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15/12

TOPOLOGY AND ORDER

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TOPOLOGY AND ORDER

by

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D. VAN NOSTRAND COMPANY, INC.
PRINCETON, NEW JERSEY
TORONTO NEW YORK LONDON

Dedico este Tese à minha mãe
LOTTY NACHBIN

D. VAN NOSTRAND COMPANY, INC.
120 Alexander St., Princeton, New Jersey (*Principal office*)
24 West 40 Street, New York 18, New York

D. VAN NOSTRAND COMPANY, LTD.
358, Kensington High Street, London, W.14, England

D. VAN NOSTRAND COMPANY (Canada), LTD.
25 Hollinger Road, Toronto 16, Canada

Published simultaneously in Canada by
D. VAN NOSTRAND COMPANY (Canada), LTD.

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE TO THE ENGLISH EDITION

In 1948 I published three notes in the Comptes Rendus de l'Académie des Sciences (Paris) containing results on the relationship between topological and order structures. In 1950 I also wrote a note along the same line for the Proceedings of the International Congress of Mathematicians (Cambridge, Mass.). Most of these results were developed in a monograph entitled TOPOLOGIA E ORDEM which I wrote in Portuguese while I was at the University of Chicago and which was printed by the University of Chicago Press in 1950. This monograph was a thesis that I submitted to the Faculdade Nacional de Filosofia, Universidade do Brasil (Rio de Janeiro), in 1950, as a candidate for a vacant chair in analysis. Many of my results in this direction have been reobtained and used in the past 15 years, mostly by mathematicians interested in the structure theory of topological semigroups, in the classification of closed semialgebras of continuous real-valued functions, and in dynamical systems. My Portuguese monograph being inaccessible to these readers, its translation into English is hereby presented. An appendix contains the English translation of the three notes written in French and the note in English mentioned above; it also contains an article in English in the same field that I published in 1950 in the journal Summa Brasiliensis Mathematicae (Rio de Janeiro). These papers are included here for the reader's convenience; they contain additional information belonging to the same area that is not developed in my Portuguese monograph.

I take pleasure in thanking Professor R. P. Halmos, one of the editors of the series VAN NOSTRAND MATHEMATICAL STUDIES, for kindly offering to me the possibility of publishing my monograph in this series. I should also like to thank Dr. Lulu Bechtolsheim for her efficiency in carrying out the present translation.

Leopoldo Nachbin

August 10, 1964

Rochester, New York

PREFACE

The purpose of this monograph is to present results obtained by the author in his research on spaces which are, at the same time, equipped with a topological structure and an order structure, research which was initiated in 1947. Some of these results were communicated, without the respective proofs, to the Académie des Sciences de Paris, and a summary of these results appears in the Comptes Rendus of this Academy for the early part of 1948 (see the Bibliography at the end of the present monograph). Part of the results of this monograph were also the subject of a communication by the author to the International Congress of Mathematicians, Harvard University, 1950.

Starting out from the fundamental, now classical, work of P. Urysohn and A. Weil in the field of general topology, the present monograph introduces the concepts of a normally ordered space, a compact ordered space, and a uniform ordered space, and generalizes the most outstanding results and aspects of the theory developed by Urysohn and Weil to these spaces.

The monograph is, essentially, divided into two parts. The first consists of the Introduction and contains purely preparatory material. It establishes the terminology adopted, which, incidentally, with few exceptions, agrees with that followed by N. Bourbaki in the so far published fascicles of his well-known treatise "Éléments de Mathématiques." In addition, this introductory chapter pursues the pedagogical purpose of presenting the concepts that will be used systematically and giving their historical development, thus making the reading of this paper easier for a person who is not versed in modern mathematics. We must, however, stress that the entire contents of the Introduction may be found in detailed, often excellent, exposition in various books that are indicated at the end of each paragraph and that offer the reader a more comprehensive discussion both of the material covered and of its ramifications. A few results and definitions which are later used in a purely incidental manner are not included in the Introduction but may be found in the references mentioned.

The second part comprises Chapters I, II, and III and is devoted to the actual exposition of the theory of topological ordered spaces and uniform ordered spaces and of results concerning locally convex ordered vector spaces. Each chapter is preceded by a summary describing its contents.

Paragraph 1 of the Introduction contains various bibliographical references selected from the many which may be given to facilitate the task of the reader who wishes to follow more closely the historical development of the mathematical ideas that led to the problems here under investigation. From Paragraph 2 of the Introduction on, however, we shall refer to the Bibliography at the end of this monograph, indicating for this purpose, the name of the author in question followed by a key number.

Leopoldo Nachbin

August 10, 1950
Chicago, Illinois

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INTRODUCTION

§1. Historical outline.

1. Considerations of a topological nature, depending on the concepts of limit and continuity, originated together with the oldest problems of geometry and mechanics, such as the calculation of areas and the movement of figures.

In the hands of eminent mathematicians and during a long period, infinite series were a tool used in an entirely formal manner, that is, without regard for convergence considerations. Gauss seems to have been the first to think about the legitimacy of the use of infinite processes, such as the series expansion of Newton's binomial with an arbitrary exponent, which at times led to surprising absurdities. To Abel and Cauchy, however, is due the credit for having defined the concepts of a convergent series and sequence and the concept of a continuous function with the rigor that is so familiar to us today.

The first mathematician who attempted to isolate the idea of a topological space and who sensed its far reaching importance was Riemann.[1] However, in order for the expansion of topology in this direction to become possible, it was indispensable that this new discipline should have at its disposal experience and information concerning important particular cases.

Then came Cantor's investigations of 1874, meeting with opposition from many of his contemporaries because of their complete novelty. These investigations were in part inspired by the desire to analyze the difficult questions concerning the convergence of Fourier series. Simultaneously, the theory of real numbers was erected on a solid foundation by

[1] B. Riemann, "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," *Gesammelte Mathematische Werke*, Leipzig, 1892, pp. 272-287.

Dedekind and Cantor. The systematic study of the concept of a set, of an accumulation point, etc. are linked to the work of Cantor.[2]

Parallel to the investigations on the topology of the line and of p-dimensional Euclidean space, it was attempted to make use of the same methods, not only with respect to point sets in the sense of elementary geometry but also to sets whose elements were curves, surfaces, and, above all, functions. The pioneers in this period of infancy of functional analysis were Ascoli, Pincherle, and principally Volterra.[3] To the latter we owe a systematic study (1887) of line functions (or functionals according to the terminology adopted since Hadamard) and of the infinitesimal calculus of functionals.

An epoch-making step of progress was achieved, at the beginning of our century, by the introduction of the so-called Hilbert spaces, later defined axiomatically by von Neumann (1927). These spaces are, without doubt, the most important and fertile example of topological spaces of an infinite number of dimensions among all the examples of such spaces known today. By their rich structure which includes the concepts of the sum of vectors, the product of a scalar and a vector, and the scalar product of two vectors, these Hilbert spaces unite with their geometrical elegance an impressive variety of possible analytical applications.

