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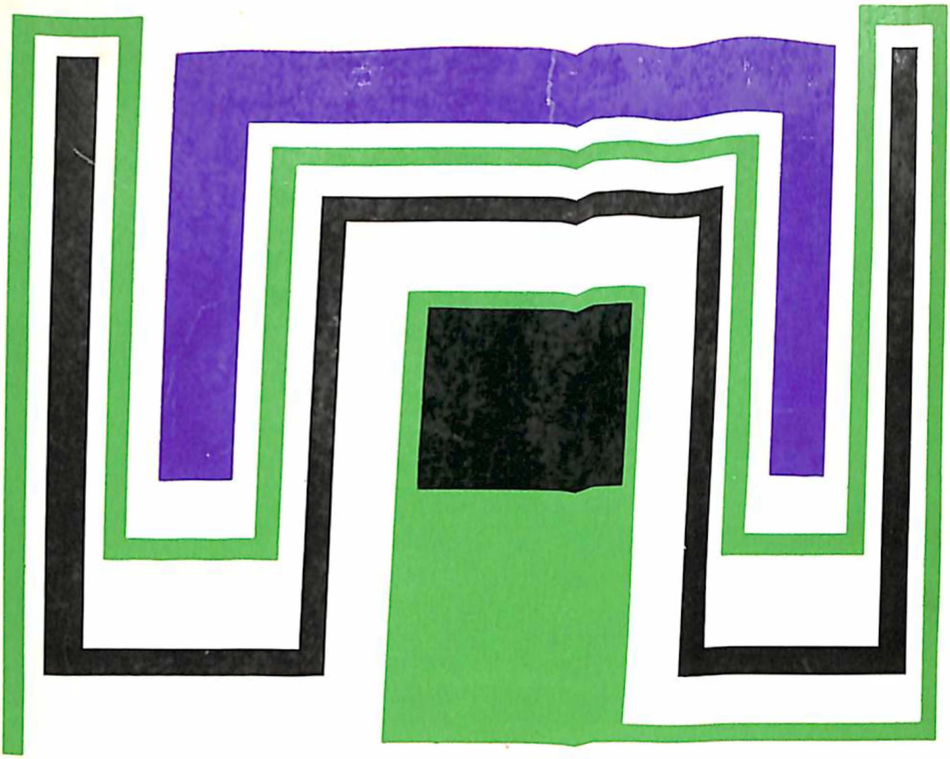

CONTEMPORARY GEOMETRY

ANDRÉ DELACHET

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CONTEMPORARY GEOMETRY

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Foreword

GEOMETRY IS CERTAINLY the branch of mathematics which most attracts the layman. Nowhere with so much power as in geometry do truth and beauty appear so intimately connected. Was it not Fénelon who said: *Defy the bewitchments and devilish charms of geometry!*

Attracted first of all by the supreme harmony of forms, the mind soon lets itself be captivated by the wonderful chain of ratiocination to which they have given rise. It wishes to pursue more profoundly its excavations in this monument of pure classic beauty: mathematics.

Ancient Greece, until Euclid, regarded mathematics as an art more than a science, but although there often are fascinating aesthetic satisfactions which stimulate contemporary mathematicians to cultivate their beloved science, the modern Occident has not sanctioned this point of view.

It is to the cultured man of the twentieth century who has known how to appreciate this beauty but whose business has kept him away from the "Temple of Mathematics" that we wish to dedicate this work.

After making a hasty survey to recall the condition of geometry in the last century, we shall endeavor to show the influence of the notion of "group" in geometry and we shall seek to guide the reader, in the simplest possible manner, from the concrete notions of elementary geometry to the modern conceptions of "abstract spaces."

The kindness of M. Fréchet, professor at the Sorbonne, and M. Ky Fan, doctor of mathematics, who have been very willing to lavish on us their enlightened advice, has permitted us to devote the last part of this work to topology, that modern branch of mathematics in which Poincaré perceived the very essence of geometry.

Finally, under the heading "Applications of Topology," we have wished to pay tribute to the very interesting classes in "higher geometry" which G. Bouligand teaches at the Sorbonne, and to introduce the reader to that "finite geometry" which Montel said in a recent lecture required more excellence of imagination than of knowledge.

The translator wishes to express here his gratitude to many colleagues at the City College—and in particular to Prof. Bennington P. Gill—who gave generous help and encouragement.

Contents

Historical Introduction

As early as the beginning of the nineteenth century, as Chasles observed (*Rapports sur les progrès de la Géométrie*, 1870), mathematics, considered independently of its applications, divided into two distinct branches which gave one another mutual help: analysis and geometry.

From the point of view of method, geometry presented in the nineteenth century two different directions, which took their point of departure principally in the works of Gaspard Monge and Carnot: in the *Traité de l'Application de l'Analyse à la Géométrie* and the *Géométrie descriptive* of Monge, and in the *Géométrie de position* and the *Théorie des Transversales* of Carnot.

It was in the Ecole Polytechnique, almost from its inception, that these discoveries were made. Indeed, from the beginning, this great establishment was able to breed eminent scholars, for at the same time that pupils were prepared there for admission to the Ecoles d'Applications, they were taught the latest advances in science.

1. Infinitesimal Geometry. The great treatise by Monge on the application of analysis to geometry was written first under the title *Feuilles d'analyse appliquée à la Géométrie*, and as early as 1799 the author taught his new theory to the Polytechnic students. This work is based on the use of Cartesian coordinates,[1] but the profound considerations of the author concerning the generation of surfaces defined by properties based on *curvature*, and the conclusions which he deduced from them regarding the integration of partial differential equations, constituted an entirely new body of knowledge, allied at the same time with the most promising questions of general geometry and with the most difficult theories of the integral calculus. Thanks to the work of Monge, the conceptions of Descartes and Fermat, the inventors of analytic geometry,

[1] See the definition of these coordinates in one of the following works: M. Boll, *Les Etapes des mathématiques* ("Que sais-je?", No. 42); P. Marchal, *Histoire de la Géométrie* ("Que sais-je?", No. 109). See also p. 13.

have regained the position, alongside the infinitesimal calculus of Leibniz and Newton, which they had been allowed to lose during the course of the eighteenth century and which they should never have ceased to occupy. Thanks to it also, mathematicians have understood that the alliance of geometry and analysis is useful and fertile and that this alliance is a condition of success for both. Analysis of this work of Monge can have no place in the elementary study which we are undertaking. Nevertheless, without being too technical, we can try to make comprehensible the important notion of *curvature*:

Let us consider a circle with center at I and of radius R. It is evident that the greater the radius of this circle, the more nearly the arc will assume the appearance of a line; thus, the greater the radius of the circle to which we are comparing a curve, the smaller the curvature; "hair-pin" curves are those which have a very small *radius of curvature*. It is therefore natural to define the curvature of a circle as the reciprocal of its radius: 1/R.

Let us now try to give an account of the curvature at a point O of any plane curve C whatever (Fig. 1). Let us suppose that this curve has a tangent at O which we take as the axis Ox. Let Oy be the *normal* to C at O, that is, the perpendicular to the tangent Ox. The axes Ox and Oy form a Cartesian reference system with respect to which C has an equation $f(x, y) = 0$ which expresses the necessary and sufficient condition which the "coordinates" $\overline{\text{KM}} = x$ and $\overline{\text{OK}} = y$ of a point M in the plane must satisfy in order to be on this curve. Through this point M passes a circle I tangent at O to Ox, whose center I is therefore on Oy (and on the perpendicular bisector of the segment OM). This circle I or rather the arc $\overset{\frown}{\text{OM}}$ of this circle fits the contour of C better and better the nearer M is

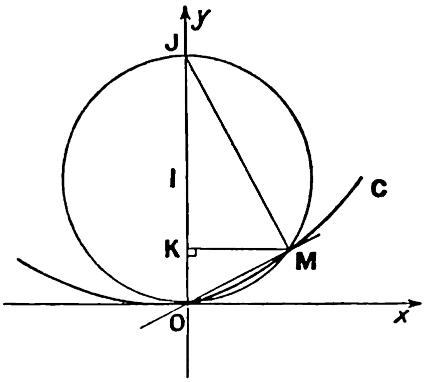

FIG. 1

to O. Consequently, the nearer M is to O, the closer the curvature of C is to that of I along the arc $\widehat{OM}$. We are thus led to make the point M approach the point O and to define the curvature at O of this curve as that of the circle I_O, the limit of circle I.

Now, angle $\widehat{OMJ}$, inscribed in a semicircle, is a right angle, and, as a result of an elementary relation in the right triangle: $\overline{OM}^2 = \overline{OK} \cdot \overline{OJ}$; let $x^2 + y^2 = 2Rx$ (R being the radius of the circle I). Therefore: $R = y^2/2x + x/2$.

When M approaches O, x and y approach *zero*, whence R approaches a limit R_O equal to the limit of $y^2/2x$, if this last exists. The circle I therefore has for its limiting position the circle I_O tangent at O to Ox, with center I_O, such that $\overline{OI_O} = R_O$. This circle is called the *osculating circle* of the curve C at the point O. It assumes virtually the shape of the curve C along a small arc which includes the point O. Its radius R_O is called the radius of curvature at O of the curve C. The curvature of this curve at this point has the value $1/R_O$.

These properties of plane curves are extended without difficulty to *skew* curves (that is, curves not contained in a plane), and are generalized for surfaces.

In his *Mémoire sur la courbure des surfaces*, read to the Académie in 1776, Meusnier already gave a complete theory of the curvature of surfaces at a point, entirely distinct from that which Euler had made known in 1760 in his *Recherches sur la courbure des surfaces* (Mémoire de l'Académie de Berlin). In his memoir, Meusnier studied, among other things, the radius of curvature of the sections of a surface made by a plane passing through a point of it. He thus obtained the very simple properties discovered by Euler, and, moreover, the beautiful theorem which shows the relation of the radius of curvature of oblique sections to that of normal sections:

Let a surface be cut by a plane Π passing through the line Mx tangent to this surface at a point M. Then the center of curvature ω of the section Γ is the orthogonal projection on Π of O, the center of curvature at M of the section C of the surface made by the plane P normal to the surface and passing through Mx (Fig. 2); so that if Π is revolved about Mx, ω describes a circle, with diameter MO, in the plane perpendicular to Mx at M.

Meusnier's memoir is remarkable as much for its simplicity as for its fecundity. Up to nearly the end of the nineteenth century, it gave rise to numerous investigations by making intuitive certain results obtained by the scholars of the eighteenth century.

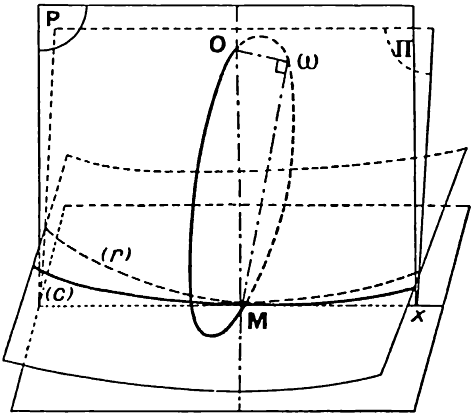

FIG. 2

Among the pupils of Monge, we must very particularly single out Baron Charles Dupin, whose works have had the greatest influence on the progress of science, as much in research in pure geometry as in mechanics and mathematical physics.

Dupin, a naval architect who left the Ecole Polytechnique in 1803, revealed his capacities as early as the first year of his attendance at the Ecole by his solution of the problem of *spheres tangent to three others*, and by the discovery of the admirable theorems to which this question had led him. Among his discoveries, many of which have become classic topics taught in classes in advanced mathematics in our high schools and colleges, we shall mention the important theory of the *indicatrix of curvature* of a surface at each point, which sums up and elucidates from a new and promising point of view, the first results of Euler and Meusnier relating to the radius of curvature of normal sections of a surface, and which lends itself to the most fruitful applications and developments.

Dupin had had the inspired idea of measuring off on the tangent Mx at M to the normal section C of the surface (Fig. 2) the lengths MN = MN′ = $\sqrt{\text{MO}}$ (the radius of curvature of this section), and had proven that the geometric locus of these points N and N′, when P rotates about the normal at M to the surface, is a certain conic I which he called the *indicatrix of curvature*. In addition to giving a first approximation of the shape of the surface in the neighborhood of the point M (for it is very obviously similar to the section of this surface by a plane parallel to and close to its tangent plane at M), this indicatrix of curvature reveals new theories and new aspects of older theories. The *lines of curvature of a surface*, curves traced on the surface which have an *extremal* (maximum or minimum) radius of curvature at each point, appear as the curves whose tangents at each point are the axes of the indicatrix.

Dupin studied two new systems of curves which he called *asymptotic curves* because a curve of such a system is tangent at each point to an asymptote of the indicatrix of the surface. These asymptotic curves, which were slow to attract the attention of geometers, have since been included in the majority of their works on surfaces, and even in our times, Bouligand has seen fit to deal with them in generalizing them by the notion of *asymptotiques d'option.*

Dupin was led, by all his preliminary labors, to the following celebrated theorem: *Three sets of orthogonal surfaces always intersect along their lines of curvature*, a theorem which has become the basis of a host of investigations of surfaces, whether one is studying their general properties or whether one has to consider their occurrence in questions of mathematical physics.

It can be said that every principal result of inquiries to which Dupin's energetic mind had been attracted was constantly met with again during the course of the nineteenth century and often even in our times in the studies of geometers. The fact is that the thoughts of the author were not directed to problems picked out at random and without prospects: the sense of beauty and utility and an intelligent enthusiasm for science never ceased to inspire him. Thus it was that he was led to the *theory of normal congruences* through the practical problem of the reflection of light rays. This celebrated geometer illustrates the fact that only the cooperation of theory and practice can make science progress.

Under the influence of the works of the pupils of Monge, among whom, in addition to Dupin, we ought to mention Lancret, author of admirable discoveries about skew curves, infinitesimal geometry regained the position in all research which Lagrange had wanted to take away from it forever.[2]

The geometric methods thus re-established were to receive the liveliest impetus after the publication of *Disquisitiones generales circa superficies curvas*, brought out by Gauss in 1827. Beginning with the publication of this treatise, which from the very first seemed connected with the purest analysis, the infinitesimal method assumed a scope in France hitherto unknown. Frenet, Bertrand, Möbius, J.-A. Serret, Bouquet, Puiseux, Ossian, Bonnet, and Paul Serret developed the theory of skew curves. Liouville, Chasles, and Minding joined with them in order to carry on the systematic study of Gauss's treatise. Jacobi, integrating the differential equation of the *geodesic lines* of the ellipsoid, inspired a large number of investigations.

Initially, the shortest path from a point A to another point B of a surface was called a *geodesic line* of the surface. Thus the geodesics of a plane are the straight lines of this plane, those of the sphere, the great circles of that sphere. The problem of investigating the geodesics of a surface is one of the first minimum problems which requires the determination of an unknown function. This problem is part of the branch of functional analysis called the *calculus of variations*.[2] Geometers have been able to show that these geodesics of a surface are nothing but curves drawn on the surface possessing at each point an *osculating plane* normal to the surface.[3]

At the same time, the problems studied in the *Application de l'Analyse* of Monge were fully developed. Gabriel Lamé, using the results obtained by Dupin and Binet, became the creator of an entirely new theory destined to receive the most varied applications in mathematical physics.

2. Synthetic Geometry. It is to a pupil of Monge—the

[2] Cf. *L'Analyse mathématique* ("Que sais-je?", No. 378).

[3] We call the osculating plane to a curve at a point M the limit, if it exists, of the plane passing through the tangent at this point to this curve and through a point, M′, infinitesimally close, when M′ approaches M. Cf. Delachet, *Calcul vectoriel et calcul tensoriel* ("Que sais-je?", No. 418), p. 58.

French mathematician Poncelet—that synthetic geometry owes the fertile ideas which regenerated it. Taken prisoner by the Russians in 1813 in the crossing of the Dnieper, and interned at Saratoff, Poncelet employed the leisure which his captivity afforded him in proving the principles which he had developed in the *Traité des propriétés projectives des figures*, published in 1822, and in the great memoirs on *polar reciprocals* and on *harmonic means*, which date from almost the same period. His ideas, much debated by French analysts, and most particularly by Cauchy, were stated precisely by Gergonne, to whom geometry owes the famous *principle of duality*.[4] Later, Chasles and Steiner, who devoted their entire lives to investigating pure geometry, embracing in the main, if not in detail, Poncelet's predilections, proposed to establish an independent theory, rivaling Descartes' analytics.

Chasles has set forth his ideas in two works of great importance, the *Traité de Géométrie Supérieure* (1852) and the *Traité des Sections Coniques*, unfortunately unfinished and only the first part of which was published in 1865.

The three fundamental points of his doctrine are:

1. The introduction of the principle of signs, which simplifies at one and the same time the statements and the proofs, and which accords the fullest possible import to Carnot's analysis of transversals.

This principle is often poorly understood by pupils of our secondary schools, notwithstanding its being taught as early as the third class of the *lycée*. We are reminded of it when applying the *relation of Chasles* to evaluate the algebraic length of a line segment $\overline{AB}$ lying along the axis $x'Ox$:

$$\overline{AB} = \overline{OB} - \overline{OA}$$

This principle was not quite so new as Chasles believed when he wrote his *Traité de Géométrie Supérieure*; Möbius, in his *Calcul Barycentrique*, and Grassmann had already made use of it.

2. The introduction of *imaginaries*, which took the place of Poncelet's *principle of continuity* and provided proofs as general as those of analytic geometry.

[4] See Marchal, *Histoire de la Géométrie* ("Que sais-je?", No. 109). See also p. 17.

This method is truly new. Chasles knew how to illustrate it by examples of great interest, well known to "*taupins*."* But Chasles introduced imaginaries only in terms of their symmetric functions, so that he was not able to define the *anharmonic ratio* of four elements when these cease to be real, in whole or in part.[5]

Four years later von Staudt (1856) established the complete method of calculating the most general anharmonic ratios of imaginary elements in the *Beiträge der Geometrie der Lage*. This very original extension of the method of Chasles, although rigorous, is laborious and very abstract: it is admirable chiefly for the ingenuity its author has displayed in achieving it.

3. The simultaneous proof of propositions which are correlative, that is to say, which correspond to one another by virtue of the principle of duality.

We meet such properties as early as the elementary mathematics class; for example, if two points are collinear on line D, the polars of these points with respect to a circle C are concurrent at a point P, the pole of the line D with respect to the circle C.

Like Chasles, Steiner followed the path of pure geometry, but he neglected to give us a complete account of the methods upon which he depended. These methods seemed to rest in part on the introduction of *elementary geometric forms*, already considered by Desargues.

Let us give a simple example to illustrate what is meant by elementary geometric forms:

Given a complete quadrilateral (Fig. 3), that is, the figure formed by four lines (the sides of the quadrilateral) which intersect by twos in six points A, B, C, D, E, F (the six vertices

* Trans. note: *Taupins*, literally "click beetles," is a term used to refer to students preparing for the entrance examination to the Ecole Polytechnique.

[5] The idea of anharmonic, or cross, ratio is an extension of the notion of harmonic division. Given four collinear points A, B, C, D, we call the anharmonic ratio of these four points the number

$$\frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$$

If this number is equal to -1, the ratio is called harmonic.

of this quadrilateral). The lines joining the vertices in pairs and which are not the sides, i.e., AD, EB, CF, are the diagonals.

If from any point whatever, S, of space, we project this figure on a plane parallel to FC, for example, the projections of the sides of this quadrilateral are lines parallel in pairs. They form a parallelogram, which is an *elementary geometric form* of the complete quadrilateral.

The projectivity of the anharmonic ratio permits us to deduce immediately from the known theorem "The diagonals of a parallelogram bisect each other" the new theorem "Each diagonal of a complete quadrilateral is divided harmonically by the other two."

