side OA along the real axis, and so that a point starting from O and moving along OA would begin to describe the polygon in the positive sense. Let the exterior angles of the polygon be $\alpha\pi$, $\beta\pi$, . . . $\lambda\pi$ in order, so that $\alpha + \beta + \ . \ . \ . + \lambda = 2$.

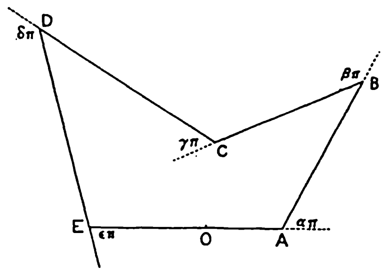

FIG. 3.

Now the relation (9) will transform the real axis of the z-plane into a polygon with exterior angles $\alpha\pi$, $\beta\pi$, . . . $\lambda\pi$, whatever be the values of a, b, . . . l, in this order. It must therefore be shown that these values can be chosen so that the polygon thus obtained is similar to the given one.

To show this, we first notice that three of the values a, b, . . . l can be chosen arbitrarily (e.g., we can take three of them to be 0, 1, ∞ if these are likely to be convenient). For we can make a bilinear transformation of the form $zz_1 + pz + qz_1 + r = 0$, which, by a suitable choice of p, q, r, will transform the real z-axis into the real z_1-axis and any three points on the former into three arbitrary points on the latter, without any reference to the polygon in the w-plane.

If n is the number of sides of the polygon, we then have $n - 3$ of the letters a, b, . . . l left over, and these can be chosen so as to make the ratios of $n - 2$ of the sides of the polygon obtained from (9) agree with those of the given polygon. All the angles being given, the two polygons will then be similar. If we then replace (9) by

$$w = \int_{z_0}^{z} \frac{M dt}{(a - t)^{\alpha}(b - t)^{\beta} \ . \ . \ . \ (l - t)^{\lambda}} + N \quad . \quad (12)$$

where M and N are complex constants, then M can be chosen to make the size and orientation of the polygon determined by (12) agree with the given polygon; and N can be chosen so that the translation which it represents will carry the one polygon into coincidence with the other.

§ 6. The following details may be noted:

It will often be convenient to use the same letter for the variable of integration and for the upper limit in such an integral as the one that appears in (12).

Any change in the lower limit z_0 is equivalent merely to a change in the constant N.

Whether we write, e.g., $(z - a)^\alpha$ or $(a - z)^\alpha$ to begin with is immaterial if the constant M is still to be chosen.

Since the representation is conformal at every point on the contour C, it follows that, corresponding to every right angle which occurs on C, there will be a right angle, in the same sense, on the contour described by w.

§ 7. Representation of a half-plane upon a rectangle. Examples. Examples will be given of the representation of the upper half of the z-plane upon a rectangle in the w-plane. After what has been said, the figures will explain themselves in the first four examples. Corresponding points in the z- and w-planes are indicated by the same letter.

Ex. 1.

$$w = \int_{z_0}^{z} \frac{dz}{(a-z)^{\frac{1}{2}}(b-z)^{\frac{1}{2}}(c-z)^{\frac{1}{2}}(d-z)^{\frac{1}{2}}}, \quad (-\infty < z_0 < a) \qquad (13)$$

Here $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\delta = \frac{1}{2}$, so that the direction of motion of the point w turns through a right angle to the left as z passes from left

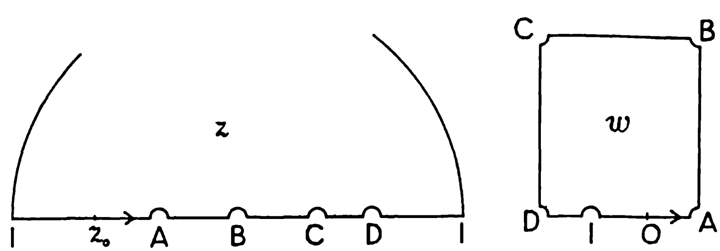

FIG. 4.

to right of each of the points a, b, c, d. Since $\sigma = \alpha + \beta + \gamma + \delta = 2$, there is no change in the direction of motion of w when z passes from the segment DI to the segment IA by way of the large semicircle, which is not shown completed in Fig. 4. The lower limit z_0 is taken to the left of the point $z = a$. The point O in the w-plane corresponds to z_0 in the z-plane.

Ex. 2.

$$w = \int_{z_0}^{z} \frac{dz}{(a - z)^{\frac{1}{2}}(b - z)^{\frac{1}{2}}(c - z)^{\frac{1}{2}}}, \quad (-\infty < z_0 < a) \quad . \quad (14)$$

In this case, we have $\sigma = \alpha + \beta + \gamma = \frac{3}{2}$, so that $2 - \sigma = \frac{1}{2}$ and the direction of motion of the point w turns through a right angle to the left when z passes from the segment CI to the segment IA; see Fig. 5.

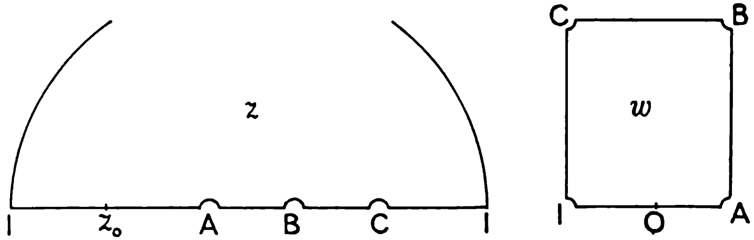

FIG. 5.

Ex. 3.

$$w = \int_0^{z} \frac{dz}{(1 - z^2)^{\frac{1}{2}}(1 - k^2 z^2)^{\frac{1}{2}}} = \text{sn}^{-1} z, \quad z = \text{sn}\, w \quad . \quad (15)$$

See Fig. 6. Compare V, Figs. 1, 2.

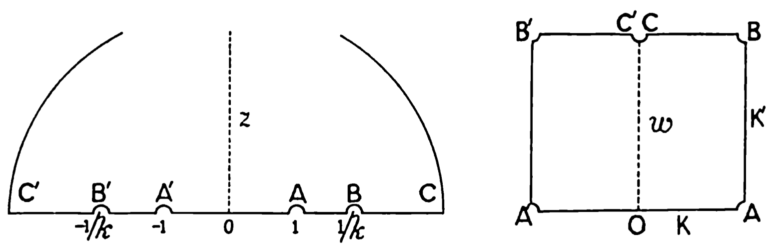

FIG. 6.

Ex. 4.

$$w = \int_0^{z} \frac{dz}{\{4z(1 - z)(1 - k^2 z)\}^{\frac{1}{2}}} = \text{sn}^{-1} \sqrt{z}, \quad z = \text{sn}^2 w \quad (16)$$

If we introduce an intermediate variable ζ, this transformation can be written (see Fig. 7)

$$\zeta = \sqrt{z}, \quad w = \int_0^{\zeta} \frac{d\zeta}{(1 - \zeta^2)^{\frac{1}{2}}(1 - k^2\zeta^2)^{\frac{1}{2}}} = \text{sn}^{-1} \zeta \quad . \quad (17)$$

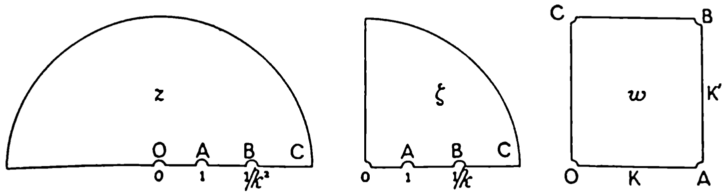

FIG. 7.

Ex. 5. Given a, b, c, d $(a < b < c < d)$, to find the sides of the rectangle in the w-plane into which the upper half t-plane is transformed by

$$w = \int_a \frac{dt}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}(c-t)^{\frac{1}{2}}(d-t)^{\frac{1}{2}}} \quad . \quad (18)$$

We first make a bilinear transformation of the t-plane into a z-plane, such that pairs of corresponding points are $t = a$, $z = 0$; $t = b$, $z = 1$; $t = d$, $z = \infty$; and we let $z = 1/k^2$ correspond to $t = c$. The transformation can be written

$$z = \frac{d-b}{b-a}\frac{t-a}{d-t}, \quad \frac{1}{k^2} = \frac{d-b}{b-a}\frac{c-a}{d-c} \quad . \quad . \quad (19)$$

(See Fig. 8.) From it follows

$$1 - z = \frac{d-a}{b-a}\frac{b-t}{d-t}, \quad 1 - k^2 z = \frac{d-a}{c-a}\frac{c-t}{d-t} \quad . \quad (20)$$

$$dz = \frac{d-b}{b-a}\frac{(d-a)dt}{(d-t)^2} \quad (21)$$

and hence, after a little reduction,

$$w = \frac{2}{\{(c-a)(d-b)\}^{\frac{1}{2}}} \int_0^z \frac{dz}{\{4z(1-z)(1-k^2 z)\}^{\frac{1}{2}}} \quad (22)$$

If we put $m = \frac{1}{2}\{(c-a)(d-b)\}^{\frac{1}{2}}$, then

$$mw = \text{sn}^{-1}\sqrt{z} \quad . \quad (23)$$

or

$$z = \text{sn}^2\, mw = \text{sn}^2\, w_1, \quad w_1 = mw \quad . \quad . \quad (24)$$

The sides of the rectangle in the w_1-plane are K, K', and hence the sides of the rectangle in the w-plane are K/m, K'/m; where

$$k^2 = \frac{b-a}{d-b}\frac{d-c}{c-a} = \frac{AB\,.\,CD}{BD\,.\,AC} \quad . \quad . \quad (25)$$

Here A, B, C, D may refer to either the t-plane or the z-plane, since cross-ratio is unaltered by bilinear transformation.

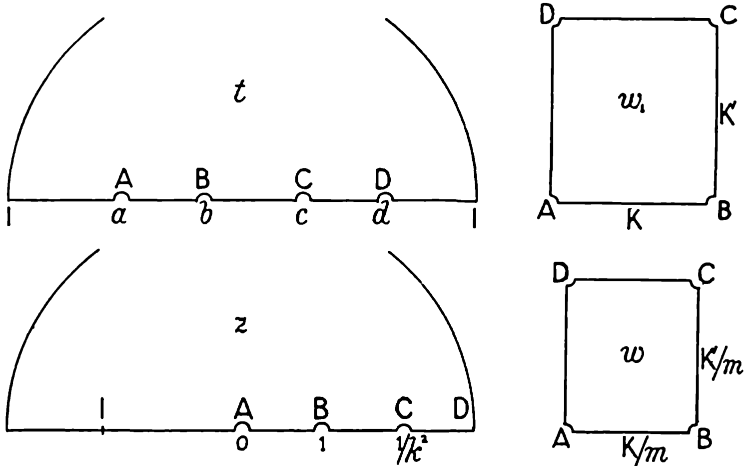

FIG. 8.

Ex. 6. Given a rectangle in the w-plane, to transform it into a half-plane.

Let the lengths of the sides of the given rectangle be l, l'. Let it be placed with one corner at $w = 0$ and one side of length l along the real axis (Fig. 9).

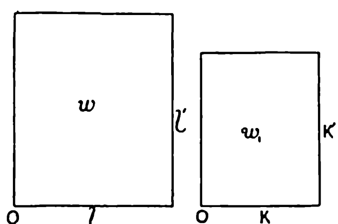

FIG. 9.

Let k be such that $K'/K = l'/l$. Then k can be found from tables (see § 8 below) or by calculation (see, e.g., Copson, p. 412), and hence K and K' can be found. If we put further

$$m = K/l = K'/l', \quad \text{and} \quad w_1 = mw \quad . \quad . \quad (26)$$

then

$$mw = w_1 = \int_0^z \frac{dz}{\{4z(1-z)(1-k^2z)\}^{\frac{1}{2}}} \quad . \quad . \quad . \quad (27)$$

or $z = \text{sn}^2\, mw$, will transform the rectangle in the w-plane into the upper half of the z-plane. This can be transformed into other half planes by means of bilinear transformations.

§ 8. Given the ratio K'/K, to find k from the tables. In the theory of elliptic functions, it is usual to put $q = e^{-\pi K'/K}$.

In Milne-Thomson's tables, the number q is tabulated against m, where $m = k^2$. Hence, if K'/K is given, q can be calculated and m, or k^2, taken from the tables.

In the tables of Jahnke and Emde (e.g., 4th edition, p. 49, Dover Publications) values of $\log_{10} q$ are tabulated against values of α, where $k = \sin \alpha$. If K'/K is given, $\log_{10} q$ can be calculated from the formula $\log_{10} q = -\pi(\log_{10} e). K'/K \doteqdot -1{\cdot}3644K'/K$, and then α, and hence k, found from the tables.

If k is small ($k < 0{\cdot}2$ or $K'/K > 2$) the approximation of II, § 6 will be good enough for many practical purposes.

In the tables at the end of *Anwendung der Elliptischen Funktionen in Physik und Technik*, by F. Oberhettinger and W. Magnus (Springer, Berlin, 1949), values of K'/K are tabulated against k^2. A separate table is given for small values of k.

In certain cases the values of K'/K and k can both be expressed in simple finite forms (see IX, § 7). These values often serve as useful guides in problems involving numerical applications.

§ 9. **Curved boundaries.** For articles on the extension of the Schwarz–Christoffel transformation to the representation of a half-plane upon a curvilinear polygon, the reader is referred to: (i) W. M. Page, *Proc. London Math. Soc.*, 11, 1912–13, p. 313; (ii) J. G. Leathem, *Phil. Trans.*, A, 215, 1915, p. 439.

EXAMPLES VI

1. Show that the transformation

$$z = \int_0^t \{t(1-t)(4-2t)\}^{-\frac{1}{2}}\, dt$$

represents the upper half t-plane upon a square in the z-plane; and that the length of a side of the square is $K(1/\sqrt{2}) \doteqdot 1{\cdot}854$.

2. Show that the lengths of the sides of the rectangle in the w-plane upon which the upper half t-plane is represented by the transformation

$$w = \int_0^t \{t(1-t)(3-t)(9-t)\}^{-\frac{1}{2}}\, dt$$

are $K/\sqrt{6}$ and $K'/\sqrt{6}$, ($k = \frac{1}{2}$), or $0{\cdot}6882$ and $0{\cdot}8804$ approx.

3. Show that the transformation

$$z = a\int_0^t t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}\, dt$$

represents the upper half t-plane upon a right-angled isosceles triangle in the z-plane. By putting $t = u^2$, $u = v^2$, $1 - v^2 = w^2$, show that the length of the hypotenuse of the triangle is $4aK(1/\sqrt{2})$.

4. Show that the transformation

$$z = \int_0^t \frac{dt}{t^{\alpha}(1-t)^{1-\alpha}(1-k^2t)^{\alpha}}, \; (0 < \alpha < 1,\, 0 < k^2 < 1)$$

represents the upper half t-plane on a parallelogram, one of the angles of which is $\alpha\pi$.

5. Show that the transformation

$$z = \int_0^t t^{-\frac{2}{3}}(1-t)^{-\frac{2}{3}}\, dt$$

represents the upper half t-plane upon an equilateral triangle OAB, such that

$$OA = \int_0^1 t^{-\frac{2}{3}}(1-t)^{-\frac{2}{3}}\, dt, \quad AB = \int_1^{\infty} t^{-\frac{2}{3}}(t-1)^{-\frac{2}{3}}\, dt$$

Verify that $OA = AB$.

6. Show that the transformation

$$z = \int_0^t t^{-\frac{2}{3}}(1-t)^{-\frac{1}{2}}(1-k^2t)^{-\frac{1}{2}}\, dt, \quad (k^2 < 1)$$

represents the upper half t-plane on a trapezium, the internal angles of which are 60°, 90°, 90°, 120°.

7. If $0 < \alpha < 1$, $0 < \beta < 1$, $a < b < c$, show that the transformation

$$z = \int_a^t \frac{dt}{(t-a)^{1-\alpha}(b-t)^{1-\beta}(c-t)^{\alpha+\beta}}$$

represents the upper half t-plane upon a triangle of angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ in the z-plane, where $\gamma = 1 - \alpha - \beta$.

Show that the same is true of the transformation

$$z = \int_a^t \frac{dt}{(t-a)^{1-\alpha}(b-t)^{1-\beta}}, \quad (a < b)$$

8. Show that the transformation

$$z = \int_0^t \frac{dt}{t^{\alpha}(1-t^2)^{\beta}}, \; (0 < \alpha < 1,\; 0 < \beta < 1,\; 1 < \alpha + 2\beta < 2)$$

represents the upper half t-plane upon a convex quadrilateral in the z-plane, with external angles $\alpha\pi$, $\beta\pi$, $(2 - \alpha - 2\beta)\pi$, $\beta\pi$.

9. Show that the transformation $t = \operatorname{sn} w$, $z = (1 + it)/(1 - it)$ will represent the rectangle $-K < u < K$, $0 < v < K'$ upon the unit circle $|z| = 1$.

10. Show that the transformation $z = \operatorname{sn} w \operatorname{dn} w / \operatorname{cn} w$ will represent the rectangle $-\frac{1}{2}K < u < \frac{1}{2}K$, $-\frac{1}{2}K' < v < \frac{1}{2}K'$ on the unit circle $|z| = 1$.

CHAPTER VII

APPLICATIONS

§ **1.** The conformal representation of one plane upon another by an equation of the form $w = f(z)$, where z and w denote complex variables, has found many applications to two-dimensional problems in electric and magnetic fields, in hydrodynamics and aerodynamics, to problems of torsion and flexure in the theory of elasticity, etc. Elliptic functions arise naturally in these applications when the equation $w = f(z)$ takes the form of a Schwarz–Christoffel transformation in which the integral is elliptic. We shall first recall how the study of such problems leads to equations of the type $w = f(z)$.

§ **2. Flow of electricity.** When electricity is flowing through an isotropic solid, the *current density vector* at any point P is a vector in the direction of flow at P, i.e., normal to the equipotential surface, and of magnitude C given by

$$C = -\sigma\partial\phi/\partial\nu \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where σ denotes the *specific conductivity* ($1/\sigma =$ the *specific resistance*), ϕ is the potential at P, and $\partial/\partial\nu$ indicates differentiation in the direction of the normal to the equipotential. The current density vector has the property that its component in any given direction is equal to the current per unit area flowing through an element dS of surface normal to the given direction. This component is equal to $-\sigma\partial\phi/\partial n$ where $\partial/\partial n$ indicates differentiation in the given direction.

§ **3. Steady flow in plane metal sheets.** In two dimensions the components of the current density vector parallel to co-ordinate axes Ox, Oy are

$$C_x = -\sigma\partial\phi/\partial x, \quad C_y = -\sigma\partial\phi/\partial y \quad . \quad . \quad . \quad (2)$$

When the flow is steady we find, by considering that the net flow of electricity into a rectangular element at (x, y) per unit time per unit thickness is then zero, that C_x, C_y must satisfy the equation

$$\partial C_x/\partial x + \partial C_y/\partial y = 0 \quad . \quad . \quad . \quad . \quad (3)$$

from which it follows that a function ψ exists such that

$$C_x = -\sigma\partial\psi/\partial y, \quad C_y = \sigma\partial\psi/\partial x \quad . \quad . \quad . \quad (4)$$

From (2) and (4) we see that ϕ and ψ satisfy the Cauchy–Riemann equations

$$\partial\phi/\partial x = \partial\psi/\partial y, \quad \partial\phi/\partial y = -\partial\psi/\partial x \quad . \quad . \quad (5)$$

and hence that we can put

$$\chi = \phi + i\psi = F(x + iy) = F(z) \quad . \quad . \quad (6)$$

where $F(z)$ denotes an analytic function of z. The function ψ is called the *conjugate* of ϕ. The curves ϕ = const. are the equipotential curves. The curves ψ = const., orthogonal to the curves ϕ = const., are the lines of flow or stream lines.