The existence of so many examples of spaces like the Euclidean spaces and their subspaces and the various function spaces in which topological considerations find natural applications gave rise to the desire or, rather the necessity of a synthesizing approach which would permit the study of the properties held simultaneously by all these spaces and would, consequently, bring about a better comprehension of the peculiar aspects of each one of them.

Thus general topology originated with the introduction, in 1906, of metric spaces by Fréchet[4] and with the elaboration of an autonomous theory

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[2] G. Cantor, *Gesammelte Abhandlungen*, Berlin, 1932.

[3] V. Volterra et J. Perés, *Théorie générale des fonctionnelles*, Paris, 1936.

[4] M. Fréchet, "Sur quelques points du calcul fonctionnel," Rendiconti del Circolo Matematico di Palermo, vol. 22, 1906, pp. 1-74.

of abstract topological spaces by Hausdorff in 1914[5]; the merits of Hausdorff's achievement are recognized by the association of his name with the so-called Hausdorff spaces. From this time on, the steps of progress of the new discipline followed rapidly one upon the other.

During the period in which the topology of the line developed, the discovery of the compactness criteria of Bolzano-Weierstrass and Borel-Lebesgue stood out at once. To this group of results there belongs a theorem, due to Weierstrass, according to which every continuous function on a bounded and closed interval there attains a minimum. Weierstrass' observation that the application of an analogous principle in function spaces is not always valid but must be based on previous justification met with response in Hilbert's proof of the existence of a minimum for the integral $\iint[(\partial f/\partial x)^2 + (\partial f/\partial y)^2]\, dxdy$ and the subsequent solution of the classical Dirichlet problem concerning harmonic functions. This remark of Weierstrass is one of the sources from which the concepts of semi-continuity and of compactness in function spaces draw their interest. Semi-continuity was discovered by Baire in the case of real variables and was later utilized by Tonelli in the calculus of variations. We owe to Fréchet the formulation of the idea of compactness in metric spaces in the sequential manner of Bolzano-Weierstrass and the recognition of its equivalence, in this case, to the property of Borel-Lebesgue. The concept of a compact space as it is now considered in topology became the object of systematic study, based on the criterion of Borel-Lebesgue, as recently as 1929 and originated with Alexandroff and Urysohn.[6]

Normal spaces, the importance of which derives, to a considerable extent, from the extension theorem for continuous real-valued functions, were introduced by Tietze. First established by Lebesgue for the case of functions defined on a subset of the plane and by Tietze for the case of functions defined on a metric space, this theorem attained a general form

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[5] F. Hausdorff, *Grundzuege der Mengenlehre*, Leipzig, 1914.

[6] P. Alexandroff and P. Urysohn, "Mémoire sur les espaces topologiques compacts," Verhandelingen der Akademie van Wetenschappen te Amsterdam, vol. 14, 1929, pp. 1-96.

in a basic paper by Urysohn.[7] The related category of completely regular spaces was brought out by Tychonoff's work[8] on the compactification of topological spaces and includes, according to a theorem of Pontrjagin, all topological groups.

Continuity is a purely local phenomenon; the corresponding global phenomenon we call uniform continuity today. The first trace of the idea of uniform spaces in mathematics is found in Cauchy's general criterion for the convergence of a series or sequence. Under the influence of Weierstrass and Heine, the ideas of a uniformly convergent series and of a uniformly continuous function entered the domain of mathematical analysis.

To Fréchet and Hausdorff we owe the concept of a complete metric space (one in which Cauchy's convergence criterion is satisfied), the concept of a uniformly continuous function on a metric space, and the possibility of completing every metric space by a construction analogous to that employed by Cantor in order to define the real numbers on the basis of the rational numbers. One of the fruits of this order of ideas is the Riesz-Fischer theorem according to which the space of square integrable functions in the sense of Lebesgue is complete.

With the definition of topological groups by Schreier in 1925 and of compact spaces by Alexandroff and Urysohn in 1929, the concept of a uniformly continuous function came to have significance in a greater number of cases. Finally, in 1937, A. Weil[9] introduced uniform spaces thus encompassing in one single theory various aspects common to the theories of metric spaces, topological groups, and compact spaces.

In the panorama of the foundations of topology just outlined, we have failed to mention the ideas which, originating in Riemann's work, were subsequently developed by Betti and Poincaré, leading to the analysis situs or algebraic topology of today. The reason is that these ideas belong to

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[7] P. Urysohn, "Ueber die Maechtigkeit der zusammenhaengenden Mengen," Mathematische Annalen, vol. 94, 1925, pp. 262-295.

[8] A. Tychonoff, "Ueber die topologische Erweiterung von Raeumen," Mathematische Annalen, vol. 102, 1929, pp. 544-561.

[9] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale,* Paris, 1937.

a direction of research distinct from that in which we shall be interested in the present monograph.

2. The general concept of order (or partial order) has its origin both in logic and mathematics. It seems to have been isolated for the first time in the course of the nineteenth century, although older roots may be traced to work preceding that period. The important case of total order, which from the point of view of modern mathematics only represents a special example, is as old as the ideas of number and time, and its origin is lost in the fog of the past.

Chronologically, the most important investigations in this direction, dating back to 1847, are those of G. Boole concerning the mathematical analysis of logic and the laws of thought. His name is, thus, indissolubly associated with the so-called Boolean algebras which, for a long period of time, represented the only known example of an algebraic system the elements of which were devoid of any trace of numerical significance. These algebras have, accordingly, come to enjoy considerable mathematical interest at the present time.

Axioms defining the concept of an ordered set are found in the work of C.S. Pierce on the algebra of logic, dating back to 1880; in 1890 such axioms were also studied systematically by Schroeder, but his studies, too, were still carried out from the point of view of the needs of logic.

To Dedekind, actually, is due the credit for having been the first to observe that the concept of an ordered set occurs with such frequency in mathematics that it deserves to be studied as an autonomous subject.[10] This point of view, expressed in 1897, was later advocated by Hausdorff in his book on the foundations of set theory and by Emmy Noether in her work on algebra.

The notion of a lattice, or, more precisely, the definition of the concepts of supremum and infimum on the basis of an order relation, goes back to Pierce. But Pierce was erroneously of the opinion that all lattices are distributive. This mistake was corrected by Schroeder who isolated the

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[10] R. Dedekind, "Ueber Zerlegungen von Zahlen durch ihre grossten gemeinsamen Teiler," *Gesammelte Mathematische Werke*, vol. II, Braunschweig, 1931, pp. 103-147.

concept of a distributive lattice.