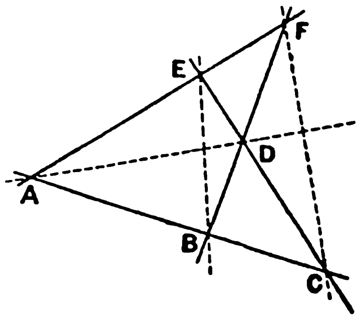

FIG. 3

Steiner studied in particular the construction of curves and surfaces of higher degree, with the aid of pencils or networks of curves of lower order.

3. Composite Methods. Paralleling the labors of the pure geometers, Gergonne, Bobillier, Sturm and, above all, Plücker had perfected the geometry of Descartes and established an analytic system suited to the discoveries of geometers. It was Plücker who established by his works the foundations of modern analytic geometry. We owe to him *tangential coordinates*, *homogeneous* (trilinear) *coordinates*, and, finally, the use of *canonical forms* whose validity was discovered by the method—sometimes so fallacious, but so fruitful—of *counting constants*.

At this time a brilliant period opened for geometric research of every nature. Analysts interpreted every result geometrically. Geometers strove to discover in each question some general principle, most often undemonstrable by geometric techniques, in order to make a host of particular deductions flow from it effortlessly, all firmly bound to one another and to the principle from which they were derived.

This was the beginning of the *composite method* which was to take on its full worth with mathematicians such as Otto

Hesse, who knew how to give the method of Plücker its full power; Boole who uncovered the first notion of *covariant* in the works of Bobillier; Cayley, Sylvester, Hermite, Brioschi, who created the *theory of forms*, a theory which was to assume its complete fullness with Aronhold, Clebsch, Gordan . . . and which Elie Cartan was to generalize at the beginning of the twentieth century in the notion of *exterior form*.

This was also the dawn of *algebraic geometry* with Cayley, Salmon, Cremona, Kummer, Moutard, Laguerre, Clebsch, Magnus. However, the prime motive power for algebraic geometry was not only modern geometry, but also Legendre's theory of elliptic functions, too much neglected by the French geometers of the nineteenth century, but developed and augmented by Abel and Jacobi, and later by Weierstrass and Riemann.

Pure geometers did not remain inactive. Poinsot, creator of the theory of torque, cleared up by a synthetic method problems of solid mechanics which the investigations of d'Alembert, Euler and Lagrange seemed to have definitely settled. Chasles and Steiner also brought their contribution to bear upon mechanics via this medium of pure geometry. It was in this period that Chasles conceived his principle of correspondence between two variable quantities which was to be so fruitful. For several years this celebrated postulate was accepted without objection; numerous geometers thought they had established it incontestably. But, as Zeuthen then said, it is indeed difficult in proofs of this type to recognize whether there is not some small point which has eluded its authors. It fell to Halphen to consummate definitely all this research by pointing out precisely the range of validity of Chasles's postulate.

In infinitesimal geometry we meet again the two predilections which we have just discussed in finite geometry. Some, with J. Bertrand and O. Bonnet, wished to establish an independent method based directly on the use of infinitesimals; others on the contrary, like Lamé and Beltrami, followed the customary analytic course.

Since the work of Ribaucour, geometers seem to have been won over to the composite method. The rectangular axes of analytic geometry are preserved but made *variable* by relating them in what seems the most convenient manner to the system which we wish to study. This method of the *moving trihedron* makes the majority of the objections to the analytic method disappear. It reunites the advantages of what we call *intrinsic geometry* (the study of a geometric entity in itself) to those which result from the use of classical analysis. The complications of calculation which this method entails usually disappear if we make use of the notions of *invariants* or *co-variants* of quadratic differential forms[6] which we owe to the works of Lipschitz and of Christoffel, inspired by Riemann's studies in non-Euclidean geometry.[7]

At the beginning of the twentieth century, Elie Cartan was able to give this method its full scope in his inestimably valuable research on the geometry of Riemann spaces.

[6] Cf. E. Cartan, *Notions sur les invariants intégraux* (Hermann) 1922.
[7] Cf. E. Cartan, *op. cit.*

Part I
THE NOTION OF GROUP

CHAPTER ONE

The Algebraic Origin of Group

THERE IS A THEORY which in a span of sixty years has had an extraordinary success among the branches of mathematics, and which is today beginning to be introduced in certain physical theories: the theory of groups.

Without wishing to undertake here a systematic exposition of this abstract (because very general) notion, we shall still give this important geometrical theory the attention it merits. First of all, what is its origin?[1]

It was a young French geometer, Evariste Galois (1811–1832), killed in a duel at the age of twenty, who first introduced this notion in connection with the solution of algebraic equations. The life of Galois, although very short, is replete with romantic episodes which have no place in this work.[2] His theory of the solution of algebraic equations, which he had hurriedly summed up in sixty pages during the night preceding his duel, was much debated and for a long time unappreciated. Toward the end of the nineteenth century, Hermite was to explicate its full importance and especially to show how great was the genius of this young man who had been able to create a new and valuable method whose import he had only imperfectly foreseen.

This notion of group has attained its full generality in the theory of sets. Nowadays, we call a group any set G of elements which satisfies the following axioms:

1. There exists a *law of combination* which associates to every pair of elements (x, y) of G, taken in that order, an element z of G, called its *combination*. Such a correspondence is generally written:

$$x \,\&\, y = z$$

2. This law of combination is associative, that is,

$$(x \,\&\, y) \,\&\, z = x \,\&\, (y \,\&\, z)$$

[1] The reader interested in thoroughly examining this notion should refer to M. Queysanne and A. Delachet, *L'Algèbre moderne* ("Que sais-je?", No. 661).

[2] Cf. E. T. Bell, *Les Grands mathématiciens*, 1929.

the parentheses denoting, as is always the case in algebra, the *combinations* effected.

3. There exists a *unit* (or *null*) element, e, such that

$$x \mathbin{\&} e = e \mathbin{\&} x = x$$

for any x.

4. Every element x possesses an *inverse* element x^{-1} such that

$$x \mathbin{\&} x^{-1} = x^{-1} \mathbin{\&} x = e$$

Defined in so general a manner, this notion can understandably be applied to very varied situations. Let us give a few examples:

The set of positive rational numbers (integers and fractions) forms a group if ordinary multiplication is given as the rule of combination. The unit element of this group is the number 1. This set indeed satisfies the four preceding axioms (as shown in elementary mathematics classes), but it is less general than a group defined by these axioms, for it satisfies a fifth one: multiplication is *commutative*, that is: $x \cdot y = y \cdot x$.

Such a group is called *commutative* or *Abelian*. *Abelian* groups possess properties of their own. Similarly, the set of all rational numbers (positive, negative, and zero) forms an *Abelian* group having ordinary addition as the law of combination. The unit here is 0; the inverse of a number x is $-x$. This group is Abelian, for addition is not only associative, but also commutative.

The two examples we have just given are examples of particular groups—*Abelian* groups. It is easy to give a more general example of them: the very same one which enabled E. Galois to construct his theory of the solution of an algebraic equation, the group of *substitutions on a finite number of letters.*[3]

To simplify our account, let us content ourselves with studying the case of three letters: a, b, c.

We say we have effected a *substitution* on the set of the three letters when we have made an interchange among them, involving either all at once or only some of them. These *substitutions* are six in all: the number of ways in which we can write these three letters in different orders.[4] Each of these

[3] Cf. *L'Algèbre moderne.*

[4] Or the number of "linear permutations" of these three letters.

substitutions is a certain *operation* carried out on the elements of the set of letters a, b, c which associates with some one of these elements (the letter a, for example) another element of the set which then replaces it. Writing underneath each of the letters a, b, c its transform by the operation being considered, we can define the six substitutions by the following table:

$$u = \begin{vmatrix} a & b & c \\ a & b & c \end{vmatrix} \qquad v = \begin{vmatrix} a & b & c \\ b & c & a \end{vmatrix} \qquad w = \begin{vmatrix} a & b & c \\ c & a & b \end{vmatrix}$$

$$x = \begin{vmatrix} a & b & c \\ a & c & b \end{vmatrix} \qquad y = \begin{vmatrix} a & b & c \\ c & b & a \end{vmatrix} \qquad z = \begin{vmatrix} a & b & c \\ b & a & c \end{vmatrix}$$

Let us show that the set of these six substitutions forms a group. For this purpose, we must first of all define a law of combination for these elements u, v, w, x, y, z. It is natural to say that the *combination* of two of these elements is the unique substitution which exchanges the letters a, b, c among themselves in the same way as the *successive application* of the two substitutions represented by these elements does.

Ever since Cauchy, we call the *combination* of two substitutions their *product*. Let us study, for example, the product of

$$x = \begin{vmatrix} a & b & c \\ a & c & b \end{vmatrix} \quad \text{and of } y = \begin{vmatrix} a & b & c \\ c & b & a \end{vmatrix}$$

x changes a into a, which y transforms into c;
x changes b into c, which y transforms into a;
x changes c into b, which y transforms into b.

Thus the product x & y changes (a, b, c) into (c, a, b), which we shall write as

$$x \,\&\, y = \begin{vmatrix} a & b & c \\ c & a & b \end{vmatrix} = w$$ [5]

which is indeed a substitution of the set.

This law of combination is *associative* from its very definition.

There exists a unit element, the substitution u, whose *left-* or *right-hand* application to a substitution of the set alters nothing, since it changes the letters a, b, c into themselves.

[5] Let us note here that x & y denotes the product of x first, then y.

We generally name such a unit substitution the *identity* substitution.

Finally, every element possesses an *inverse* element whose combination with it on the *left* or the *right* gives the identity substitution. Thus we have

$$v \,\&\, w = w \,\&\, v = u, \quad \text{or that} \quad x \,\&\, x = u$$

On the other hand, this group is not *Abelian*, for its law of combination is not commutative. For example,

$$x \,\&\, y = \begin{vmatrix} a & b & c \\ a & c & b \end{vmatrix} \;\&\; \begin{vmatrix} a & b & c \\ c & b & a \end{vmatrix} = \begin{vmatrix} a & b & c \\ c & a & b \end{vmatrix} = w$$

while

$$y \,\&\, x = \begin{vmatrix} a & b & c \\ c & b & a \end{vmatrix} \;\&\; \begin{vmatrix} a & b & c \\ a & c & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \end{vmatrix} = v$$

As with multiplication of real numbers, we can set up a table for this law of combination, which is read like the table of Pythagoras.

2nd factor / 1st factor	u	v	w	x	y	z
u	u	v	w	x	y	z
v	v	w	u	y	z	x
w	w	u	v	z	x	y
x	x	z	y	u	w	v
y	y	x	z	v	u	w
z	z	y	x	w	v	u

This table exhibits a law of combination among the three elements u, v, w which shows that this set is a group. This group is, moreover, Abelian, as we see from the table. It belongs to the total group of substitutions on three letters which we have just been studying. We say that it constitutes a *subgroup* of it. It can be shown that the necessary and sufficient condition for a set of elements g of a group G to form a subgroup of this group is that "if v and w are any two elements whatever of g, the element (v & w^{-1}) is also an element of g."[6] This theorem, of use in a great number of cases, is not indispensable here, for the very simple *multiplication table* of the set u, v, w immediately reveals this set as an Abelian group.

The study of certain particular groups has led contemporary mathematicians to the idea of *group structure* from which they have developed the theory independently. This theory is *abstract*[7] in the sense that it does not concern itself with the particular entities which it considers but rather with the relations which it defines among these objects and which do not depend upon their nature. It is thus that two mathematical entities as dissimilar in appearance as the equation of the fifth degree and the regular polyhedron with twenty faces, called the *icosahedron*, when examined closely lead to the same group structure, as Klein showed about 1884. These entities are called *isomorphic*.

Every undergraduate is acquainted with at least one example of two *isomorphic* groups: the group of real, positive numbers (with multiplication as the law of combination) and the group of real numbers, positive, negative, or zero (with addition as the law of combination). We can pass from one to the other with the help of a kind of dictionary: *the logarithm table*.

Any two isomorphic groups whatever are always capable of such a correspondence. As a result of this, in order to study a group, we can do it in a purely abstract way, or, on the contrary, with the aid of a simple concrete representation of this group. Conversely, a concrete theory will be all the better understood if we have been able to describe a group structure with which it is isomorphic. The knowledge of the properties of this group leads the investigation of corresponding concrete theorems back to a simple translation.[8]

The generality of group structure is such that it is encountered in almost all branches of mathematics. We are going to see its importance in geometry.

[6] Cf. E. Cartan, *op. cit.*, §27.
[7] Cf. *L'Analyse mathématique*; and E. Cartan, *op. cit.*
[8] The new programs of advanced mathematics classes now reserve a choice position for this notion.

CHAPTER TWO

Geometry and the Theory of Groups

GROUPS IN ELEMENTARY GEOMETRY

1. The Group of Displacements. The most elementary geometry—the one whose need is thrust upon us in the first place—is the study of figures as they appear in nature to the untutored perception of our senses. It is for this reason that we are led, in good teaching, to define the line as the idealized image of a stretched string, when we give this definition to a young child who has not yet made contact with the abstract world of mathematics; for him, a point will for a long time be the imprint left on a sheet of paper by the application of a sharp pencil. These definitions, borrowed from the world of our perceptions, are necessary to geometry, which could not be established without them.

Nevertheless, we make haste to establish *relations* between these elements which we have defined (for example, "two points determine one and only one line"), and it is these *relations* which we make use of in our proofs, rather than the concrete definitions from which we have started.

Since elementary geometry is the study of the relations existing between the elements, the figures, which comprise it, we can expect, in studying the underlying architecture of these figures, to discover a group structure which characterizes them.

The most important problem which arises here, as in the study of any set, is the division of this set into *equivalence classes*, that is, into a certain number of collections of elements which have the same properties relative to the theory under consideration.[1]

In elementary geometry, we consider figures to have the same properties if they are equal, in the usual sense of the word, that is, if one can make them coincident. For example, to prove that two triangles ABC and A′B′C′ are equal, we show that we can place A, B, C on A′, B′, C′, respectively. This

[1] Cf. *L'Algèbre moderne.*

definition of equal figures leads to the hypothesis of the invariance of geometric figures under any movement whatever of these figures. This hypothesis derives from a property of solids, of which these figures are the idealized images, a property which seems obvious to our senses.

We are thus naturally led to define a relation between two equal figures. This relation should be independent of the concrete operation which permits them to be made coincident—of the *movement*: we give it the name *displacement*.

How shall we study these displacements? The most natural method seems to be to copy the concrete property which has given birth to them. We select from movements those which seem to us the simplest: the movement by *translation* first of all, which is characterized by the fact that two positions of the same vector $\overrightarrow{AB}$ (or directed line segment) of our solid which is being moved in space maintain the same direction and the same sense. We then state precisely the definition of this particular displacement called *translation*, meanwhile taking great care to avoid terms which could introduce a confusion between the concrete movement which we wish to idealize, and the abstract transformation which we are defining between the points of two figures—which we call homologous under this transformation.[2]

Expressing thus in mathematical language more and more complicated movements, we ascertain that any one whatever of the displacements considered results from the successive application of a limited number of them: rotation in the plane, translation, and rotation around an axis in space.

We define thus a law of *inner combination* among these transformations of displacements: the set of two displacements successively carried out on a figure F and on its homologue F′ under the first of these is again a displacement, called the product of the first two.

From its very nature, this law of combination is *associative*.

Clearly there exists a *unit* displacement, or *identity trans-*

[2] The word homologous, still often used in such a case, appears badly chosen to us; it seems not to distinguish the initial figure from its transform. It is preferable to reserve this name for figures transformed one into the other by a reciprocal transformation (also called an involution), such as the rotation through 180° around a point in the plane.

formation: the displacement which associates with each point of a figure that very same point.

Finally, these displacements being pointwise one-to-one transformations, that is, associating to every point A of a figure F one and only one point A′ of its homologue, there corresponds to each of them an *inverse displacement*—the one which transforms F′ back to F.

Thus we see that displacements form a group. This group characterizes the figures of elementary geometry, for it permits us to arrange them in classes of superposable figures.

A study made in the sense which we have just indicated might appear satisfactory if we require of our geometry a simple, practical purpose. But in this case, why so much effort to evolve its definitions from the physical world? Why not admit a great number of properties intuitively evident and make an experimental geometry quite adequate for practical purposes? Let us give a simple example: we prove to the young pupils of our secondary schools the Pythagorean theorem:

> Given a triangle ABC, with a right angle at A, the lengths BC = a, CA = b, AB = c of the sides satisfy the relation
>
> $$a^2 = b^2 + c^2$$

Let us now provide ourselves with a draftsman's triangle which every elementary pupil will identify with our right triangle, and with a double-decimeter scale. Let us measure the sides of the triangle. We can hope to find only a good approximation to a by the formula of Pythagoras.

If we apply a theory to the physical world, we must supplement it by a concrete interpretation determined by the idealized things whose relations the theory studies.

Thence, to satisfy our intellect, why not go a step further and separate our theory completely from the concrete world which has given rise to it, with the hope, if we do want to apply it to this physical domain, of being able to determine the law of correspondence between the objects and their abstract images?

The notion of equality of two figures has a physical character which we can remove from it by adopting an axiomatic point of view. We propose to define displacements in terms of simple axioms capable of permitting us to derive from them the properties of geometric figures which we have made prominent in the elementary theory.

Let us call a *displacement* every one-to-one, pointwise transformation satisfying the following axioms:

1. Displacements form a group.
2. A half-line Ox is transformed into a half-line O$'x'$, O and O$'$ being images by at least one displacement.
3. If M and M$'$ are two image points of the half-lines Ox and O$'x'$, respectively, with respect to a displacement which associates these half-lines, they are images under every displacement which transforms Ox into O$'x'$.
4. There exists one and only one displacement which transforms an oriented half-plane xOy into a second half-plane x'Oy', the axes XOx and X$'$Ox' being images.

Using this definition, we can derive all the properties of elementary metric geometry. For example, the equality of two line segments OM and O$'$M$'$ will be defined in the following way: Two segments are equal if they are images with respect to a displacement.

From axiom No. 3, they are therefore images with respect to every displacement which associates the half-lines OM and O$'$M$'$ on which they lie.

We see making an appearance here a method of defining axiomatically a geometry with the help of its principal group. This procedure has the advantage over the elementary method of being readily generalized.

2. The Principal Group of Metric Geometry. This group of displacements is not the principal group of metric geometry, for in order that two figures be equal, it not only requires that homologous segments be equal, but also that the plane angles or the dihedral angles of the figure have the same orientation.

Elementary metrics considers two figures equal if the corresponding lengths are preserved, without any hypothesis on orientation. For example, if we are looking at ourselves in a mirror, we are not superposable with our image, but we are none the less equal to it from the point of view of elementary metrics.

We must therefore complete the group of displacements by the transformations of symmetry in order to obtain the principal group of this geometry. The product of a displacement and a symmetry is called a *reflection*.[3] The set of displacements and reflections clearly forms a group characterized by the preservation of lengths. The study of properties invariant with respect to the transformations of this group constitutes elementary metrics.

3. The Principal Group of Euclidean Geometry. Euclid did not restrict his study of geometry to equal figures. He also considered similar figures. Let us recall that two triangles ABC and A′B′C′ of space are said to be similar if they have the same angles. We can then define the displacement D which transforms the triangle ABC into a triangle A′B″C″ (Fig. 4) in such a way that B″ and C″ are on the half-lines A′B′ and A′C′, and that B″C″ is parallel to B′C′. It suffices for example to define D as the product of the translation $\overrightarrow{AA'}$ and of a certain rotation around an axis passing through A′. The two triangles A′B′C′ and A′B″C″ then correspond to each other under a one-to-one pointwise transformation which we call a *homothetic* transformation.