The complex variable χ is called the *complex potential*. The flow in the z-plane can be regarded as a transformation of uniform flow parallel to the real axis in the χ-plane, by means of the equation $\chi = F(z)$ or $z = F^{-1}(\chi)$. Since

$$\frac{\partial\chi}{\partial x} = \frac{d\chi}{dz}\frac{\partial z}{\partial x} = \frac{d\chi}{dz} \qquad (7)$$

therefore

$$-\sigma d\chi/dz = -\sigma(\partial\phi/\partial x + i\,\partial\psi/\partial x) = C_x - i\,C_y \quad . \quad (8)$$

and consequently $-\sigma d\chi/dz$ can be interpreted as the current density vector reflected in the axis of x; it follows in particular that

$$\sigma|d\chi/dz| = \sqrt{(C_x^2 + C_y^2)} = C \qquad (9)$$

the resultant current density.

If $\partial/\partial n$, $\partial/\partial s$ denote differentiations in any two directions at right angles, ds making a positive right angle with dn, we see from (5), by supposing the co-ordinate axes taken in these directions that

$$\partial\phi/\partial n = \partial\psi/\partial s, \quad \partial\phi/\partial s = -\,\partial\psi/\partial n \quad . \quad (10)$$

§ 4. The equivalent resistance of a region of the z-plane bounded by lines of flow and equipotentials. Let $ABCD$ be a region of the

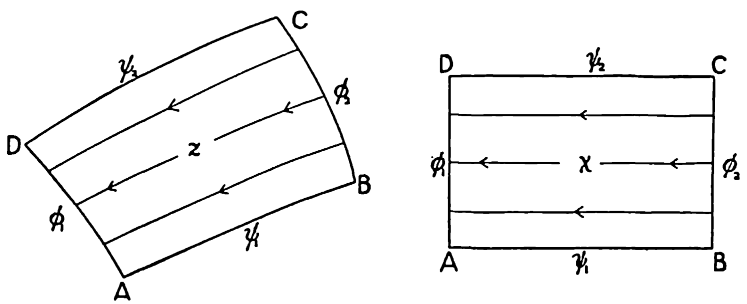

FIG. 1.

z-plane bounded by lines of flow AB, CD and equipotentials AD, BC. It may be regarded as the conformal representation of a rectangle $ABCD$ in the χ-plane (Fig. 1) on the z-plane.

Let $\partial/\partial\nu$ denote differentiation in the direction of the outward normal to BC, and let ds be an element of arc of BC measured so that s increases from B to C. Then the current per unit thickness flowing *into* the region $ABCD$ across BC will be

$$\int_B^C \sigma \frac{\partial\phi}{\partial\nu} ds = \int_B^C \sigma \frac{\partial\psi}{\partial s} ds = \sigma(\psi_2 - \psi_1)$$

by (1) and (10), and hence the equivalent resistance will be

$$\sigma^{-1}(\phi_2 - \phi_1)/(\psi_2 - \psi_1) \quad . \quad . \quad . \quad . \quad (11)$$

This is the same as the resistance of the rectangle $ABCD$ in the χ-plane provided the specific resistance σ^{-1} is the same. Consequently, when the flow in the z-plane is regarded as a conformal transformation of a uniform flow in the χ-plane, the resistance of any part of the plane bounded by two equipotentials and two lines of flow will be conserved. The fraction

$$(\phi_2 - \phi_1)/(\psi_2 - \psi_1), \text{ or } AB/BC \text{ in the } \chi\text{-plane} \quad . \quad (12)$$

may be called the *geometric resistance.*

COR. If one rectangle be represented conformally upon another, the corners corresponding in order in the same sense, then the two rectangles will be similar.

§ 5. Equation (6) may be written in the form

$$\psi - i\phi = i^{-1}F(z) = F_1(z) \quad . \quad . \quad . \quad (13)$$

consequently ψ is the real part of an analytic function $F_1(z)$ and $-\phi$ is then the conjugate of ψ. It follows that from any given field of current we can infer the existence of a second possible field such that the equipotentials of the second field are the stream lines of the first, and the stream lines of the second are the equipotentials of the first.

We note that, in Fig. 1, if the curves $\psi =$ const. are taken as the equipotentials, then the geometric resistance is BC/AB, the reciprocal of that in (12).

§ 6. Hydrodynamics. Steady irrotational flow of liquid in two dimensions. Let u, v be the components of velocity of the liquid at any point (x, y). The flow is said to be *irrotational* when the condition

$$\partial v/\partial x = \partial u/\partial y \quad . \quad . \quad . \quad . \quad . \quad (14)$$

is satisfied at every point in the liquid. Mathematically, this is the necessary and sufficient condition that a function ϕ, the *velocity potential,* should exist such that

$$u = -\,\partial\phi/\partial x, \quad v = -\,\partial\phi/\partial y \quad . \quad . \quad (15)$$

When, in addition, the flow is steady, the components u, v satisfy the equation

$$\partial u/\partial x + \partial v/\partial y = 0 \quad . \quad . \quad . \quad . \quad (16)$$

called the *equation of continuity*, from which it follows that there exists a function ψ, the *stream function*, such that

$$u = -\ \partial\psi/\partial y, \quad v = \partial\psi/\partial x \quad . \quad . \quad . \quad . \quad (17)$$

From (15) and (17) it follows that ϕ and ψ satisfy equations (5), so that we can again put

$$\chi = \phi + i\psi = F(x + iy) = F(z) \quad . \quad . \quad (18)$$

where $F(z)$ is an analytic function. The flow in the z-plane can be regarded as a conformal transformation of uniform flow parallel to the real axis in the χ-plane. The variable χ may be called the *complex velocity potential*, and since

$$-\ d\chi/dz = -\ \partial\chi/\partial x = -\ \partial\phi/\partial x - i\ \partial\psi/\partial x = u - iv \ . \quad (19)$$

it follows that $-\ d\chi/dz$ is the velocity vector reflected in the real axis. The resultant velocity q is given by

$$q = \surd(u^2 + v^2) = |d\chi/dz| \quad . \quad . \quad . \quad (20)$$

§ 7. Examples. Below will be given a few examples of applications which involve elliptic functions.

Ex. 1. As we have seen, the transformation

$$z = \operatorname{sn}\chi, \text{ or } \chi = \operatorname{sn}^{-1} z = \int_0^z \frac{dz}{(1 - z^2)^{\frac{1}{2}}(1 - k^2z^2)^{\frac{1}{2}}}$$

represents the rectangle $0 < \phi < K$, $0 < \psi < K'$ conformally upon the positive quadrant of the z-plane (see VI, § 7, Ex. 3 and V, § 2). It follows that if electric current flows into an infinite quadrantal plane plate (Fig. 2), passing into the plate across an electrode which occupies an interval AB of one infinite edge, and passing out across an electrode which extends over the whole of the other infinite edge OC, then the geometric resistance will be K/K', where K is formed from the modulus k given by $OA = 1$, $OB = 1/k$; that is,

$$\text{geometric resistance} = K/K', \text{ where } k = OA/OB \qquad (21)$$

In particular, if $OB = \surd 2$, then $k = 1/\surd 2$, $K = K'$, the rectangle in the χ-plane is a square, and the geometric resistance is unity. In other words, if the infinite quadrantal plate were of unit thickness, its equivalent resistance would then be equal to the specific resistance.

Ex. 2. If we take $k = 1/\surd 2$ in Ex. 1 and put $z_1 = f(z)$, then the infinite quadrant in the z-plane (Fig. 2), where now $OB = \surd 2$, will be transformed into a region of the z_1-plane. This new region, being another transformation of the square $OABC$ in the χ-plane (Fig. 3), will also be such that its geometric resistance is unity. E.g., if $z_1 = z^2$ we obtain the upper half of the z_1-plane (Fig. 4) with one electrode

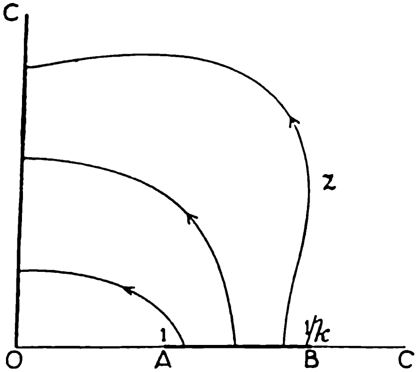

FIG. 2.

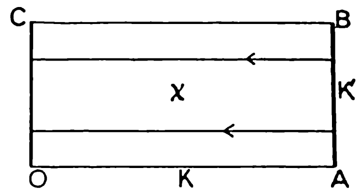

FIG. 3.

extending over the real axis from $z_1 = 1$ to $z_1 = 2$, and the other from $z_1 = -\infty$ to $z_1 = 0$; the geometric resistance of this half-plane is then unity.

Ex. 3. To the last case (Fig. 4) apply a bilinear transformation which transforms the upper half z_1-plane into the upper half of a z_2-plane, with the real axes corresponding. If the points O, A, B, C are thereby

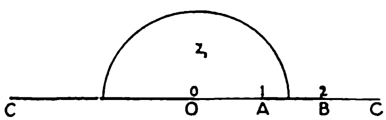

FIG. 4.

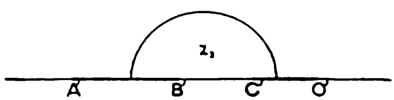

FIG. 5.

transformed into O', A', B', C', then the geometric resistance of the upper half z_2-plane with $A'B'$ and $C'O'$ as electrodes (Fig. 5) will again be unity. But the points O, A, B, C in the z_1-plane form a harmonic range. Consequently, the points O', A', B', C' in the z_2-plane form a harmonic range, since cross-ratios are unaltered by bilinear transformation. Thus, when the points O', A', B', C' form a harmonic range, the geometric resistance of the infinite half plane is unity.

§ 8. **An example due to H. F. Moulton.** The following example occurs in a paper by Moulton * :

Given a rectangle $OABC$ and any four points P, Q, R, S on the

* H. F. Moulton, *Current Flow in Rectangular Conductors*, P.L.M.S., Series 2, Vol. III, p. 104; Jeans, *Electricity and Magnetism*, § 391.

perimeter: to find the geometric resistance when PQ and RS are electrodes (Fig. 6).

It is necessary to find a transformation $\chi = F(z)$ which will transform the given rectangle $OABC$ in the z-plane into a rectangle

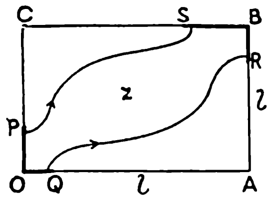

FIG. 6.

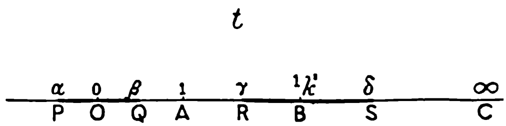

FIG. 7.

$PQRS$ in the χ-plane, with the corners P, Q, R, S corresponding to the extremities of the electrodes in the z-plane. Use is made of intermediate half planes. Let $l = OA$, $l' = AB$ and let

$$K'/K = l'/l = \text{ratio of sides of the given rectangle.}$$

Then, as in VI, § 7, Ex. 6, k can be found and the rectangle $OABC$ will be represented on a half t-plane (Fig. 7) by

$$t = \text{sn}^2\,(mz, k), \quad (m = K/l = K'/l')$$

Let $z = z_1$ at P, z_2 at Q, z_3 at R, z_4 at S, and in the t-plane let $t = \alpha$ at P, β at Q, γ at R, δ at S, so that $\alpha = \text{sn}^2\,mz_1$, $\beta = \text{sn}^2\,mz_2$, $\gamma = \text{sn}^2\,mz_3$, $\delta = \text{sn}^2\,mz_4$.

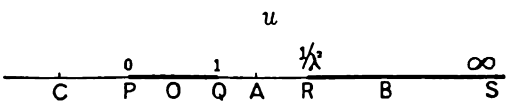

FIG. 8.

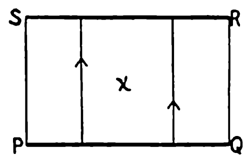

FIG. 9.

As in VI, § 7, Ex. 5, we now transform the half t-plane into a rectangle $PQRS$ in the χ-plane (Fig. 9) by way of an intermediate half u-plane (Fig. 8), putting

$$u = \frac{\delta - \beta}{\beta - \alpha} \cdot \frac{t - \alpha}{\delta - t}, \quad \lambda^2 = \frac{\beta - \alpha}{\delta - \beta} \cdot \frac{\delta - \gamma}{\gamma - \alpha} \quad . \quad (22)$$

$$u = \text{sn}^2\,(\chi, \lambda) \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The geometric resistance is then equal to L'/L, where L, L' are complete elliptic integrals of the first kind with moduli λ, λ'.

A few examples are worked out below. Moulton gave the numerical results shown in Fig. 10 for a square in the four cases depicted. Each electrode extends over a length equal to one-fifth of a side of the square.

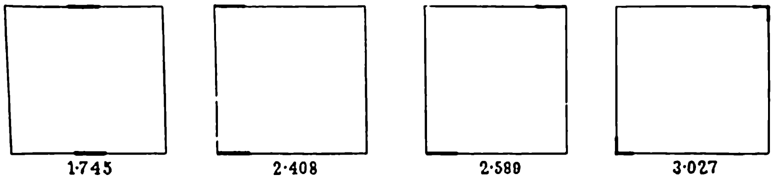

FIG. 10.

Ex. 1. If the extremities P, Q, R, S of the electrodes are given by $z_1 = \frac{1}{2}il'$, $z_2 = x$, $z_3 = l + \frac{1}{2}il'$, $z_4 = x + il'$, then the geometric resistance is unity (Fig. 11).

Here, since $m = K/l = K'/l'$, we have $\alpha = \text{sn}^2 \frac{1}{2}iK' = -1/k$, $\beta = \text{sn}^2 mx$, $\gamma = \text{sn}^2 (K + \frac{1}{2}iK') = 1/k$, $\delta = \text{sn}^2 (mx + iK') = 1/(k^2 \text{sn}^2 mx)$, and therefore, by (22), we find $\lambda^2 = \frac{1}{2}$. Consequently, $L' = L$ and the geometric resistance is unity.

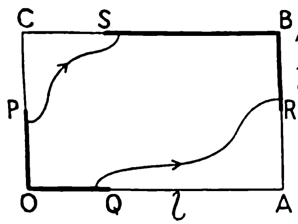

FIG. 11.

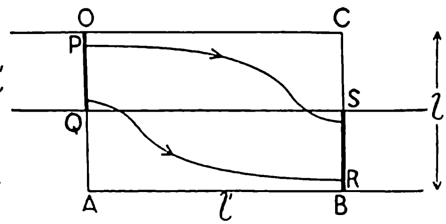

FIG. 12.

Ex. 2. Find the geometric resistance when $z_1 = 0$, $z_2 = \frac{1}{2}l$, $z_3 = l + il'$, $z_4 = \frac{1}{2}l + il'$.

This case has been applied to the problem of finding the resistance of an overlapping joint * (see Fig. 12).

We have now

$$\alpha = 0, \qquad \beta = \text{sn}^2 \tfrac{1}{2}K = 1/(1 + k'),$$
$$\gamma = \text{sn}^2 (K + iK') = 1/k^2, \qquad \delta = \text{sn}^2 (\tfrac{1}{2}K + iK') = 1/(1 - k'),$$

and by (22) we find $\lambda^2 = \frac{1}{2}(1 - k')$, which gives λ when k is known. To find k, we have $K'/K = l'/l$ and hence, by VI, § 8, we can find k for a given value of l'/l. When λ has been calculated the geometric resistance is L'/L. A few approximate values are given in the following table:

* R. M. Wilmotte, *J.I.E.E.*, Vol. 64, 1926, p. 1089.

l'/l .	0	$\frac{1}{2}$	1	2	3	4	∞
L'/L .	1	1·08	1·47	2·44	3·44	4·44	∞

Ex. 3. Let $z_1 = 0$, $z_2 = x$, $z_3 = x + il'$, $z_4 = il'$. Then

$$\alpha = 0, \quad \beta = \text{sn}^2\, mx, \quad \gamma = \text{sn}^2\,(mx + iK') = 1/(k^2\,\text{sn}^2\, mx), \quad \delta = \infty.$$

By (22), with $\delta = \infty$, we find $\lambda^2 = \beta/\gamma = k^2\,\text{sn}^4\, mx$, $\lambda = k\,\text{sn}^2\, mx$. This becomes Moulton's second example (see Fig. 10) when we put

$$l = l', \quad K = K' = 1{\cdot}8541, \quad k^2 = k'^2 = \tfrac{1}{2}, \quad mx = \tfrac{1}{5}K = 0{\cdot}3708.$$

From Legendre's tables (see II, § 10), we then find

$$\text{sn}\,(0{\cdot}3708, \sin 45°) = \sin 21°\,1' = 0{\cdot}3585,$$

from which follows

$$\lambda = (1/\sqrt{2})(0{\cdot}3585)^2 = \sin 5°\,13'$$

and hence the geometric resistance L'/L is found to be 2·408.

§ 9. Source and sink at points on the perimeter of a rectangle. Consider the two transformations

$$t = \text{sn}^2\,(w, k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\chi = -\log\{(t - p)/(t - q)\} \quad . \quad . \quad . \quad (25)$$

The first of these converts the rectangle $OABC$ in the w-plane, with its corners at 0 K, $K + iK'$, and iK', into the upper half t-plane (Fig. 13). The second, if p and q are real, converts the upper half t-plane into an infinite strip of the χ-plane, of width π.

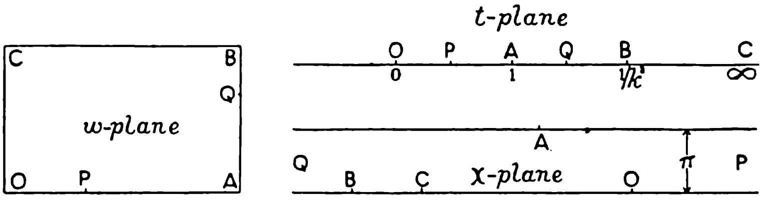

FIG. 13.

Expressed in terms of the steady irrotational flow of liquid in two dimensions, the second transformation converts a uniform stream in the χ-plane, of width π and unit velocity, into the flow in the upper half t-plane due to a source and sink of equal strengths at two points P and Q on the real t-axis. The first transformation then converts this flow into the flow inside a rectangle of sides K, K', caused by a source and sink at points on the perimeter corresponding to P and Q.

§ 10. It should be noted that any constant may be added to χ, because the flow depends only upon the derivative of χ; so that (25) can be replaced by $\chi = -\log\{c(t-p)/(t-q)\}$, where c is a constant.

To obtain the case in which the sink is at infinity, we may suppose that q and c both tend to infinity in such a way that c/q tends to a finite limit c_1. We shall then have, for a source at a finite point p and a sink of equal strength at infinity,

$$\chi = -\log\{c_1(t-p)\}, \text{ or, equally well, } \chi = -\log(t-p),$$

remembering that any constant can be added to χ.

§ 11. When the rectangle $OABC$ has sides of lengths l and l' and lies in the z-plane with its corners at 0, l, $l + il'$, and il', it is only necessary: (i) to put $K/K' = l/l'$ and find k from this equation to ensure that the rectangle in the w-plane will be similar to that in the z-plane; (ii) having found k and thence K and K', to change the scale by putting $z/w = l/K = l'/K'$. Let ρ be the common value of these fractions; then, if there is a source at the point $z = z_1$ and an equal sink at $z = z_2$, the complex velocity potential χ will be given by

$$z = \rho w, \quad t = \text{sn}^2(w, k), \quad \chi = -\log\frac{t-p}{t-q} \qquad (26)$$

if p and q are finite, where

$$p = \text{sn}^2(z_1/\rho, k) \quad q = \text{sn}^2(z_2/\rho, k) \quad . \quad . \qquad (27)$$

If $z_2 = il'$, so that the sink is at infinity in the t-plane, the last of equations (26) must be replaced by $\chi = -\log(t-p)$.