The first investigations of ordered and lattice-ordered Abelian groups were initiated by Dedekind who proved among other things, that every lattice-ordered group is distributive. This fact was later rediscovered by Freudenthal[11] in a paper in 1936, which had great influence on the more recent development of the theory of vector lattices and its applications to functional analysis, and which was inspired by ideas of F. Riesz.[12]

In 1900, Dedekind[13] introduced the concept of a modular lattice later called Dedekind lattice, a concept which was not so easy to isolate as that of the distributive lattice; these Dedekind lattices later came to play an important role in the axiomatization of projective geometry.

Systematic studies on the theory of lattices and its applications, dating from approximately 1930, were carried on by F. Klein, Menger, G. Birkhoff, Ore, von Neumann, and M. H. Stone, to cite just a few of the names linked with the most important fundamental concepts of the subject.

Among the most recent advances stands out, on the one side, the formulation of the maximal principle[14] which actually goes back to Hausdorff and represents a welcome substitute for the method of transfinite induction due to Cantor; on the other side, the introduction by von Neumann,[15] in 1936, of continuous geometries which generalize finite dimensional projective space and which, in addition to their purely geometrical aspects, have interesting relations to the chapter of modern analysis devoted to the theory

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[11] H. Freudenthal, "Teilweise geordnete Moduln," Verhandelingen der Akademie van Wetenschappen te Amsterdam, vol. 39, 1936, pp. 641-651.

[12] F. Riesz, "Sur la décomposition des opérations fonctionnelles linéaires," Atti del Congresso Internazionale dei Matematici, vol. 3, Bologna, 1928, pp. 143-148.

[13] R. Dedekind, "Ueber die von drei Moduln erzeugte Dualgruppe," *Gesammelte Mathematische Werke*, vol. II, Braunschweig, 1931, pp. 236-271.

[14] N. Zorn, "A remark on method in transfinite algebra," Bulletin of the American Mathematical Society, vol. 41, 1935, pp. 667-670.

[15] J. von Neumann, *Lectures on Continuous Geometry*, Princeton, 1936-1937.

of algebras of operators in Hilbert spaces.

From the point of view of functional analysis, we must cite the work of Kantorovitch[16] and Kakutani[17] and mention that the study of the spectral decomposition of self-adjoint operators in Hilbert space constitutes an important example of a field in which the methods and ideas of vector lattices prove their fertility.

The study of the general concept of order is, without doubt, a valuable instrument for the comprehension of the foundations of various branches of modern mathematics and of the aspects they have in common. It is, however, not superfluous to stress that such a study must not be undertaken for its own sake, but that the mathematical value of the concept of order resides in the applications which it admits.

§2. Set theory

1. This second paragraph of the Introduction is devoted to the presentation of the most important notational symbols to be used, among which classical ones now occur such as $\in$, $\subset$, $\cup$ and $\cap$. Thus, for example, $x \in X$ (or $X \ni x$) will mean that the point x belongs to the set X, and $X \subset Y$ (or $Y \supset X$) will mean that the set X is contained in the set Y. The symbol $\cup$ represents the union of sets: for example, $X \cup Y$ represents the union of the sets X and Y, and $\bigcup_{n=1}^{\infty} X_n$ represents the union of the sets $X_1, \ldots, X_n, \ldots$. Analogous remarks hold for the intersection $\cap$. The empty subset of a set is indicated by $\emptyset$. The difference, in the sense of set theory, of X and Y is indicated by $X - Y$. [In the theory of additive groups, the symbol $X - Y$ has a different meaning; in a case in which the interpretation is doubtful, the reader will be warned.] From the point of view of notation, we shall not distinguish between a point and the set that reduces to that point.

If C indicates a condition, or a series of conditions, involving a point x, we represent by $[x; C]$ the set of all points x which satisfy

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[16] L. Kantorovitch, "Lineare halbgeordnete Raeume," Recueil Mathématique, vol. 2. 1937, pp. 121-168.

[17] S. Kakutani, "Concrete representation of abstract M-spaces," Annals of Mathematics, vol. 42, 1941, pp. 994-1024.

the condition, or series of conditions, C. For example

$$[x;\ x \in E,\quad f(x) = 0]$$

indicates the set of all those points x which belong to the set E and for which the function f vanishes.

Let us consider two sets X and Y. A *function* f on X into Y is a correspondence which associates with every point $x \in X$ a point $y = f(x) \in Y$. The set X is called the *domain* of the function f and Y its *range-space*. If $A \subset X$, we represent by $f(A)$ the *direct image* of A under f, that is, the set of points y of the form $f(x)$ for which $x \in A$. Similarly, if $B \subset Y$, we represent by $f^{-1}(B)$ the *inverse image* of B under f, that is, the set of points $x \in X$ for which $f(x) \in B$.

If f_1 and f_2 represent two functions whose domains are respectively X_1 and X_2, we shall say that f_1 is an *extension* of f_2 provided $X_1 \supset X_2$ and we have $f_1(x) = f_2(x)$ for every $x \in X_2$.

Let us consider two sets E_1 and E_2. The *Cartesian product*, or simply *product*, of E_1 by E_2 is, by definition, the totality of all ordered pairs (x_1, x_2) where $x_1 \in E_1$ and $x_2 \in E_2$. We shall represent this product by $E_1 \times E_2$. (See Fig. 1.) In particular, the *Cartesian square*, or simply *square*, of a set E is the product of E by itself, that is, the totality of all ordered pairs (x, y) where $x, y \in E$. We shall use the notation E^2 to represent this square.

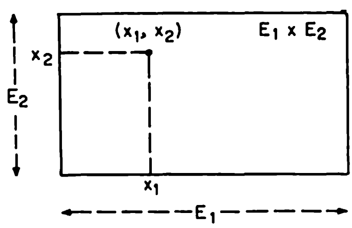

Fig. 1

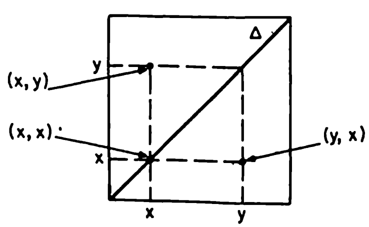

Fig. 2

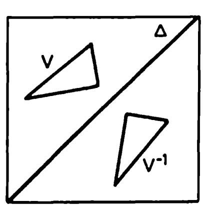

Fig. 3

The *diagonal* Δ of E^2 is the set of pairs of the form (x, x) where $x \in E$. The point of the square *symmetric* to a point (x, y) with respect to the diagonal is the point (y, x). (See Fig. 2.) If V indicates a subset of E^2, the symmetric subset V^{-1} with respect to the diagonal is the totality of all points symmetric to the points of V with respect to the diagonal. (See Fig. 3.)

If (x, y) and (x', y') designate two pairs of E^2 such that $y = x'$, the point (x, y') will be called the *composite* of (x, y) with (x', y'). (See Fig.4.) Consider, now, two subsets V and W of E^2.