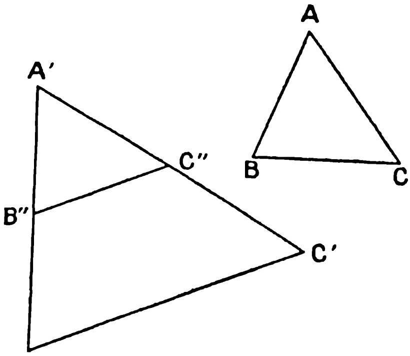

FIG. 4

The homothetic transformation can be defined in the following way: It is a one-to-one pointwise transformation which associates to every set of three points *a*, *b*, *c* the set *a*′, *b*′, *c*′ in such a way that the sides of the triangles *abc* and *a*′*b*′*c*′ are parallel.

Two figures of Euclidean space are considered equivalent if they are *similar*. We say that they are images with respect to the *similarity*, a product of the displacement and of the homothetic transformation which we have exhibited.

[3] G. Thovert suggests the word *antidisplacement*, which we find particularly well chosen for the reasons given by A. Chauvin (*Bull. de l'Association des professeurs de mathématiques*, September, 1949).

It is easy to see that the set of similarities forms a group, the principal group of Euclidean geometry. This geometry therefore studies the properties invariant with respect to the transformations of the group of similarities.[4]

THE ANALYTIC ASPECT OF THE THEORY OF GROUPS

1. Cartesian Space. Let us now discuss how this idea of group which conquered elementary geometry can invade the Cartesian domain.

Perhaps it is well first of all to recall some very elementary ideas in analytic geometry.

Let us consider (Fig. 5) a trirectangular trihedron Oxyz, i.e. the figure formed by the three axes Ox, Oy, Oz, all mutually perpendicular. To every point M of space at a finite distance corresponds a set of three numbers: its *abscissa* x, which measures its distance from the plane yOz, taken positively in the direction of Ox; its *ordinate* y, which measures its distance from the plane xOz, taken positively in the direction of Oy; its *altitude* z, which measures its distance from the plane xOy, taken positively in the direction of Oz.

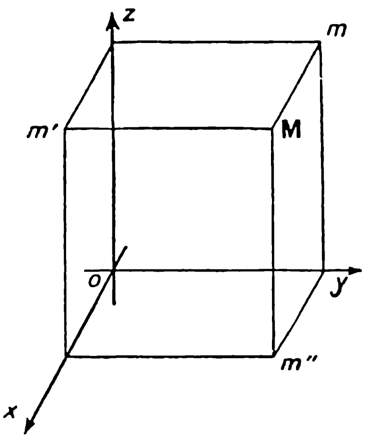

FIG. 5

Conversely, given three finite numbers—real, positive, negative, or zero—a unique point M is defined. These numbers x, y, z are called the *Cartesian coordinates* of point M. The space thus defined, called *Cartesian space*, is therefore an infinite set of elements which are points at a finite distance, or, what amounts to the same thing, sets of three finite and ordered numbers (x, y, z).

How is the principal group of metric geometry expressed for such a space? Naturally by relations among the coordinates

[4] The foregoing considerations assume that we call a similarity the product of a displacement and of a positive or negative homothetic transformation.

(x, y, z) and (x', y', z') of two points M and M′, homologous with respect to a transformation of the group.

These relations are linear, that is, of the type

$$x = x_0 + ax' + a'y' + a''z'$$
$$y = y_0 + bx' + b'y' + b''z'$$
$$z = z_0 + cx' + c'y' + c''z'$$

where the coefficients $a, a', a'', b, b', b'', c, c', c''$ satisfy certain equalities which express the fact that the trihedron $Oxyz$ is trirectangular.[5]

Metric geometry appears thus as the set of properties invariant under the transformations so defined.

It is necessary to make only a slight and quite obvious modification in the formulas thus obtained in order that they express analytically the *group of similarities*.[6]

2. Desarguesian Space. Let us continue our generalizations. We have left aside one whole set of points of elementary geometry: infinitely remote points. How can we bring them in? Their distances from our coordinate planes xOy, yOz, and zOx become infinitely great, so that their coordinates x, y, z approach infinity.

The notion of limit is going to come to our assistance: let us introduce four numbers X, Y, Z, T, the last assumed for the moment to be different from zero, by the formulas:

$$X = xT \qquad Y = yT \qquad Z = zT$$

Our point M is thus as well defined by these numbers as by its coordinates x, y, z, but these four new numbers are defined only within a factor of proportionality; that is to say, if (X, Y, Z, T) represent a point M of Cartesian space, (aX, aY, aZ, aT)—a being any real number whatever—represent the same point. These numbers X, Y, Z, T are the *homogeneous Cartesian coordinates* of the point M.

Let us imagine that the point M moves on a line defined by the points A and B with the coordinates (X′, Y′, Z′, T′) and (X″, Y″, Z″, T″).

[5] These formulas are nothing more than the formulas of change of axes demonstrated in advanced mathematics classes.

[6] Cf. on this subject, as in other respects, for everything concerning the application of the theory of groups to geometry: L. Godeaux, *Les Géométries* (Armand Colin).

We easily show (in a course of advanced mathematics) that M can be defined by the relation:

$$\frac{\overline{\mathrm{MA}}}{\overline{\mathrm{MB}}} = -k\frac{\mathrm{T}''}{\mathrm{T}'}$$

The homogeneous coordinates of M are expressed then as functions of k by the formulas:

$$\mathrm{X} = \mathrm{X}' + \mathrm{X}''k \qquad \mathrm{Y} = \mathrm{Y}' + \mathrm{Y}''k$$

$$\mathrm{Z} = \mathrm{Z}' + \mathrm{Z}''k \qquad \mathrm{T} = \mathrm{T}' + \mathrm{T}''k$$

To each position of M corresponds a well-determined value of k, and if M recedes infinitely on AB, the ratio $\overline{\mathrm{MA}}/\overline{\mathrm{MB}}$ is as close to 1 as we wish; *it approaches 1.* Consequently, k approaches $-\mathrm{T}'/\mathrm{T}''$ and T approaches 0, in which case X, Y, Z cannot all be zero, for otherwise X′, Y′, Z′, T′ and X″, Y″, Z″, T″ would be proportional, which is impossible since A and B are not coincident.

In the same way, to every value of k different from $-\mathrm{T}'/\mathrm{T}''$ corresponds a point M at a finite distance on AB, and to this particular value of k there does not correspond any point at a finite distance. We are thus led to complete Cartesian space by its points at infinity, which are those whose fourth homogeneous coordinate T is zero, while the other three, X, Y, Z, are not all zero.

This augmented space is called *Desarguesian space*, from the name of the French geometer Desargues who first introduced the notion of the point at infinity, but in an entirely different form. Desarguesian space thus appears as the set of ordered groups of four numbers (X, Y, Z, T) not all zero. Such a group of four numbers (X, Y, Z, T) or a group composed of four numbers respectively proportional to X, Y, Z, T is called a *point*. If T = 0, this point is called, by convention, the *point at infinity*.

We can extend our generalization still further and suppose that the numbers X, Y, Z, T are not necessarily *real*, but can be *imaginary* in whole or in part. If they are not proportional to real numbers, their set is called an *imaginary point*. This imaginary point is said to be the *point at infinity* if T = 0.

It is easy to obtain the analytic expression for the principal group of the geometry of Desarguesian space, or the *group of affine transformations*; this expression is derived from that of the group of similarities by the introduction of homogeneous coordinates, whose fourth coordinate we maintain, while the points at infinity must be interchanged between themselves.

(x, y, z) and (x', y', z') of two points M and M′, homologous with respect to a transformation of the group.

These relations are linear, that is, of the type

$$x = x_0 + ax' + a'y' + a''z'$$
$$y = y_0 + bx' + b'y' + b''z'$$
$$z = z_0 + cx' + c'y' + c''z'$$

where the coefficients $a, a', a'', b, b', b'', c, c', c''$ satisfy certain equalities which express the fact that the trihedron Oxyz is trirectangular.[5]

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The homogeneous coordinates of M are expressed then as functions of k by the formulas:

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$$Z = Z' + Z''k \qquad T = T' + T''k$$

To each position of M corresponds a well-determined value of k, and if M recedes infinitely on AB, the ratio $\overline{MA}/\overline{MB}$ is as close to 1 as we wish; *it approaches 1.* Consequently, k approaches $-T'/T''$ and T approaches 0, in which case X, Y, Z cannot all be zero, for otherwise X′, Y′, Z′, T′ and X″, Y″, Z″, T″ would be proportional, which is impossible since A and B are not coincident.

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The only condition which the coefficients must satisfy expresses the fact that the coordinate trihedron exists.

An affine transformation is a very simple one which we generally study in elementary mathematics by means of its metric properties. Its most important property is that of transforming two parallel lines a and b into two parallel lines a' and b'. However, it does not preserve angles or distances. If we impose on an affine transformation the requirement that it preserve angles, it becomes a similarity, and if we impose the requirement that it preserve distances, it reduces to a reflection or to a displacement.

3. Projective Space. In Desarguesian space, the points at infinity, which we can consider as lying in the same plane, play a special role. If we agree to regard this plane as an ordinary plane, we obtain a generalization of Desarguesian space: *projective space.*

In such a space, a point is accordingly the ordered set of four numbers (x_1, x_2, x_3, x_4) not all zero. These numbers, called the *projective coordinates* of the point, are defined within a factor of proportionality.

In projective coordinates, any plane whatever is represented by a linear and homogeneous relation among the coordinates of its points:

$$X_1x_1 + X_2x_2 + X_3x_3 + X_4x_4 = 0$$

and every equation of this type represents a plane.

Knowledge of the coefficients X_1, X_2, X_3, X_4 determines a plane. These four numbers, which cannot simultaneously be zero, and which are clearly defined within a factor of proportionality, are called *tangential* coordinates of the plane. The set of planes of projective space can therefore be considered as the set of ordered groups of four numbers (X_1, X_2, X_3, X_4) not simultaneously zero and defined within a factor of proportionality.

We catch a glimpse here of a complete parallelism between projective space considered as a *set of points* and the same space considered as a *set of planes*: if we have carried out a certain number of calculations with the designation of the projective coordinates of points by *lower-case letters* and those of planes by *upper-case letters*, and if we have our calculation interpreted by a person who uses the *opposite notation*, this person would arrive at a theorem whose statement can be inferred from that of ours by interchanging the words *point* and *plane*. Such is the principle known as the *principle of duality*. Every graduate student recognizes an example of this principle in the relation between poles and polars with respect to a circle. However, in this theory, the center of the circle and the line at infinity of its plane play a special role; this results from the fact that in elementary classes it is the metric point of view which is adopted, not the projective.

The principle of duality is not applicable to properties of affine, Euclidean, or metric geometry which involve the plane at infinity.

We can define between the *points* of projective space, considered as a *point set*, a set of one-to-one transformations called *collineations*. This set possesses the structure of a *group*. To points in the same plane it associates points which are themselves coplanar. However, since projective space may be considered as a *set of planes*, it is natural to ask oneself whether one cannot establish a one-to-one correspondence between its point elements and its plane elements. This is indeed possible and leads to the set of *correlations*, but this set does not form a group.

The set of collineations and correlations forms a group, called the *projective group*, which constitutes the principal group of projective geometry. The group of collineations is a subgroup of it. The set of properties invariant under the collineation transformations comprises *restricted projective geometry*. The group of affine transformations is a subgroup of the group of collineations, so that affine geometry is a *doubly restricted projective geometry*.

Part II

GEOMETRIES AND ABSTRACT SPACES

CHAPTER ONE

The Concept of Abstract Geometry

THE PRECEDING considerations, added to the elaboration by Plücker of line geometry, the study of space considered as a set of lines, have led geometers to the concept of *abstract geometry*.

In all the preceding examples, we have begun by defining the scope of application for the geometry contemplated: Cartesian space for metric and Euclidean geometries; Desarguesian space for affine geometry; projective space for projective geometry. Then we defined a principal group of these geometries.

Let us conceive of:

1. A variety V of elements which we shall agree to call points.
2. A set of transformations, carried out on these points, forming a group G.

The geometry of the variety V having G for its principal group is the set of properties of V invariant under the transformations of this group.

It was Sophus Lie (1842–1899) and Felix Klein (1849–1925) who first thought of this very general concept.

Sophus Lie was a celebrated Norwegian geometer who lived for a long time in France associated with Felix Klein, a pupil of Plücker, with whom he had become friendly around 1869. In Paris, these two scholars made the acquaintance of Camille Jordan and Gaston Darboux, the latter almost exactly the same age as Lie. It was in this period that Lie made his finest discovery which, in conjunction with the notion of *transformation group*, whose importance he demonstrated, was to lead him to the general conception of geometry which we have indicated.

This discovery of Lie is the famous transformation which bears his name and which establishes a most unexpected relation between lines and spheres of space on the one hand, and between the asymptotic lines and the lines of curvature of surfaces on the other hand.

Obliged to leave France in 1870 by Germany's declaration of war, Lie reached Christiania and received his doctorate in 1871, while Klein returned to Berlin.

As early as 1872, Felix Klein set forth in the famous Erlangen Program the fundamental role which the notion of group plays in geometry.

Lie never forgot that the source of his discoveries was due to the influence of the great French geometer Gaspard Monge, and always considered himself as the successor of the French mathematician Evariste Galois. Therefore, toward the end of his career, he resumed contact with the young mathematicians of the Ecole Normale Supérieure, one of whose recent directors, Ernest Vessiot, has, by his works, greatly contributed to making known the masterly work of Lie.

Until some thirty years ago, this concept remained unchanged although geometers had considered more and more general varieties of elements and groups of transformations. The genius of Elie Cartan was required to extend the field of geometry still further. We cannot treat the study of this extension here; at the very most, we shall be able to give some indications of it at the end of Part Two.

Subordinate Geometries and Equivalent Geometries. We have seen that the principal group of metric geometry is a subgroup of that of similarities. Metric geometry, which studies the properties invariant under the transformations of this subgroup of the principal group of Euclidean geometry, is said to be subordinate to the latter. In the same manner, Euclidean geometry is subordinate to affine geometry, which in turn is subordinate to projective geometry.

In a more general way, if we can find a subgroup G′ of the principal group G of the geometry of a variety V, the properties of this geometry which are invariant under the transformations of this group G′ form a *subordinate geometry* of the first one.

Often the fact that one geometry is subordinate to another is not apparent. Thus, projective properties such as those expressed by the theorems of Pascal and Brianchon concerning the hexagon inscribed in or which circumscribes a circle are demonstrated in metric form in elementary mathematics. Their projective character appears only upon the introduction of the elements at infinity.

Once an abstract geometry has been defined, we can devise as many subordinate geometries as there exist subgroups of its principal group.

In another category of ideas, we ask ourselves whether it is not possible to substitute for the study of a new geometry which we are defining a geometry already known to us. This is the problem of *equivalent geometries*.

We have met a simple example of these geometries: the application of the principle of duality whenever it is possible. By virtue of this principle, to every property of a set of lines and points of the point space of projective geometry, corresponds a property of the tangential space of that geometry which we can translate word for word by replacing point by plane, coplanar points by planes having a common point, collinear points by planes having a line in common, etc.

Geometers have traced from particular examples such as the latter the underlying reasons for this equivalence.

Let us imagine two abstract geometries: geometry I of a variety V' with principal group G', and geometry II of a variety V'' with principal group G''. Let us suppose that there exists a one-to-one transformation T between the elements of V' and V''. Then let A' be an element of V', B' its image under a transformation T' of G'; A'' and B'' the images of A' and B', under T. There exists a transformation T'' with respect to which the element B'' of the variety V'' is the image of the element A'' of this variety; T'' is the product of the inverse transformations of T' and T which exist because the first belongs to a group and the second is one-to-one. If we suppose that T'' belongs to G and describes it when T' describes G', then, the transformations being one-to-one, we shall be able to interchange the role of the two varieties and of their groups.

Hence, to every property of geometry I we shall be able to correlate a property of geometry II by the procedure described, and vice versa. In this case we shall say that geometries I and II are *equivalent*.

This idea of *equivalent geometries* is very fruitful, for not only does it permit easy solution of problems by referring them to problems already known, but, moreover, it makes possible a wider application of these problems to new ones.

In this fashion, Felix Klein, studying line geometry of three-dimensional space in the footsteps of his teacher, Plücker, constructed a geometry equivalent to that in which he was seeking the properties of five-dimensional space. A first extension of projective geometry had just been born thanks to the idea of continuous groups of transformations.

Since then, there have been many more of them. We are going to study the most important ones, omitting however non-Euclidean geometries, of which much has already been said.[1]

[1] *L'Analyse mathématique.*

CHAPTER TWO

The Extension of Projective Geometry

1. Algebraic Geometry. In expanding the concept of homographic transformations, we obtain an extension of projective geometry which has remained until our times a very important chapter of mathematics: *algebraic geometry*.

In the most simple case of three-dimensional Euclidean space to which we shall limit ourselves, the homographic transformation, or collineation, establishes a one-to-one algebraic relation of the first degree between the coordinates of two homologous points. Can we preserve the essential character of the collineation, that is, of being an algebraic, one-to-one transformation between points, without the analytic relations which express this correspondence having the form of polynomials of the first degree? A simple example, familiar to every graduate, will answer this question in the affirmative.

Given a fixed point O in space and a constant number k, we know that the point-wise, one-to-one transformation which correlates to every point M of space the point M′ of the line OM defined by the relation

$$\overline{\mathrm{OM}} \cdot \overline{\mathrm{OM}'} = k$$

is called an *inversion*.

Let us now consider three coordinate axes forming a trirectangular trihedron Oxyz. Let (x, y, z) be the coordinates of M and (x', y', z') those of M′. Application of the Pythagorean theorem leads to the relations:

$$x' = x \frac{k}{x^2 + y^2 + z^2}$$

$$y' = y \frac{k}{x^2 + y^2 + z^2}$$

$$z' = z \frac{k}{x^2 + y^2 + z^2}$$

which are the analytic expressions of the inversion transformation. We deduce at once from these relations the following:

$$x = x' \frac{k}{x'^2 + y'^2 + z'^2}$$

$$y = y' \frac{k}{x'^2 + y'^2 + z'^2}$$

$$z = z' \frac{k}{x'^2 + y'^2 + z'^2}$$

which verifies that our transformation is one-to-one and has an inverse.

This correspondence shows that there exist in projective space transformations of the type sought: they are called *birational* or *Cremona transformations*, from the name of the Italian geometer Cremona (1830–1903) who was the first to study them in their full generality.

Contemporary geometers give the name of *birational transformations* to still more general transformations which are not birational in space, but are so only from curve to curve (or between two surfaces).

It is easy to show the existence of such transformations. Let us consider in the rectangular Cartesian plane the transformation:

$$\begin{aligned} X &= x^2 \\ Y &= y \end{aligned} \qquad (1)$$

It correlates to a point m of the plane with coordinates (x, y) one and only one point M with coordinates (X, Y), but conversely, to a point M with coordinates (X, Y) [with $X > 0$, for simplicity], it correlates two points m and m', symmetric with respect to Oy; for while their ordinate y is unique, their abscissa can take on the two values $+\sqrt{X}$ and $-\sqrt{X}$.

Now let us consider any curve c whatever, assumed for greater simplicity to be in the first quadrant xOy of the axes, and all points of which have an abscissa greater than 1 (Fig. 6). The transformation (1) makes a curve C correspond to it and to every point m of c corresponds a point M of C.

Conversely, to a point M of C corresponds the point m of c

of which M is the image with respect to the preceding transformation, and its symmetric point m' with respect to Oy, which is not on c. Such a transformation, not birational in the plane, is nevertheless birational relative to the two curves c and C.

We can show that the only transformations birational from curve to curve are represented by relations of the type

$$X = f(x, y)$$
$$Y = g(x, y)$$

where $f(x, y)$ and $g(x, y)$ are rational functions in x and y.