§ 12. Examples. The first of the two examples that follow refers to the rectangle $OABC$ of sides l, l'. The modulus k is given by

$$K/K' = l/l' \qquad (28)$$

while

$$w = z/\rho, \text{ where } \rho = l/K = l'/K' \quad . \quad . \quad . \qquad (29)$$

Ex. 1. Source at $z = a$ $(0 < a < l)$, and a sink of equal strength at $z = a + il'$.

Let $u = a/\rho$. Then $u + iK' = (a + il')/\rho$, and χ is given by

$$\chi = -\log\frac{t-p}{t-q} = -\log\frac{\text{sn}^2 w - \text{sn}^2 u}{\text{sn}^2 w - \text{sn}^2(u + iK')}.$$

Now $\text{sn}(u + iK') = 1/(k \,\text{sn}\, u)$, and hence, after adding a constant and using Examples I (*b*), (4), we have

$$\chi = -\log\frac{\text{sn}^2 w - \text{sn}^2 u}{1 - k^2 \text{sn}^2 w\, \text{sn}^2 u} = -\log\{\text{sn}(w+u)\,\text{sn}(w-u)\} \quad (30)$$

We notice three particular cases:

(i) Source at O and an equal sink at C. Putting $u = 0$, we have

$$\chi = -\log \mathrm{sn}^2 w = -2 \log \mathrm{sn}\, w \qquad (31)$$

(ii) Source at A and an equal sink at B.

By putting $u = K$, and since $\mathrm{sn}\,(w + K) = \mathrm{cn}\, w/\mathrm{dn}\, w$, we have, ignoring an additive constant,

$$\chi = -\log (\mathrm{cn}^2 w/\mathrm{dn}^2 w) = -2 \log (\mathrm{cn}\, w/\mathrm{dn}\, w) \qquad (32)$$

(iii) Sources at O and A, sinks at B and C, all of the same strength. By superposing the flows given by (31) and (32), we see that

$$\chi = -2 \log (\mathrm{sn}\, w\, \mathrm{cn}\, w/\mathrm{dn}\, w) \qquad (33)$$

Ex. 2. Let G be the midpoint of OA, and H the midpoint of BC so that $OG = CH = \frac{1}{2}l$ (Fig. 14). Consider the flow in the rectangle

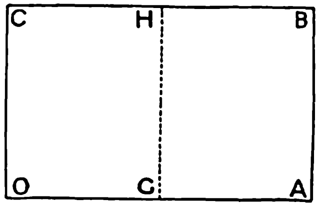

FIG. 14.

$OGHC$ due to a source at O and an equal sink at C. By (31), χ will be given by

$$\chi = -2 \log \mathrm{sn}\,(\zeta, \lambda) \qquad (34)$$

where λ is determined by the equation

$$L/L' = \tfrac{1}{2}l/l' \qquad (35)$$

L and L' being the complete elliptic integrals of the first kind to moduli λ and λ'; while, by (29),

$$\zeta = z/\sigma, \text{ where } \sigma = \tfrac{1}{2}l/L = l'/L' \qquad (36)$$

§ 13. Application to Landen's transformation. Consider (33) and (34). It is evident from physical considerations * that in these two cases the flow in the rectangle $OGHC$ is the same (Fig. 14), and hence that the corresponding expressions for χ can only differ by a constant. Therefore, we can put

$$M\, \mathrm{sn}\,(\zeta, \lambda) = \mathrm{sn}\, w\, \mathrm{cn}\, w/\mathrm{dn}\, w \qquad (37)$$

where M is a constant. Putting $\zeta = z/\sigma = \rho w/\sigma$ and equating coefficients of first powers of w, we get $M\rho/\sigma = 1$ and hence

$$1/M = \rho/\sigma = 2L/K = L'/K' \qquad (38)$$

* Cf. Greenhill, *Elliptic Functions*, p. 305.

Also, at G, $z = \frac{1}{2}l$, $\zeta = L$, $w = \frac{1}{2}K$. Substituting in (37), we get, by Examples I (*b*), 2,

$$M = \operatorname{sn} \tfrac{1}{2}K \operatorname{cn} \tfrac{1}{2}K/\operatorname{dn} \tfrac{1}{2}K = 1/(1 + k') \quad . \quad . \quad (39)$$

Further, at H, $z = \frac{1}{2}l + il'$, $\zeta = L + iL'$, $w = \frac{1}{2}K + iK'$, and hence, after substituting in (37), we find

$$\lambda = (1 - k')/(1 + k') \quad . \qquad (40)$$

Equation (37) now becomes

$$\operatorname{sn} \{(1 + k')w, \quad (1 - k')/(1 + k')\} = (1 + k') \operatorname{sn} w \operatorname{cn} w/\operatorname{dn} w.$$

This is the first of the following three formulæ, known as the formulæ of *Landen's transformation*:

$$\operatorname{sn} \{(1 + k')w, \; (1 - k')/(1 + k')\} = (1 + k') \operatorname{sn} w \operatorname{cn} w/\operatorname{dn} w \quad . \quad (41)$$

$$\operatorname{cn} \{(1 + k')w, \; (1 - k')/(1 + k')\} = \{1 - (1 + k') \operatorname{sn}^2 w\}/\operatorname{dn} w \quad . \quad (42)$$

$$\operatorname{dn} \{(1 + k')w, \; (1 - k')/(1 + k')\} = \{1 - (1 - k') \operatorname{sn}^2 w\}/\operatorname{dn} w \quad . \quad (43)$$

Equation (40), called the *modular equation* of the transformation, may be written in any one of the forms

$$\lambda = \frac{1 - k'}{1 + k'}, \; \lambda' = \frac{2\sqrt{k'}}{1 + k'}, \; k = \frac{2\sqrt{\lambda}}{1 + \lambda}, \; k' = \frac{1 - \lambda}{1 + \lambda} . \quad (44)$$

By (38) and (39), the corresponding complete elliptic integrals are connected by the equations

$$\frac{K}{2L} = \frac{K'}{L'} = \frac{1}{1 + k'} = \frac{1 + \lambda}{2} \quad . \qquad (45)$$

$$K/K' = 2L/L' \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

§ 14. Gauss's transformation. The following three formulæ are called the formulæ of *Gauss's transformation* or *Landen's second transformation*:

$$\operatorname{sn} \{(1 + k)w, \; 2\sqrt{k}/(1 + k)\} = (1 + k) \operatorname{sn} w/(1 + k \operatorname{sn}^2 w) \qquad (47)$$

$$\operatorname{cn} \{(1 + k)w, \; 2\sqrt{k}/(1 + k)\} = \operatorname{cn} w \operatorname{dn} w/(1 + k \operatorname{sn}^2 w) \quad . \qquad (48)$$

$$\operatorname{dn} \{(1 + k)w, \; 2\sqrt{k}/(1 + k)\} = (1 - k \operatorname{sn}^2 w)/(1 + k \operatorname{sn}^2 w) \quad . \quad (49)$$

The corresponding modular equation can be expressed in any one of the forms

$$\gamma = \frac{2\sqrt{k}}{1 + k}, \ \gamma' = \frac{1 - k}{1 + k}, \ k = \frac{1 - \gamma'}{1 + \gamma'}, \ k' = \frac{2\sqrt{\gamma'}}{1 + \gamma'} \quad . \quad (50)$$

The corresponding complete elliptic integrals are connected by the equations

$$\frac{K}{\Gamma} = \frac{K'}{2\Gamma'} = \frac{1}{1 + k} \quad . \quad (51)$$

$$K'/K = 2\Gamma'/\Gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

The elliptic functions on the L.H.S. of Gauss's transformation are expressed in terms of those with a smaller modulus on the R.H.S. In Landen's transformation the reverse was the case.

Gauss's transformation can be obtained from Landen's transformation by replacing w by iw (see also Examples VII, 9).

§ **15.** If we replace K by $K(k)$ and put $L = K(\lambda)$ in (45) and $\Gamma = K(\gamma)$ in (51), we obtain

$$2K(\lambda)/(1 + k') = K(k) = K(\gamma)/(1 + k) \quad . \quad (53)$$

where $\lambda = (1 - k')/(1 + k') < k < \gamma = 2\sqrt{k}/(1 + k)$; and the line of equalities (53) may evidently be continued at either end, the moduli of the complete elliptic integrals increasing from left to right.

§ **16. Transformation in general.** What is called the general theory of the transformation of elliptic functions is concerned with the problem: Given an elliptic differential in which x is the variable, to transform it into another elliptic differential in which y is the variable, by means of an *algebraic* relation between x and y. When the algebraic relation is of the form $y = U(x)/V(x)$, where $U(x)$ and $V(x)$ are polynomials, one of degree n, the other of degree n or $n - 1$, then the transformation is said to be of order n. For example, from (42) and (43) we have

$$\frac{\text{cn}\,(w/M, \lambda)}{\text{dn}\,(w/M, \lambda)} = \frac{1 - (1 + k')\,\text{sn}^2\,w}{1 - (1 - k')\,\text{sn}^2\,w}$$

that is,

$$y = \frac{1 - (1 + k')x^2}{1 - (1 - k')x^2} \quad (54)$$

where

$$y = \frac{\text{cn}\,(w/M, \lambda)}{\text{dn}\,(w/M, \lambda)} = \text{sn}\left(\frac{w}{M} + L, \lambda\right)$$

$$x = \text{sn}\,(w, k), \ M = 1/(1 + k'), \ \lambda = (1 - k')/(1 + k')$$

and hence

$$\frac{dw}{M} = \frac{dy}{\{(1-y^2)(1-\lambda^2y^2)\}^{\frac{1}{2}}}, \quad dw = \frac{dx}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}}$$

Consequently, the algebraic relation (54) leads to

$$\frac{dy}{\{(1-y^2)(1-\lambda^2y^2)\}^{\frac{1}{2}}} = \frac{(1+k')dx}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}}, \quad \left(\lambda = \frac{1-k'}{1+k'}\right)$$

a transformation of the second order.

The theory of transformation can be discussed algebraically for small values of n, but the general problem requires more advanced methods.*

EXAMPLES VII

1. Check the numerical results given in Fig. 10 for Moulton's first, third, and fourth examples.

2. Show that

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1-\frac{3}{4}\sin^2\phi)}} = \tfrac{4}{3}K(\tfrac{1}{3})$$

3. Show that

$$K(\tan^2 \tfrac{1}{2}a) = \cos^2 \tfrac{1}{2}a \; K(\sin a)$$

4. Show that

$$K(\tfrac{1}{9}) = \tfrac{9}{10}K(\tfrac{3}{5}) = \tfrac{9}{10}K(\sqrt{15}/4)$$

5. Show that

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{\{1-(\sqrt{2}-1)^2\sin^2\phi\}}} = \frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}\pi} \frac{d\psi}{\sqrt{\{1-2(\sqrt{2}-1)\sin^2\psi\}}}$$

6. Write out the formulæ of Landen's transformation for $k = \sqrt{3}/2$.

7. By putting $L = K'$, $L' = K$ in (46), and correspondingly $\lambda = k'$, $\lambda' = k$ in (44), show that $k = \sqrt{2} - 1 = \tan \frac{1}{8}\pi$ when $K/K' = 1/\sqrt{2}$.

8. By using (46) and (44), show that

(i) $k' = (\sqrt{2}-1)^2 = \tan^2 \frac{1}{8}\pi$ when $K/K' = 2$,
(ii) $k' = \{(\sqrt[4]{2}-1)/(\sqrt[4]{2}+1)\}^2$ when $K/K' = 4$,
(iii) $k' = \{\sqrt{2}+1-\sqrt{(2\sqrt{2}+2)}\}^2$ when $K/K' = 2\sqrt{2}$.

9. Obtain the formulæ of Gauss's transformation by considerations similar to those of § 13, dividing the rectangle $OABC$ into two halves by joining the mid points of the sides OC, AB.

* See, e.g., the books by Cayley and Greenhill, and especially Fricke, *Die Elliptischen Funktionen*, Vol. II.

10. Cubic transformation. Let $OABC$ be the rectangle in the z-plane with its corners at $z = 0$, $z = l$, $z = l + il'$, $z = il'$. Let A', A'' be the two points $z = \frac{1}{3}l$, $z = \frac{2}{3}l$, and B', B'' the two points $z = \frac{1}{3}l + il'$, $z = \frac{2}{3}l + il'$, respectively.

Write down the complex potential for the flow inside the rectangle $OABC$ due to a unit source at O, a unit sink at C, a double source at A'', and a double sink at B''. Also write down the complex potential for the flow inside the rectangle $OA'B'C$ due to a unit source at O and

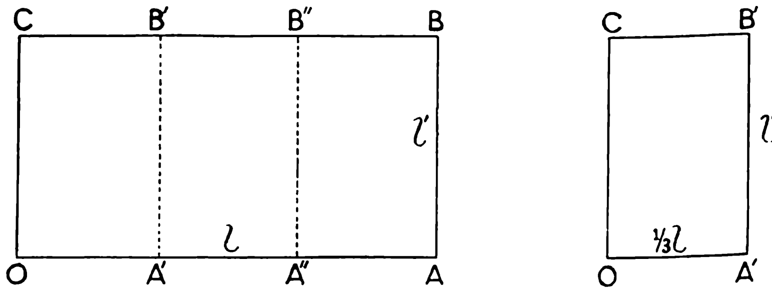

FIG. 15.

a unit sink at C, the side $A'B'$ being now part of the rigid boundary (Fig. 15). By considering that these two flows are identical in the rectangle $OA'B'C$, show that

$$y = x\frac{s(2 - s) - x^2}{s^2 - (2s - 1)x^2}$$

where

$$x = \operatorname{sn}(w, k), \qquad y = \operatorname{sn}(w/M, \lambda)$$
$$s = \operatorname{sn}\tfrac{1}{3}K, \qquad 1/M = 3L/K = L'/K'$$
$$k^2 = \frac{2s - 1}{s^3(2 - s)}, \qquad k'^2 = \frac{(1 - s)^3(1 + s)}{s^3(2 - s)}$$
$$\lambda^2 = \frac{(2s - 1)^3}{s(2 - s)^3}, \qquad \lambda'^2 = \frac{(1 - s)(1 + s)^3}{s(2 - s)^3}$$

Deduce the *modular equation* of the third order $\sqrt{(k\lambda)} + \sqrt{(k'\lambda')} = 1$.

(i) If $K/K' = \sqrt{3}$, show that $L'/L = \sqrt{3}$, and hence that $k = \lambda'$. Deduce that $k = \sin 75°$, $s = \sqrt{3} - 1$.

(ii) If $K/K' = 3$, show that $L = L'$, and hence that $\lambda = \lambda' = 1/\sqrt{2}$ and that $2kk' = (2 \sin 15°)^4$.

(iii) If $K/K' = \sqrt{6}$, prove that $k' = (\sqrt{3} - \sqrt{2})(2 - \sqrt{3})$.

CHAPTER VIII

CONFORMAL REPRESENTATION (*cont.*)

§ 1. Elliptic integrals of the second kind. The elliptic integrals of the second kind, $E(w)$ and $Z(w)$, were introduced in the second chapter with real values of w. In particular, we saw that $Z(w)$ is an odd function and periodic with period $2K$. We return to these integrals to consider, first, their values when w is complex, and secondly, a few examples of conformal representation in which they appear. By definition and IV (10), we have

$$E(iv, k) = \int_0^{iv} \mathrm{dn}^2\, u\, du = i \int_0^{v} \frac{\mathrm{dn}^2\,(t, k')}{\mathrm{cn}^2\,(t, k')}\, dt \quad (u = it)$$

and hence, by carrying out the integration (Examples II (*a*), 1, xiv),

$$E(iv, k) = i\{v - E(v) + \mathrm{sn}\, v\, \mathrm{dn}\, v/\mathrm{cn}\, v\} \quad (\mathrm{mod}\ k') \ . \qquad (1)$$

Again, since $\quad Z(w) = E(w) - (E/K)w$

we have $\quad Z(iv, k) = E(iv, k) - (E/K)iv$

and $\quad Z(v, k') = E(v, k') - (E'/K')v$

and by multiplying the second of these by i and adding, we get

$$Z(iv, k) + i\,Z(v, k') = E(iv, k) + i\,E(v, k') - iv(EK' + E'K)/KK'$$

and hence, by (1) and Legendre's formula (Examples II (*b*), 3),

$$Z(iv, k) = -\,i\left\{Z(v) + \frac{\pi v}{2KK'} - \frac{\mathrm{sn}\, v\, \mathrm{dn}\, v}{\mathrm{cn}\, v}\right\} \quad (\mathrm{mod}\ k'). \qquad (2)$$

By putting iv for u in II, (26), we can now express $E(iv + K)$ in a similar way.

By putting $v + K'$ for v in (1), we find $E(iv + iK')$, and thence $E(u + iK')$ by putting u for iv.

By putting $u + K$ for u, we then find also $E(u + K + iK')$, and hence $E(iv + K + iK')$ by putting iv for u.

In this way are found the entries relating to the function $E(u)$ in Tables II and III at the end of this book; and in a similar way can be found the entries relating to $Z(u)$. To obtain some of the entries in the forms given, it will be necessary to use Legendre's formula (p. 25), and some may be more readily obtained by using the addition formulæ of II, §§ 7, 8.

§ 2. Conformal representation by elliptic integrals of the second kind. Consider the transformation

$$z = \int_0^t \frac{(a - t)dt}{\{4t(1 - t)(1 - k^2 t)\}^{\frac{1}{2}}}$$

$$= \int_0^u \frac{(a - u^2)\,du}{\{(1 - u^2)(1 - k^2u^2)\}^{\frac{1}{2}}} \quad (t = u^2) \ . \quad (3)$$

$$= \int_0^\chi (a - \text{sn}^2\,\chi)d\chi \qquad (u = \text{sn}\,\chi) \ . \quad (4)$$

The figures in the t, u, and χ planes are shown in Fig. 1. The figure in the z plane depends upon the value of a. The change in

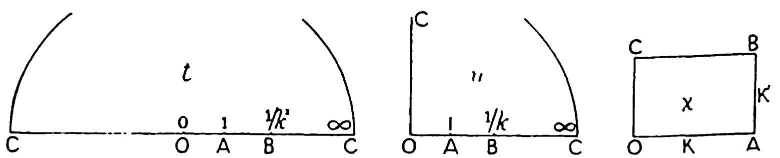

FIG. 1.

shape of this figure as a varies from $-\infty$ to $+\infty$ is indicated in Fig. 2. In the various cases shown there, except those in which $a = 0$, $a = 1$, and $a = 1/k^2$, the direction of motion of the point z turns through an angle $\frac{1}{2}\pi$ to the left as the point t, moving from left to right along its own real axis, passes each of the points $t = 0$, $t = 1$, $t = 1/k^2$. As the point t passes $t = a$, the direction of motion of the point z turns through an angle π to the right, i.e., the direction of motion of the point z is completely reversed: this is in accordance with the rule at the end of VI, § 4, when $\alpha = -1$. When $a = 0$, $a = 1$, or $a = 1/k^2$, the direction of motion of the point z turns through an angle $\frac{1}{2}\pi$ to the right at the corresponding point $t = 0$, $t = 1$, or $t = 1/k^2$.

Further, in the notation of VI, § 4, we have here $\sigma = \frac{1}{2}$ (except when $a = \infty$). Consequently, when t is large,

$$z \doteqdot H \int t^{-\frac{1}{2}} dt = 2Ht^{\frac{1}{2}}$$

approximately, where H is a constant. It follows that, when the point t describes the infinite semicircle in the t plane, the point z describes an infinite quadrant of a circle (the quadrant is not drawn in the diagrams of Fig. 2).