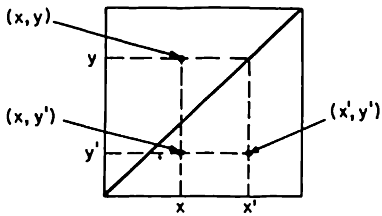

Fig. 4

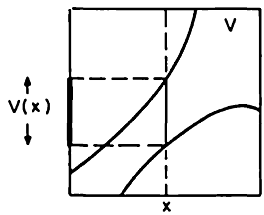

Fig. 5

By definition, the collection formed of all the composites of a point of V with a point of W shall be called the composite of V and W and denoted by $V \circ W$. There is no difficulty in verifying that the operation on subsets of the square so described is associative, that is,

$$U \circ (V \circ W) = (U \circ V) \circ W \quad ;$$

thus expressions of the type $U \circ V \circ W$ have a clear meaning. We also note that

$$V \circ \Delta = \Delta \circ V = V$$

For every subset $V \subset E^2$ and every point $x \in E$, we indicate by $V(x)$ the set of those points $y \in E$ such that $(x, y) \in V$. (See Fig.5.)

The product concept extends to the case of a finite or even an infinite number of sets. However, we have no need for this more general case (except for that of the cube $E^3 = E \times E \times E$ which will occur in an incidental manner).

References: Bourbaki, 1.

§3. Topological spaces

1. In this paragraph we shall present the ideas of general topology which are used systematically in all that follows.

We consider a set E. A *topology* on E is a set of subsets of E which are called open according to this topology and which are such that

1) E and $\emptyset$ are open subsets.
2) The intersection of a non-empty finite collection of open subsets is open.
3) The union of a non-empty collection of open subsets, whether finite or infinite in number, is open.

A *topological space* is a set *equipped* with a topology, that is, a set on which a topology is given.

The set R of real numbers will always be considered as being equipped with its *natural topology* which is defined as follows. A subset $X \subset R$ is said to be open if X is empty, or, in the contrary case, if for every point $a \in X$ there exists a number $r > 0$ such that the open interval $]a - r, a + r[$ is contained in X. (See Fig. 6.) In a more general manner, we consider p-dimensional Euclidean space. A point of this space R^p is by definition a finite sequence:

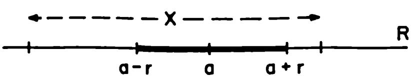

Fig. 6

$$x = (x_1, \ldots, x_p)$$

of p real numbers $x_1, \ldots, x_p$. The Euclidean distance between two points

$$x = (x_1, \ldots, x_p), \quad y = (y_1, \ldots, y_p)$$

is defined by the formula

$$d(x, y) = [(x_1 - y_1)^2 + \cdots + (x_p - y_p)^2]^{\frac{1}{2}}$$

This distance concept possesses, among others, the following properties:

1) $d(x, x) = 0$ and $d(x, y) > 0$ if $x \neq y$;
2) $d(x, z) \leq d(x, y) + d(y, z)$;
3) $d(x, y) = d(y, x)$.

The set of points $x \in R^p$ such that $d(x, a) < r$, where $a \in R^p$ and $r > 0$, is called the *open ball* of center a and radius r. In the following the space R^p is always considered as a topological space relative to its *natural topology:* A set $X \subset R^p$ is said to be open in this topology if X is empty, or, in the contrary case, if for every point $a \in X$ it is possible to determine a number $r > 0$ such that the open ball of center a and radius r is contained in X.

According to Fréchet, one designates as a *metric space* every set E in which, for any two points x and y of E, is defined a distance $d(x, y)$ whose values are real numbers and which has the above properties 1), 2), and 3). The function $(x, y) \to d(x, y)$ is called a *metric*. E can, then, be equipped with a topology defined in the same manner as in the case of R^p. Many important examples of topological spaces are included in the category of metric spaces.

In a topological space E, we call a subset $X \subset E$ *closed* if the complementary subset $E - X$ is open. In consequence of the axioms concerning open subsets, we have the following properties for closed subsets:

1) E and $\emptyset$ are closed subsets;
2) The union of a non-empty finite collection of closed subsets is closed;
3) The intersection of a non-empty collection of closed subsets, whether finite or infinite in number, is closed.

If we compare the properties of open subsets with the corresponding properties of closed subsets, we recognize the duality of these two concepts.

We designate as the *closure* of a subset X of a topological space E the smallest closed subset containing X; this closure is represented by $\bar{X}$. We note the following properties:

1) $\bar{\emptyset} = \emptyset$, 2) $X \subset \bar{X}$,
3) $\overline{X \cup Y} = \bar{X} \cup \bar{Y}$, 4) $\bar{\bar{X}} = \bar{X}$.

The dual notion is that of the interior of X. It is the largest open subset of X.

Given a point or a subset of a topological space, every subset which contains the given point or subset in its interior is called a *neighborhood* of this point or subset. We note the following properties of neighborhoods:

1) A subset which contains a neighborhood is also a neighborhood.

2) The intersection of a non-vanishing finite number of neighborhoods is also a neighborhood.

These properties hold for the neighborhoods both of a point and of a set.

A *base or fundamental system for the neighborhoods* of a point or a subset in a topological space E is a collection of neighborhoods which are called *basic* and which are such that every neighborhood of this point or subset contains at least one basic neighborhood.

An *open base* of a topological space E is a collection of open subsets of E which are called *basic* and which are such that every non-empty open subset of E may be expressed as the union of a non-empty collection of basic open subsets.

An *open subbase* for a topological space E is a collection of open subsets of E which are called *subbasic* and which are such that the collection of the intersections of a non-vanishing finite number of subbasic open subsets constitutes an open base.

Among the spaces which prove to be most useful in the applications are included those which have the property that the neighborhood systems of their points have countable bases, and, among these, more especially, the metric spaces. For each of these spaces there obtains a simple interdependence between the topology of, and the so-called convergent sequences in, the space. A sequence $x_1, \ldots, x_n, \ldots$ of points in a topological space E is said to *converge* to a point $x \in E$ if, for every neighborhood V of x, it is possible to determine an integer $N \geq 1$ such that $x_n \in V$ for every $n \geq N$. We then write $x_n \to x$.

2. Let us consider two topological spaces E and F. A function f on E to F is said to be *continuous at a point* $a \in E$, if, for each neighborhood W of $b = f(a)$ on F, a neighborhood V of a on E can be determined such that $f(x) \in W$ for every $x \in V$. (See Fig. 7.) The function f is said to be

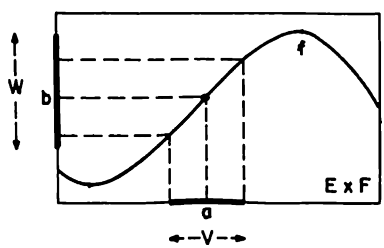

Fig. 7.

continuous on the space E if it is continuous at every point of this space. We note that a necessary and sufficient condition in order that f be continuous on E is that the inverse image under f of every open subset of F be open in E; or that the inverse image under f of every closed subset of F be closed in E. In the case of a real function of real variables, the continuity of the function is expressed in the usual ε and δ language due to Cauchy and extended by Fréchet to metric spaces.