Algebraic geometry dates essentially from the famous memoir of Brill-Nöther (1874), "Über die algebräischen Funktionen und ihre Anwendung an der Geometrie" (*Math. Annalen*, VII). It is, in principle, the set of properties invariant with respect to birational transformations clearly forming a group; but it is, in fact, the study of geometric properties of curves or surfaces obtained by a synthesis of geometry, algebra, topology, and even sometimes of arithmetic.

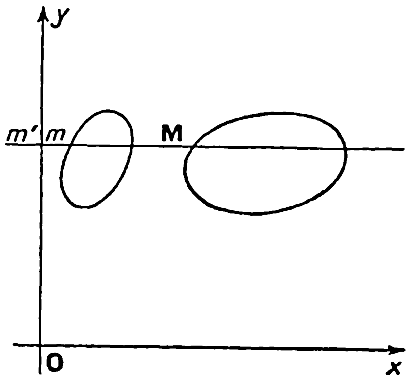

FIG. 6

The first problems which arise are of an *enumerative* character. The most simple, which geometers had solved as early as Euclid's elements, is that of determining the number of points common to two circles. Every problem of enumeration can be reduced to a problem of intersection of two varieties in a space of higher dimensions. The first general theorem we meet in this aspect of the theory is Bezout's—well known by "taupins"*—"two plane curves of degrees m and n (that is, intersected by a line in m or n points, real or imaginary, distinct or coincident) intersect in $m \cdot n$ points." This theorem, almost obvious in the case of simple curves, acquired its full generality and complete rigor thanks only to the discoveries of the French mathematician Halphen.

* Cf. translator's note, p. xvi.

A somewhat unexpected and very amusing application of this theorem of Bezout has been made recently, by Choquet and Kreweras, to the study of *holometric* spaces, so named by their creators to abbreviate their definition: "metric point spaces in which the distance between two points is a positive integer."

It is easy to give an idea of the beautiful theorem which Choquet and Kreweras have established concerning these spaces if for greater simplicity we limit ourselves to the plane.

Let us imagine a set of points in the plane so situated that their mutual distances are measured by integers (the unit of length having been chosen once and for all). If the number of these points is infinite, they are necessarily situated on the same line.

Let us designate by E the metric space composed of this infinite set of points. Let A and B be two points of E. Every point M of E is such that

$$|MA - MB| = k$$

k being an integer less than or at most equal (if M is on the prolongation of AB) to the integer a which measures the length of the line segment AB. Therefore M lies on one of the hyperbolas with foci at A and B, and whose transverse axis has as its length any one of the values whatever (finite in number) which the integer k can take on—i.e., $0, 1, 2, \ldots, a$. This family of hyperbolas F, which includes the perpendicular bisector of AB ($k = 0$) and the half lines composed of the prolongations of the line segment AB ($k = a$), can be considered as an algebraic curve of finite degree p. Let C be a third fixed point of E. In the same fashion as above, we can associate with the couple (A, C), say, a family of hyperbolas F′ on one of which M must be located, and which can be considered as a factorable algebraic curve of finite degree p'. If C is not on the line AB, F and F′ cannot have a common member and therefore intersect in $p \cdot p'$ points. This is impossible since E contains an infinitude of points necessarily belonging to these two families of hyperbolas. Therefore C is of necessity on the line AB, and E has an infinitude of points on AB and a finite number of points not on this line.

Let us now show that there cannot exist any point of E other than those on the line AB. Let O be one of these points. By hypothesis, the line AB contains an infinite number of points of E; hence points as far away as we wish (since their mutual distance is integral). Let M and N be two points (Fig. 7) of

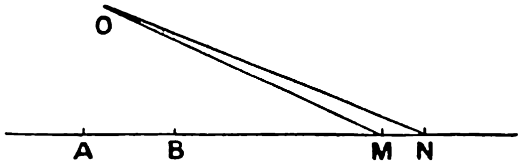

FIG. 7

AB belonging to E, both of which we can always choose as far as we please on the half line which extends the segment AB beyond B. In the triangle OMN we always have the relation

$$OM + MN > ON$$

whence

$$OM + MN - ON > 0$$

or, since we are concerned with an inequality involving integers,

$$OM + MN - ON \geqslant 1$$

Now, when M and N recede indefinitely, N remaining to the right of M, the triangle OMN approaches more and more a collapsed triangle and ON can therefore be made as close to OM + MN, whence OM + MN − ON approaches 0, which is in contradiction with the preceding inequality. O is therefore necessarily on the line AB.

This proof is easily generalized to the case of a holometric space immersed in an Euclidean space of a finite-numbered dimension, and leads to the same result.

The principal direction of current research in algebraic geometry is that which Godeaux calls the problem of the *uniformization of algebraic functions.*[1]

Let us consider for example an ellipse referred to the axes $x'Ox$, $y'Oy$ whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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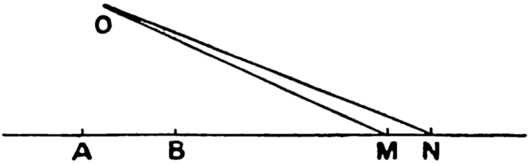

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We obtain a parametric representation of the ellipse in terms of the functions:

$$x = a \cos t \qquad y = b \sin t$$

Geometers have asked themselves if it were possible to find a similar representation for every algebraic curve. This problem is connected with the analysis of functions of complex variables.[2] It was Poincaré who solved it in the general case by showing that the coordinates of points on an algebraic curve can be expressed by *fuchsian** functions of a variable, these functions corresponding to the same group of substitutions isomorphic to the group of displacements in the Lobachevskian plane. Guided by the notion of *invariant* introduced into geometry by the idea of group, mathematicians then sought for the conditions under which the *uniformization* of a curve C′ reduces to that of a curve C already *uniformized*. The solution of this problem leads to dividing curves into equivalence classes, invariant under every birational transformation. This question has been handled by numerous methods which cannot be made the subject of discussion in this work, which is not meant for specialists.

Geometers have sought to extend to algebraic surfaces the results established for curves; but this question is much more difficult and still poses numerous problems. It was started by Nöther. The French school of algebraic geometry (Humbert, Painlevé, Poincaré, Picard and Garnier) has especially devoted its efforts to the study of integrals on surfaces. It is principally the Italian geometers who are investigating these problems, Castelnuovo, Enriques, and Severi, who recently has introduced a new idea (the notion of group of points) which has made a substantial advance in this matter.

The passage from algebraic surfaces to algebraic hypersurfaces, that is to say, to algebraic varieties of more than two dimensions imbedded in a space of more than three dimensions brings up essentially different difficulties. Nevertheless it is in this direction that contemporary geometers, and Godeaux in particular, are directing their efforts.

2. From Three-dimensional Space to Abstract Spaces. Projective geometry is capable of a different extension from that

[2] *L'Analyse mathématique.*

* Translator's note: Also known as automorphic functions.

which we have just been studying. In that geometry we have given the name "point" to an ordered set of four numbers (x_1, x_2, x_3, x_4) not all zero, fixed within a factor of proportionality. This consideration permits of an immediate generalization: let us call "point" an ordered set of $(n + 1)$ numbers not simultaneously zero $(x_1, x_2, \ldots, x_{n+1})$. Let us agree that two such sets of proportional numbers represent the same point. The set of these points constitute projective space of n dimensions. We can define a group structure for it, generalizing the "collineation-correlation" group which is the principal group of this geometry. This space is called *linear*, for the properties of its points are expressed by linear relations among their projective coordinates $(x_1, x_2, \ldots, x_{n+1})$.

3. The Notion of Vector Space. The linear character of the preceding notion explains at the same time the simplicity and importance of it. In treating concrete problems, quite different in appearance from one another, physicists and mathematicians often report that they are led to formally identical calculations. Geometers have therefore understood the necessity of contriving a theory sufficiently general to unite all these problems, and sufficiently simple to provide a tool for the physicist. Such is the source of the notion of *vector space.*

To explain this notion, the idea of group is not sufficient. For greater simplicity, we shall first of all dispose of the case of the three-dimensional space of elementary classical geometry.

Let E be a set of vectors $\vec{X}$, $\vec{Y}$, etc., of this space, having a fixed origin I.

1. To two of these vectors, vector addition assigns a third vector $\vec{X} + \vec{Y}$, called their sum, and possessing the following properties:

(a) $\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$ (commutativity)

(b) $\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{X} + \vec{Y}) + \vec{Z}$ (associativity)

(c) Given a vector $\vec{X}$ and a vector $\vec{Y}$, there is one and only one vector $\vec{Z}$ such that

$$\vec{X} = \vec{Y} + \vec{Z}$$

Besides, there exists a vector $\vec{O}$ (having its origin and its extremity coincident with I) such that for any $\vec{X}$ whatever

$$\vec{X} + \vec{O} = \vec{O} + \vec{X} = \vec{X}$$

and to every vector $\vec{X}$ we can associate its opposite, $-\vec{X}$, such that

$$\vec{X} + (-\vec{X}) = \vec{O}$$

2. To a vector $\vec{X}$ and a real number a we can make correspond the vector $a\vec{X}$ which is called the product of $\vec{X}$ by the number a and which has the following properties:

(a') $1 \cdot \vec{X} = \vec{X}$ (associativity)

(b') $a(b\vec{X}) = (ab)\vec{X}$

(c') $(a + b)\vec{X} = a\vec{X} + b\vec{X}$ (distributivity)

$a(\vec{X} + \vec{Y}) = a\vec{X} + a\vec{Y}$

The structure of vector space is a generalization of properties which we have just recalled from the set of vectors of ordinary space associated with the set of real numbers.

Let us imagine that we can define for a set K two laws of combination, the first *additive*, analogous to the addition of real numbers, that is, possessing a *commutative group* structure, the second *multiplicative*, simply possessing a *group* structure, but *distributive* with respect to the first. We will say that this set K constitutes a *field*.[3]

Now let us consider a set E for which we can define a law of combination possessing a *commutative group* structure analogous to the first property of vectors of ordinary space, and a law of *external* combination—since elements external to E may be brought in—between the elements of E and of K possessing the structure of multiplication of a vector by a real number.

We shall say that E is a vector space with respect to the field K.

It can be shown that the elements of such a vector space are

[3] Cf. *L'Algèbre moderne*, §30.

linearly expressible as a function of a finite number n of them, which constitute the base of this space; n is then the dimension of this space.

In generalizing the notion of vector space we come to the notion of *tensor* so useful in contemporary physics.[4] Finally, we shall content ourselves with pointing out the recent extension of these questions to spaces much more general, such as Hilbert spaces, which are function spaces of infinite dimensionality, or, in a more geometric domain, general Riemannian spaces studied by Elie Cartan and his pupils.

4. The Generalization of the Notion of Distance. The notion of space has assumed today a very general significance (of which we have just given an idea). We so refer to every set in which we can define *neighboring* elements or points. An important particular case (of which we have already considered some aspects) is that of metric spaces in which *distance* can be defined.

The intuitive notion which we have of distance comes from an elementary property of the triangle. Let us consider a triangle ABC:

1. The distance AB between the points A and B is equal to the distance BA between B and A.
2. Distances satisfy the inequality

$$AB \leqslant AC + CB$$

the equality holding only if the point C is on the line segment AB between A and B.

We are thus led to call the distance between any two elements a, b, whatever, a positive or zero number (a, b) satisfying the following conditions:

1. $(a, a) = 0$
2. $(a, b) = (b, a)$
3. $(a, b) \leqslant (a, c) + (c, b)$

Such a *descriptive* definition is independent of the nature of the elements being considered, which can be points of ordinary space, curves, surfaces, functions, etc. It is this notion of generalized distance which is the origin of the theory of abstract

[4] Cf. *Calcul vectoriel et calcul tensoriel.*

spaces, introduced for the first time by Fréchet, about 1904, and which has since undergone a broad extension.

But the possibility of defining the notion of distance over a set is too restrictive. Classical geometry does not restrict its domain to metric properties. Continuity plays a large role in the most important properties of figures—the *topological* properties. We shall see in Part Three of this work that the essential notion is then that of the proximity of two points. It is in studying this question that contemporary geometers have been led to consider spaces of an infinite number of dimensions, and even to distinguish these spaces among themselves.

What makes these notions of great interest is that they are not restricted to being an intellectual exercise, but that they are applicable in a great number of domains, the most important of which is, perhaps, that of quantum physics.

Part III
TOPOLOGY

INTRODUCTION

What is Topology?

TOPOLOGY, CREATED BY Riemann, under the name of *analysis situs*, is the study of continuity in geometry, of its preservation under transformations, and of the corresponding invariants. We owe its development to Poincaré. This great geometer explains[1] in language devoid of all technical terms and in a remarkably intuitive way just what this new geometry is. We could not do better than to quote him in order to provide a definition of this science for the reader interested in geometry, but unfamiliar with the often very special terminology of mathematics:

> Geometers distinguish ordinarily two kinds of geometry; the first they call metric, the second projective. Metric geometry is based on the idea of distance; in it, two figures are regarded as equivalent when they are equal in the sense mathematicians give to this word. Projective geometry is based on the idea of the straight line. In order for two figures to be considered as equivalent, it is not necessary that they be equal; it is sufficient that we be able to pass from one to the other by a projective transformation, that is, that one be perspective to the other. We have often called this second body of knowledge qualitative geometry; it is indeed that, if one contrasts it with the first: it is clear that measure, quantity, play here a less important role. Yet this is not entirely so. The case for a line to be straight is not purely qualitative; we could not be assured that a line was straight without making measurements, or without sliding along this line an instrument called a ruler, which is a kind of measuring tool.
>
> But there is a third geometry from which quantity is completely banished and which is purely qualitative; it is *analysis situs*. In this discipline, two figures are equivalent whenever we can pass from one to the other by a continuous deformation, whatever, moreover, may be the rule of this deformation, provided that it preserves continuity. Thus a circle is equivalent to an ellipse or to any closed curve whatever, but it is not equivalent to a line segment because this segment is not closed; a sphere is equivalent

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to any convex surface whatever; it is not to a torus because there is a hole in the torus and there isn't any in a sphere. Let us imagine any model whatever and a copy of this same model made by a clumsy draughtsman; the proportions are altered, lines drawn by a trembling hand have undergone unfortunate swervings and introduce untoward curvatures. From the point of view of metric geometry, as well as that of projective geometry, the two figures are not equivalent; they are so, on the contrary, from the point of view of *analysis situs.*

Analysis situs is a very important science for geometry; it gives rise to a series of theorems as self-consistent as those of Euclid; and it is on this set of propositions that Riemann has erected one of the most remarkable and most abstract theories of pure analysis. I shall quote two of these theorems in order to make its nature understood: two closed, plane curves intersect in an even number of points; if a polyhedron is convex, that is, if we cannot trace any closed curve on its surface without cutting it in two, the number of edges is equal to that of the vertices, plus that of the faces, less two; and this remains so when the faces and the edges of this polyhedron are curves.

Now this is what interests us in this *analysis situs*: it is here that geometric intuition truly comes into play. When, in a theorem of metric geometry, we appeal to this intuition, it is because it is impossible to study the metric properties of a figure by making an abstraction of its qualitative properties, that is, those which are the very object of *analysis situs.* It is often said that geometry is the art of careful reasoning from badly drawn figures. This is not just a witty remark; it is a truth worthy of reflection. For what is a badly drawn figure? It is one which our clumsy draughtsman, of whom we were just now speaking, might have done: he changes the proportions more or less grossly; his straight lines have alarming zigzags; his circles have disagreeable bumps; all this makes no difference, it will not at all disturb the geometer, it will not hinder careful reasoning.

But the inexperienced artist must not represent a closed curve by an open one, three lines intersecting in a single point by three lines which have no point in common, a surface with a hole by one without. Then we could no longer make use of his figure, and reasoning would become impossible. Intuition would not have been impeded by the defects in the drawing which are of concern only in metric or in projective geometry; it will become impossible, however, when these defects have reference to *analysis situs.*

This very simple observation shows us the true role of geometric intuition; it is to further this intuition that geometry needs to draw figures, or at the very least to portray them mentally. But if the metric or projective properties of these figures are thought little of, or if we are interested only in their purely qualitative properties, it is then only that geometric intuition truly intervenes. Not that I wish to say that metric geometry is based on pure logic, that no intuitive truth is involved in it; but these are intuitions of another kind, analogous to those which play the essential role in arithmetic and in algebra.

Topology is the most recent branch of the geometric sciences. Gauss was able to say, in 1833:

> Concerning the geometry of position which Leibniz foresaw and at which only two geometers, Euler and Vandermonde, were destined to cast a slight glance, we know and we are acquainted with, after a century and a half, little more than nothing.

Today this science is divided into several chapters: *combinatorial* or *algebraic topology*, which is concerned with algebraic and geometric ideas and enters into the theory of equations; and *general topology* (or the study of abstract topological spaces) of which one chapter is the *topology of sets*.

The Development of Topology. As we have already said, it was Riemann who, in 1851, gave the first applications of *combinatorial topology* to classic mathematics by investigating the underlying relations between the theory of surfaces and the theory of functions. The work of Möbius, Jordan, Schläfli, Dyck, Betti and Kronecker proceeded to establish the first important results of this science. But it is to the five memoirs which H. Poincaré published on combinatorial topology that we owe its most notable progress. These memoirs, about 1895, perfecting the systematic theory of combinatorial topology as we understand it today, were the starting point of a great number of investigations among which are those of Brouwer, Lebesgue, Veblen, Alexander, Lefschetz, Alexandroff and Hopf.

Independently of combinatorial topology, Georg Cantor founded, in 1879, *set topology* with his theory of sets.[2] He was the first to define the fundamental topological notions in

[2] *L'Analyse mathématique.*

Euclidean space of *n* dimensions. Cantor's theory was widely used and propagated by the French school of the theory of functions. Then the ideas of Cantor were generalized to sets of curves and surfaces under the influence of Ascoli, Volterra and Hadamard in 1884. This generalization was, moreover, closely connected with the creation of the functional calculus by Volterra, in 1887.[3] About 1904, Fréchet realized that the nature of the elements of the set (points, curves, functions, etc.) were of little importance and that the essential fact was the topological structure[4] of the elements of the set.

Thus, in determining the topological properties common to sets of points and of functions, Fréchet was led to generalize the concept of space and to introduce the *topology of abstract spaces*, spaces whose points are abstract elements of any nature whatever. Since then, set topology has taken a new development to become what is called *general topology*, which continues to be enriched by the works of Choquet and Lichnerowicz.

A good part of recent topological research has been devoted to the merging of combinatorial topology and set topology. From June 26 to July 2, 1947, a colloquium of specialists in algebraic topology was organized by the Centre National de la Recherche Scientifique, in Paris. Under the presidency of Arnaud Denjoy, the American Whittney, the Englishmen Hodge and Whitehead, the Belgian Hirsh, the Dutchman Freudenthal, the Swiss Hopf, de Rham and Stiefel, the Frenchmen Henri Cartan, Leray and Erehsmann brought forward the results of their latest works. The role of topology in the theory of integration and in enumerative geometry, the study of varieties in four dimensions and the rotations of the sphere in *n* dimensions, and the methods of calculating Betti numbers were brought up. Leray introduced and studied the new notion of *homology ring* of a representation. This notion plays an important role in various theories, notably in that of *fiber spaces*, the subject of the lecture by Erehsmann. Henri Cartan considered the recent theories of Alexander and of Leray in a new form, which enabled him to obtain extremely general theorems.

[3] *Ibid.* [4] *Ibid.*

Topology is a scientific field in full ferment where the contribution of young mathematicians such as Koszul (Ecole Normale Supérieure class of 1941) ought to be especially mentioned.