§ 3. Particular cases of the above transformation are, after multiplication by a constant factor k^2,

$$a = 0, \qquad z = -k^2 \int_0^\chi \operatorname{sn}^2 \chi \, d\chi = E(\chi) - \chi \quad . \quad . \quad (5)$$

$$a = 1, \qquad z = k^2 \int_0^\chi \operatorname{cn}^2 \chi \, d\chi = E(\chi) - k'^2\chi \quad . \quad (6)$$

$$a = 1/k^2, \quad z = \int_0^\chi \operatorname{dn}^2 \chi \, d\chi = E(\chi). \quad (7)$$

Equation (6) was applied by Bickley to the electric field outside a charged rectangular conductor.*

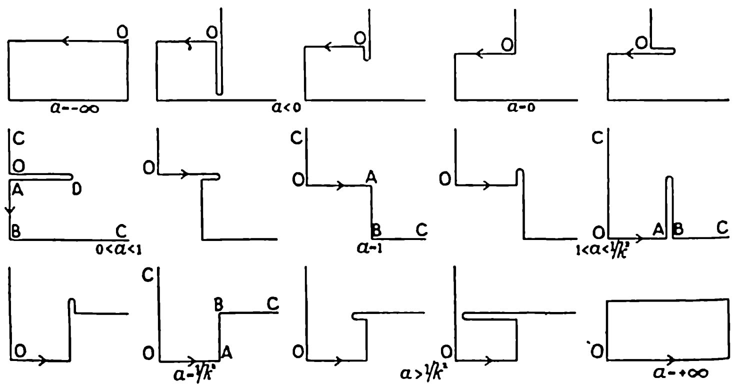

FIG. 2.

Equations (5) and (7) give equivalent representations, as we see from the fourth and twelfth diagrams in Fig. 2. Considering (7) in more detail, we see for example that in this case

at A, $\chi = K$, $\quad z = E$
at B, $\chi = K + iK'$, $\quad z = E + i(K' - E')$

and hence that, in the twelfth diagram,

$$AB/OA = (K' - E')/E \quad . \quad . \quad . \quad . \quad (8)$$

The ratio AB/OA being given in the z-plane, (8) is an equation that determines the value of k in order that (7) may represent the rectangle in the χ-plane upon a region similar to that in the z-plane.

* W. G. Bickley, *Proc. London Math. Soc.*, 1934, **37** (2), 82.

It may be noticed that to the line $\chi = \frac{1}{2}K + iv$ corresponds a curve along which $|d\chi/dz|$ is constant; for, at any point on this curve,

$$|d\chi/dz| = |dz/d\chi|^{-1} = |\mathrm{dn}^2\,(\tfrac{1}{2}K + iv)|^{-1} = 1/k'$$

by Examples IV, 5 (ii).

§ 4. The parallel plate condenser in two dimensions. The sixth and tenth diagrams in Fig. 2 are effectively the same from our present point of view. Either of them can be applied to the problem of the parallel plate condenser in two dimensions,* by two reflexions, one in the real axis, the other in the imaginary axis. If we replace (4) by the equivalent equation

$$z = Z(\chi) + \alpha\chi \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where α is a constant and Z denotes the Jacobian Zeta function, then the sixth diagram is given by

$$\alpha = 0, \quad z = Z(\chi) \qquad (10)$$

and the tenth by

$$\alpha = \frac{\pi}{2KK'}, \quad z = Z(\chi) + \frac{\pi\chi}{2KK'} \quad . \qquad (11)$$

Considering (10) further, we see that on the sixth diagram OD represents the length of half of one of the plates of the condenser, and that AB represents half the distance between the plates. Now, at D, $Z(\chi)$ has a maximum value Z_m, say, which can be interpolated from Milne-Thomson's tables of the Zeta function when k is known. At A, $\chi = K$, $Z(\chi) = 0$; while at B $\chi = K + iK'$, $Z(\chi) = -\pi i/2K$. Consequently,

$$AB/OD = \pi/(2KZ_m) \quad . \quad . \quad . \quad . \quad (12)$$

The ratio AB/OD being given, this is an equation which determines k. It can be solved by approximate methods.

§ 5. Jacobi's elliptic integral of the third kind. By II, (5) and (7), in Legendre's notation,

$$\begin{aligned} \Pi(k, n, \phi) &= \int_0^\phi \frac{d\phi}{(1 + n\sin^2\phi)(1 - k^2\sin^2\phi)^{\frac{1}{2}}} \\ &= F(\phi, k) - \int_0^\phi \frac{n\sin^2\phi\, d\phi}{(1 + n\sin^2\phi)(1 - k^2\sin^2\phi)^{\frac{1}{2}}} \\ &= u - \int_0^u \frac{n\,\mathrm{sn}^2 u}{1 + n\,\mathrm{sn}^2 u}\,du \end{aligned}$$

* A. E. H. Love, *Proc. London Math. Soc.*, 1924, **22** (2), 337; F. Bowman, *ibid.*, 1935, **39** (2), 205.

where $\operatorname{sn}(u, k) = \sin \phi$. In the Jacobian notation, it is usual to put $n = -k^2 \operatorname{sn}^2 a$, and then to denote the standard integral of the third kind by $\Pi(u, a)$ and to define it by the equation

$$\Pi(u, a) = \int_0^u \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} du \quad . \quad . \quad (13)$$

§ 6. **Interchange of argument and parameter.** The function $\Pi(u, a)$ satisfies the relation

$$\Pi(u, a) - \Pi(a, u) = u\, Z(a) - a\, Z(u) \quad . \quad . \quad (14)$$

called the formula of *interchange of argument and parameter.*

Proof. By II, (29), and since $Z(u)$ is odd, we have

$$Z(u + v) = Z(u) + Z(v) - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u + v)$$
$$Z(u - v) = Z(u) - Z(v) + k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u - v)$$

By subtraction,

$$\begin{aligned} Z(u + v) - Z(u - v) &= 2Z(v) - k^2 \operatorname{sn} u \operatorname{sn} v\{\operatorname{sn}(u + v) + \operatorname{sn}(u - v)\} \\ &= 2Z(v) - 2k^2 \operatorname{sn}^2 u \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v \\ &= 2Z(v) - 2(\partial/\partial u)\Pi(u, v) \end{aligned}$$

By differentiation with respect to v,

$$\operatorname{dn}^2(u + v) + \operatorname{dn}^2(u - v) - 2\operatorname{dn}^2 v = -2(\partial^2/\partial u \partial v)\Pi(u, v)$$

By interchanging u and v, subtracting, and dividing by 2,

$$(\partial^2/\partial u \partial v)\{\Pi(u, v) - \Pi(v, u)\} = \operatorname{dn}^2 v - \operatorname{dn}^2 u$$

By integration with respect to u,

$$(\partial/\partial v)\{\Pi(u, v) - \Pi(v, u)\} = u \operatorname{dn}^2 v - E(u) + V$$

where V is independent of u. Putting $u = 0$, we see that $V = 0$, since $\Pi(0, v) \equiv 0$ and $\Pi(v, 0) \equiv 0$. By a further integration, with respect to v,

$$\Pi(u, v) - \Pi(v, u) = uE(v) - vE(u) + U$$

where U is independent of v. Putting $v = 0$ shows that $U = 0$. Hence, replacing v by a, we have

$$\Pi(u, a) - \Pi(a, u) = uE(a) - aE(u)$$

which is equivalent to (14).

§ 7. **Conformal representation by the Jacobian elliptic integral of the third kind.** Let $\nu = 1/(k^2 \operatorname{sn}^2 a)$, and put

$$z = \Pi(u, a) = \int_0^u \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} du \quad . \quad (15)$$

$$= \frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a} \int_0^u \frac{\operatorname{sn}^2 u\, du}{\nu - \operatorname{sn}^2 u} \quad . \quad . \quad . \quad (16)$$

Putting $t = \operatorname{sn}^2 u$, we have also

$$z = \frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a} \int_0^t \frac{t\,dt}{(\nu - t)\{4t(1-t)(1-k^2t)\}^{\frac{1}{2}}} \tag{17}$$

Suppose first that $0 < a < K$, so that $\operatorname{sn} a$ is real and $\nu > 1/k^2 > 1$. Then equation (16) represents a rectangle in the u-plane upon a region of the z-plane, and (17) gives the representation on an intermediate half t-plane. The figures in the u, t, and z planes are shown in Fig. 3.

Let the point t move along its real axis from left to right, starting at $t = 0$. Between $t = 0$ and $t = 1$, dz is real and positive, and so the corresponding point z describes a piece of its real axis. At the point A, and again at B, the direction of motion of the point z turns through an angle $\frac{1}{2}\pi$ to the left. Between B and N, the point z is moving in the negative direction

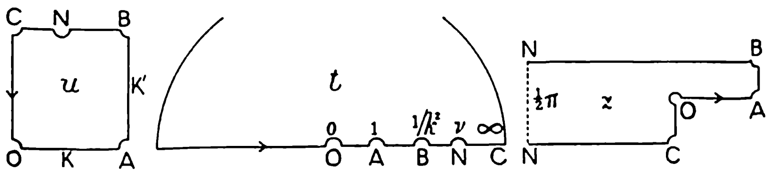

FIG. 3.

of the real axis (dz is real and negative). As $t \to \nu$ the point z behaves in a way which has not appeared in any previous example, for the integral in (17) tends to infinity, and the point z tends to an infinite distance along the line BN in the z-plane.

At the point $t = \nu = 1/(k^2 \operatorname{sn}^2 a)$, we have $(4t)^{\frac{1}{2}} = 2/(k \operatorname{sn} a)$; $(1-t)^{\frac{1}{2}} = (t-1)^{\frac{1}{2}}e^{-\frac{1}{2}\pi i} = -i(t-1)^{\frac{1}{2}} = -i \operatorname{dn} a/(k \operatorname{sn} a)$, by VI, (6); $(1-k^2t)^{\frac{1}{2}} = (k^2t-1)^{\frac{1}{2}}e^{-\frac{1}{2}\pi i} = -i(k^2t-1)^{\frac{1}{2}} = -i \operatorname{cn} a/\operatorname{sn} a$, by VI, (6). Consequently, at this point the residue of the integrand in (17), including the constant factor $\operatorname{cn} a \operatorname{dn} a/\operatorname{sn} a$ before the integral sign, is

$$\frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a} \cdot \frac{1/(k^2 \operatorname{sn}^2 a)}{-(2/k \operatorname{sn} a)(-i \operatorname{dn} a/k \operatorname{sn} a)(-i \operatorname{cn} a/\operatorname{dn} a)} = \frac{1}{2}$$

By a fundamental result in contour integration, the value of the integral (17), taken in the positive sense round the circumference of a small circle surrounding the point $t = \nu$, is therefore $2\pi i \times \frac{1}{2} = \pi i$; and when the point t describes the small semicircle round the point $t = \nu$ in the negative sense, the value of the integral will be $-\frac{1}{2}\pi i$. Corresponding to this small semicircle in the t-plane, the point z will therefore move through a distance $\frac{1}{2}\pi$ in the negative direction of the imaginary axis.

Beyond $t = \nu$, we have $\nu < t < \infty$, and

$$dz = \frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a} \frac{t\,dt}{-(t-\nu)(2t^{\frac{1}{2}})\{-i(t-1)^{\frac{1}{2}}\}\{-i(k^2t-1)^{\frac{1}{2}}\}}$$

which shows that dz is real and positive, so that the direction of motion of the point z has been completely reversed.

Thus, corresponding to the factor $\nu - t$ in the denominator of the integrand, the point z goes to infinity along a certain straight line. Then, at right angles to its direction of motion, it "jumps" a distance $\frac{1}{2}\pi$ to the left, reverses its direction of motion, and returns along a parallel line.

When t is large, the integral behaves like

$$H\int t^{-\frac{3}{2}}\,dt = -2Ht^{-\frac{1}{2}}$$

where H is a constant; from which it follows that while t describes an infinite semicircle in the positive sense, the point z describes an infinitesimal quadrant in the negative sense. Finally, while t completes its circuit by moving from $-\infty$ to 0, the point z completes the figure in the z-plane as shown.

§ 8. The shape of the figure in the *z*-plane. We first notice the three formulæ

$$\Pi(K, a) = KZ(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\Pi(K+iK', a) = KZ(a) + i\{K'Z(a) + \pi a/(2K)\}. \quad (19)$$

$$\Pi(iK', a) = -i\{\tfrac{1}{2}\pi - K'Z(a) - \pi a/(2K)\} \quad . \quad (20)$$

The first of these follows by putting $u = K$ in (14), since $\operatorname{cn} K = 0$ and $Z(K) = 0$. The second follows by putting $u = K + iK'$ in (14), since $\operatorname{dn}(K+iK') = 0$ and

$$Z(K+iK') = -\pi i/(2K).$$

Since $u = K$ at A, we have therefore in the z plane

$$OA = KZ(a) \quad . \quad . \quad (21)$$

and since $u = K + iK'$ at B,

$$AB = K'Z(a) + \pi a/(2K) \quad . \quad . \quad . \quad . \quad (22)$$

Also, from the conformal representation, we have

$$CO + AB = \tfrac{1}{2}\pi$$

and therefore

$$CO = \tfrac{1}{2}\pi - AB = \tfrac{1}{2}\pi - K'Z(a) - \pi a/(2K) \quad . \quad (23)$$

from which also follows (20), since $u = iK'$ at C.

When a and k are given, equations (21), (22), (23) determine the lengths of CO, OA, AB such that $CO + AB = \frac{1}{2}\pi$.

In practice, however, the figure in the z-plane would usually be given, with $CO + AB \neq \frac{1}{2}\pi$, and it would be necessary to find the values of k and a by some method of approximation. We could, for instance, proceed as follows:

In the z-plane let $\alpha = OA$, $\beta = AB$, $\gamma = CO + AB$, and put $r = \alpha/\gamma$, $r' = \beta/\gamma$. Then, by (21), (22), (23),

$$r = KZ(a)/(\tfrac{1}{2}\pi) \qquad (24)$$

$$r' = \{K'Z(a) + \tfrac{1}{2}\pi a/K\}/(\tfrac{1}{2}\pi) \qquad (25)$$

which can be replaced by

$$\pi r = 2KZ(a) \qquad (26)$$

$$a = Kr' - K'r \qquad (27)$$

Given r and r', we can now plot a as a function of k from (27). Then we can plot r as a function of k from (26). By interpolation we can then find the value of k for the given value of r, and calculate the corresponding value of a. With these values of k and a, the transformation $z = \Pi(u, a)$ will lead to a figure similar to the given one in the z-plane, but with $CO + AB = \frac{1}{2}\pi$. Its dimensions must therefore be changed in the ratio $\gamma/(\frac{1}{2}\pi)$. The final transformation will thus be $z = (2\gamma/\pi)\Pi(u, a)$.

§ 9. **Parameter imaginary.** If a is replaced by ia in (15), (16) (17), we get

$$z = \Pi(u, ia) = \int_0^u \frac{k^2 \operatorname{sn} ia \operatorname{cn} ia \operatorname{dn} ia \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 ia \operatorname{sn}^2 u}\, du \qquad (28)$$

$$= \frac{i \operatorname{dn}(a, k')}{\operatorname{sn}(a, k') \operatorname{cn}(a, k')} \int_0^u \frac{\operatorname{sn}^2 u \, du}{\nu + \operatorname{sn}^2 u} \qquad (29)$$

$$= \frac{i \operatorname{dn}(a, k')}{\operatorname{sn}(a, k') \operatorname{cn}(a, k')} \int_0^t \frac{t\, dt}{(\nu + t)\{4t(1 - t)(1 - k^2 t)\}^{\frac{1}{2}}} \qquad (30)$$

where $t = \operatorname{sn}^2 u$, $\nu = \operatorname{cn}^2(a, k')/\{k^2 \operatorname{sn}^2(a, k')\}$, and we suppose now that $0 < a < K'$. The figures in the u, t, and z planes are shown in Fig. 4. It may be verified, as in § 7, that the residue

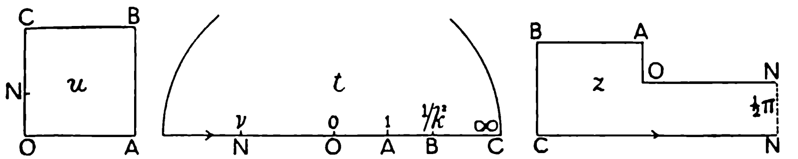

FIG. 4.

of the integral (30) at $t = -\nu$ is $-\frac{1}{2}$, and hence that the "jump" in the z-plane at $z = \infty$ is $\frac{1}{2}\pi$. The values of z at the points A, B, C are given respectively by

$$\Pi(K, ia) = KZ(ia) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\Pi(K + iK', ia) = KZ(ia) + i\{K'Z(ia) + \pi ai/(2K)\} \quad (32)$$

$$\Pi(iK', ia) = -\tfrac{1}{2}\pi i + i\{K'Z(ia) + \pi ai/(2K)\} \quad (33)$$

where $Z(ia)$ is a pure imaginary, given by (2). From these formulæ the lengths of OA, AB, BC can be found when required.

§ 10. Two examples of the application of conformal representation by the elliptic integral of the third kind will be mentioned here.

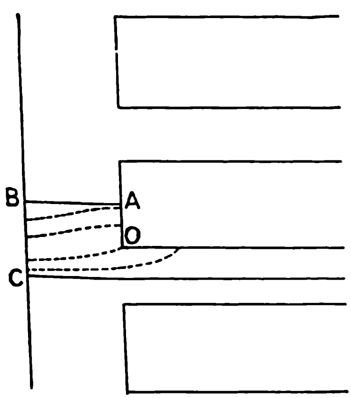

FIG. 5.

I. In the *Journal of the Institution of Electrical Engineers*, 1926, **64**, 1115, F. W. Carter discussed the magnetic field of the dynamo-electric machine. (See also J. D. Cockcroft, *J.I.E.E.*, 1928, **66**, 385.) In particular, in the case of a number of equal teeth separated by equal slots, with all the teeth at the same potential (Fig. 5), he showed how to calculate the reluctance of the magnetic field per half tooth. [The *reluctance* of a magnetic field corresponds to the equivalent resistance of a field of electric current of the same shape; in fact, the definition of "geometric resistance" in VII, § 4, would apply also to "geometric reluctance."]

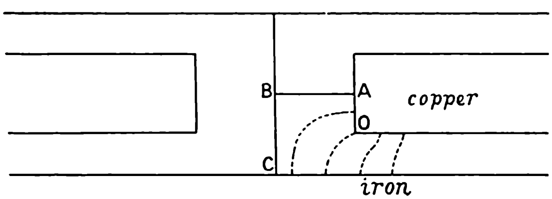

FIG. 6.

II. In the *A.S.E.A. Journal*, 1929, **6** (6), H. Lundkvist discussed the relation between the temperature read by a thermocouple between the conductors and the true copper temperature, in a slot containing two conductors. Mathematically stated, the problem would be as follows, with reference to Fig. 6: Given the temperature difference between B and C, measured by a thermocouple, to find the temperature difference between A and C.

EXAMPLES VIII

1. Show that

$$\begin{aligned} E(u + 2iK') &= E(u) + 2i(K' - E') \\ Z(u + 2iK') &= Z(u) - \pi i/K \\ E(u + 2K + 2iK') &= E(u) + 2E + 2i(K' - E') \\ Z(u + 2K + 2iK') &= Z(u) - \pi i/K \end{aligned}$$

2. Prove that

$$E(u + iv) = E(u) + \frac{k^2 scds_1^2}{1 - s_1^2 d^2} + i\left\{v - E(v, k') + \frac{s_1 c_1 d_1 d^2}{1 - s_1^2 d^2}\right\}$$

$$Z(u + iv) = Z(u) + \frac{k^2 scds_1^2}{1 - s_1^2 d^2} - i\left\{Z(v, k') + \frac{\pi v}{2KK'} - \frac{s_1 c_1 d_1 d^2}{1 - s_1^2 d^2}\right\}$$

where $s = \operatorname{sn}(u, k)$, $s_1 = \operatorname{sn}(v, k')$, etc.