Let us again consider two topological spaces E and F. We can define a topology on the product $E \times F$ in the following way. We say that a subset X of $E \times F$ is open if X is empty or, in the contrary case, if, for every point $(a, b) \in X$, it is possible to determine a neighborhood V of a on E and a neighborhood W of b on F such that $V \times W \subset X$. The topology thus obtained on $E \times F$ is called the *product* of the topologies of E and F. Whenever we consider the product of two topological spaces as a topological space, it shall be understood that we refer to the product topology.

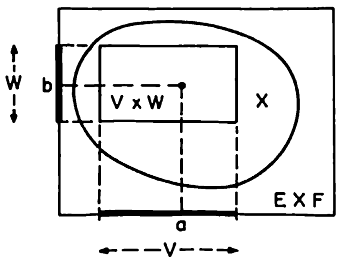

Fig. 8.

The concept of a topological product space defined above permits the formulation of the concept of continuity for functions of two variables. This concept may be extended to the case of a product of a finite or infinite number of topological spaces.

We consider a topological space E and let $F \subset E$ be one of its subsets. We can define a topology on F in the following way. We say that a subset $X \subset F$ is open, if an open subset T of the topological space E can be found so that $T \cap F = X$. (See Fig. 9.) In this fashion we obtain a topology on the set F which is said to be *induced* by the topology of E. F is then designated as a topological *subspace* of E. Whenever we consider a topology on a

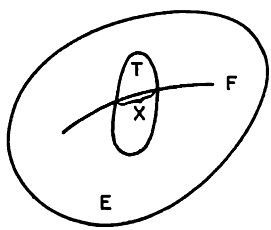

Fig. 9.

subset of a topological space, it shall be understood that this topology is the induced topology.

The processes which consist of forming products of topological spaces, or of taking subspaces of topological spaces are very important in that they permit us to construct new examples of topological spaces from already known ones.

3. An *open covering* of a topological space E is a collection (X_i) of open subsets X_i of E whose union is identical with the space E, that is,

$$E = \bigcup_i X_i \quad .$$

A topological space E is said to be *compact* if it possesses the following *property of Borel-Lebesgue:*

BL. Every open covering (X_i) of E contains a finite open subcovering; that is, one can determine a finite number of indices $i_1, \ldots, i_n$ such that

$$E = X_{i_1} \cup \cdots \cup X_{i_n}$$

A subset of a topological space is said to be *compact* if this subset is a compact space relative to the induced topology.

The *theorem of Borel-Lebesgue* affirms that a subset of p-dimensional Euclidean space is compact if and only if it is bounded (that is, contained in at least one closed ball), and closed.

According to a theorem of Fréchet, a subset X of a metric space E is compact if and only if it possesses the following *property of Bolzano-Weierstrass:*

BW. Every sequence of points of X contains a subsequence which converges to a point of X.

4. In order that a topological space be really useful in the applications, it must satisfy some axiom of separation which permits one to distinguish its various points topologically from one another. Among the familiar axioms of separation, that of Hausdorff stands out. A *Hausdorff space* is a topological space which satisfies the following *Hausdorff axiom of separation:*

H. If $a, b \in E$ are distinct points, it is possible to determine a neighborhood V of a and a neighborhood W of b which are disjoint. (See Fig. 10.)

Every metric space is a Hausdorff space.

In the applications, one sometimes has to do with topological spaces that do not satisfy the Hausdorff axiom of separation. In such cases one proceeds as follows. One defines, in terms of the original spaces, (by a method which we shall not describe since we shall not use it) certain related spaced called quotient spaces which, then, satisfy the axiom in question.

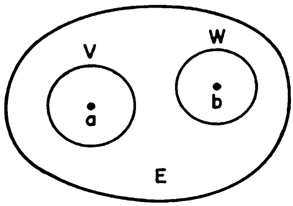

Fig. 10.

A topological space E is said to be *normal* if it satisfies the following separation condition:

N. If F_0 and F_1 are disjoint closed subsets of E, it is possible to determine an open subset $A_0 \supset F_0$ of E and an open subset $A_1 \supset F_1$ of E which are disjoint.

We note the analogy between the axioms H and N.

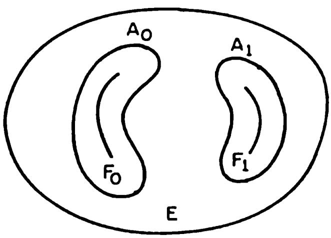

Fig. 11.

The most important examples of normal spaces are metric spaces and compact Hausdorff spaces.

The importance of normal spaces for general topology rests on two fundamental results due to Urysohn.

The first, known as *Urysohn's separation theorem*, states that a topological space E is normal if and only if, for any two disjoint closed subsets F_0, $F_1 \subset E$, it is possible to determine a continuous real-valued function f on E such that $0 \leq f \leq 1$ (that is, f assumes only values between 0 and 1), $f(x) = 0$ if $x \in F_0$ and $f(x) = 1$ if $x \in F_1$ (see Fig. 12); or, as we are accustomed to say in a suggestive and abbreviated manner, if it is possible to separate two disjoint closed subsets by means of a continuous function.

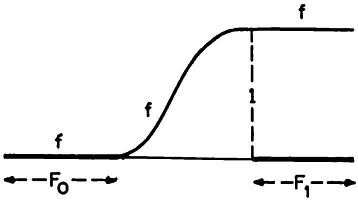

Fig. 12.

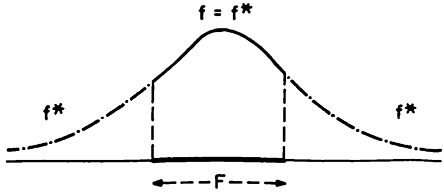

Fig. 13.

The second result, known as *Urysohn's extension theorem,* tells us that a topological space is normal if and only if, for every closed subset $F \subset E$, any real-valued function f which is continuous on F can be extended to a real-valued function f^* which is continuous on the entire space E. (See Fig. 13.)

An important category of topological spaces is that of *completely regular* or *Tychonoff* spaces. A topological space E is said to be *completely regular* if E is a Hausdorff space that has the following *Tychonoff property:*

T. If $a \in E$ and V is a neighborhood of a, it is possible to determine a continuous real-valued function f on E, where $0 \leq f \leq 1$, such that $f(a) = 1$ and $f(x) = 0$ for every point $x \in E - V$. (See Fig. 14.)

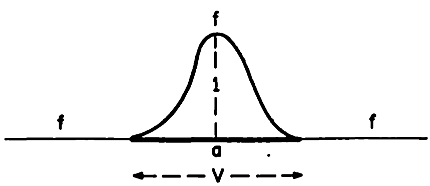

Fig. 14.