This considerable development of contemporary topology arises from the fact that it is currently impossible to imagine a theory in analysis which would not be based on a previous topological study. Thus topology, in spite of the apparently very vague results to which it leads, is closely linked with the most precise mathematical questions. Since Riemann this contribution of topology to analysis has only made it grow. First, there was Poincaré and his investigation of differential equations and dynamic systems; its application to the *Calculus of Variations* by Birkhoff, Morse, Lusternik, Schnirelmann; its new occurrence in the *Theory of Differential and Functional Equations* by Birkhoff, Kellogg, Schauder; its usefulness was disclosed in algebraic geometry by Lefschetz, Severi, Van der Waerden; Bouligand and Marchaud have introduced it in differential geometry, following the mathematician Juel, and have made use of it with great success in considerably different channels.

Finally, according to Georges Valiron, one of the most recent applications of topology is the study of classes of deformation of functions of complex variables, under the impetus of the Americans Marston Morse and Heins. The little book of Morse, *Topological Methods in the Theory of Functions of a Complex Variable*, which appeared in 1947, gives a first idea of this new point of view.

CHAPTER ONE

Continuity in Geometry [1]

LET US TRY to state precisely our ideas about this fundamental branch of geometry, while remaining on the most elementary level possible. It is a consequence of the intuitive definition given by H. Poincaré that topology is, first of all, the study of qualitative geometric properties.

In elementary geometry, the greater part of the properties studied are metric; such, for example, are the equality of two triangles, the Pythagorean relation between the hypotenuse of a right triangle and the two legs of its right angle, the condition that a quadrilateral be a rhombus, a square. . . . In this geometry, curves seem to be made of a *rigid*, *non-deformable* material. We can only displace them in the plane or in space, and under this displacement, our curves retain their *shape*, their *size*. A line segment remains a line segment and the distance between its extremities is *unchanged*. We are led to assign to the same category all superposable curves and only those. By extension, we can assign to the same category figures symmetric to a given figure with respect to a plane or with respect to a point, which although generally not superposable, preserve this property of superposition segment by segment. We recognize the group of *reflections*, the principal group of elementary metric geometry. Nevertheless, there exist figures of this geometry which, if they do not have equivalent *quantitative* properties, do have certain *qualitative* properties intuitively identical. Let us consider, for example, a circle, a square and a circular ring (the portion of a plane included between two circles). Common intuition lets us imagine properties common to the first two not possessed by the third. For example, whatever closed convex polygon we have in the interior of the circle or of the square, the surface which it bounds is completely interior to the surface bounded

[1] The sources of this chapter are principally: L. Godeaux, *Les Géométries*, and Fréchet and Ky Fan, *Introduction à la Topologie Combinatoire, I, Initiation* (Vuibert), 1946.

by the circle or the square. It is clear that this property does not hold for the circular ring (Fig. 8). Relative to this *qualitative* property, the interior of the circle and the interior of the square are equivalent; but the interior of a circular ring is not equivalent to them.

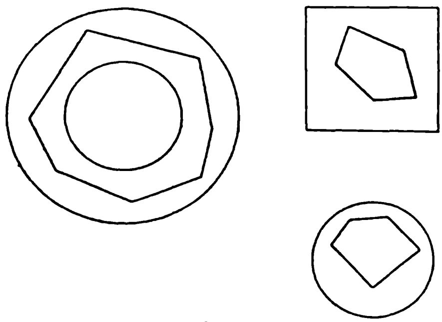

FIG. 8

The circumstance seems to be related to the fact that we can pass from the circumference of a circle to the perimeter of a square by a *continuous deformation*, whereas we are intuitively aware of the impossibility of such a passage from these figures to the two circles bounding the circular ring. We are thus led to invent a geometry in which curves will be composed of a deformable and elastic material, such as rubber strands which we would be able to deform, to expand or contract at will. From this point of view, our square and our circle are the same curve. We can no longer distinguish an ellipse from a circle or from a simple (non-self-intersecting) polygon.

All these curves—if we restrict ourselves for the moment to plane curves—possess the property we have just described. They possess many others of which the principal one is to divide the plane into two parts: two points of the same part can always be joined by a plane polygonal line not intersecting the boundary curve, while two points taken respectively in each of the parts do not have this property. This is the famous theorem of Jordan; we shall state precisely the conditions for its validity at the end of this chapter.

1. The Concept of Curve. Now that intuition has permitted us to abstract the essential feature that we require of our new curves, we are in a position to define these curves axiomatically, and all their topological properties must result from these axioms and from them alone.

If we deform an elastic string, we obtain a new form more or less elongated or contracted, more or less straight or sinuous, but since we are always operating with the same string, distribution of points and continuity will be preserved. We are thus led to conjecture that our curves satisfy the axioms of *order* and of *continuity* postulated for lines by Hilbert and Dedekind.[2] We must, moreover, extend these axioms in such a way that they define simultaneously an *unbounded* curve obtained by deformation of an elastic string initially stretched along a Euclidean line, and a *closed* curve, which possesses the following property of the projective line: in traversing it always in the same sense we end up by returning to the point of departure.

Considering with F. Enriques[3] a curve unbounded in both senses, containing an infinity of points and not intersecting itself, we state that it satisfies the axioms of distribution of the Euclidean (Hilbert) line:

I. If A, B, C are three points of a curve and if B is between A and C, it is also between C and A.

II. If A and C are two points of a curve, there is at least one point B of this curve lying between A and C, and one point D of this curve such that C is between B and D.

III. Of three points A, B, C of a curve there is always one and only one lying between the other two.

IV. Four points A, B, C, D of a curve can always be arranged in such a way that B is between A and C at the same time as between A and D, and that C lies between A and D as well as between B and D.

These axioms show that on a curve we can define two opposite senses of traversal. Continuity will be defined thanks to an axiom analogous to Dedekind's axiom for the projective line:

V. If AB is a segment of a curve and if this segment is divided into two parts such that:

1. Each point of the segment belongs to one of the parts;

[2] L. Godeaux, *Les Géométries*.

[3] "Les Principes de la Géométrie," *Encyclopédie des Sciences Mathématiques*.

2. A belongs to the first part and B to the second;
3. In the sense of traversal from A to B, any point of the first part precedes any point of the second;

Then there exists a point C of the segment of the curve AB (which can belong to one of the parts) such that every point of the segment which precedes C belongs to the first part, and every point of the segment which follows C belongs to the second.

Now let us consider a curve bounded by two points A and B. If we omit these two points we obtain a curve unbounded in the two senses (by acknowledging that there is always a point between A and any point M whatever of the curve, and between M and B). For the unbounded curve thus obtained we stipulate the five preceding axioms. It then follows that there exist on AB *two senses of transversal*, one going from A to B, the other from B to A.

Let us now consider two bounded curves, AB and CD, such that C coincides with B and D with A. The set of these two curves will be called a closed curve. It is easy to see that these closed curves satisfy all the axioms of the projective line.

But we shall no longer stress these arduous matters, which we have wanted to treat here only in order to show how much these seemingly intuitive problems require a very great effort of abstraction. Let it be sufficient for us to retain that we have been able to define axiomatically three types of curves: *unbounded* (or *open*) curves, *bounded* curves, and *closed* curves. On each of them there exist two opposite and continuous orientations.

2. The Concept of Surface. Let us return to one of the examples considered at the beginning of the chapter: if we consider the surface of a circle or the plane surface bounded by the perimeter of a square, every convex polygonal curve drawn in the interior of one of these surfaces bounds an area which is entirely contained within it. Let us suppose the interior surface of the circle made of some deformable and extensible material, say rubber, like the filament that bounds it. We can completely deform it and proceed to lay it over the surface of the square. We thus have a conception of surfaces which preserve the qualitative property we are considering. If we even deform this surface in such a way as to apply it *without tearing and without overlapping* to the surface of a sphere, the property still remains valid.

There are many other properties which are unchanged by such a deformation, and we shall give several examples of them in the following chapter. All these properties which are invariant under such transformations of figures are called *topological*; it is these which topology studies.

Now, our rubber surface can be considered as formed by two pencils of rubber filaments which cross each other, and whose common points are the points of the surface. We are thus led to consider a surface as the set of points belonging to two pencils of curves F and F′ satisfying the following conditions:

1. A point of the surface belongs to a curve F and to a curve F′.
2. A curve F and a curve F′ have in common one and only one point of the surface.
3. On any two curves whatever of the pencil F (or F′) the curves of the pencil F′ (or F) cut out series of points which succeed one another in the determined directions.

These two last conditions express in particular the non-overlapping of the elastic surface which we are deforming. The set of pencils F and F′ constitute what we call a *network* analogous to that of the parallels to the coordinate axes in the Cartesian plane.

3. The Principal Group of Topology. If we deform an elastic figure without tearing or overlapping, we can prove that there are remarkable relations between the initial figure F′ and its final state F″:

1. To every point of F′ corresponds one and only one point of F″.
2. To every point of F″ corresponds one and only one point of F′.
3. To two neighboring points of F′ correspond two neighboring points of F″.
4. To two neighboring points of F″ correspond two neighboring points of F′.
5. If F′ is a curve, F″ is a curve, and to three points A′, B′, C′ of F′ there correspond three points A″, B″, C″ of F″, arranged in the same order.

6. If F″ is a curve, F′ is a curve, and to three points A″, B″, C″ of F″ there correspond three points A′, B′, C′ of F′, arranged in the same order.

The first two properties express the fact that the transformation T which associates F′ and F″ is a *pointwise one-to-one* transformation. It consequently has an inverse transformation T^{-1}, itself *pointwise* and *one-to-one*.

The third property expresses the *continuity* of T, and the fourth (which results moreover by virtue of the first three) that of T^{-1}. In the same way the last two, which involve each other reciprocally by virtue of the first two (and which, moreover, are consequences of the first four), express the fact that the transformation considered is *ordered.* This transformation T is called *homeomorphic* or even a *topological transformation.*

It follows that in order to define our transformation T axiomatically, we can choose among the six properties we have just stated those independent properties which would seem the most convenient to us and from which the others would be deduced. The definition which seems to us the most intuitive is the one given by Fréchet and Ky Fan in their *Introduction à la Topologie Combinatoire*, *I*, *Initiation*:

> A homeomorphism between two figures (or two sets of points) is a correspondence such that to every point of one of the two figures corresponds one and only one point of the other, and that to two neighboring points of one correspond two neighboring points of the other.

To employ more "mathematical" language, we call every *one-to-one* and *bicontinuous* transformation a homeomorphism.

Nevertheless, it is interesting for us to give an account of the manner in which we can arrive at this point of view by making the idea of *order* play the essential role. Let us consider with Godeaux (*Les Géométries*) a pointwise transformation T between the points of space of elementary geometry, for example, satisfying the following conditions:

1. It is one-to-one.
2. It makes the points of a curve correspond to the points of a curve.
3. To two pairs of points AB, CD separating one another on a curve, it associates two pairs of points A′B′, C′D′ which separate one another on the corresponding curve.

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1. It is one-to-one.
2. It makes the points of a curve correspond to the points of a curve.
3. To two pairs of points AB, CD separating one another on a curve, it associates two pairs of points A′B′, C′D′ which separate one another on the corresponding curve.

This last axiom has as its purpose the preservation of *order*: if we define an orientation on one curve K, there corresponds to it on its homologue K′ a well-determined orientation. Given two points A and B of K, we can therefore define an orientation on this curve, such as for example the one in which A precedes B. Let us separate the set of points of the arc of the curve AB into two classes I and S such that:

1. Each point of the arc AB belongs to one of the classes;
2. The point A belongs to I and the point B to S; and
3. Every point of I precedes every point of S.

According to the fifth (or Dedekind's) axiom stated for curves, there exists a point C of the arc $\widehat{AB}$ such that every point preceding C belongs to I and every point following C belongs to S. Let A′, B′, C′ be the corresponds of A, B, C, (respectively) under the transformation T defined by the conditions of one-to-oneness and of order. To the orientation ACB of K corresponds the orientation A′C′B′ of K′. To every point of arc $\widehat{AC}$ corresponds a point of arc $\widehat{A'C'}$, and to every point of arc $\widehat{CB}$ a point of arc $\widehat{C'B'}$. Moreover, every point of arc $\widehat{A'B'}$ is the correspond of one and only one point of arc $\widehat{AB}$. Consequently the points of arc $\widehat{A'B'}$ are divided into two classes I′ and S′ having the same properties as I and S. These two classes are separated by the point C′. Hence T is *continuous*. The continuity of T^{-1} evidently results from the one-to-oneness of T.

Reasoning analogous to that which we have just developed would show that if a transformation is one-to-one and bi-continuous, it is ordered. Whatever be the axioms of definition of this transformation T, it is clear that T^{-1} satisfies them too, as well as the product of several transformations of the same type. Therefore, homeomorphisms form a *group*: this is the *principal group of topology*.

4. The Insufficiency of Intuition. We have been led to the abstract notion of homeomorphism by the intuitive consideration of the deformation of an elastic figure without tearing or overlapping. It must not be thought that such a procedure is the only one which can be employed in order to produce a homeomorphism in actual fact.

Let us consider, for example, a *torus*, that is, the surface generated by a circle revolved about an axis located in its plane, and not intersecting it (Fig. 9): a doughnut, or better for the manipulations to which we are going to subject it, the rubber ring which is used to play ringtoss (we shall suppose it hollow and forming the surface of our torus). Let us cut this ring along a generating circle *ss'*; we obtain a kind of rubber tube with which we can make a knot (Fig. 9A). Let us then join again the two ends of our tube by pasting them together in such a way that the points which were coincident on the ring coincide on the surface obtained. It is clear that to each point of the ring we can make correspond a point of the surface of the knot, and vice versa. Moreover, to two neighboring points on one of these surfaces correspond two neighboring points on the other. Consequently, the ring and the knot correspond to each other with respect to a one-to-one and bicontinuous transformation, with respect to a homeomorphism. We have passed from the surface of the ring to that of the knot by tearing, deformation, and rejoining, and it is impossible to pass from one to the other by a simple deformation without tearing or overlapping.

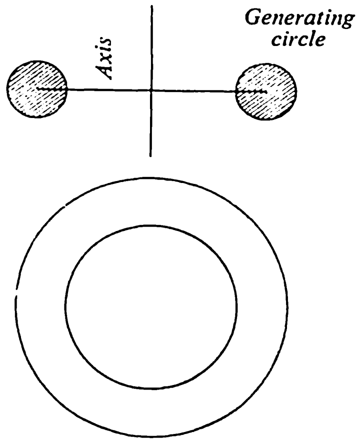

FIG. 9

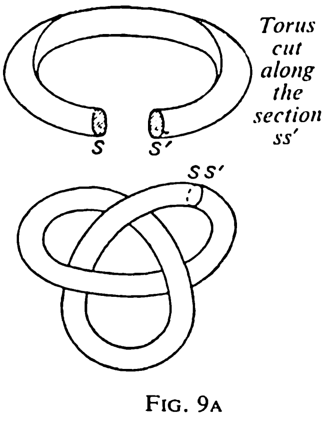

FIG. 9A

Thus a homeomorphism does not express, so it seems, exactly the concrete image which has served to develop it: the geometry of elastic figures. This is due to the fact that these figures are immersed in a particular space from which we cannot separate them; in the passage from one figure to its transform, the homeomorphism must exist at every instant so that it must make correspond to the space the space itself, in order that we can pass from the initial position of the figure to its final position by a *continuous succession* of such transformations. These particular homeomorphisms which make the correspondence depend upon figures from the surrounding environment and which themselves express more faithfully our geometry of elastic figures, are called *isotopisms*. They occur in what we shall later call *relative topology*.

When we attempt to make the surface of our ring coincide point by point with that of the knot, we begin by cutting it along a generating circle. From this, we obtain a tube with two ends, so that the correspondence between the ring and this tube ceases to be one-to-one: to one point of the generating circle before the tearing two points correspond, one on each end. The space in which our figure is immersed is no longer homeomorphic to itself; the procedure of passing from the ring to the knot cannot be reduced to a continuous succession of homeomorphisms.

We have used the expression *homeomorphic figures*; it is clear that we mean by it figures which correspond one to another under a one-to-one and bicontinuous transformation. As a result, if two sets of points E and F are homeomorphic to a same third one G, they are homeomorphic to each other. Moreover, we have already said that homeomorphisms form a group. Homeomorphic sets are very numerous: two equal figures, two similar figures are evidently homeomorphic, so that the group of displacements, the group of similarities can be considered as subgroups of the group of homeomorphisms.

The surface of a sphere from which we have removed a point is homeomorphic to the entire plane. In fact, we can always bring it about by a displacement that the plane will be tangent to the sphere at B, the point diametrically opposite the removed point A. From then on, a stereographic projection with center at A[4] establishes a one-to-one and bicontinuous correspondence between the two surfaces.

A point and a line segment are clearly not homeomorphs since we cannot establish a one-to-one correspondence between

[4] *L'Analyse mathématique.*

them. It seems intuitive that it is the same with a line segment and the surface of a square. This matter, however, is not obvious.

It is, nevertheless, possible to show in a very elementary way—that is, within reach of a bachelor of science—that there exists a one-to-one correspondence between the points of a line segment and those of the surface of a square. For this purpose, we shall appeal to arithmetic:

We can always suppose that the line segment considered is "parametrized" by a number t varying continuously between 0 and 1. To a fixed value of t we shall be able in general to associate a unique sequence:

$$a_1 a_2 a_3 \ldots a_n \ldots \tag{S}$$

such that every a_n is either 0 or 1, so that

$$0 \cdot a_1 a_2 a_3 \ldots a_n \ldots$$

is the representation of t in the system of numerical notation with base 2.

Indeed, we can always determine the a_n in a unique manner by carrying out in the decimal system the set of operations:

$$\begin{array}{rlr} t &= a_1/2 + t_1 & t_1 < 1/2 \\ t_1 &= a_2/2^2 + t_2 & t_2 < 1/2^2 \\ t_2 &= a_3/2^3 + t_3 & t_3 < 1/2^3 \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ t_{n-1} &= a_n/2^n + t_n & t_n < 1/2^n \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

It is clear that these operations can be carried on indefinitely in a unique manner as long as we do not encounter a t_n equal to 0 (n clearly being greater than or equal to the index of the first non-zero a_p). If we do encounter such a t_n, it is easy to see that t is a *diadic* fraction, for it is written:

$$\frac{2k-1}{2^n}$$

and there then correspond to it two possible representations in the system with 2 as base:

$$0.a_1 a_2 \ldots a_{n-1}\, 1\,0\,0\,0 \ldots \tag{S'}$$

or

$$0.a_1 a_2 \ldots a_{n-1}\, 0\,1\,1\,1 \ldots \tag{S''}$$

since $1/2^n$ is the sum of the geometric series:

$$1/2^{n+1} + 1/2^{n+2} + \dots$$

Conversely, to a sequence:

$$a_1a_2a_3 \dots a_n \dots \qquad \text{(S)}$$

always corresponds a single number t belonging to the interval (0, 1) which is expressed in the binary system as:

$$0.a_1a_2a_3 \dots a_n \dots$$

If therefore we remove from the set of real numbers included between 0 and 1 the diadic fractions (which clearly form a denumerable set[5]) and from the set of sequences (S) those from the sequences (S′) and (S″) which also form a denumerable set, the correspondence established between the points of the segment (0, 1) and the sequences (S) is one-to-one. Now, the set of sequences (S′) and (S″) being denumerable, the same is true of their *union*. Therefore we can complete this correspondence by a one-to-one correspondence between the union set of (S′) and (S″) and that of the diadic fractions.

The result is that the *linear continuum* has the same *power* as the set of sequences (S). Our segment (0, 1) is in a one-to-one correspondence with the set of these sequences.