3. In the case $a = 1$ in Fig. 2, show that

$$OA/AB = (E - k'^2 K)/(E' - k^2 K')$$

If $k^2 = \frac{1}{2}$, show that the geometric resistance is unity when OA and BC are regarded as electrodes.

4. In the case $a = 1/k^2$ in Fig. 2, discussed in § 3, show that the co-ordinates x and y of any point on the curve that corresponds to $\chi = \frac{1}{2}K + iv$ are given in terms of the parameter v by

$$x + iy = \frac{E}{2} + \frac{1 - k'}{2}\frac{1 + k's^2}{1 - k's^2} + i\left\{v - E(v, k') + \frac{k'scd}{1 - k's^2}\right\}$$

where $s = \operatorname{sn}(v, k')$, $c = \operatorname{cn}(v, k')$, $d = \operatorname{dn}(v, k')$.

5. The transformation $z = \Pi(u, a)$, with $0 < a < K$, was considered in § 7; and $z = \Pi(u, ia)$, with $0 < a < K'$, in § 9. Show that no essentially new diagram is obtained in the z-plane when K or $K + iK'$ or iK' is added to a in the first case, or to ia in the second case.

6. Consider the variation in shape of the representation of the upper half t plane upon the z plane by the transformation

$$z = \int_0^t \frac{\mu - t}{\nu - t}\frac{dt}{\{4t(1 - t)(1 - k^2 t)\}^{\frac{1}{2}}} \quad . \quad . \quad . \quad (34)$$

when μ varies from $-\infty$ to $+\infty$ while ν is fixed, in the four cases: (i) $0 < \nu < 1$, (ii) $1 < \nu < 1/k^2$, (iii) $1/k^2 < \nu < \infty$, (iv) $-\infty < \nu < 0$.

In each case consider the limiting sequence of shapes when $k \rightarrow 0$ and when $k \rightarrow 1$.

[If $t = \operatorname{sn}^2 u$, the transformation (34) is of the form $z = Au + B\Pi(u, a)$, where A, B are constants.]

CHAPTER IX

REDUCTION TO THE STANDARD FORM

§ 1. In the second chapter the reduction of the integral

$$\int dx/\sqrt{X} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where X denotes a quartic in x, to Legendre's standard form, was deferred. Methods of effecting this reduction will now be considered.

We suppose the coefficients in X to be all real. Then, by the theory of equations, the roots of the equation $X = 0$ will be (i) all real, (ii) two real and the other two a pair of conjugate complex numbers, or (iii) two pairs of conjugate complex numbers. The quartic X can therefore be expressed as the product of two quadratics with real coefficients: thus,

$$X = X_1 X_2 = (ax^2 + 2bx + c)(a'x^2 + 2b'x + c') \quad . \quad (2)$$

where a, b, c, a', b', c' are real.

The case in which X is a cubic can be included by supposing one root to be infinite.

§ 2. Cayley's method of reduction. In the integral (1) make the substitution

$$x = \frac{\lambda t + \mu}{t + 1}, \quad dx = \frac{(\lambda - \mu)dt}{(t + 1)^2} \quad . \quad . \quad (3)$$

then

$$X_1 = \{a(\lambda t + \mu)^2 + 2b(\lambda t + \mu)(t + 1) + c(t + 1)^2\}/(t + 1)^2 \quad (4)$$

$$X_2 = \{a'(\lambda t + \mu)^2 + 2b'(\lambda t + \mu)(t + 1) + c'(t + 1)^2\}/(t + 1)^2 \quad (5)$$

We can now choose λ and μ so that the coefficients of the first powers of t in the numerators of X_1 and X_2 will vanish. This requires

$$a\lambda\mu + b(\lambda + \mu) + c = 0 \quad . \quad . \quad (6)$$

$$a'\lambda\mu + b'(\lambda + \mu) + c' = 0 \quad . \quad . \quad (7)$$

and hence

$$\frac{\lambda\mu}{bc' - b'c} = \frac{\lambda + \mu}{ca' - c'a} = \frac{1}{ab' - a'b} \quad . \quad (8)$$

and so λ and μ must be the roots of the quadratic equation

$$(ab' - a'b)\theta^2 - (ca' - c'a)\theta + bc' - b'c = 0 \quad . \quad (9)$$

Except in one case, this equation will have real roots. To prove this, it will be sufficient to show that $(\lambda - \mu)^2 > 0$; for then $\lambda - \mu$ will be real, and since $\lambda + \mu$ is real from (8), it will follow that λ, μ will both be real. Now let

$$X_1 = a(x-\alpha)(x-\beta), \quad X_2 = a'(x-\gamma)(x-\delta) \quad . \quad (10)$$

then we can replace (8) by

$$\frac{\lambda\mu}{\alpha\beta(\gamma+\delta)-\gamma\delta(\alpha+\beta)} = \frac{\lambda+\mu}{2(\alpha\beta-\gamma\delta)} = \frac{1}{\alpha+\beta-\gamma-\delta} \quad (11)$$

and hence we find

$$\frac{(\lambda-\mu)^2}{4} = \frac{(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)}{(\alpha+\beta-\gamma-\delta)^2} \quad . \quad (12)$$

from which it may at once be verified that $(\lambda-\mu)^2 > 0$ when α, β are a conjugate pair, or when α, β and γ, δ are both conjugate pairs, or when α, β, γ, δ are all real except in the one case in which the pairs of real roots α, β and γ, δ interlace, i.e., one of each pair lies between the other pair. In this exceptional case, the four roots may be divided into two other pairs which do not interlace, and will then lead to real values of λ and μ.

We may therefore suppose that real values of λ and μ have been found such that the substitution (3) will result in equations of the form

$$X_1 = (pt^2+q)/(t+1)^2, \quad X_2 = (p't^2+q')/(t+1)^2$$

and hence

$$\frac{dx}{\sqrt{X}} = \frac{dx}{\sqrt{(X_1X_2)}} = \frac{(\lambda-\mu)dt}{(t+1)^2} \Big/ \frac{(pt^2+q)^{\frac{1}{2}}(p't^2+q')^{\frac{1}{2}}}{(t+1)^2}$$

Then, according to the signs of the real coefficients p, q, p', q', it will remain to reduce to the standard form differentials of the following six types, in which, of course, a and b no longer have the same meanings as in (2):

$$du_1 = \{(a^2-t^2)(b^2-t^2)\}^{-\frac{1}{2}}dt \quad (t^2 < a^2 < b^2) \quad . \quad . \quad \text{I}$$

$$du_2 = \{(t^2-a^2)(t^2-b^2)\}^{-\frac{1}{2}}dt \quad (a^2 < b^2 < t^2) \quad . \quad . \quad \text{II}$$

$$du_3 = \{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}dt \quad (a^2 < t^2 < b^2) \quad . \quad . \quad \text{III}$$

$$du_4 = \{(a^2-t^2)(b^2+t^2)\}^{-\frac{1}{2}}dt \quad (t^2 < a^2) \quad . \quad . \quad . \quad \text{IV}$$

$$du_5 = \{(t^2-a^2)(b^2+t^2)\}^{-\frac{1}{2}}dt \quad (a^2 < t^2) \quad . \quad . \quad . \quad \text{V}$$

$$du_6 = \{(a^2+t^2)(b^2+t^2)\}^{-\frac{1}{2}}dt \quad (a^2 < b^2) \quad . \quad . \quad . \quad \text{VI}$$

§ 3. Consider these six types in turn:

I. Put $t = ay$, $dt = ady$; then we find

$$du_1 = b^{-1}\{(1-y^2)(1-k^2y^2)\}^{-\frac{1}{2}}dy \quad (k^2 = a^2/b^2 < 1)$$

which is of the standard Legendre form.

II. Put $t = 1/y$, $dt = -dy/y^2$, $\alpha = 1/a$, $\beta = 1/b$; then

$$du_2 = -\alpha\beta\{(\alpha^2 - y^2)(\beta^2 - y^2)\}^{-\frac{1}{2}}\, dy$$

which belongs to type I.

III. Put $b^2 - t^2 = y^2$, $-t\,dt = y\,dy$; then we find

$$du_3 = -\{(b^2 - y^2)(b^2 - a^2 - y^2)\}^{-\frac{1}{2}}\, dy$$

which belongs to type I.

IV. Put $a^2 - t^2 = y^2$, $-t\,dt = y\,dy$; then we find

$$du_4 = -\{(a^2 - y^2)(b^2 + a^2 - y^2)\}^{-\frac{1}{2}}\, dy$$

which belongs to type I.

V. Put $t = 1/y$, $dt = -dy/y^2$, $\alpha = 1/a$, $\beta = 1/b$; then

$$du_5 = -\alpha\beta\{(\alpha^2 - y^2)(\beta^2 + y^2)\}^{-\frac{1}{2}}\, dy$$

which belongs to type IV, just considered.

VI. Put $b^2 + t^2 = y^2$, $t\,dt = y\,dy$; then

$$du_6 = \{(y^2 - b^2 + a^2)(y^2 - b^2)\}^{-\frac{1}{2}}\, dy$$

which belongs to type II, considered above.

§ 4. Modification of Cayley's method. If, in (2), the roots of the equation $X_1 \equiv ax^2 + 2bx + c = 0$ are not real, then $ac - b^2 > 0$ and we can put

$$X_1 = a\{(x + b/a)^2 + h^2\}, \quad h^2 = (ac - b^2)/a^2 \quad . \quad (13)$$

and if we now put $x + b/a = hy$, $dx = h\,dy$, the differential $dx/\sqrt{X}$ will take the form

$$\frac{dx}{\sqrt{X}} = \frac{dx}{\sqrt{(X_1X_2)}} = \frac{dy}{\{a(y^2 + 1)(Ay^2 + 2By + C)\}^{\frac{1}{2}}} \quad (14)$$

where A, B, C will be real. If we now put

$$y = \frac{\lambda t + 1}{t - \lambda}, \quad dy = -\frac{(\lambda^2 + 1)dt}{(t - \lambda)^2} \quad . \quad . \quad . \quad (15)$$

then $dx/\sqrt{X}$ will reduce to the form

$$\frac{dx}{\sqrt{X}} = \frac{M\,dt}{\{(t^2 + 1)(Pt^2 + Q)\}^{\frac{1}{2}}} \quad . \quad . \quad (16)$$

where M, P, Q are real constants, provided that λ is chosen to satisfy the quadratic equation

$$B\lambda^2 - (A - C)\lambda - B = 0 \quad . \quad . \quad . \quad (17)$$

which plainly has real roots. The differential $dx/\sqrt{X}$ will consequently depend upon one of the last three types of § 2.

Similarly, if the roots of the equation $X_1 = 0$ are real, we can begin by putting X in the form

$$X = X_1X_2 = a(y^2 - 1)(Ay^2 + 2By + C) \quad . \quad (18)$$

and if the roots of $X_2 = 0$ are also real, then X will be of the form

$$X = X_1 X_2 = aA(y^2 - 1)(y - y_1)(y - y_2) \quad . \quad (19)$$

where y_1, y_2 are real. If we now put $y = (\lambda t + 1)/(t + \lambda)$, it will be found that X has the form

$$X = (\text{const.})(t^2 - 1)(t^2 - \tau^2)/(t + \lambda)^4 \quad . \quad . \quad (20)$$

provided that λ satisfies a certain quadratic equation the roots of which will be real if y_1^2 and y_2^2 are both greater than or both less than unity, i.e., provided that the pairs $-1, +1$ and y_1, y_2 do not interlace. The differential $dx/\sqrt{X}$ will in this way reduce to one of the first three types of § 2.

§ 5. Other methods. Cayley's method may not be the most appropriate in any particular case. For example, as we see from VI, § 7, Ex. 5, the integral

$$w = \int_a^t \frac{dt}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}(c-t)^{\frac{1}{2}}(d-t)^{\frac{1}{2}}}$$

in which the roots of the quartic are all real, is at once converted, by means of the bilinear substitution

$$z = \frac{d-b}{b-a}\,\frac{t-a}{d-t}, \qquad \frac{1}{k^2} = \frac{d-b}{b-a}\,\frac{c-a}{d-c}$$

into

$$w = \frac{2}{(c-a)^{\frac{1}{2}}(d-b)^{\frac{1}{2}}}\,\mathrm{sn}^{-1}\left(\frac{d-b}{b-a}\,\frac{t-a}{d-t}\right)^{\frac{1}{2}}$$

In other cases a trigonometric substitution may be convenient (see § 6, Ex. 4 below).

§ 6. Examples.

Ex. 1. Evaluate

$$I = \int_0^2 \frac{dx}{\{(2x - x^2)(3x^2 + 4)\}^{\frac{1}{2}}}$$

Using Cayley's method, we put

$$x = (\lambda t + \mu)/(t + 1), \quad dx = (\lambda - \mu)dt/(t + 1)^2$$
$$2x - x^2 = \{2(\lambda t + \mu)(t + 1) - (\lambda t + \mu)^2\}/(t + 1)^2$$
$$3x^2 + 4 = \{3(\lambda t + \mu)^2 + 4(t + 1)^2\}/(t + 1)^2$$

and choose λ, μ to satisfy the equations

$$\lambda + \mu - \lambda\mu = 0, \quad 3\lambda\mu + 4 = 0$$

These give $\lambda = \frac{2}{3}$, $\mu = -2$ or $\lambda = -2$, $\mu = \frac{2}{3}$. We take

$$x = (-2t + \tfrac{2}{3})/(t + 1), \quad dx = (-\tfrac{8}{3})/(t + 1)^2$$
$$t = -(3x - 2)/(3x + 6)$$

then t decreases from $\frac{1}{3}$ to $-\frac{1}{3}$ while x increases from 0 to 2 (Fig. 1).

Hence we find

$$I = \sqrt{6}\int_0^{\frac{1}{3}} \frac{dt}{\{(1-9t^2)(1+3t^2)\}^{\frac{1}{2}}}$$

$$= \sqrt{6}\int_0^1 \frac{\frac{1}{3}\,du}{\{(1-u^2)(1+\frac{1}{3}u^2)\}^{\frac{1}{2}}} \quad (3t = u)$$

$$= \frac{1}{\sqrt{2}}\int_0^1 \frac{dv}{\{(1-v^2)(1-\frac{1}{4}v^2)\}^{\frac{1}{2}}} \quad (1 - u^2 = v^2)$$

$$= (1/\sqrt{2})K(\tfrac{1}{2}) \doteqdot (1/\sqrt{2})(1{\cdot}6858) \doteqdot 1{\cdot}192$$

We might equally well have taken $\lambda = \frac{2}{3}$, $\mu = -2$; but then the variation of t would have needed a little more attention, because t would increase from 3 to $+\infty$ and then from $-\infty$ to -3, while x increased from 0 to 2.

FIG. 1.

Ex. 2. Given that

$$u = \int_1^x \frac{dx}{(5x^2 - 4x - 1)^{\frac{1}{2}}(12x^2 - 4x - 1)^{\frac{1}{2}}}$$

express x as an elliptic function of u.

Putting $x = 1/y$, we find

$$u = \int_1^{1/x} \frac{-\,dy}{(5 - 4y - y^2)^{\frac{1}{2}}(12 - 4y - y^2)^{\frac{1}{2}}}$$

$$= \int_3^{2+1/x} \frac{-\,dz}{(9 - z^2)^{\frac{1}{2}}(16 - z^2)^{\frac{1}{2}}} \quad (y + 2 = z)$$

$$= \int_1^{(1+2x)/3x} \frac{-\,3\,dt}{3(1-t^2)^{\frac{1}{2}}\,.\,4(1-k^2t^2)^{\frac{1}{2}}} \quad (z = 3t)$$

where $k^2 = 9/16$. Hence

$$4u = \int_{(1+2x)/3x}^1 \frac{dt}{(1-t^2)^{\frac{1}{2}}(1-k^2t^2)^{\frac{1}{2}}} = \mathrm{sn}^{-1}\,1 - \mathrm{sn}^{-1}\frac{1+2x}{3x}$$

and therefore

$$\frac{1+2x}{3x} = \mathrm{sn}\,(K - 4u) = \frac{\mathrm{cn}\,4u}{\mathrm{dn}\,4u}$$

and

$$x = \frac{\operatorname{dn} 4u}{3 \operatorname{cn} 4u - 2 \operatorname{dn} 4u} \quad (k = \tfrac{3}{4})$$

Ex. 3. Evaluate the integrals

$$I_1 = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(\sin\theta)}}, \quad I_2 = \int_0^{\frac{1}{2}\pi} \sqrt{(\sin\theta)}d\theta$$

Putting $\sin\theta = x = y^2$, we find

$$I_1 = \int_0^1 \frac{dx}{x^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}} = \int_0^1 \frac{2\, dy}{\{(1 - y^2)(1 + y^2)\}^{\frac{1}{2}}}$$

and hence, with $1 - y^2 = z^2$,

$$I_1 = \int_0^1 \frac{\sqrt{2}dz}{\{(1 - z^2)(1 - \frac{1}{2}z^2)\}^{\frac{1}{2}}} = \sqrt{2}K \quad (k = 1/\sqrt{2})$$

Similarly, after the same substitutions, we find

$$I_2 = \sqrt{2}\int_0^1 \frac{(1 - z^2)\, dz}{\{(1 - z^2)(1 - \frac{1}{2}z^2)\}^{\frac{1}{2}}}$$

and putting $z = \operatorname{sn} u$,

$$I_2 = \sqrt{2}\int_0^K (1 - \operatorname{sn}^2 u)du = \sqrt{2}(2E - K) \quad (k = 1/\sqrt{2})$$

Ex. 4. If $T = 1 + 2t^2 \cos 2\alpha + t^4$, show that

$$\int_0^x \frac{dt}{\sqrt{T}} = \int_{1/x}^\infty \frac{dt}{\sqrt{T}} = \tfrac{1}{2} \operatorname{sn}^{-1} \frac{2x}{1 + x^2} \quad (k = \sin\alpha) \quad . \quad (21)$$

where $0 < x < 1$, $0 < \alpha < \frac{1}{2}\pi$.

That the two integrals are equal follows by making the substitution $t' = 1/t$ in either of them.

Let u denote the first integral. Putting $t = \tan\theta$, followed by $y = \sin 2\theta$, we find, after a little algebra,

$$u = \int \frac{d\theta}{(1 - \sin^2\alpha \sin^2 2\theta)^{\frac{1}{2}}} = \int \frac{\frac{1}{2}\, dy}{\{(1 - y^2)(1 - k^2y^2)\}^{\frac{1}{2}}}$$

between the appropriate limits of integration; (21) follows.

In particular, by putting $x = 1$, we deduce that

$$\int_0^\infty \frac{dt}{\sqrt{T}} = \int_0^1 \frac{dt}{\sqrt{T}} + \int_1^\infty \frac{dt}{\sqrt{T}} = \tfrac{1}{2}K + \tfrac{1}{2}K = K \quad (k = \sin\alpha) \quad (22)$$

and if $x > 1$ we have

$$\int_0^x \frac{dt}{\sqrt{T}} = \int_0^\infty \frac{dt}{\sqrt{T}} - \int_x^\infty \frac{dt}{\sqrt{T}} = K - \tfrac{1}{2} \operatorname{sn}^{-1} \frac{2x}{1 + x^2} \quad (23)$$

We note the following two cases of (22) :

$$\int_0^\infty \frac{dt}{\sqrt{(1 + t^2\sqrt{3} + t^4)}} = K \quad \left(k = \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}\right) \quad (24)$$

$$\int_0^\infty \frac{dt}{\sqrt{(1 - t^2\sqrt{3} + t^4)}} = K \quad \left(k = \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}\right) \quad (25)$$

Second method. Without using a trigonometric substitution, we should reach the same result by putting

$$t + t^{-1} = u = 1/v = 2/y$$

Ex. 5. X = **a cubic.** If X is a cubic, it will have at least one real linear factor. If this be $x - a$, then X will be of the form

$$X = (x - a)(ax^2 + 2bx + c)$$

To integrate $dx/\sqrt{X}$ in this case, it may be convenient to apply Cayley's method, or to begin by putting $x - a = y^2$.