Every normal Hausdorff space is completely regular.

Taking into account Urysohn's separation theorem, we note that there is an analogy between normal spaces and completely regular spaces, or better, between normal spaces and uniformizable spaces; we designate as such topological spaces that have the Tychonoff property, irrespective of whether they do or do not satisfy the Hausdorff axiom.

The use of the term uniformizable as introduced above will be justified shortly when, after defining the concept of a uniform structure on a set, we state the necessary and sufficient condition to be satisfied by a topological space in order that it may be endowed with a uniform structure.

5. Consider a set C. A *filter* on C is a collection F of subsets of C which has the following properties:

1) $C \in F$ but $\emptyset$ does not belong to F ;
2) if $X, Y \in F$, then $X \cap Y \in F$;
3) if $X \subset Y \subset C$ and $X \in F$, then $Y \in F$.

A *base of a filter* F is a collection of subsets of C belonging to F which are called basic and which are such that every subset of C belonging to F contains at least one basic subset.

A *uniform structure* on a set E is a filter F of subsets of E^2, each of which is called a *surrounding* on E and which are such that:

1) if $V \in F$, then $\Delta \subset V$;
2) if $V \in F$, then there exists a subset $W \in F$ such that $W \circ W \subset V$;
3) if $V \in F$, then $V^{-1} \in F$.

A *uniform space* is a set *equipped* with a uniform structure, that is, a set on which a uniform structure is given.

Metric spaces furnish an important example of uniform spaces. Indeed, let E be a metric space and d the corresponding distance function. For every number $\varepsilon > 0$, we indicate by d_ε the set of points (x, y) of E^2 such that $d(x, y) \leq \varepsilon$. Let F be the collection of those subsets of E^2 which contain at least one of the sets d_ε where $\varepsilon > 0$. This collection F constitutes a uniform structure on E.

If we consider the set of real numbers R equipped with the distance function defined by $d(x, y) = |x - y|$, every set d_ε with $\varepsilon > 0$ is a strip of the plane R^2 bounded by two straight lines which are parallel and symmetric to but distinct from the bisector of the first and third quadrants. The collection of those subsets of R^2 which each contain such a strip is a uniform structure on R and is called the *natural* uniform structure of R.

Every uniform structure F on E determines a topology on E in the following manner. A subset $X \subset E$ is said to be open, if it is empty, or, in the contrary case, if, for every point $a \in X$, it is possible to determine $V \in F$ such that $V(a) \subset X$.

The uniform structure is said to be a *Hausdorff uniform structure* if the topology that it defines is a Hausdorff topology. For this it is necessary and sufficient that the intersection of all $V \in F$ be equal to the diagonal Δ.

Let us consider a topological space E. A uniform structure on E is said to be *compatible* with the topology on E if the topology determined by this uniform structure is identical with the given topology of E. According to a theorem of Weil, the necessary and sufficient condition which the topological space E must satisfy in order that a uniform structure compatible with the topology of E exists is that E be a uniformizable space.

Every compact Hausdorff space admits one and only one uniform structure which is compatible with its topology: the filter which defines this uniform structure is the filter of the neighborhoods of Δ in E^2.

We have already mentioned that, if on a given set a distance function is defined, it determines a uniform structure on that set. According to a theorem of Weil, a given uniform structure is determined by a distance function if and only if the filter on E^2 defining this structure has a countable base.

Let us consider two uniform spaces E_1 and E_2 and a function f on E_1 to E_2. Let F_1 and F_2 stand for the filters which define the uniform structures of these spaces. The function f is said to be uniformly continuous if, for every $V_2 \in F_2$, it is possible to determine $V_1 \in F_1$ such that, if $x \in E_1$ and $y \in V_1(x)$, then $f(y) \in V_2[f(x)]$.

In the case of the natural uniform structure of the line, this concept coincides with the usual concept of uniform continuity of a real-valued function of a real variable.

The concept of uniform continuity may also be formulated for functions of several variables.

References: Bourbaki, 2, 4; Lefschetz 1, Chap. I; Alexandroff and Hopf, 1, Part I; Weil, 1.

§4. Ordered sets.

1. We now proceed to indicate the ideas relative to ordered sets which we shall have occasion to use.

Let E designate a certain set. Whenever, for ordered pairs of elements of E there is defined a concept of *less than or equal to*, we say that this concept establishes a *preorder* on E; we indicate that the point $x \in E$ is less than or equal to the point $y \in E$ by writing $x \leq y$, and we always assume that the preorder is reflexive and transitive so that it possesses the following properties:

1) if $x \in E$, then $x \leq x$;

2) if $x, y, z \in E$, and $x \leq y$, $y \leq z$, then $x \leq z$.

An *order* on E is a preorder which is *antisymmetric*. that is:

3) if $x, y \in E$ and both $x \leq y$ and $y \leq x$, then $x = y$

A *total order* on E is an order which is *decisive*, that is:

4) if $x, y \in E$, then either $x \leq y$ or $y \leq x$.

A *preordered set* is a set *equipped* with a preorder, that is, a set on which a preorder is given. The concepts of an *ordered set* and a *totally ordered set* or *chain* are defined analogously.

Preordered sets constitute a more general category than ordered sets; this degree of generality is, in a certain sense, comparable to that of arbitrary topological spaces relative to Hausdorff topological spaces.

The set of real numbers will always be considered as equipped with its well-known *natural total order*.

Consider a set E. Define $x \leq y$ by $x = y$. In this fashion we clearly obtain an order relation on E, which we call the *discrete order*. Logically there is no distinction between the discrete order and the relation of equality. The advantages of looking at the equality relation as an order relation derive from the fact that many of the results relative to non-ordered sets can then be considered as particular cases of results on ordered sets. [For example, Urysohn's separation theorem, stated in §3, can be considered as a particular case of Theorem 1, §2, Chapter I as soon as the preorder referred to in the latter is interpreted as the discrete order.]

Given a set E, the *graph* of a preorder on E is the subset of the square E^2 formed by the points (x, y), where $x, y \in E$ such that $x \leq y$. If we designate this graph by G, the properties of a preorder on E assume the following form:

$$\Delta \subset G, \quad G \circ G \subset G \quad .$$

In the case of the natural order of the real numbers, this graph is the half-plane situated above the bisector of the first and third quadrants. (See Fig. 15.)

In a preordered set, we write $x \geq y$ in order to indicate that $y \leq x$. There is a natural duality between the relations $\leq$, $\geq$ which often relieves one of formulating a definition or theorem concerning one of these relations after an analogous formulation concerning the other relation.