Now, what is our square C? A set of points whose coordinates (x, y) with respect to two of the sides of C taken as axes belong to the interval (0, 1), if we take for the unit of length the length of the side of C. Therefore, we can place the points of the surface of this square into a one-to-one correspondence with the set of the union of the two sequences (S).

$$x \rightleftarrows 0.a_1a_2a_3 \dots a_n \dots \qquad (S_1)$$

$$y \rightleftarrows 0.b_1b_2b_3 \dots b_n \dots \qquad (S_2)$$

Now, the set of the union sets of (S_1) and (S_2) can clearly be put into a one-to-one correspondence with the set of the sequences (S) itself. It is sufficient for this to associate with it the sequence:

$$0.c_1c_2c_3 \dots c_{2n-1}c_{2n} \dots \qquad (\Sigma)$$

where

$$\begin{array}{ll} c_1 = a_1 & c_2 = b_1 \\ c_3 = a_2 & c_4 = b_2 \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ c_{2n=1} = a_n & c_{2n} = b_n \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \end{array}$$

[5] *Ibid.*

a correspondence which is clearly reversible and which thus associates a single group (S_1) (S_2) to a sequence (Σ).

The sequences (Σ) being in one-to-one correspondence with the points of the segment (0, 1), we have the result that this is also so of the points of the square.

This proof can be generalized at once to space, so that there exists a one-to-one correspondence between the points of a cube, or, more generally, of any volume whatever, and those of a line segment. We can make a still further extension to the case of a Euclidean space with a finite number, or even simply a denumerable number of dimensions.

The preceding demonstration shows us that there are just as many points on a line segment as in the surface of a square, but, if it does not prove for us that these figures are homeomorphic, neither does it prove the contrary. The conceived transformation is not a homeomorphism, but we do not know that it would not be possible to imagine a different one from it which, itself, would be a homeomorphism. Often, in topology, we meet evident facts which are very difficult to prove, or which are even incorrect. For example, intuition demurs at admitting that there exists in ordinary space a set of points whose projection onto a plane covers the entire surface of a square and which are homeomorphic to a line segment. Nevertheless such a set exists!

5. Jordan's Theorem. A remarkable example of a topological property intuitively evident and very difficult to prove is the one established by Jordan in 1893.

Let us try to represent mathematically the idea which we have of a simple curve. If this curve is bounded, it is necessarily homeomorphic to a line segment, that is to say, it can be laid down upon a line segment in such a way as to preserve the order of the points and continuity. We can represent the points of the line segment with the aid of a number t, measuring, for example, the distance of this point from one extremity of the segment. This number will be a parameter which varies continuously between 0 and the length of the segment, or in a more general manner between the abscissas t_0 and t_1 of the extremities of this segment which we suppose marked off on an axis.

If we attempt to define our simple curve in ordinary Euclidean space, for example, we can refer this space to three coordinate axes and make correspond to each point of the segment AB a unique point by expressing its coordinates as functions of the parameter t; let them be

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

But in order to take account of the homeomorphism of our curve and the line segment AB, we are led to make some hypotheses concerning the functions $f(t)$, $g(t)$, $h(t)$:

1. To a value of t belonging to the interval (t_0, t_1) corresponds a single point of space, that is to say, the functions $f(t)$, $g(t)$, $h(t)$ are *uniform*.
2. A point of the curve corresponds to a single value of the interval (t_0, t_1).
3. The functions $f(t)$, $g(t)$, $h(t)$ are *continuous*.

The second hypothesis, which, with the first, expresses the one-to-one correspondence between our curve and a line segment, shows that this curve will not have any multiple points, that it will not present any self-intersecting branches, unless their point of intersection should correspond to as many values of t as there are curves passing through this point. The third hypothesis expresses the continuity of the curve and, thanks to the second, the bicontinuity of its correspondence with the segment (t_0, t_1).

Indeed, to two neighboring points of this segment, and hence to two neighboring values of t, correspond two points of the curve which are interior to a parallelepiped with dimensions all the smaller as the points chosen on the segment are still closer, whence these points can be chosen as nearby as we desire provided that we choose the values of t sufficiently close. The converse is also easily seen.

If in addition we impose on these functions the following conditions:

$$f(t_0) = f(t_1) \qquad g(t_0) = g(t_1) \qquad h(t_0) = h(t_1)$$

to each extremity of the segment there corresponds the same point, whence the curve is *closed*. If we consider in particular such a closed curve C, simple and continuous, our intuition tells us that it separates the plane into two regions such that:

1. Two points of the same region can always be joined by a polygonal line not crossing the curve; and

2. Two points not belonging to the same region cannot be joined in this manner, nor by any continuous curve whatever.

We say that C divides the plane into two *domains*: an *interior* domain and an *exterior* domain. It is this intuitive topological property which constitutes Jordan's famous theorem. Its very difficult proof cannot be made sufficiently elementary to be discussed here. Consequently, every simple, closed, continuous curve with which we are acquainted—the ellipse, the circle, all nonintersecting polygons—are curves topologically equivalent from the point of view of this property. They are homeomorphic.

6. Peano's Curve. If we no longer impose on our continuous curve the condition that it be homeomorphic to a line segment, it no longer satisfies Jordan's theorem. In order to convince ourselves of this, let us give the simple example of a curve which covers an entire square: *Peano's curve.*

Let us imagine a square ABCD and let us divide it into nine equal squares by parallels to the sides (Fig. 10). Peano notes that we can join the points A and C by following the path 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, describing diagonally the small partial squares. Let us now consider one of these partial squares (the one which is at the lower left of the large square). Let us divide it likewise into nine equal squares. We can join the points 1, 2 by the same procedure as the preceding. Having arrived at 2, we can employ the same procedure to join 2, 3, and so on up to the point C.

We thus obtain a figure containing 9×9 squares, and in this figure we have drawn a continuous trace joining AC and passing through the vertices common to every group of four adjacent squares (drawing a diagonal in each little square). We already notice that this continuous curve covers a large part of the square ABCD. We can repeat this operation indefinitely and divide the large square successively into $9, 9^2, 9^3, \ldots, 9^n$, smaller and smaller squares in which we shall draw similar paths. It is easily understood that we can thus reach every point of the square by the continuous limit curve of all the preceding paths when n increases indefinitely. Peano space, for example, we can refer this space to three coordinate axes and make correspond to each point of the segment AB a unique point by expressing its coordinates as functions of the parameter t; let them be

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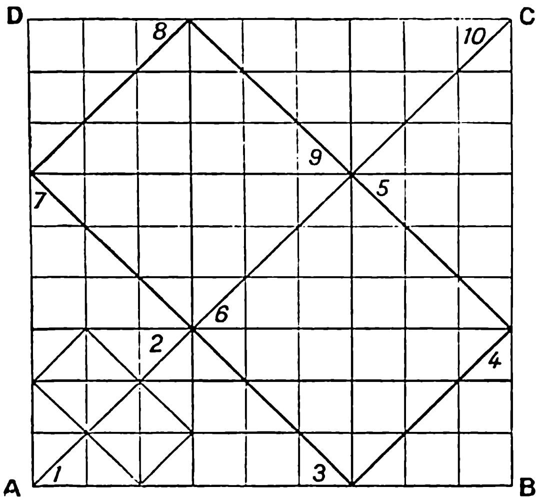


FIG. 10. The reader can follow the path indicated with a pencil point to realize the continuity of this graph, then complete for himself the first complementary drawing starting in the square from the bottom and to the left; finally, undertake a second complementary drawing by starting at A. He will quickly see that this curve fills the entire square ABCD.

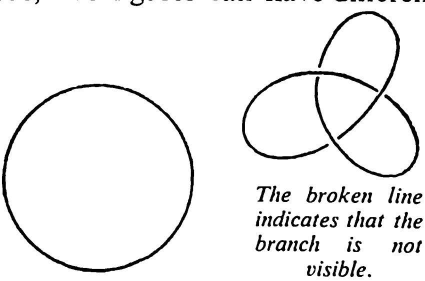
7. Relative Topology. The majority of the properties studied in topology are *topological invariants*, that is to say, properties unaltered by a homeomorphism: homeomorphism plays the same role in topology as equality does in elementary geometry. From this point of view, two equal figures have the same properties: properties invariant with respect to any displacement whatever. It is owing to this that displacements comprise the principal group of elementary geometry. Likewise two homeomorphic figures will have the same topological invariants.

[6] The reader will be able to find a complete proof of this theorem in the course of mathematical analysis of G. Valiron, *Théories des fonctions* (Masson), 1942, pp. 27 and 28.

They are topologically equal or, better, equivalent. Moreover, homeomorphisms possess the three properties required by an equivalence relation:[7]

1. *Reflexitivity*: since every set is self-homeomorphic;
2. *Symmetry*: since if a set E is homeomorphic to a set F, F itself is homeomorphic to E; and
3. *Transitivity*: since if a set E is homeomorphic to a set F and F is homeomorphic to a set G, then the sets E and G are homeomorphs.

However, topological invariants are not the only topological properties of figures when we study the relative properties of these figures. In other words, two figures can have different topological properties if we study them not in themselves, but relative to the surrounding space. Thus the example we have given of a torus and the surface of a knot exhibits two homeomorphic figures which cannot be put into correspondence with one another under a homeomorphism of whole space into itself.



The broken line indicates that the branch is not visible.

FIG 11

In the same category of ideas, let us consider (Fig. 11) a circle and a skew curve forming a knot, Hoppe's curve, for example, which is defined in trirectangular coordinates by the equations:

$$x = \cos t\,(3\cos t + 1)$$

$$y = 5\cos t \sin t$$

$$z = \sin t\,(25\cos^2 t - 1)$$

The representation of our circle (of radius 1, for example) being:

$$x = \cos t \qquad y = \sin t$$

it is clear that these two curves are homeomorphic. Nevertheless, we cannot pass from one to the other by a homeomorphism of whole space into itself.

[7] Cf. *L'Algèbre moderne*, §33.

These two figures do not have topologically the same position in space, so that one cannot pass from one to the other by a continuous sequence of homeomorphisms.

Relative topological properties are those which depend upon the position of the figure in space. From this point of view Antoine[8] shows that we can consider three types of homeomorphic figures in the plane. If we represent these figures by F and F′:

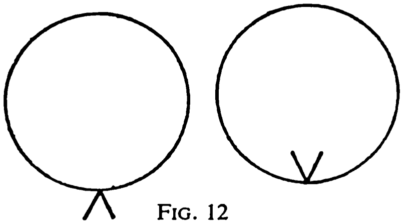


FIG. 12

1. There exists a homeomorphism of the whole plane into itself associating F and F′. This case occurs for example with two simple and continuous closed curves: a circle and an ellipse.

2. There does not exist a homeomorphism of the whole plane into itself associating F and F′, but we can associate these figures by a homeomorphism between a suitably chosen neighborhood of F and a suitably chosen neighborhood of F′. Here we must understand by neighborhood a plane set containing the figure, such that every point of this figure can be the center of a circle belonging entirely to this set. This is the case of Figure 12.

3. There does not exist a homeomorphism associating F and F′ and making the neighborhoods of these figures correspond, but there does exist one between them. Such for example is the case of Figure 12A.

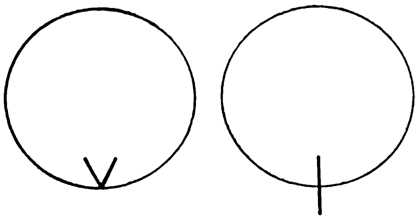


FIG. 12A

In the first case F and F′ have the same position in the plane. We say that they are *isotopes*. In the other two cases this is no longer so.

The transformation which characterizes figures having the same topological position in space is hence an *isotopism*, that

[8] "Sur l'homéomorphie de deux figures et de leurs voisinages," (*Journ. math. pures et appl.*, IV[8], pp. 221–325).

is to say, a homeomorphism which transforms space into itself. As we have already pointed out, we still say that two figures are isotopes if we can pass from one to the other by a continuous sequence of homeomorphisms.

CHAPTER TWO

Various Aspects of Topology

GENERAL TOPOLOGY

THANKS TO THE efforts of contemporary geometers, topology is divided, as we have already pointed out, into two branches: *general* or *set topology* and *combinatorial* or *algebraic topology*.

The first studies very general sets of points and obtains results often far removed from intuition. Notwithstanding this generality, set topology has numerous applications today in mathematical analysis. By its very nature, this chapter of topology calls for delicate reasoning whose abstract quality is such that it is difficult to give an idea of it in this work.

Nevertheless, we can borrow from Kérékajártó (*Vorlesungen über Topologie*, I, Berlin, Julius Springer, 1923) a type example which once again baffles our intuition: *there exist in the plane three different domains having the same frontier*.

It is necessary first of all to state precisely what we mean by *domain* and by *frontier* of a domain. The intuitive idea which we have of a plane domain is that of a set of all the interior points of a simple and continuous plane closed curve: the set of points in the plane interior to a square, a circle, an ellipse, and, in a general way, to a Jordan curve. Such a domain is a *simply connected* domain: it is first of all a domain, because to each of its points we can associate a circle having this point as center and all of whose points, interior and circumference included, belong to the set; it is simply connected because it contains all the interior points of every simple closed polygonal path which is interior to it.

If we now consider a point A which does not belong to our domain D, it can occupy two positions:

1. In every circle with center at A, there exists at least one point of D, and therefore infinitely many (as we can see by considering infinitely many circles with center A and decreasing radius). A is then called a frontier point of D. This corres-

ponds well to the intuitive idea we have of the frontier of a domain, which here will be the set of points like A.

2. There exist circles with center at A which contain no point of the domain. In this case, it is natural to say that A is exterior to the domain.

It is easy to demonstrate that if a domain does not extend to infinity, it has at least one frontier point. This is consistent with our intuition, and we believe that the reader will not see any objection in admitting it. The set of frontier points of D constitutes its frontier F.

We are now in a position to understand the example of Kérékajártó, but to make his explanation more elementary, we shall restrict ourselves to proving this theorem for the set of points of a square, and we shall borrow from G. Bouligand (*Les Aspects intuitifs de la Mathématique*, Gallimard, 1944) the allegorical form of his explanation:

> A monarch has three sons, the Black Prince, the Red Prince, and the Green Prince. He apportions to them progressively the lands of his kingdom, which assumes the shape of a square ABCD (Fig. 13).
>
> The Black Prince, his favorite, receives a square (colored solid black in the figure) in the first deed of gift. In this apportionment, the Red Prince receives the diagonally-striped strip formed by the junction of three rectangles, and the Green Prince the dotted band formed by the junction of seven rectangles. The monarch then proceeds with a second distribution, of which we have only indicated in still finer diagonal stripes that which falls to the Black Prince. We start the diagram of this new band by starting in the lower left hand region of the square, and we prescribe for ourselves that we remain at an equal distance from the two sides of the original square and from the segments which form the last two allotments: this second fief of the Black Prince thus separates the first fief of the Red Prince from that of the Green Prince. It comes to an end with a rectangle surrounding his own initial fief as the figure indicates, for in each new donation, it is understood that each of the realms remains connected.
>
> According to the same rule, we can obtain the fief bequeathed to the Red Prince in the second donation, starting from a point such as E in the upper left and making its path proceed between the left edge of the original square and the first fief of the Black Prince, augmented by the second donation. We shall end up

so as to just unite the first red fief. The second portion assigned to the Green Prince is determined by the same procedure. We shall settle the boundaries of what the princes received in the third donation by applying the same conditions, and so forth.

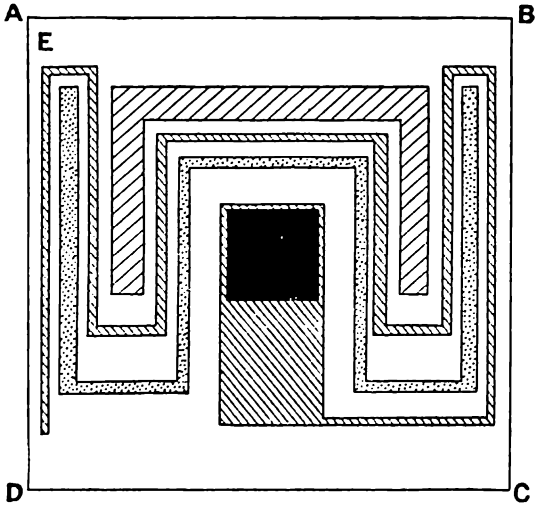


FIG. 13

The reader will be able further to put into concrete form this method by coloring Fig. 13 in black, red, and green.

Finally, the monarch has distributed his entire domain to his sons, with the exception of certain points which each of the princes can have access to at his pleasure without getting off his own domain. In fact, there exist many such sets, not void, since each domain has at least one frontier point, and every circle having this point as center encroaches upon the three domains, for the conventions adopted for the distribution of these realms show that however small such a connected domain (here our circle) interior to the original square may be, there exist infinitely many points of the three portions which belong to it. These three realms have indeed, therefore, the same frontier.

COMBINATORIAL TOPOLOGY[1]

IN ORDER TO obtain the prolific methods applicable to the

[1] The examples and the figures of this section were borrowed from the very interesting work of Fréchet and Ky Fan, *op. cit.*

greatest possible number of propositions, geometers have confined the number of figures studied in topology to a few simple figures. Such for example are curves, surfaces, and more generally *complexes* which are generalizations of surfaces. The study of these figures form the subject of combinatorial topology. They are not then regarded as point sets but as sets topologically equivalent to polygons or to polyhedra. These figures are susceptible of a combinatorial schema so that the phase of topology which studies their properties can be algebraized, the study of these schemas depending upon linear algebra or the theory of groups. We are going to state this notion precisely with the aid of several very simple examples taken from among the problems which were proposed at the beginnings of this science, but which have still not been solved.

1. The Problem of Coloring Geographical Maps. Every geographic map presents itself as a set of countries separated by boundaries. In order to distinguish these countries from one another at a single glance, it is convenient to color their representations in different colors if two countries have a common boundary; but in order not to multiply the number of colors, if two countries have only one boundary point in common, we can color them the same.

The problem which arises then is the investigation of the *minimum* number of colors necessary to color a given map, and more generally, every map. This number is evidently a topological invariant. If a map contains only one country, it is clear that a single color will be sufficient to satisfy our conditions. The map is then called *monochromatic*. If it contains two countries having a common boundary, evidently two colors are necessary to differentiate these countries: the map is *bichromatic*. But we can imagine bichromatic maps which have many more than two countries. It is enough that these countries be arranged in such a way that three of them never have a common boundary (Fig. 14). It is easy to imagine *trichromatic* maps: it is sufficient for this that there exist on this map three countries, any two of which have a common frontier not reducible to a point (Fig. 15). It is even possible to imagine a *tetrachromatic* map: it is sufficient that there exist on this map four countries any two of which have a common frontier not reducible to a point (Fig. 16). Moreover, this is the most general case. It can be noticed, for example, on the map of France that this is the case of the region containing the departments of Seine-et-Oise, Oise, Seine-et-Marne, Loiret, Eure-et-Loir, and Eure, which surround the department of the Seine. Although this map contains *seven* regions (Fig. 16A), *four* colors suffice to distinguish them. The set of these departments is topologically equivalent to the polygonal diagram drawn at the side of the map. People have sought to construct maps requiring more than four colors, without success to date.

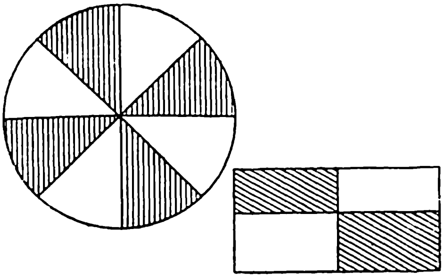


FIG. 14

Thus, whereas four colors are necessary to distinguish the colors in a map as simple as that of the Parisian region, or even still simpler, of the type of the polygonal diagram of Fig. 16, it appears that *five* colors are enough to handle the same problem whatever be the complexity of the map.