Consider the three integrals

$$I_1 = \int_0^1 \frac{dx}{(1 - x^3)^{\frac{1}{2}}}, \quad I_2 = \int_1^\infty \frac{dx}{(x^3 - 1)^{\frac{1}{2}}}, \quad I_3 = \int_{-\infty}^0 \frac{dx}{(1 - x^3)^{\frac{1}{2}}}$$

To evaluate I_2, put $x - 1 = y^2$, $x = 1 + y^2$, $dx = 2ydy$; then we find

$$I_2 = \int_0^\infty \frac{2\,dy}{(y^4 + 3y^2 + 3)^{\frac{1}{2}}} = 2 \,.\, 3^{-\frac{1}{4}} \int_0^\infty \frac{dt}{(t^4 + t^2\sqrt{3} + 1)^{\frac{1}{2}}}$$

where $y = 3^{\frac{1}{4}}t$. Hence, by (24),

$$I_2 = 2 \,.\, 3^{-\frac{1}{4}} K \quad (k = \sin 15^\circ) \quad . \quad . \quad . \quad . \quad (26)$$

It is less easy to evaluate I_1 and I_3. The ratios they bear to I_2 can be found by means of a contour integral; thus:

Integrate $dz/(1 - z^3)^{\frac{1}{2}}$ round a contour consisting of: (i) the real

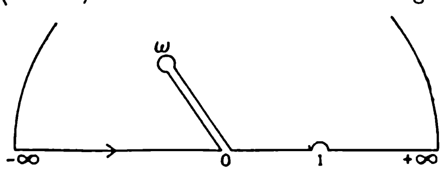

FIG. 2.

axis from $-\infty$ to 0, (ii) a loop in the negative sense round the branch point $z = \omega = e^{\frac{1}{3}\pi i}$ and back to $z = 0$, (iii) the real axis from 0 to $+\infty$, indented at $z = 1$, (iv) the infinite semicircle in the upper half z-plane (see Fig. 2).

The contributions to the integral from the indentation and from the infinite semicircle vanish in the limit.

On the path from 0 to ω, we have $z = \omega r$, $dz = \omega dr$, $(1 - z^3)^{\frac{1}{2}} = (1 - r^3)^{\frac{1}{2}}$. On the return path from ω to 0, after the circuit of the branch point, both $(1 - z^3)^{\frac{1}{2}}$ and dz have changed sign. Hence, the contribution from the whole loop is twice that from 0 to ω.

From $z = 0$ to $z = 1$, after the change of sign round $z = \omega$, we have $(1 - z^3)^{\frac{1}{2}} = -(1 - x^3)^{\frac{1}{2}}$; and beyond the indentation $(1 - z^3)^{\frac{1}{2}} = i(x^3 - 1)^{\frac{1}{2}}$. Consequently, since there are no residues, the sum

$$\int_{-\infty}^{0} \frac{dx}{(1 - x^3)^{\frac{1}{2}}} + \int_0^1 \frac{2\omega dr}{(1 - r^3)^{\frac{1}{2}}} - \int_0^1 \frac{dx}{(1 - x^3)^{\frac{1}{2}}} - \int_1^\infty \frac{i\, dx}{(x^3 - 1)^{\frac{1}{2}}}$$

is equal to zero. That is,

$$I_3 + (-1 + i\sqrt{3})I_1 - I_1 - iI_2 = 0$$

By equating real and imaginary parts, we have

$$I_3 = 2I_1, \quad I_2 = I_1\sqrt{3} \quad . \quad . \quad (27)$$

and hence, using (26),

$$I_1 = 2\,.\,3^{-\frac{3}{4}}K, \quad I_2 = 2\,.\,3^{-\frac{1}{4}}K, \quad I_3 = 4\,.\,3^{-\frac{3}{4}}K \quad . \quad (28)$$

where $k = \sin 15°$.

Further, we have

$$(I_1 + I_3)/I_2 = (I_1 + 2I_1)/(I_1\sqrt{3}) = \sqrt{3} \quad . \quad . \quad (29)$$

Now, putting $1 - x = z^2$, we find

$$I_1 + I_3 = \int_{-\infty}^{1} \frac{dx}{(1 - x^3)^{\frac{1}{2}}} = \int_0^\infty \frac{2\,dz}{(3 - 3z^2 + z^4)^{\frac{1}{2}}}$$

$$= 2\,.\,3^{-\frac{1}{4}} \int_0^\infty \frac{dt}{(1 - t^2\sqrt{3} + t^4)^{\frac{1}{2}}}$$

and consequently, by (25),

$$I_1 + I_3 = 2\,.\,3^{-\frac{1}{4}}K\ (k = \sin 75°) = 2\,.\,3^{-\frac{1}{4}}K'(k = \sin 15°)$$

and hence, by (29) and (26), the surprising result

$$K'/K = \sqrt{3}, \quad (k = \sin 15°) \quad . \quad . \quad . \quad . \quad (30)$$

which also follows easily from the cubic transformation (Examples VII, 10. See also Examples IX, 14).

Ex. 6. Evaluate the integrals

$$I = \int_0^1 \frac{dx}{(x - x^4)^{\frac{1}{2}}}, \quad J = \int_0^1 \frac{dx}{(1 - x^6)^{\frac{1}{2}}}$$

Putting $x = 1/y$ in I, we find, from Ex. 5,

$$I = \int_1^\infty \frac{dy}{(y^3 - 1)^{\frac{1}{2}}} = 2\,.\,3^{-\frac{1}{4}}K(\sin 15°) \quad . \quad (31)$$

Putting $x^2 = z$ in J we find

$$J = \int_0^1 \frac{dz}{2(z - z^4)^{\frac{1}{2}}} = \tfrac{1}{2}I = 3^{-\frac{1}{4}}K(\sin 15°) \quad (32)$$

Note. In general, the integral $\int dx/\sqrt{X}$, when X is a sextic, is not expressible in terms of elliptic integrals: it is said to be hyperelliptic; but in special cases, as in J, the integral is elliptic (see also X, § 1).

Ex. 7. Find the area between the curve $x^3 + y^3 = a^3/\sqrt{2}$ and its asymptote (Fig. 3).

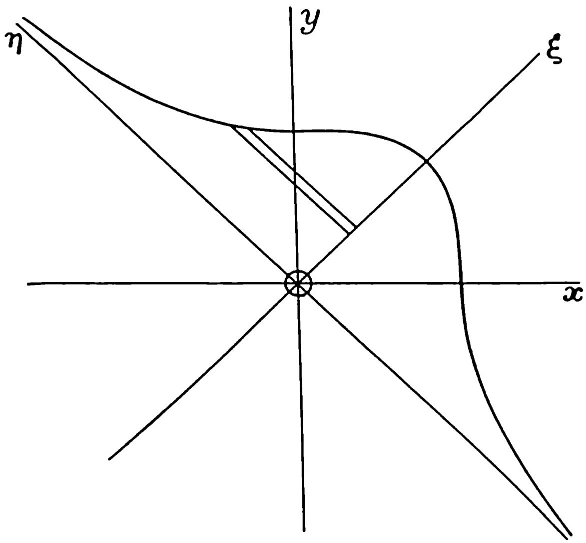

FIG. 3.

Transform the equation of the curve to a new pair of rectangular axes $O\xi$, $O\eta$ making 45° with the old pair, by putting

$$x = (\xi - \eta)/\sqrt{2}, \quad y = (\xi + \eta)/\sqrt{2}$$

We get $\xi^3 + 3\xi\eta^2 = a^3$. Hence the area A required is given by

$$A = 2\int_0^a \eta d\xi = 2\int_0^a \left(\frac{a^3 - \xi^3}{3\xi}\right)^{\frac{1}{2}} d\xi$$

$$= \frac{2a^2}{\sqrt{3}} \int_0^1 \frac{(1 - t^3)dt}{(t - t^4)^{\frac{1}{2}}} \qquad (\xi = at)$$

Putting $1 - t^3 = \frac{1}{4}(3 + 1 - 4t^3)$ in the numerator, we find, using (32),

$$A = \frac{a^2\sqrt{3}}{2} \int_0^1 \frac{dt}{(t - t^4)^{\frac{1}{2}}} + \frac{a^2}{\sqrt{3}} \Big[(t - t^4)^{\frac{1}{2}}\Big]_0^1$$

$$= a^2 \,.\, 3^{\frac{1}{2}} \,.\, 3^{-\frac{1}{4}} K = 3^{\frac{1}{4}} K a^2 \qquad (k = \sin 15°)$$

§ 7. The ratio K/K' for special values of k. In (30) we saw that $K'/K = \sqrt{3}$ when $k = \sin 15°$. This is a particular case of a general theorem, due to Abel, that k is a root of an algebraic equation with integral coefficients whenever K/K' is of the form $(a + b\sqrt{n})/(c + d\sqrt{n})$, where a, b, c, d, n are integers. The proof of the theorem is beyond the scope of this book. A few simple cases are set out below. It is sufficient to suppose $K/K' \geqslant 1$, because the reciprocal of K/K' is obtained by replacing k by k'.

$$K/K' = 1, \quad k = 1/\sqrt{2} = \sin 45^\circ \quad . . . . . . \quad (33)$$
$$= \sqrt{2}, \quad k' = \sqrt{2} - 1 = \tan 22\tfrac{1}{2}^\circ \quad . . . . . \quad (34)$$
$$= \sqrt{3}, \quad k = \sin 75^\circ = (\sqrt{3} + 1)/(2\sqrt{2}),\ 2kk' = \tfrac{1}{2} \quad (35)$$
$$= 2, \quad k' = (\sqrt{2} - 1)^2 = \tan^2 22\tfrac{1}{2}^\circ \quad . . . \quad (36)$$
$$= 2\sqrt{2}, \quad k' = \{\sqrt{2} + 1 - \sqrt{(2\sqrt{2} + 2)}\}^2 \quad . . \quad (37)$$
$$= 3, \quad 2kk' = (2 \sin 15^\circ)^4 = (2 - \sqrt{3})^2 \quad . . \quad (38)$$
$$= 2\sqrt{3}, \quad k' = (\sqrt{3} - \sqrt{2})^2(\sqrt{2} - 1)^2 = \tan^2 7\tfrac{1}{2}^\circ \quad (39)$$
$$= 4, \quad k' = (\sqrt[4]{2} - 1)^2 / (\sqrt[4]{2} + 1)^2 \quad . . . \quad (40)$$

Of these, (33) is self-evident and (35) follows from (30). The proof of (38) requires a transformation of the third order (see Examples VII, 10). The rest follow with the aid of VII, (44), (46).

§ 8. Greenhill advocated the construction of tables of elliptic functions in which the ratio K/K' would have a succession of values corresponding to such values of k. He calculated twelve tables for the British Association (*B.A. Reports*, 1911, 1912, 1913) for the following values of K/K': 1, $\sqrt{2}$, $\sqrt{3}$, 2, $3/\sqrt{2}$, $2\sqrt{2}$, 3, $2\sqrt{3}$, 4, $3\sqrt{2}$, 5, $3\sqrt{3}$.

EXAMPLES IX

1. If $u = \displaystyle\int_0^x \frac{dx}{(6 - 5x^2 + x^4)^{\frac{1}{2}}}$, show that

$$x = \sqrt{2}\,\mathrm{sn}\,(u\sqrt{3}) \quad (k^2 = \tfrac{2}{3})$$

2. If $u = \displaystyle\int_0^x (1 + t^2 - 2t^4)^{-\frac{1}{2}}\, dt$, show that

$$x\sqrt{3}\,\mathrm{dn}\,(u\sqrt{3}) = \mathrm{sn}\,(u\sqrt{3}) \quad (k^2 = \tfrac{2}{3})$$

3. If $\dfrac{u}{\sqrt{2}} = \displaystyle\int_{\sqrt{x}}^1 \frac{dt}{(1 + 6t^2 + t^4)^{\frac{1}{2}}}$, show that

$$x = (1 - \mathrm{sn}\,2u)/(1 + \mathrm{sn}\,2u) \quad (k^2 = \tfrac{1}{2})$$

4. Show that

$$\int_0^2 \frac{dx}{\{(2x - x^2)(4x^2 + 9)\}^{\frac{1}{2}}} = \frac{2}{\sqrt{15}} K \quad \left(k = \frac{1}{\sqrt{5}}\right)$$

5. If $\frac{1}{4}u = \int_1^x \frac{dx}{(15x^2 - 2x - 1)^{\frac{1}{2}}(3x^2 - 2x - 1)^{\frac{1}{2}}}$, show that

$$x = \text{dn}\, u/(2\, \text{cn}\, u - \text{dn}\, u) \quad (k = \tfrac{1}{2})$$

6. If $0 < x < a$, show that

$$\int_x^a \frac{dt}{(a^4 - t^4)^{\frac{1}{2}}} = \int_a^{a^2/x} \frac{dt}{(t^4 - a^4)^{\frac{1}{2}}} = \frac{1}{a\sqrt{2}}\, \text{cn}^{-1} \left(\frac{x}{a}, \frac{1}{\sqrt{2}}\right)$$

7. Show that

$$\int_2^\infty \frac{dz}{\sqrt{(z^3 - 4z)}} = K\left(\frac{1}{\sqrt{2}}\right)$$

8. Show that

$$\int_0^x \frac{dt}{\{t(1 - t)(2 - t)(3 - t)\}^{\frac{1}{2}}} = \text{sn}^{-1} \left(\frac{2x}{3 - x}\right)^{\frac{1}{2}} \quad (k = \tfrac{1}{2})$$

9. Show that

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{(x^2 - 2x + 2)^{\frac{1}{2}}(3 + x - 2x^2)^{\frac{1}{2}}} = \tfrac{3}{5}K\left(\frac{1}{\sqrt{10}}\right)$$

10. Show that

(i) $$\int_0^\infty (1 + t^4)^{-\frac{1}{2}}\, dt = K\left(\frac{1}{\sqrt{2}}\right)$$

(ii) $$\int_0^\infty (1 + t^2 + t^4)^{-\frac{1}{2}}\, dt = K(\tfrac{1}{2})$$

(iii) $$\int_0^\infty (1 - t^2 + t^4)^{-\frac{1}{2}}\, dt = K\left(\frac{\sqrt{3}}{2}\right)$$

11. Show that, if $0 < \alpha < \pi$,

(i) $$\int_1^\infty \frac{dx}{(x - 1)^{\frac{1}{2}}(x^2 + 2x \cos \alpha + 1)^{\frac{1}{2}}} = (2 \sec \tfrac{1}{2}\alpha)^{\frac{1}{2}} K(\sin \tfrac{1}{4}\alpha)$$

(ii) $$\int_{-\infty}^1 \frac{dx}{(1 - x)^{\frac{1}{2}}(1 + 2x \cos \alpha + x^2)^{\frac{1}{2}}} = (2 \sec \tfrac{1}{2}\alpha)^{\frac{1}{2}} K(\cos \tfrac{1}{4}\alpha)$$

(iii) $$\int_0^\infty \frac{dx}{(1 + 2x^2 \cosh 2\beta + x^4)^{\frac{1}{2}}} = \text{sech}\, \beta\, K(\tanh \beta)$$

12. Show that

$$\int_1^x \frac{dt}{\sqrt{(t - 1)}\sqrt{(t^2 + 1)}} = 2^{-\frac{1}{4}}\, \text{sn}^{-1} \frac{2^{\frac{5}{4}}\sqrt{(x - 1)}}{x - 1 + \sqrt{2}} \quad (k = \sin 22\tfrac{1}{2}^\circ)$$

13. By putting $x^2 = y$, $y = 1/t$, show that

$$\int (a + bx^2 + cx^4 + dx^6)^{-\frac{1}{2}}dx = -\tfrac{1}{2}\int (at^3 + bt^2 + ct + d)^{-\frac{1}{2}}\,dt$$

14. Sketch the curve $3x^2y = 4 - x^3$. Show that

$$27x^6(1 - y^3) = (x^3 - 1)(x^3 + 8)^2$$

and deduce that

$$\int_{-\infty}^{1} (1 - y^3)^{-\frac{1}{2}}dy = \sqrt{3}\int_{1}^{\infty} (x^3 - 1)^{-\frac{1}{2}}\,dx$$

15. Show that the length of one loop of the lemniscate $r^2 = a^2 \sin 2\theta$ is

$$\int_0^a \frac{2a^2dr}{\sqrt{(a^4 - r^4)}} = a\sqrt{2}K\left(\frac{1}{\sqrt{2}}\right)$$

If s denotes the length of the arc measured from the pole to the point (r, θ), where $0 < \theta < \frac{1}{4}\pi$, show that

$$r = a \operatorname{cn}(K - s\sqrt{2}/a) \quad \left(k = \frac{1}{\sqrt{2}}\right)$$

16. Show that the length of one loop of the curve $r^3 = a^3 \sin 3\theta$ is

$$\int_0^1 \frac{2adt}{\sqrt{(1 - t^6)}} = 2a\,.\,3^{-\frac{1}{4}}K(\sin 15^\circ)$$

17. Show that the length of the arc of the hyperbola

$$x = c(t^{-1} + t\cos\alpha), \quad y = ct\sin\alpha$$

measured from the point $t = 1$, is given by

$$s = c\int_1^t (1 - 2t^{-2}\cos\alpha + t^{-4})^{\frac{1}{2}}\,dt$$

Put $u = t + t^{-1}$ and evaluate the integral.

18. Prove that the radius of curvature ρ of the curve $\sinh y = \cos x$ is given by $\rho = \sqrt{2}\sec x$. If s is the length of the arc of the curve measured from the point where $x = 0$, show that the "natural" equation of the curve is

$$s = \int_{\sqrt{2}}^{\rho} \frac{2d\rho}{\sqrt{(\rho^4 - 4)}}$$

which reduces to $\rho \operatorname{cn} s = \sqrt{2}$, $(k = 1/\sqrt{2})$.

19. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ can roll without slipping on the underside of the curve $y/a = \operatorname{dn}(x/b)$ so that its centre describes the axis of x, provided that the modulus k is equal to the eccentricity of the ellipse.

Deduce that, if $k = \sqrt{(a^2 - b^2)}/a$, the length of the curve $y/a = \operatorname{dn}(x/b)$ from $x = 0$ to $x = Kb$ is aE. Verify this result without reference to the rolling ellipse.

20. Show that the length s of the arc of the curve $3a^2y = x^3$, from the origin to the point (x, y), where $0 < x < a$, is given by

$$s = \left(\frac{x^2}{9} + y^2\right)^{\frac{1}{2}} + \frac{a}{3}\,\mathrm{sn}^{-1}\frac{2ax}{a^2 + x^2} \quad \left(k = \frac{1}{\sqrt{2}}\right)$$

What is the length when $x > a$? Show that $s - y \to \frac{2}{3}Ka$ when $x \to \infty$.