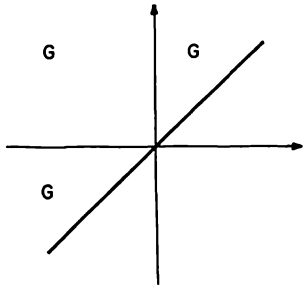

Fig. 15.

Let E designate a preordered set. A subset $X \subset E$ is said to be *decreasing* if $a \leq b$ and $b \in X$ imply $a \in X$. Every subset $X \subset E$ determines in a unique fashion, a decreasing set $d(X)$ which is the smallest one among the decreasing sets containing X; a point a belongs to $d(X)$ if and only if it is possible to determine a point $b \in X$ such that $a \leq b$. Dually one defines the concept of an *increasing* set and of the smallest increasing set $i(X)$ containing a given subset $X \subset E$.

A subset $X \subset E$ is said to be *convex* whenever $a \leq b \leq c$ and a, $c \in X$ imply $b \in X$. [This concept must not be confused with that of a convex subset in the vector sense introduced later.] Every subset $X \subset E$ determines, in a unique manner, a convex subset $c(X)$ which is the smallest one among the convex subsets containing X: a point b belongs to $c(X)$ if and only if it is possible to determine two points $a, c \in X$ such that $a \leq b \leq c$.

Every subset of a preordered set may also, in a natural manner, be considered as a preordered set relative to the *induced preorder*.

A subset X of an ordered set E is said to be *bounded from above* if it is possible to determine a point $a \in E$ such that $x \leq a$ for every $x \in X$; such a point is then called an *upper bound* of X. We say that X has a *supremum* in E if X is bounded from above and if, among its upper bounds, there exists one which is less than or equal to all of them. This bound is, then, unique and is designated as the *supremum* of X in E. We shall use the symbol $\vee$ or sup in order to represent a supremum. For example, $\vee_i x_i$ or $\sup_i x_i$ will indicate the supremum of a family (x_i). The concepts of a set that is *bounded from below*, of a *lower bound* and of an *infimum* are defined in a dual manner and the symbol $\wedge$ or inf is used

A *sup-lattice* is an ordered set in which any two elements x and y have a supremum $x \vee y$. The notion of an *inf-lattice* is defined in the dual fashion. A *lattice* is an ordered set which is, at the same time, a sup- and an inf-lattice.

3. Let us consider two preordered sets E_1 and E_2. A function f on E_1 to E_2 is said to be *increasing* if $x, y \in E_1$ and $x \leq y$ imply $f(x) \leq f(y)$. The function is said to be *decreasing* if $x, y \in E_1$ and $x \leq y$ imply $f(x) \geq f(y)$. These definitions apply, in particular, when E_2 is the ordered set of real numbers.

References: Glivenko, 1; Birkhoff, 1; Bourbaki, 1.

§5. Groups and vector spaces.

1. An *Abelian* or *commutative group* is a set E on which is given the concept of *addition*; this addition associates with every ordered pair of elements x, y of E, and element $x + y$ of E (their *sum*) in such a way that the following axioms are satisfied:

1) addition is commutative, that is, $x + y = y + x$; and associative, that is, $x + (y+z) = (x+y) + z$;
2) there exists an element of E, represented by 0 and called the *zero* of the group, such that $0 + x = x$ for every $x \in E$. This element 0 is necessarily unique;
3) given an element $x \in E$, there exists an element of E, represented by $-x$ and called the element *symmetric* to x, such that $x + (-x) = 0$. For every x, the element $-x$ is necessarily unique.

The set of real numbers constitutes an Abelian group relative to the familiar concept of addition.

The *difference* of two elements x, y in an Abelian group E is defined by $x - y = x + (-y)$.

If X and Y designate two subsets of an Abelian group, the set of elements of the form $x + y$ where $x \in X$ and $y \in Y$, will be represented by $X + Y$; and similarly for $X - Y$. [This meaning of the symbol $X - Y$ must not be confused with that introduced in set theory.] The set of elements of the form $-x$, where $x \in X$, will be represented by $-X$.

2. A *real vector space* or, simply, a *vector space* is a set E the elements of which are called vectors and on which are defined the concepts of addition and multiplication; the addition makes E an Abelian group; the multiplication associates with every real number or scalar ξ and every vector $x \in E$ a vector $\xi x \in E$ in such a manner that the following conditions are satisfied:

1) multiplication is distributive; that is, $\xi(x+y) = \xi x + \xi y$ and $(\xi+\zeta)x = \xi x + \zeta x$;

2) multiplication is associative; that is, $\xi(\zeta x) = (\xi\zeta)x$;

3) $1x = x$.

(In these conditions ξ and ζ are scalars, x and y vectors, and 1 the number one.)

The zero element 0 of a vector space is often called the *origin*.

A subset X of a vector space E is said to be *convex* if,* for any two vectors, $a, b \in X$ and any scalar ξ where $0 \leq \xi \leq 1$, we have

$$\xi a + (1 - \xi)b \in X$$

[This concept must not be confused with that of a convex subset in the sense of order.]

A *linear transformation* from a vector space E_1 to another vector space E_2 is a function T on E_1 to E_2 such that

$$T(x+y) = T(x) + T(y), \quad T(\xi x) = \xi T(x)$$

where $x, y \in E_1$ and ξ is a scalar.

A *vector subspace* of a vector space E is a subset $F \subset E$ such that

1) $0 \in F$;

2) if $x, y \in F$, then $x + y \in F$;

3) if $x \in F$, then $\xi x \in F$.

3. A *preordered*, respectively *ordered*, *Abelian group* is, by definition, an Abelian group which is, at the same time, a preordered, respectively ordered, set in such a manner that $x \leq y$ implies $x + z \leq y + z$, where x, y, z are elements of the group. An element x of the group is said to be *positive* if $x \geq 0$.

A *lattice-ordered group* is an ordered group which is, at the same time, a lattice. The elements

$$x \vee 0, \quad (-x) \vee 0, \quad x \vee (-x)$$

are, respectively, called the *positive part*, the *negative part*, and the *absolute value* of the element x and are represented by x_+, x_-, and $|x|$.

An Abelian group is *directed* if every element can be expressed as the difference of two positive elements. Every lattice-ordered group is directed since $x = x_+ - x_-$.

4. An *ordered*, respectively *preordered* vector space is, by definition a vector space which is, at the same time, a preordered, respectively ordered, Abelian group in such a manner that $x \leq y$ and $\xi \geq 0$ imply $\xi x \leq \xi y$.

The concepts relating to preordered Abelian groups apply, in particular, to preordered vector spaces. Thus we can speak of a *lattice-ordered vector space* or a *vector lattice*, etc.

In every preordered vector space, the set of positive elements has the following properties:

1) if x and y are positive, then $x + y$ is positive;

2) if x is positive and $\xi \geq 0$, then ξx is positive,

and is, therefore, called a *cone* of positive elements.

A linear transformation T between two preordered vector spaces is said to be positive if $x \geq 0$ implies $T(x) \geq 0$.