This number of colors necessary to color a geographical map is called the *chromatic number* of the map. For a long time, mathematicians had thought it proven that this number was 4. Thus it was that when Cayley had attributed, in 1878, the statement of the four-color theorem to De Morgan, Frederick Guthrie asserted in 1880 that his brother, Francis Guthrie, had given a proof of this theorem as early as 1850. Since then, there have been numerous mathematicians who have believed this

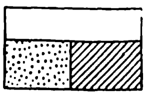


FIG. 15

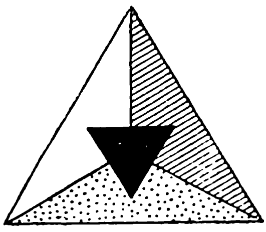


FIG. 16

theorem proven, but their proofs are all false. It has been possible to prove, nevertheless, that five colors are enough.[2]

As often happens, the investigations which have been made with the object of solving this problem, if they have not been completely successful, have contributed to the discovery of a great number of results, interesting as much for topology as for many other branches of mathematics.

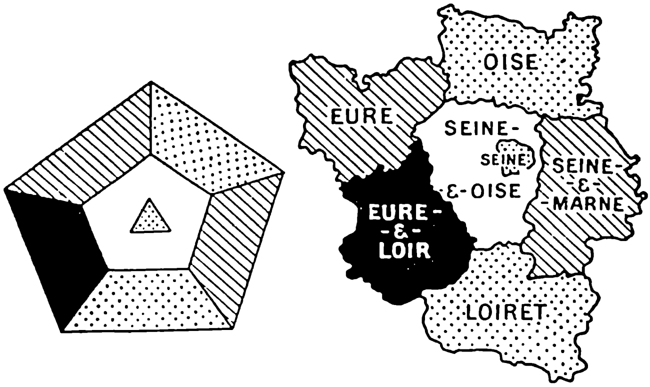


FIG. 16A

2. Generalization of the Map Problem. It is noteworthy that the determination of this chromatic number does not exist for the plane which is the simplest surface (or for the sphere which, being homeomorphic to a plane if we remove one of its points, possesses the same chromatic number), although we have been able to determine it for much more complicated surfaces.

Let us imagine, for example, a map drawn on the surface of a rubber ring. We have been able to prove that the chromatic number of this map is 7. To remain on an elementary level, we shall prove only that seven colors are necessary, that is to say, that we can draw on a ring a geographical map requiring at least seven colors to distinguish its various countries. Let us cut this ring along a generating circle (Fig. 17); the surface obtained is a kind of twisted cylindrical tube. Let us straighten

[2] The reader who would like to sift this question to the bottom can consult the following works: A. Errera, *Du Coloriage des cartes et de quelques questions d'A. S.* (G.-Villars), 1928; P. J. Heawood, "Map Colour Theorem," *Quarterly J. Math.*, XXIV (1890), pp. 332–338.

it out and cut it along a generatrix. Then let us spread out this surface on the plane. We obtain a rectangle homeomorphic to our ring under the condition that we regard as identical the points of the sides of this rectangle which are opposite one another. Then let us lay out the arrangement of the regions of Fig. 17A. In reconstructing our ring by deformation and pasting together without overlapping, we obtain a torus on the surface of which will be drawn seven regions requiring the use of seven different colors, since any two of them whatever have a common boundary.

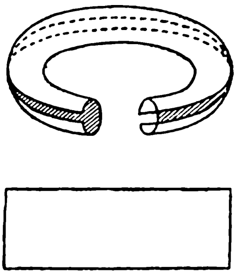


FIG. 17

Therefore the chromatic number of a map drawn on every surface homeomorphic to a torus is at least equal to seven.

An analogous problem on the subject of three-dimensional domains has been set up, but Frederick Guthrie has shown its impossibility. It is indeed possible to construct for any positive integer n three-dimensional domains such that any two whatever of them have contact along a whole surface. Hence in order to color them, one must have as many colors as domains. Let us borrow once more from Fréchet and Ky Fan a very simple example to illustrate this fact:

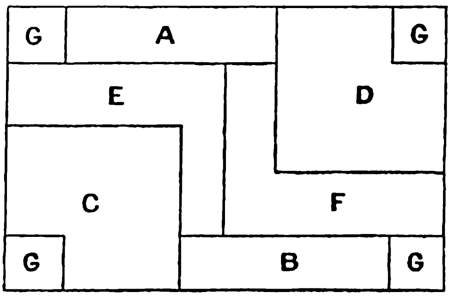


FIG. 17A

Cut a parallelepiped by a plane parallel to the base (Fig. 18). We thus obtain two parallelepipeds. We divide the first into n parallelepipeds (numbered from 1 to n) by planes parallel to a direction of side surface, and the other into n parallelepipeds (numbered from 1 to n) by planes parallel to the other direction of the side surfaces. We can thus construct n domains numbered from 1 to n, domain no. p containing the two small parallelepipeds of the same number. We then declare that these domains taken two by two in any manner whatever always have a common boundary surface.

3. Descartes' Theorem. In the intuitive explanation of the notion of topology given by Poincaré which we have quoted in the introduction to Part Three of this work, this great geometer pointed out by way of illustration the following theorem:

> If a polygon is convex, that is to say if we can draw a closed curve on its surface without cutting it in two, the number of edges is equal to that of the vertices plus that of the faces, minus 2.

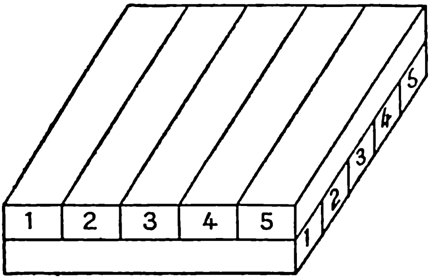


FIG. 18

This theorem often attributed to Euler is in fact due to Descartes, as D. Hilbert and S. Cohn-Vossen (*Anschauliche Geometrie*, Berlin, Julius Springer, 1932, p. 254) pointed out. These geometers, moreover, gave a very intuitive proof of it which seems to us interesting to reproduce.

First of all let us state precisely that by a polyhedron is understood a set of polygons (the faces of the polyhedron, their sides being its edges) satisfying the following conditions:

1. Any two faces whatever have no common interior point.
2. Any two faces whatever have one and only one common edge. If we remove one face of a pyramid, for example, we no longer have a polyhedron.
3. Any two faces whatever can be joined by a succession of faces having, two by two, a common edge.
4. Two faces having a common vertex can be arranged in such an order that two consecutive faces have a common edge passing through this vertex.

The theorem of Descartes is not valid for every polyhedron satisfying the preceding definition, but it is applicable to the most important of them, to *simple* polyhedra, that is to say, homeomorphic to the surface of a sphere.

We can now give a precise statement of this theorem:

Given a simple polyhedron having F faces, S vertices and A edges, there exists among these numbers the relation*

$$S - A + F = 2 \qquad (1)$$

* Translator's note: V is normally used for vertices and E for edges in American mathematical works.

Let us imagine this simple polyhedron made of rubber. Let us cut out a face of this polyhedron. The set of polygons remaining is homeomorphic to a sphere, with one of its points removed, and therefore to a plane. Consequently, we can lay it out on a plane without tearing or overlapping, in such a way as to constitute a network of polygons, each element of this network having the same number of vertices as the corresponding face, and in such a way that two plane polygons arising from the deformation of two faces having an edge or a vertex in common, have a side or a vertex in common. We have drawn (Fig. 19) the network corresponding to a cube. In such a network, the number of vertices and sides is the same as that of the vertices and edges of the polyhedron from which it is taken, while that of the number of faces has decreased by one. Consequently, it is necessary for us to prove that for this network:

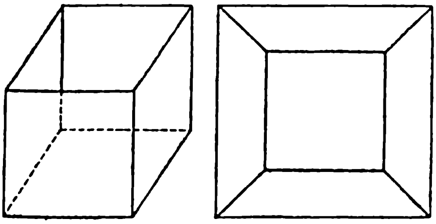


FIG. 19

$$S - A + F = 1$$

S, A, and F having the same meaning as for the polyhedron.

Let us remark first of all that we can always suppose the network formed exclusively of triangles, for if this is not so, we can reduce the problem to this case by dividing the polygons into triangles, using their diagonals. Drawing a diagonal does not modify the number S − A + F, for it increases F and A at the same time by one, without changing S. Moreover, we can without changing the number S − A + F reduce this network to a single triangle by a finite number of operations of the two following kinds:

1. We cancel a triangle by removing a side; this does not then change S, but it decreases F and A by one (Fig. 20).

2. We cancel a triangle by removing a non-contiguous vertex and the two sides which terminate there; hence, F and S decrease by 1, but A diminishes by 2 (Fig. 21).

Fig. 22 shows the end of the transformation by combining the two operations we have just described.

As a result of these operations, the number S − A + F has the same value for the network considered as for a single triangle; now, in this case, S = 3, A = 3, F = 1, hence:

$$S - A + F = 1$$

which proves the relation (1) for every simple polygon.

The *applications* of Descartes' Theorem are very numerous. One of the simplest is the determination of all regular polyhedra (in the sense of elementary geometry); this application shows that the topological pro-

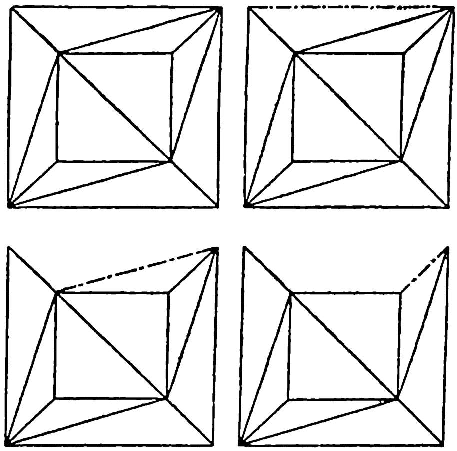


FIG. 20

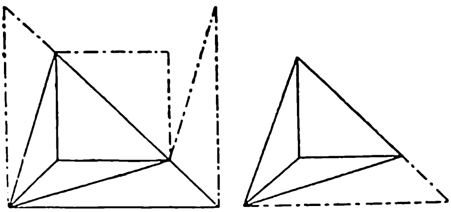


FIG. 21

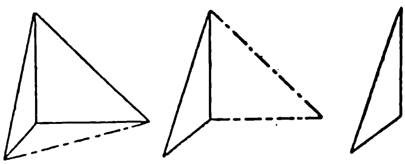


FIG. 22

perties of figures are the most characteristic of their nature.

Designating by a the (constant) number of edges terminating at a vertex, and by c the (constant) number of sides of each face, we readily see that

$$S \cdot a = 2A = F \cdot c$$

and, as a result of formula (1), that

$$\frac{1}{A} = \frac{1}{a} + \frac{1}{c} - \frac{1}{2}$$

Taking account of the obvious inequalities

$$a \geqslant 3 \quad \text{and} \quad c \geqslant 3$$

we can easily deduce from this the only possible values of a, c, A, S, F which indeed correspond to the simple regular polyhedra which we know how to construct, and which are therefore the only ones which exist.

Regular polyhedra	a	c	A	S	F
Tetrahedron	3	3	6	4	4
Octahedron	4	3	12	6	8
Icosahedron	5	3	30	12	20
Cube	3	4	12	8	6
Dodecahedron	3	5	30	20	12

This proof does not use the equality of the faces of these polyhedra, but only the fact that the edges are equally distributed around each vertex and in each face, so that they concern a much more general class of simple polyhedra than the regular polyhedra already known by Euclid.

4. The Principal Problem of the Topology of Surfaces. But the theorem of Descartes has much more important applications than those which consist in discovering the underlying reason for various properties, already known to the Greeks, of the figures of elementary geometry. It is this theorem in fact which leads to the solution of the principal problem of the theory of "closed surfaces": to discover whether two given surfaces are homeomorphic.

Characteristic of a Surface. Let us consider any simple polyhedron whatever whose surface is made of rubber, and let us deform it so as to spread it out over the surface of a sphere without tearing or overlapping. We thus obtain a division

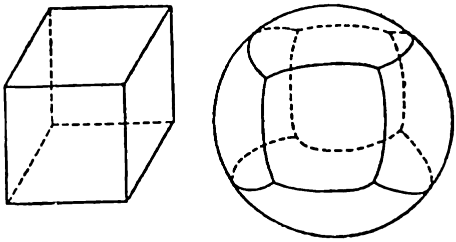


FIG. 23

of the sphere into a set of curvilinear polygons having two by two, one and only one common edge. Conversely, to every division of the sphere into curvilinear polygons having this property corresponds a simple polyhedron which is homeomorphic to it (Fig. 23). Consequently, Descartes' theorem holds for every division of the sphere according to the procedure indicated. The number:

$$S - A + F = 2$$

thus exhibited, which depends neither on the form of the curvilinear polygons employed nor on their (finite) number, is called the *characteristic* of the sphere. It is accordingly the characteristic of every surface homeomorphic to the sphere.

If we now consider a surface which is not homeomorphic to a sphere, a torus for example, it is easy to prove that its characteristic is not equal to 2. It is clear that a torus is homeomorphic to a polyhedron obtained (Fig. 24) by hollowing out from a parallelepiped a parallelepiped of the same height, but of a smaller base. But this polyhedron is not homeomorphic to a sphere, for it admits of a hole which will persist under every continuous deformation, while the sphere does not admit of any.

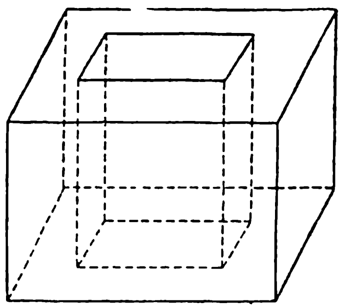


FIG. 24

Now, for this polyhedron, S = 16, A = 32, F = 16, whence S − A + F = 0.

Accordingly, there exists a division of a torus into curvilinear polygons having, by twos, one and only one common edge, for which the number S − A + F = 0. The theorem of Descartes is not proved for the torus, but we prove that whatever be the (finite) number of curvilinear polygons which make up this torus, if these polygons have, by twos, one and only one common edge, the foregoing relation will hold. Thus the torus, and every surface homeomorphic to it, has a characteristic equal to 0.

This characteristic which can be defined for every closed surface is a topological invariant. However, if two homeomorphic surfaces have the same characteristic, the converse is not obvious. It is even false, and in order completely to solve the problem of the topology of surfaces, we are going to be obliged to introduce a new notion: the orientability of surfaces.

5. Orientable and Non-orientable Surfaces. Let us consider a plane P. It is possible to define on every circle drawn in this plane a sense of traversal which would be the same for an observer standing on the plane, his feet at the center of each of the circles, who would look at an insect moving on them; the sense usually adopted is that from right to left or the direction opposite to the hands of a watch which we imagine placed on this plane, the hands turned toward the observer. This sense orients the plane. We call it the trigonometric sense, because it is of use in defining oriented angles in elementary geometry and permits the study of simple functions of these angles: the trigonometric functions.

Let us try to extend this idea to any surface whatever. However complicated a surface studied in combinatorial topology may be, since it must be continuous, we willingly suppose that in bringing our attention to bear on a sufficiently reduced portion of this surface, we can consider this portion homeomorphic to the surface of a triangle. Now, this triangle possesses the same property as the plane in its entirety: two circles drawn on its surface can be oriented in the counterclockwise sense for an observer who is looking at them both at the same time. This sense can be defined, moreover, by giving the vertices of this triangle in a fixed order. For example (Fig. 25), in naming the triangle ABC, we indicate by the order in which we encounter the points A, B, C on its circumscribed circle, the sense of orientation of every circle interior to this triangle, the sense of orientation of the surface of the triangle.

Let us now consider a tetrahedron ABCD (Fig. 26). We can successively orient each face ABC, ACD, ADB, BDC by the arrows indicated on the figure in such a way that an individual moving all around the tetrahedron will see the hands of a watch placed before him on the corresponding face turning always in the same direction: the sense opposite that indicated by the arrows. We shall say that the surface of a tetrahedron is orientable. It is clearly the same with the surface of every

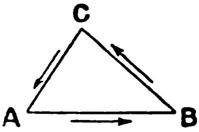


FIG. 25

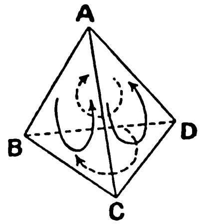


FIG. 26

simple polyhedron, if we notice that such a polyhedron can be cut up into a set of triangles still comprising a polyhedron in the sense in which we have defined such a surface in §3.

This orientation of a polyhedron can be made with the help of a simple rule called the *rule of Möbius*. Let us notice in Fig. 26, that the edges are traversed in the opposite sense according as we consider them as forming part of one face of the polyhedron or of the adjacent face. In laying the surface of the tetrahedron ABCD over a sphere, without tearing or overlapping, we can preserve this rule about the vertices of the tetrahedral division of the sphere which is obtained, and consequently orient this sphere (Fig. 27).

In a general manner, we can find on a surface a polygonal division such that any two polygons whatever have in common one and only one edge, except—if the surface is not closed—for the edges which form its border, and which then belong to only a single polygon. If it is possible to assign a sense of traversal to each face in such a way that each edge common to two faces has two different senses according as it is considered as belonging to one or the other face, then we say that the surface is orientable. It is intuitive that this notion of orientability constitutes an invariant. Fréchet and Ky Fan give a proof of this in their work cited at the beginning of the chapter.

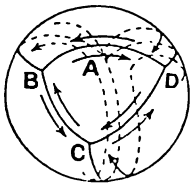


FIG. 27

Simple surfaces which we are accustomed to see daily are usually orientable, so that the existence of non-orientable surfaces is not absolutely evident. We shall prove this existence by inviting the reader to expose himself to an amusing experience:

Cut a sheet of paper into a rectangle whose width is relatively small with respect to its length. You thus obtain a kind of strip ABCD (Fig. 28). Twist this strip through 180° by holding the edge CD between two fingers, and place the segment CD on the segment AB, C falling on A and D on B, meantime agreeing to identify the points of these segments which coincide with one another (practically, you can paste the edges AB and CD one to the other). You thus

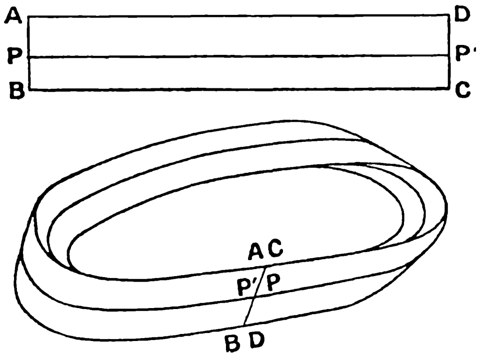


FIG. 28

construct a surface R known by the name of a "Möbius strip" which possesses remarkable properties. We proceed to examine some of them.

1. If, starting from the point A, we follow the edge of this strip with the point of a pencil, always going in the same direction, we describe at first the former side AB of the original

rectangle; then, continuing our way without lifting the point of the pencil, the side BC, since B and D have been identified; C being identified with A, we return to A. This strip has therefore only one edge.

If we use a dull pencil, colored in blue on one side and in red on the other, and if we move it, holding it very nearly normal to the strip, we can prove that if at the start the strip was to the right of the blue side, it remains there the entire time during our experiment. Therefore, not only is this edge a closed curve, but also this curve has but one "border."

2. Before twisting our rectangle ABCD, let us draw on it the line PP′ which joins the midpoints of the sides AB and CD. After the twisting and the rejoining, P and P′ are on both sides of the strip, but must be considered as coincident, since they are the points of AB and CD which have come into coincidence.

If we move the point of our pencil along the former line PP′, we can thus pass from one side of our strip to the other, without crossing its edge. This is impossible for a surface such as a hemisphere, for example; to pass in a continuous way from the exterior to the interior of a hemisphere, we must necessarily cross at one point the great circle which bounds this hemisphere. If we continue our walk on the line PP′ which we see transparently on our strip, we finish by returning to the point of departure P.