21. Show that the length s of the curve $4a^3y = x^4$, from the origin to the point (x, y), provided that $0 < x < a/\sqrt{(\sqrt{3} - 1)}$, is given by

$$s = \left(\frac{x^2}{16} + y^2\right)^{\frac{1}{2}} + \frac{3^{\frac{3}{4}}a}{8}\,\mathrm{sn}^{-1}\frac{2\,.\,3^{\frac{1}{4}}\,.\,x(a^2 + x^2)^{\frac{1}{2}}}{x^2\sqrt{3} + a^2 + x^2} \quad (k = \sin 75^\circ)$$

Also show that, when $x \to \infty$,

$$s - y \to 3^{\frac{3}{4}}a[\tfrac{1}{4}K - \tfrac{1}{8}\,\mathrm{sn}^{-1}\{3^{\frac{1}{4}}(\sqrt{3} - 1)\}]$$

22. Two pendulums have the same period of oscillation. One, of length l, swings through an arc of 60° ($\alpha = 30°$); the other, of length l', swings through an arc of 300° ($\alpha = 150°$). Show that $l = 3l'$.

23. A particle oscillates on the axis of y about the origin under the action of a force equal to n^2y^3 per unit mass. If at $t = 0$,

(i) $y = a$, $dy/dt = 0$, show that $y = a\,\mathrm{cn}\,ant$;
(ii) $y = 0$, $dy/dt = v$, show that $y = (v\sqrt{2}/n)^{\frac{1}{2}}\,\mathrm{cn}\,(K - vt\sqrt{2})$;

where in each case $k = 1/\sqrt{2}$.

24. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + \omega^2\left(y + \frac{y^3}{a^2}\right) = 0.$$

25. (i) If $K/K' = 2/\sqrt{3}$, show that

$$k' = \tan^2 37\tfrac{1}{2}^\circ = (\sqrt{3} - \sqrt{2})^2(\sqrt{2} + 1)^2$$

(ii) If $K/K' = 2\sqrt{3}$, show that

$$k' = \tan^2 7\tfrac{1}{2}^\circ = (\sqrt{3} - \sqrt{2})^2(\sqrt{2} - 1)^2$$

26. Show that λ and μ, in § 2, are the double points of the involution determined by the pairs of roots (α, β), (γ, δ) of the equations $X_1 = 0$, $X_2 = 0$.

Indicate the positions of α, β; γ, δ; λ, μ on an Argand diagram in the three cases referred to in § 1, thus illustrating geometrically that λ and μ are real except in the one case in which the pairs of roots are both real and interlace.

CHAPTER X

A DEGENERATE HYPERELLIPTIC INTEGRAL

§ **1.** An integral of the type

$$\int \{u(1-u)(1+\kappa u)(1+\lambda u)(1-\kappa\lambda u)\}^{-\frac{1}{2}}\, du \quad . \qquad (1)$$

in which the expression under the square root sign is a particular kind of quintic, can be shown to be dependent upon elliptic integrals.* If we regard the quintic as a sextic with roots $0, \infty$; $1, 1/\kappa\lambda$; $-1/\kappa, -1/\lambda$; then this sextic is characterised by the property that the roots belong in pairs to the same involution $(zz' = 1/\kappa\lambda)$.

§ **2.** Consider † the succession of transformations

$$z = \int_0^t \frac{dt}{2t^{\frac{3}{4}}(1-t)^{\frac{1}{4}}(1-k^2 t)^{\frac{1}{4}}} \quad . \qquad (2)$$

$$= \int_0^u \frac{du}{\{u(1-u^2)(1-k^2u^2)\}^{\frac{1}{2}}} \quad (t = u^2) \qquad (3)$$

$$= \int_0^\chi d\chi/\sqrt{(\operatorname{sn}\chi)} \qquad (u = \operatorname{sn}\chi) \quad . \qquad (4)$$

By these transformations, a trapezium in the z-plane (Fig. 1) with angles $\frac{1}{4}\pi$, $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, $\frac{3}{4}\pi$ is represented first on the upper half t-plane (Fig. 2), then on the positive quadrant of the u-plane (Fig. 3), and finally upon a rectangle in the χ-plane (Fig. 4).

§ **3.** The integral (3) is of the same type as (1). It can be expressed as the sum of two elliptic integrals with complementary moduli. Following Cayley, *Elliptic Functions*, p. 360, we put

$$\lambda = \frac{1+\sqrt{k}}{\sqrt{(2+2k)}}, \quad \lambda' = \frac{1-\sqrt{k}}{\sqrt{(2+2k)}}, \quad k = \left(\frac{\lambda-\lambda'}{\lambda+\lambda'}\right)^2 . \qquad (5)$$

$$\lambda^2 + \lambda'^2 = 1, \quad \lambda^2 \geqslant \tfrac{1}{2} \geqslant \lambda'^2 \quad . \quad . \qquad (6)$$

$$\lambda + \lambda' = \frac{\sqrt{2}}{\sqrt{(1+k)}}, \quad \lambda - \lambda' = \frac{\sqrt{(2k)}}{\sqrt{(1+k)}} \quad . \qquad (7)$$

* Cayley, *Elliptic Functions*, p. 360; also, Greenhill, *Applications of Elliptic Functions*, pp. 160 *et seq.*

† F. Bowman, *Proc. London Math. Soc.*, 1935, **39**, (2), 211; and 1936, **41**, 271.

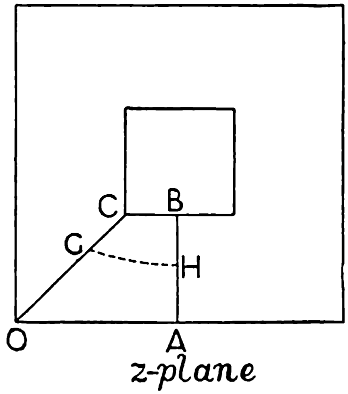

FIG. 1.

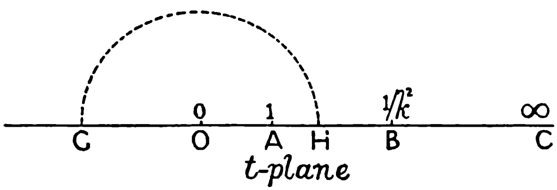

FIG. 2.

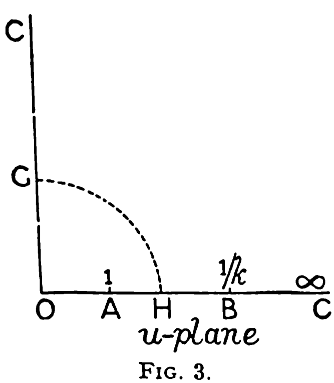

FIG. 3.

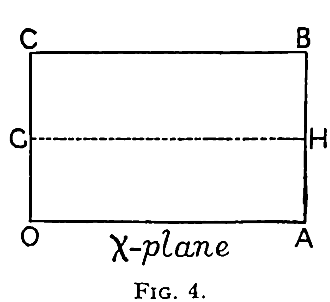

FIG. 4.

We also put

$$\sqrt{u} = \frac{(\lambda + \lambda')\zeta}{\sqrt{(1 - \lambda^2\zeta^2)} + \sqrt{(1 - \lambda'^2\zeta^2)}} = \frac{(\lambda + \lambda')\zeta}{Z + Z'} \tag{8}$$

where $Z = \sqrt{(1 - \lambda^2\zeta^2)}$, $Z' = \sqrt{(1 - \lambda'^2\zeta^2)}$. Then we find

$$(1 + u)(1 + ku) = \left\{1 + \frac{(\lambda + \lambda')^2\zeta^2}{(Z + Z')^2}\right\}\left\{1 + \frac{k(\lambda + \lambda')^2\zeta^2}{(Z + Z')^2}\right\}$$

Now

$$\begin{aligned} &(Z + Z')^4 + (1 + k)(\lambda + \lambda')^2\zeta^2(Z + Z')^2 + k(\lambda + \lambda')^4\zeta^4 \\ &= (Z + Z')^4 + 2\zeta^2(Z + Z')^2 + (\lambda^2 - \lambda'^2)^2\zeta^4 \\ &= (Z + Z')^4 + 2\zeta^2(Z + Z')^2 + (Z^2 - Z'^2)^2 \\ &= (Z + Z')^2\{(Z + Z')^2 + 2\zeta^2 + (Z - Z')^2\} \\ &= 2(Z + Z')^2(Z^2 + Z'^2 + \zeta^2) \\ &= 4(Z + Z')^2 \end{aligned}$$

and hence

$$(1 + u)(1 + ku) = 4/(Z + Z')^2 \quad . \quad . \quad . \tag{9}$$

Similarly,

$$(1 - u)(1 - ku) = 4(1 - \zeta^2)/(Z + Z')^2 \tag{10}$$

Again, by differentiation of (8),

$$\frac{1}{2\sqrt{u}}\frac{du}{d\zeta} = (\lambda + \lambda')\left\{\frac{1}{Z + Z'} + \frac{\zeta}{(Z + Z')^2}\left(\frac{\lambda^2\zeta}{Z} + \frac{\lambda'^2\zeta}{Z'}\right)\right\}$$

from which, after putting $\lambda^2\zeta^2 = 1 - Z^2$, $\lambda'^2\zeta^2 = 1 - Z'^2$, we get

$$\frac{du}{\sqrt{u}} = \frac{2(\lambda + \lambda')d\zeta}{(Z + Z')ZZ'} \quad . \quad . \quad . \quad . \tag{11}$$

Substituting from (9), (10), (11) in (3), we find

$$z = \tfrac{1}{2}(\lambda + \lambda')\int_0^\zeta \frac{(Z + Z')d\zeta}{\sqrt{(1 - \zeta^2)}\,.\,ZZ'}$$

$$= \frac{1}{\sqrt{(2 + 2k)}}\int_0^\zeta \left\{\frac{d\zeta}{(1 - \zeta^2)^{\frac{1}{2}}(1 - \lambda^2\zeta^2)^{\frac{1}{2}}} + \frac{d\zeta}{(1 - \zeta^2)^{\frac{1}{2}}(1 - \lambda'^2\zeta^2)^{\frac{1}{2}}}\right\}. \tag{12}$$

or, if $m = \sqrt{(2 + 2k)}$,

$$mz = \operatorname{sn}^{-1}(\zeta, \lambda) + \operatorname{sn}^{-1}(\zeta, \lambda') \quad . \quad . \quad . \quad . \tag{13}$$

§ 4. To express ζ in terms of u, we have, from (9) and (8),

$$(1 + u)(1 + ku)\zeta^2 = 4\zeta^2/(Z + Z')^2 = 4u/(\lambda + \lambda')^2$$

and therefore, by (7),

$$(1 + u)(1 + ku)\zeta^2 = 2(1 + k)u \quad . \tag{14}$$

From (3), (4), (13), (14), we now have

$$\int_0^\chi \frac{d\chi}{\sqrt{(\operatorname{sn}\chi)}} = \int_0^u \frac{du}{\{u(1-u^2)(1-k^2u^2)\}^{\frac{1}{2}}} = \frac{\operatorname{sn}^{-1}(\zeta, \lambda) + \operatorname{sn}^{-1}(\zeta, \lambda')}{\sqrt{(2+2k)}} \quad . \quad (15)$$

where

$$\zeta^2 = \frac{2(1+k)u}{(1+u)(1+ku)}, \quad u = \operatorname{sn}\chi \quad . \quad (16)$$

§ 5. The region of the ζ-plane that corresponds to the quadrant

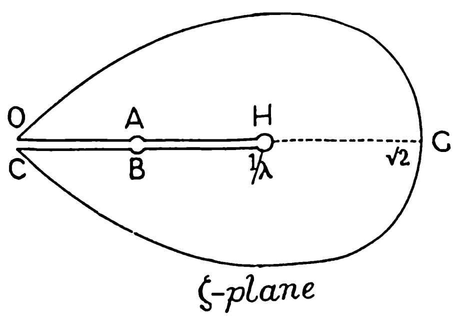

FIG. 5.

in the u-plane is shown in Fig. 5. The following two details may be noted:

Firstly, from (16), when $u = 1/\sqrt{k}$ we find $1 - \lambda^2\zeta^2 = 0$, and therefore $\zeta = \pm 1/\lambda$. Taking $\zeta = +1/\lambda$, and putting

$$\zeta - 1/\lambda = h, \quad u - 1/\sqrt{k} = \varepsilon,$$

we find approximately

$$h = A\varepsilon^2, \text{ where } A = -\tfrac{1}{2}(2+2k)^{\frac{1}{2}}k^{\frac{3}{4}}(1+\sqrt{k})^{-3} \quad . \quad (17)$$

so that when the point u describes an indentation round the point $u = 1/\sqrt{k}$ in the form of a small semicircle, then the point ζ describes a small circle in the same sense about the point $\zeta = 1/\lambda$.

Secondly, if we put $\zeta_1 = \zeta^2$, it may be verified that, to the imaginary axis OC in the u-plane, corresponds the circle $|\zeta_1 - 1| = 1$ in the ζ_1-plane (the ζ_1-plane is not depicted). The curve shown in Fig. 5 is one half of a lemniscate, the transform of this circle by $\zeta_1 = \zeta^2$.

§ 6. Now put $w = \operatorname{sn}^{-1}(\zeta, \lambda)$, $w' = \operatorname{sn}^{-1}(\zeta, \lambda')$; then (13) becomes

$$mz = w + w' \quad . \quad . \quad . \quad . \quad (18)$$

The boundary curves in the w and w'-planes are indicated in Figs. 6, 7, in which L, L' denote complete elliptic integrals of the first kind with moduli λ, λ' respectively.

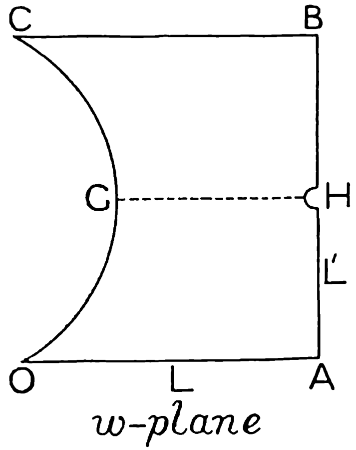

FIG. 6.

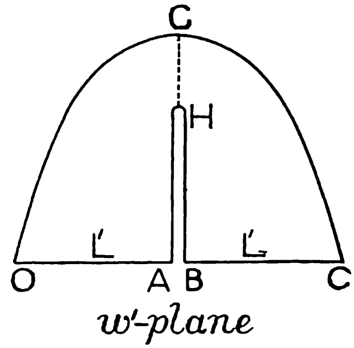

FIG. 7.

§ 7. From the dimensions of the trapezium in the z-plane we can now find λ and k. In Fig. 1, let $a = OA$, $b = CB$; then $AB = a - b$; and at B we have

$$z = a + i(a - b),\ w = L + 2iL',\ w' = L',$$

and therefore, by (18),

$$m(a + ia - ib) = L + 2iL' + L' \qquad . \quad (19)$$

By equating real and imaginary parts, we find

$$ma = L + L',\quad mb = L - L'$$

and hence

$$\frac{a}{b} = \frac{L + L'}{L - L'},\quad \frac{L}{L'} = \frac{a + b}{a - b} \qquad . \quad . \quad . \quad . \quad (20)$$

It follows that, if a/b is known, we can at once find L/L', from which λ can be found by means of tables. Then k will be given by (5), and K/K', the ratio of the sides of the rectangle in the χ-plane, can be found if required (see VI, § 8).

§ 8. **An application.** Consider a condenser consisting of two long parallel prisms, one inside the other, such that the cross-section is bounded by two concentric squares of sides $2a$, $2b$ $(a > b)$, similarly situated, as in Fig. 1. The cross-section is divisible into eight trapeziums, of which $OABC$ is typical.

In the application of conformal representation to the electrostatic field between the surfaces of a condenser, the capacity of the condenser is conserved from one plane to another (cf. VII, § 4). Consequently, in the present example, the capacity of the prismatic condenser per unit length will be eight times that of a parallel plate condenser the plates of which are represented by OA, CB in the χ-plane, Fig. 4; that is, the

$$\text{capacity} = 8 \times \frac{1}{4\pi}\frac{K}{K'} = \frac{2K}{\pi K'} \quad . \quad . \quad . \quad (21)$$

§ 9. **Numerical example.** Suppose $a = 2b$. Then from (20) we have $L/L' = 3$. We could now use tables to find λ, but it happens that the calculation is made easy by the use of the singular modulus that corresponds to $L/L' = 3$. From IX, (38), in this case,

$$2\lambda\lambda' = (2 \sin 15°)^4 = (2 - \sqrt{3})^2 = 7 - 4\sqrt{3}$$

and hence, by (5), $k^2 = \frac{3}{4}$, $k'^2 = \frac{1}{4}$. Milne-Thomson's tables now give

$$K = 2{\cdot}1565156, \quad K' = 1{\cdot}6857504$$

and with $1/\pi = 0{\cdot}3183098862$ we find the

$$\text{capacity} = 0{\cdot}814403 \quad . \quad . \quad . \quad . \quad (22)$$

This example has been used to illustrate the method of relaxation.*

§ 10. **Approximation for squares nearly equal in size.** Let $h = a - b = AB$ in Fig. 1. If h is small compared with a, or b, then L is large compared with L', by (20); therefore λ is large compared with λ'. It follows from (5) that k is large compared with k', and therefore K is large compared with K'. Hence, by II, (22), the relations

$$\lambda'^2 = 16e^{-\pi L/L'}, \quad k'^2 = 16e^{-\pi K/K'} \quad . \quad . \quad . \quad (23)$$

will be approximately true. Now $\sqrt{k} = (1 - k'^2)^{\frac{1}{4}} \doteqdot 1 - \frac{1}{4}k'^2$, and therefore, by (5),

$$\lambda' \doteqdot \tfrac{1}{2}(\tfrac{1}{4}k'^2) = \tfrac{1}{8}k'^2$$

and so

$$\log(\lambda'/4) \doteqdot \log(k'^2/32)$$

and hence, by (23),

$$\frac{K}{K'} \doteqdot \frac{L}{2L'} - \frac{1}{\pi}\log_e 2 \quad . \quad . \quad . \quad (24)$$

Now, by II, § 6, the approximation (23) will be good for practical purposes if $L/L' > 2$ and $K/K' > 2$. By (24), both

* E.g., H. and B. S. Jeffreys, *Methods of Mathematical Physics*, 1950, p. 310.

these conditions will be covered if $L/L' > 5$, or, by (20), $b > \frac{2}{3}a$, in which case we may put

$$\frac{K}{K'} = \frac{\frac{1}{2}(a+b)}{a-b} - \frac{1}{\pi}\log_e 2 \quad . \quad . \quad . \quad (25)$$

and one-eighth of the capacity of the condenser of § 8 will be given approximately by

$$\frac{1}{4\pi}\frac{K}{K'} = \frac{1}{4\pi}\frac{\frac{1}{2}(a+b)}{h} - \frac{1}{4\pi^2}\log_e 2 \quad . \quad . \quad . \quad (26)$$

that is, the capacity of a parallel plate condenser of length $\frac{1}{2}(a+b)$, corrected by the subtraction of the term $(\log_e 2)/(4\pi^2)$.