We have again obtained a closed curve drawn on this surface having only a single border. To convince ourselves of this, let us try to tint the right border (for example); we shall thus color the entire strip.

Since we are able to pass from one side of this strip to the other without crossing its edge, we cannot say that it has two sides; we call it *unilateral.*

This last property can again be made evident by a different experiment: if we cut the strip along the curve PP′, it will not be cut up, as a uninformed observer would be able to expect, but will form a doubly twisted strip made by the single border of this curve.

Innumerable studies have been made of this surface. In particular, H. Tietze ("Einige Bemerkungen über das Problem der Kartenfärbens auf einseitigen Flächen," *Jahresber. Deutsch. Math. Vereinigung*, XIX (1910), pp. 155–159) has proven that the chromatic number of the Möbius strip is 6. It is easy to see that it is at least equal to 6 by dividing the rectangle ABCD which serves to form it, as we have indicated in Fig. 29.

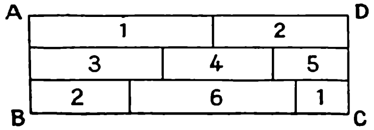


Fig. 29

Let us seek to orient this strip. We can always restore a polygonal division of its surface to a polyhedron whose faces are triangles one side of which contributes to determining the edge of the strip. We thus obtain the succession of triangles (Fig. 30):

AEF, FEG, FGH, HGI, HIJ, JIA, JAE, etc.

by observing Möbius' rule of edges. We see that this rule cannot be followed for the last triangle, the edge AE necessarily having the same orientation on the adjacent faces IJA and JAE.

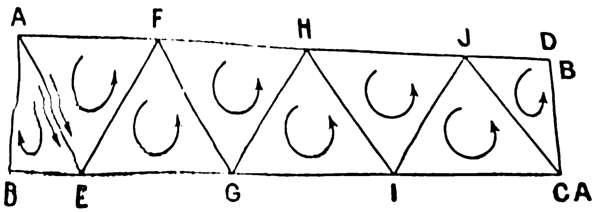


Fig. 30

The Möbius strip is therefore not an orientable surface. This is owing to the fact that this surface is unilateral. Indeed, let us consider a point P of this surface (Fig. 31); if the strip were orientable, it would be possible to shift a small circle on its surface by describing at its center a continuous curve in such a way that two neighboring positions of this circle might have the same orientation. We would arrive at the point P′ coinciding with P, but on the side of the strip "opposite" to that which contains P; at this point, for an observer who would be looking at the side

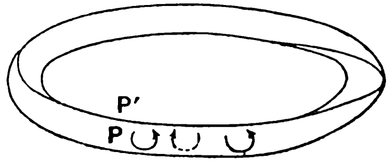


Fig. 31

containing P, the orientation of the circle with center P′ would be opposite to that of the circle containing P. This reasoning shows clearly the greater importance of the possibility of the continuous passage from one side to the other of a surface over its orientability. The fact that the Möbius strip is not a closed surface is of no concern here. Moreover, one can imagine closed unilateral surfaces. We understand that such a surface, having but one side, has neither an exterior nor an interior. It must therefore intersect itself at certain points. Such for example is the *heptahedron of C. Reinhardt* (Fig. 32).

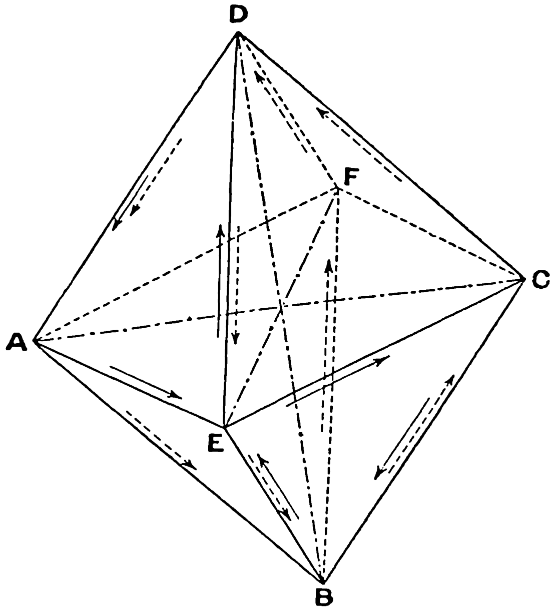


FIG. 32

Let us consider an octahedron ABCDEF. We can define this heptahedron H by the following set of polygons:

the four triangles

AED, EBC, CFD, ABF

and the three squares

ABCD, EBFD, AECF

The sides and the vertices of these seven polygons coincide with the edges and the vertices of the octahedron. The diagonals AC, BD, EF are not sides but lines where two polygons intersect each other. (We have drawn them as dotted lines.) On each of these lines, each point (except the vertices) must be regarded as the combination of two distinct points according as it appears on one or the other of the two polygons of which this line is the diagonal. The seven polygons are consequently considered as having no common interior point. Moreover, each side of H belongs to two and only two polygons. The consequence is that H is a polyhedron, therefore a closed surface.

This polyhedron is *non-orientable*. Indeed, let us fix, for example, the sense AED of one of its faces. The face having the edge ED in common with AED must be oriented according to Möbius' rule: DEBF. By working successively in the same way, we obtain the faces:

BEC, which has the edge BE in common with the preceding;

BCDA, which has the edge BC in common with the preceding one.

Hence, the edge AD is oriented twice in the same sense. H is not orientable.

These two ideas which we have just described, the *characteristic* of a surface and *orientability*, lead to the principal theorem of the topology of surfaces:

> The necessary and sufficient condition that two closed surfaces be homeomorphic is that they have the same characteristic and that they be both orientables or non-orientable.[3]

[3] Fréchet and Ky Fan, *op. cit.*, give a very interesting elementary proof of this theorem.

CHAPTER THREE

Applications of Topology

APPLICATIONS TO THE GEOMETRIC THEORY OF FUNCTIONS

WE HAVE ALREADY said that Riemann was the first mathematician to understand the usefulness of topology for analysis. Riemann surfaces play a fundamental role in the study of functions of complex variables. We cannot treat this question here, but at least we can define those among these surfaces which correspond to the simplest functions: algebraic functions; they are the surfaces homeomorphic to closed and orientable surfaces.

According to the principal theorem of the topology of surfaces which we have stated at the end of the preceding chapter, *the necessary and sufficient condition that two closed orientable surfaces be homeomorphic is that they have the same characteristic.*

It is to a more intuitive idea than that of characteristic that we must generally appeal here in order to define these surfaces. Let us consider a sphere and a closed simple curve C drawn on this sphere (Fig. 33). This curve separates the sphere into two parts, and every simple continuous curve joining two points belonging respectively to each of these parts cuts C at one point at least. This intuitive fact, we have already observed, results from the theorem of Jordan and from the homeomorphism of a sphere with a point removed, to a plane. It is, therefore, impossible to find a simple closed curve not separating the sphere into two "pieces." We say that the sphere is a surface of *genus* 0.

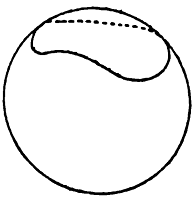


FIG. 33

On the contrary, let us consider a torus (Fig. 34): it is possible to draw on its surface simple closed curves such as a generating circle *a* or a circle *b* produced by the rotation about the axis of the torus of one of the points of the generating circle, which do not separate the torus into two "pieces." It is even possible to draw two of them: the set of a and b do not separate the torus into two "pieces." But in this case the two curves have a point in common (see Fig. 17). We can prove that it is impossible to find two simple closed curves having no common point on the torus which do not divide it into two "pieces." We say that the torus has *genus* 1.

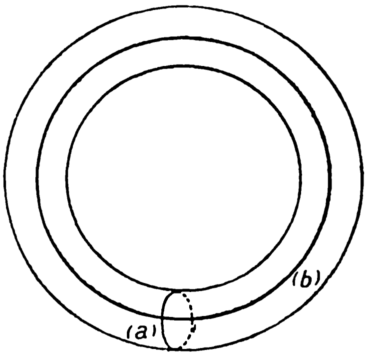


FIG. 34

We can also prove that it is impossible to find more than two simple closed curves on the torus, having or not having common points, not separating this torus into two regions. The maximum number of these curves, 2, is called the *connectivity number* of the torus.

In a more general way, we call the *genus* g of a surface the maximum number of simple closed curves with no common points that one can draw on this surface without cutting it apart, and the *connectivity number* k the maximum number of simple closed curves with or without common points having the same property.

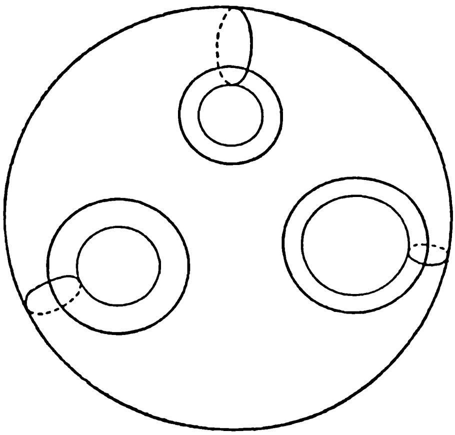


FIG. 35

If two closed orientable surfaces are homeomorphic, they clearly have the same *genus* and the same *connectivity number*. The converse is also true, so that one anticipates that these numbers g and k are linked to the characteristic c of a surface. Fréchet and Ky Fan[1] show that

$$c = 2(1 - g) \quad \text{and} \quad c = 2 - k$$

[1] *Op. cit.*

We can prove moreover that closed Riemann surfaces alone are homeomorphic with the sphere, the torus, and the generalized torus with p holes (Fig. 35). It is easy to foresee that the genus of a torus with p holes is p and that its connectivity number is $2p$. The preceding formulas make evident the fact that $k = 2g$. Fig. 35 shows clearly that the genus of a torus with three holes is 3 and that its connectivity number is 6.

APPLICATIONS TO SYNTHETIC GEOMETRY

1. Finite Geometry. The discovery of Descartes, substituting for the direct study of figures that of the analytic elements which correspond to them, was carried forward by the investigations of analytic or infinitesimal geometry for a long time without any thought of casting a backward glance to examine the foundations. Scholars were carried away by the wonderful fertility of the methods, the beauty and richness of the results. The development of topology was to change this attitude.

They had come thus far by not dissociating the geometric truth from the analytic support which made possible its establishment, without noticing that many properties belonging to algebraic or analytic figures remained valid under considerably less restrictive hypotheses. For example, through a point outside an ellipse one can draw two tangents, but this property, often related to the fact that a curve of the second degree is also of the second class, belongs to every *oval*, that is to say, to every closed convex curve.

Then a basic question arises: what properties established for algebraic or analytic elements remain true in more general cases? This is the beginning of what Darboux called *finite geometry*. This question is related to that of the investigation of the minimum number of hypotheses needed for a theorem to be meaningful.

In the problem of tangents to the oval, it is enough to assume that the curve possesses a tangent at each point. This fundamental question has been made the subject of numerous works at the head of which we must put those of the Danish mathematician C. Juel, of his precursors, such as Von Staudt, and of his school, Brunn, J. Hjelmsler, Kneser, Nagy. They were followed in France by the works of A. Marchaud. The investigations of Juel are concerned with curves or surfaces formed of "elementary pieces," such that the notion of *order* is preserved, that is to say that the number of points of intersection with a line or a plane is finite. There is found the sound idea which has given these works their success; from the analytic foundation it is sufficient to retain the notion of *order* or that of *index*, of *class* or *index of class*, the *index* being the maximum number of real points of intersection with a line, and the *index of class* the maximum number of real tangents at a point.

Marchaud has gone still further: he has reduced what was redundant in the hypotheses of Juel; also retaining the notion of order, he applies it to continuous sets of points and shows that these continuous entities are precisely curves or surfaces having tangents or osculating elements, when we make some restrictive hypotheses. Among modern geometers working in finite geometry, we further mention O. Haupt.

In order to give the reader some idea of this chapter of synthetic geometry while yet remaining elementary, we shall restrict ourselves to pointing out some particularly striking results and to giving some information about the study of plane curves, taken from a lecture given at the invitation of the Faculty of Sciences of Clermont-Ferrand by Paul Montel, in December, 1932.

The curves studied in finite geometry are those for which the element of contact varies in a continuous manner. We call an element of contact of a curve the set of one of its points and of the tangent at this point. We suppose besides that these curves have a determined order and according to the problem being studied, we introduce further some appropriate restrictive hypotheses.

The constitutive element of curves is the *elementary arc*. The elementary arc is a convex arc having a tangent which varies in a continuous way; it is of the second order since a line cuts it at most in two points, and of the second class, since from a point at most two tangents can be drawn to it, and is transformed by "duality" into an elementary arc. The combination of two arcs tangent to the same line at their common extremity gives:

The *elementary arc* (Fig. 36); the *inflection* (Fig. 37); the *cusp of the first type* (Fig. 38), corresponding to a point of inflection of the first kind for an algebraic or analytic curve; and the *cusp of the second type* (Fig. 39), corresponding to a point of inflection of the second kind.

Without laying any further stress on these definitions, let us point out that a great number of properties whose origin seemed to be algebraic in nature have been extended to the figures of finite geometry. For example, we show in the more advanced mathematics classes that:

1. every curve of the third degree without a double point has three real points of inflection;

2. the non-ruled algebraic surface of the third degree contains at most 27 real lines; and that

3. every bitangent plane to a torus cuts it along two circles (the theorem of Yvon Villarceau).

FIG. 36

FIG. 37

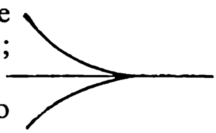


FIG. 38

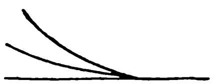


FIG. 39

To these three theorems correspond the following theorems of finite geometry:

1. Every curve of the third order without a double point has three points of inflection.

2. The (non-ruled) surface of the third order contains at most 27 lines. (Such a surface in silver, not analytic, was presented to Juel for his scientific jubilee; it had the 27 lines.)

3. Every surface of revolution of the fourth order generated by the rotation of an oval about a line in its plane but not intersecting it, is cut by a bitangent plane along two ovals.

Remarkable particular properties of certain figures have thus been disengaged from their analytic foundation. But finite geometry, although it has been the subject of considerable study, is still only in its infancy. In this branch of mathematics, imagination plays a role as important as the critical faculty, for methods to attack the problems must be created in entirety as soon as the analytic tool is abandoned. A great advance will have been made when we are able to characterize the *principal group* of this geometry, which has not yet been done.

2. Direct Infinitesimal Geometry. The one subdivision of geometry which seems most allied to analysis is surely infinitesimal geometry. Nevertheless, the development of set theory and of topology could not help but have repercussions on this part of geometry. Previously, the discovery of the Peano curve which fills the entire square, and the introduction of Jordan curves had shown that ideas which geometers had regarded until then as the most intuitive were far from being as simple as they had seemed. Nevertheless, in the same way as a continuous function without a derivative was for a long time regarded as a monstrosity unworthy of attention, geometers believed themselves right in considering only "well-behaved curves" and "well-behaved surfaces," that is to say, those admitting of definition by equations differentiable to the order necessary to make their calculations.

It was Lebesgue who first revealed the insufficiency of these considerations. As early as his admission to the Ecole Normale Supérieure in 1894, the keen critical intelligence of this celebrated mathematician revealed itself when he sought to convince his classmates of the incorrectness of an important theorem of geometry by a simple experiment which is easy to perform. We prove in "classical" infinitesimal geometry that for a surface to be fitted to a plane (with lengths preserved) it is necessary and sufficient that it be developable, that is to say that this surface be a cone, a cylinder, or the surface generated by the tangents to a skew curve. However, if like Lebesgue, we crumple a handkerchief in our hand, we obtain a surface whose irregularity is such that we have no doubt that it does not belong at all to the class of developable surfaces. Nevertheless, this surface can be fitted isometrically to a plane.

This experimental observation is the point of departure of the reform of infinitesimal geometry. But this rebirth was late, as witness the lecture given by Darboux in 1908 to the International Congress of Mathematicians at Rome—not one of the problems which he indicated for investigation by future geometers had reference to the revision of the principles of infinitesimal geometry.

A short time ago Georges Bouligand thought that the progress of modern analysis permitted a complete revision of the classical propositions in the direction of Lebesgue's remark. He has undertaken under the name of *direct infinitesimal geometry* the study of the exact correspondence between the hypotheses introduced and the geometrical properties which result from them. Bouligand has done, in a way, for infinitesimal geometry what Juel did for analytic geometry. Systematically, he has associated with each point of the variety, the set of limit elements with which he is dealing rather than the single limit element (tangent, osculating plane, etc.) which everyday geometry supposes always exists. Thence the notions of "contingent" and of "paratingent."

These notions stem at once from set topology and from "restricted topology," that is to say, from a topology in which the functions represent certain curves possessing a fixed number of successive derivatives.

Let us recall what we mean by a point of accumulation of a set E of points, infinite in number, of ordinary Euclidean space: if A is a point such that in every sphere with center at A there exists at least one point of E (and consequently an infinitude), A is called a point of accumulation of the set. To such a point of accumulation of a set E, Bouligand attaches two sets of lines:

1. The *contingent*, or set of half-lines, limits of the half-lines joining A to a point M of E as M approaches A; and

2. The *paratingent*, or set of lines, the limits of the lines MM′ joining two points M and M′ of E, when these points simultaneously approach A.

These notions of *contingent* and *paratingent* whose fundamental importance Bouligand has shown, arise from *restricted topology of the first category*, in the sense that if two curves have a common point A and the same paratingent or same contingent at this point, the same is true of the curves derived from them by a one-to-one, bicontinuous transformation, defined by functions having first derivatives with a nonvanishing "Jacobian."

It is possible to define a contingent or a paratingent of the second order or of a higher order, but the infinitesimal geometry which depends on them must then be restricted to Euclidean space.

This branch of geometry is the one most studied today (with the geometry of Riemann spaces), for Bouligand has known how to interest a whole Pleiad of young investigators in it.

Conclusion

WE COULD NOT end this little book without calling upon the reader to believe with us in the "future" of geometry, and of mathematics in general.

"I do not understand," a colleague brought up on classical philosophy said to us recently, "how mathematics, the *rational* science par excellence, can progress!"

> The fact is, as Gaston Bachelard[1] so very truly says: "Rationalism is a philosophy which strives, a philosophy that wants to extend itself, that wishes to multiply its applications" and not, as it is too often considered, "a philosophy which *recapitulates* . . . which *reduces* the richness of the multifarious to the poverty of the identical. . . . We must not confuse deduction which assures and induction which invents. Rationalism in its progressive study is eminently inductive—and is that even in mathematical thought. Hardly has a theorem been discovered when one tries to generalize it, to extend it. . . ."

To be sure, deduction is the principal tool of the mathematician, but cannot one say that it serves chiefly to convince him of facts which are presented at first to his natural intuition, then to a kind of "extended intuition" (in the sense of G. Bouligand[2])? Must experience itself be excluded from mathematical thought? We do not think so. On the contrary, we believe that the three aspects, intuition, experience, and theory are intermingled in the mind of the geometer who constantly applies what Ferdinand Gonseth[3] calls "the principle of appropriateness."

Absolute reality does not exist in mathematics, and we have been able to see in the course of this work that such and such a property is true only relative to such another chosen in an "appropriate" way.

[1] *Le Rationalisme Appliqué* (Presses Universitaires de France), 1949.

[2] *Op. cit.*

[3] *La Géométrie et le problème de l'espace* (Ed. du Griffon: Bibl. Scient.), Neuchâtel, 1947.

We wrote at the beginning of our foreword:

Nowhere with as much power as in geometry do the *true* and the beautiful appear so intimately related.

We should rather say:

Nowhere with as much power as in geometry do the appropriate and the beautiful appear so intimately related.

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