§ **11.** From (4) follows $dz/d\chi = 1/\sqrt{(\operatorname{sn}\chi)}$. Now, by Examples IV, 5, (i), we have $|\operatorname{sn}\chi| = 1/\sqrt{k}$ when $\chi = \phi + \frac{1}{2}iK'$. Consequently, along the curve in the z-plane defined by $\chi = \phi + \frac{1}{2}iK'$, we have

$$|d\chi/dz| = |\operatorname{sn}\chi|^{\frac{1}{2}} = k^{-\frac{1}{4}} = \text{const.} \quad . \quad . \quad (27)$$

This is a curve along which, in the application to the condenser problem of § 8, the electric intensity is constant. The curve is dotted in Fig. 1, and its transforms in the other planes are also dotted. Its Cartesian equation can be found as follows:

By reference to Figs. 6, 7 we see that, along GH, we can put

$$w = mX + iL', \quad w' = L' + imY \quad . \quad . \quad (28)$$

where X and Y are real. Now $\zeta = \operatorname{sn}(w, \lambda) = \operatorname{sn}(w', \lambda')$, and therefore

$$\operatorname{sn}(mX + iL', \lambda) = \operatorname{sn}(L' + imY, \lambda')$$

that is,

$$\frac{1}{\lambda \operatorname{sn}(mX, \lambda)} = \frac{\operatorname{cn}(imY, \lambda')}{\operatorname{dn}(imY, \lambda')} = \frac{1}{\operatorname{dn}(mY, \lambda)}$$

from which follows

$$\operatorname{dn}^2(mX, \lambda) + \operatorname{dn}^2(mY, \lambda) = 1 \quad . \quad . \quad (29)$$

To find the geometrical meaning of X and Y, we have

$$z = x + iy = (w + w')/m = X + L'/m + i(Y + L'/m) \quad (30)$$

and if (x_0, y_0) are the co-ordinates of the midpoint of OC, we have, from § 7,

$$x_0 = y_0 = \tfrac{1}{2}(a-b) = L'/m \quad . \quad . \quad . \quad (31)$$

and from (30) and (31) follow $x = x_0 + X$, $y = y_0 + Y$, which shows that (X, Y) are co-ordinates referred to the midpoint of OC as origin. Hence, (29) is the equation of the curve GH, with the midpoint of OC as origin (Fig. 1).

EXAMPLES X

1. By putting $k = 1$ in (15), verify that

$$\int_0^x \frac{dx}{(1 - x^2)\sqrt{x}} = \tan^{-1}\sqrt{x} + \tanh^{-1}\sqrt{x}$$

2. By putting $k = 0$ in (15), show that

$$\int_0^x \frac{dx}{\sqrt{(x - x^3)}} = \sqrt{2}\,\mathrm{sn}^{-1}\left(\frac{2x}{1 + x}\right)^{\frac{1}{2}} \quad (k = 1/\sqrt{2})$$

Also, find the integral by putting $x = y^2$ and verify that

$$K - \mathrm{sn}^{-1}\sqrt{(1 - x)} = \mathrm{sn}^{-1}\{2x/(1 + x)\}^{\frac{1}{2}} \quad (k = 1/\sqrt{2})$$

3. By putting $k = \frac{1}{3}$ in (15), show that

$$\int_0^K \{\mathrm{sn}\,(\chi, \tfrac{1}{3})\}^{-\frac{1}{2}} d\chi = \sqrt{3}\sin 75^\circ K\,(\sin 15^\circ)$$

4. Show that, when $(a + b)/(a - b) = \sqrt{3}$, the capacity of the condenser of § 8 is $2K/(\pi K')$, with $k = \frac{1}{3}$.

5. Show that, if $y = xe^{\alpha}$ and $k = e^{-2\alpha}$, then

$$\frac{dx}{\sqrt{x}\sqrt{(1 - 2x^2\cosh 2\alpha + x^4)}} = \frac{e^{-\frac{1}{2}\alpha}\,dy}{\sqrt{y}\sqrt{(1 - y^2)}\sqrt{(1 - k^2y^2)}}$$

6. By means of the substitution $x + x^{-1} = y$, show that

$$\frac{(Ax^{-\frac{1}{2}} + Bx^{\frac{1}{2}})\,dx}{\{\pm(1 - 2bx^2 + x^4)\}^{\frac{1}{2}}} = \left\{\frac{\frac{1}{2}(B - A)}{(y + 2)^{\frac{1}{2}}} + \frac{\frac{1}{2}(B + A)}{(y - 2)^{\frac{1}{2}}}\right\}\frac{dy}{\{\pm(y^2 - 2 - 2b)\}^{\frac{1}{2}}}$$

7. Integrate $(A + Bx)dx/\{x(1 - x^2)(1 - k^2x^2)\}^{\frac{1}{2}}$ by means of Ex. 6, after putting $t = x\sqrt{k}$.

8. Use Example 6 to integrate $(\mathrm{dn}\,\chi)^{\frac{1}{2}}\,d\chi$ and $(\mathrm{dn}\,\chi)^{-\frac{1}{2}}\,d\chi$.

9. Use Example 6 to integrate $dx/(1 + x^8)^{\frac{1}{2}}$.

10. Use Example 9 to find the length of the curve $5a^4y = x^5$ from the origin to the point (x, y).

11. Use Example 6 to integrate $dx/\{(x^4 - a^4)(b^4 - x^4)\}^{\frac{1}{2}}$.

12. Let s denote the length of the Cassinian oval

$$r^4 - 2a^2r^2\cos 2\theta + a^4 = b^4$$

measured from the point where $r = \sqrt{(a^2 + b^2)}$, $\theta = 0$, to the point (r, θ). Show that

$$-\frac{ds}{dr} = \frac{2b^2r^2}{\{(a^2 + b^2)^2 - r^4\}^{\frac{1}{2}}\{r^4 - (b^2 - a^2)^2\}^{\frac{1}{2}}}$$

Find: (i) the whole length of the curve when $a < b$, (ii) the length of one of the ovals when $a > b$.

13. By putting $x^{-1} - x = y$, show that

$$\int_0^1 \frac{(1 + x^2)dx}{\sqrt{(1 + x^8)}} = \int_0^\infty \frac{dy}{\sqrt{(y^4 + 4y^2 + 2)}}$$

14. Let $X = 1 + x^4$, $X_1 = 2k - k'^2x^2$, $X_2 = k'^2 + 2kx^2$, where $0 < k < 1$ and $k'^2 = 1 - k^2$. Also, let $y = x(X_1/X_2)^{\frac{1}{2}}$, and let k be the modulus of the functions $\mathrm{sn}^{-1}\, y$, $\mathrm{sn}^{-1}\,(y/k)$. Verify that

$$\frac{d}{dx}\,\mathrm{sn}^{-1}\, y = \frac{2(k - x^2)}{(XX_1X_2)^{\frac{1}{2}}}, \quad \frac{d}{dx}\,\mathrm{sn}^{-1}\,\frac{y}{k} = \pm\,\frac{2(1 + kx^2)}{(XX_1X_2)^{\frac{1}{2}}}$$

according as $x^2 <$ or $> k$. Hence evaluate the integral

$$\int \frac{A + Bx^2}{\{(1 + x^4)(2k - k'^2x^2)(k'^2 + 2kx^2)\}^{\frac{1}{2}}}\,dx$$

If $k = \tan\alpha$ $(0 < \alpha < \tfrac{1}{4}\pi)$, show that

$$\frac{(\sin 4\alpha)^{\frac{1}{2}}}{2\sqrt{2}}\,\frac{d}{dx}\left(\tan\alpha\,.\,\mathrm{sn}^{-1}\, y \pm \mathrm{sn}^{-1}\,\frac{y}{\tan\alpha}\right)$$
$$= (1 + x^4)^{-\frac{1}{2}}(1 - 2x^2\cot 4\alpha - x^4)^{-\frac{1}{2}}$$

Consider, in particular,* $\alpha = \tfrac{1}{8}\pi$, showing that in this case

$$\int_0^1 (1 - x^8)^{-\frac{1}{2}}\,dx = \frac{K(\sqrt{2} - 1)}{\sqrt{2}}$$

15. Prove that the length of one half of one loop of the curve $r^4 = a^4 \sin 4\theta$ is

$$a \times K(\sqrt{2} - 1)/\sqrt{2} = 1{\cdot}16362a$$

[$K(\sqrt{2} - 1) = 1{\cdot}645608$, from Greenhill's tables.]

16. In the notation of § 3, if $U = (1 - u^2)(1 - k^2u^2)$ and

$$y = (1 - \zeta^2)^{\frac{1}{2}}(1 - \lambda^2\zeta^2)^{\frac{1}{2}}, \quad y' = (1 - \zeta^2)^{\frac{1}{2}}(1 - \lambda'^2\zeta^2)^{\frac{1}{2}}$$

show that

$$\frac{2u^{\frac{1}{2}}\,du}{\sqrt{U}} = \frac{(\lambda + \lambda')^2}{\lambda - \lambda'}\left(\frac{1}{y} - \frac{1}{y'}\right)d\zeta$$

$$\frac{2u^{\frac{3}{2}}\,du}{\sqrt{U}} = \frac{(\lambda + \lambda')^2}{(\lambda - \lambda')^3}\left\{\frac{4}{\zeta^2}\left(\frac{1}{y} - \frac{1}{y'}\right) - \frac{1 + 2\lambda^2}{y} + \frac{1 + 2\lambda'^2}{y'}\right\}d\zeta$$

Hence integrate $(\mathrm{sn}\,\chi)^{\frac{1}{2}}\,d\chi$ and $(\mathrm{sn}\,\chi)^{\frac{3}{2}}\,d\chi$.

17. Consider the representation of the upper half t-plane on the z-plane by the transformations

$$\text{(i)}\ z = \int_0^t \frac{t^{\frac{1}{4}}\,dt}{\sqrt{T}}, \quad \text{(ii)}\ z = \int_0^t \frac{t^{\frac{3}{4}}\,dt}{\sqrt{T}}, \quad \text{(iii)}\ z = \int_0^t \frac{t^{\frac{5}{4}}\,dt}{\sqrt{T}}$$

where $T = 4t(1 - t)(1 - k^2t)$. Show how to work out the integrals.

---

* See Greenhill, *Elliptic Functions*, p. 164, where minor corrections are necessary.

18. Discuss the representation of the upper half t-plane on the z-plane by the transformation

$$z = \int_0^t \frac{\mu - t}{\nu - t} \frac{dt}{t^{\frac{1}{2}} \sqrt{T}}$$

where $T = 4t(1 - t)(1 - k^2 t)$. Show how the shape of the boundary in the z-plane varies while μ increases from $-\infty$ to $+\infty$, and ν remains constant, in the four cases:

(i) $-\infty < \nu < 0$, (ii) $0 < \nu < 1$, (iii) $1 < \nu < 1/k^2$, (iv) $1/k^2 < \nu < \infty$.

Consider also the limiting shapes when $k = 0$ and $k = 1$.

19. Take the integral $\int dz/(1 + z^8)^{\frac{1}{2}}$ round the closed contour composed of the boundary of the infinite first quadrant and loops from the origin round the two branch points $z = e^{\frac{1}{8}\pi i}$, $z = e^{\frac{3}{8}\pi i}$. Hence show that (see Ex. 14)

$$\int_0^\infty (1 + x^8)^{-\frac{1}{2}} dx = 2\sqrt{2} \sin \tfrac{1}{8}\pi \int_0^1 (1 - x^8)^{-\frac{1}{2}} dx$$

20. Take the integral $\int dz/(1 - z^8)^{\frac{1}{2}}$ round the closed contour composed of the boundary of the infinite upper half-plane indented at $z = 1$ and $z = -1$, together with loops round the branch points $z = e^{\frac{1}{4}\pi i}$, $z = i$, $z = e^{\frac{3}{4}\pi i}$. Deduce that (see Ex. 14)

$$\int_1^\infty (x^8 - 1)^{-\frac{1}{2}} dx = \int_0^1 x^2 (1 - x^8)^{-\frac{1}{2}} dx = (\sqrt{2} - 1) \int_0^1 (1 - x^8)^{-\frac{1}{2}} dx$$

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TABLES OF FORMULÆ

Table I

$$s^2 + c^2 = 1. \quad k^2s^2 + d^2 = 1. \quad k^2c^2 + k'^2 = d^2. \quad k'^2s^2 + c^2 = d^2.$$

$$k = 0, \ \text{sn}\, u = \sin u, \quad \text{cn}\, u = \cos u, \quad \text{dn}\, u = 1.$$

$$k = 1, \ \text{sn}\, u = \tanh u, \quad \text{cn}\, u = \text{dn}\, u = \text{sech}\, u.$$

$$\text{sn}\,(u + v) = (s_1c_2d_2 + s_2c_1d_1)/(1 - k^2s_1^2s_2^2)$$

$$\text{cn}\,(u + v) = (c_1c_2 - s_1s_2d_1d_2)/(1 - k^2s_1^2s_2^2)$$

$$\text{dn}\,(u + v) = (d_1d_2 - k^2s_1s_2c_1c_2)/(1 - k^2s_1^2s_2^2)$$

$$S = \text{sn}\, 2u = 2scd/(1 - k^2s^4)$$

$$C = \text{cn}\, 2u = (1 - 2s^2 + k^2s^4)/(1 - k^2s^4)$$

$$D = \text{dn}\, 2u = (1 - 2k^2s^2 + k^2s^4)/(1 - k^2s^4)$$

$$s^2 = \frac{1 - C}{1 + D}, \quad c^2 = \frac{D + C}{1 + D}, \quad d^2 = \frac{D + C}{1 + C}$$

$$\text{sn}\,(u + iv) = (sd_1 + icds_1c_1)/(1 - d^2s_1^2)$$

$$\text{cn}\,(u + iv) = (cc_1 - isds_1d_1)/(1 - d^2s_1^2)$$

$$\text{dn}\,(u + iv) = (dc_1d_1 - ik^2scs_1)/(1 - d^2s_1^2)$$

$$|\text{sn}^2\, w| = (1 - c^2c_1^2)/(1 - d^2s_1^2) = (1 - CC_1)/(D_1 + DC_1)$$

$$|\text{cn}^2\, w| = (1 - s^2d_1^2)/(1 - d^2s_1^2) = (D + CD_1)/(D_1 + DC_1)$$

$$|\text{dn}^2\, w| = (d_1^2 - k^2s^2)/(1 - d^2s_1^2) = (D + CD_1)/(1 + CC_1)$$

where $w = u + iv$, $s = \text{sn}\,(u, k)$, $s_1 = \text{sn}\,(v, k')$, $C = \text{cn}\,(2u, k)$, $C_1 = \text{cn}\,(2v, k')$, etc.

$$|\text{sn}\, w \pm 1| = (1 \pm sd_1)/\sqrt{(1 - d^2s_1^2)}$$

$$|\text{cn}\, w \pm 1| = (1 \pm cc_1)/\sqrt{(1 - d^2s_1^2)}$$

$$|\text{dn}\, w \pm 1| = (d_1 \pm dc_1)/\sqrt{(1 - d^2s_1^2)}$$

$$|\text{sn}\, w \pm 1/k| = (d_1 \pm ks)/\{k\sqrt{(1 - d^2s_1^2)}\}$$

$$|\text{cn}\, w \pm ik'/k| = (dd_1 \mp kk'ss_1)/\{k\sqrt{(1 - d^2s_1^2)}\}$$

$$|\text{dn}\, w \pm k'| = (dd_1 \pm k'c_1)/\sqrt{(1 - d^2s_1^2)}$$

Table II

$u + iK'$	$u + K + iK'$
$\dfrac{1}{k \operatorname{sn} u}$	$\dfrac{\operatorname{dn} u}{k \operatorname{cn} u}$
$-\dfrac{i \operatorname{dn} u}{k \operatorname{sn} u}$	$-\dfrac{ik'}{k \operatorname{cn} u}$
$-\dfrac{i \operatorname{cn} u}{\operatorname{sn} u}$	$\dfrac{ik' \operatorname{sn} u}{\operatorname{cn} u}$
$E(u) + i(K' - E') + \dfrac{cd}{s}$	$E(u) + E + i(K' - E') - \dfrac{sd}{c}$
$Z(u) - \dfrac{\pi i}{2K} + \dfrac{cd}{s}$	$Z(u) - \dfrac{\pi i}{2K} - \dfrac{sd}{c}$
$\operatorname{sn} u$	$\dfrac{\operatorname{cn} u}{\operatorname{dn} u}$
$\operatorname{cn} u$	$-\dfrac{k' \operatorname{sn} u}{\operatorname{dn} u}$
$\operatorname{dn} u$	$\dfrac{k'}{\operatorname{dn} u}$
$E(u)$	$E(u) + E - \dfrac{k^2 sc}{d}$
$Z(u)$	$Z(u) - \dfrac{k^2 sc}{d}$
u	$u + K$

TABLE III

In terms which depend upon a modulus as well as upon the variable v, the modulus is k'.

$iv + iK'$	$iv + K + iK'$
$-\dfrac{i \operatorname{cn} v}{k \operatorname{sn} v}$	$\dfrac{\operatorname{dn} v}{k}$
$-\dfrac{\operatorname{dn} v}{k \operatorname{sn} v}$	$-\dfrac{ik' \operatorname{cn} v}{k}$
$-\dfrac{1}{\operatorname{sn} v}$	$-k' \operatorname{sn} v$
$i\left\{v - E(v) + K' - E' - \dfrac{cd}{s}\right\}$	$E + i\{v - E(v) + K' - E'\}$
$-i\left\{Z(v) + \dfrac{\pi(v + K')}{2KK'} + \dfrac{cd}{s}\right\}$	$-i\left\{Z(v) + \dfrac{\pi(v + K')}{2KK'}\right\}$
$\dfrac{i \operatorname{sn} v}{\operatorname{cn} v}$	$\dfrac{1}{\operatorname{dn} v}$
$\dfrac{1}{\operatorname{cn} v}$	$-\dfrac{ik' \operatorname{sn} v}{\operatorname{dn} v}$
$\dfrac{\operatorname{dn} v}{\operatorname{cn} v}$	$\dfrac{k' \operatorname{cn} v}{\operatorname{dn} v}$
$i\left\{v - E(v) + \dfrac{sd}{c}\right\}$	$E + i\left\{v - E(v) + \dfrac{k'^2 sc}{d}\right\}$
$-i\left\{Z(v) + \dfrac{\pi v}{2KK'} - \dfrac{sd}{c}\right\}$	$-i\left\{Z(v) + \dfrac{\pi v}{2KK'} - \dfrac{k'^2 sc}{d}\right\}$
iv	$iv + K$

TABLE IV

iK'	$\frac{1}{2}K + iK'$	$K + iK'$
∞	$1/\sqrt{(1-k')}$	$1/k$
∞	$-i\sqrt{k'}/\sqrt{(1-k')}$	$-ik'/k$
∞	$-i\sqrt{k'}$	0
∞	$\frac{1}{2}(E+1+k') + i(K'-E')$	$E + i(K'-E')$
∞	$\frac{1}{2}(1+k'-\pi i/K)$	$-\frac{1}{2}\pi i/K$
$\dfrac{i}{\sqrt{k}}$	$\dfrac{\sqrt{(1+k)}+i\sqrt{(1-k)}}{\sqrt{(2k)}}$	$\dfrac{1}{\sqrt{k}}$
$\dfrac{\sqrt{(1+k)}}{\sqrt{k}}$	$\dfrac{(1-i)\sqrt{k'}}{\sqrt{(2k)}}$	$-\dfrac{i\sqrt{(1-k)}}{\sqrt{k}}$
$\sqrt{(1+k)}$	$\dfrac{\sqrt{k'}}{\sqrt{2}}\{\sqrt{(1+k')} - i\sqrt{(1-k')}\}$	$\sqrt{(1-k)}$
$\frac{1}{2}i(K'-E'+1+k)$	$\frac{1}{2}(E+k+ik') + \frac{1}{2}i(K'-E')$	$E + \frac{1}{2}i(K'-E'+1-k)$
$\frac{1}{2}i\left(-\dfrac{\pi}{2K}+1+k\right)$	$-\dfrac{\pi i}{4K} + \frac{1}{2}(k+ik')$	$-\dfrac{\pi i}{4K} + \frac{1}{2}i(1-k)$
$\operatorname{sn} 0 = 0$	$1/\sqrt{(1+k')}$	1
$\operatorname{cn} 0 = 1$	$\sqrt{k'}/\sqrt{(1+k')}$	0
$\operatorname{dn} 0 = 1$	$\sqrt{k'}$	k'
$E(0) = 0$	$\frac{1}{2}(E+1-k')$	E
$Z(0) = 0$	$\frac{1}{2}(1-k')$	0
0	$\frac{1}{2}K$	K

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