5. A *topological Abelian group* is an Abelian group which is, at the same time, a topological space in such a manner that $x + y$ is a continuous function of (x, y) and that $-x$ is a continuous function of x.

Every topological Abelian group E may be converted into a uniform space in the following way. If A represents a neighborhood of 0 in the group E, we designate by A^* the subset of E^2 defined by

$$A^* = [(x, y);\ x, y \in E,\ x - y \in A]$$

Then the sets of the form A^* constitute a base for a filter on E^2 which, in turn, defines, on the space E, a uniform structure that is compatible with the topology of E.

The real numbers constitute a topological Abelian group relative to the natural definitions of addition and topology previously indicated.

Let us consider a topological Abelian group E such that the neighborhood system of every point has a countable base. A sequence $x_1, \dots, x_n, \dots$ of points of E is said to be a *Cauchy* sequence if, for every neighborhood A of 0, it is possible to determine an integer $N \geq 1$ such that $x_m - x_n \in A$ whenever $m, n \geq N$. The group is said to be *complete* in the sense of Cauchy if every Cauchy sequence is convergent.

This concept of a complete group was generalized by Weil but we shall have no occasion to make use of his more general definition.

6. A *topological vector space* is a vector space which is, at the same time, a topological Abelian group in such a manner that ξx is a continuous function of (ξ, x).

A topological vector space is said to be a *locally convex* vector space if at every point the set of convex neighborhoods constitutes a base for the neighborhood system of that point.

A *normed space* is a vector space on which is given a function that associates with every vector x a real number $\|x\|$, called the *norm* or *length* of this vector, in such a manner that

1) $\|0\| = 0$ and $\|x\| > 0$ if $x \neq 0$;
2) $\|x + y\| \leq \|x\| + \|y\|$;
3) $\|\xi x\| = |\xi| \cdot \|x\|$.

A normed space is a metric space relative to the metric defined by

$$d(x, y) = \|x - y\|$$

and is, consequently, also a topological space; it is easily verified that this space is, then, a locally convex vector space.

A *Banach space* is a normed space which is complete in the sense of Cauchy.

References: Bourbaki, 5, 6, and 3; Halmos, 1; Birkhoff, 1; Banach, 1; Dieudonné, 1 and 2; Nachbin, 4.

CHAPTER I

TOPOLOGICAL ORDERED SPACES

Summary.

In this chapter we shall be studying the principal relations of interdependence between a topology and an order. Our program will consist in generalizing the basic facts of the theories of normal spaces and of compact spaces. For this purpose, after introducing the concepts of a closed preorder relative to a topology and of a locally convex topology relative to a preorder (§1), we define the concept of a normally preordered space. This concept reduces to that of a normal space when the preorder considered is the discrete order. We establish the fundamental theorems concerning the separation of two closed sets by a continuous increasing function (Theorem 1, §2) and concerning the extension to the entire space of a continuous increasing function defined only on a closed subset of the space (Theorem 2, §2). (In the case of a discrete order these results are due to Urysohn.) Following this, we define compact ordered spaces and then discuss two useful results in order to show how interesting these spaces are. The one result refers to the ordered normality of every compact ordered space (Theorem 4, §3), and the other to the problem of the extension of continuous increasing functions (Theorem 6, §3).

§1. Closed order and convex topology

1. In this paragraph we shall discuss some of the most usual connections between a topology and an order.

Let us consider a topological space E equipped with a preorder.

We shall say that a preorder (in particular, an order) on E is *closed* if its graph in the square E^2 is a closed subset of the topological space E^2.

We shall also say that the topology of E is *locally convex* if the set of convex neighborhoods of every point of E is a base for the neighborhoods system of this point.

We begin by establishing the following result:

PROPOSITION 1. The preorder of E is closed if and only if, for every two points $a, b \in E$ such that $a \leq b$ is false, it is possible to determine an increasing neighborhood V of a and a decreasing neighborhood W of b which are disjoint. If the preorder of E is closed, then for every point $a \in E$ the sets $d(a)$ and $i(a)$ are closed.

PROOF. Suppose the preorder is closed and $a \leq b$ false. Since, then, the point (a, b) does not belong to the graph G of the preorder, and since G is closed, we can determine a neighborhood V' of a and a neighborhood W' of b such that

$$(V' \times W') \cap G = \emptyset \quad ;$$

in other words, if $x \in V'$ and $y \in W'$, then $x \leq y$ is false. We set:

$$V = i(V') \ , \quad W = d(W')$$

Since $V \supset V'$ we see that V is an increasing neighborhood of a. Similarly, $W \supset W'$ implies that W is a decreasing neighborhood of b. Moreover, V and W are disjoint. Indeed, assume there exists a point $z \in V \cap W$. Since $z \in V$, there is a point $x \in V'$ such that $x \leq z$. Analogously, $z \in W$ furnishes a point $y \in W'$ such that $z \leq y$. From $x \leq z$ and $z \leq y$ we obtain $x \leq y$. On the other hand, $x \in V'$ and $y \in W'$ imply that $x \leq y$ is false as we saw above. This contradiction proves that V and W are actually disjoint.

Conversely, if G is not closed, there exists a point of E^2 such that

(1) $$(a, b) \in \bar{G}, \quad (a, b) \in E^2 - G \quad ;$$

that is, $a \leq b$ is false. We then consider an arbitrary increasing neighborhood V of a and an arbitrary decreasing neighborhood W of b. Since $V \times W$ is a neighborhood of (a, b), the first relation in (1) implies that

$$(V \times W) \cap G \neq \emptyset \quad ;$$

that is, there exist $v \in V$ and $w \in W$ such that $(v, w) \in G$ so that $v \leq w$. From $v \in V$ and $v \leq w$ results that $w \in V$ since V is increasing; thus w belongs both to V and to W; that is, V and W are not disjoint. The first part of the proposition is thus proved.

Let us now suppose that the preorder is closed. Given a point $a \in E$; then, if $b \in E - i(a)$, $a \leq b$ is false. We apply the first part of the proposition and determine an increasing neighborhood V of a and a decreasing neighborhood W of b which are disjoint. Now $i(a) \subset V$ implies

$$W \cap i(a) = \emptyset$$

which proves that $i(a)$ is closed. We reason analogously for $d(a)$ and thereby complete the proof.

PROPOSITION 2. Every topological space E equipped with a closed order is a Hausdorff space.

PROOF. Let us consider two distinct points $a, b \in E$. Since we are concerned with an order, one of the two relations

$$a \leq b, \quad b \leq a$$

is false. Suppose that the first one is false (the case of the second is analogous). The application of Proposition 1 shows that a and b have disjoint neighborhoods as we desired.

2. We terminate this paragraph with the following result:

PROPOSITION 3. If a topological space is equipped with a preorder such that the set consisting of the open decreasing and the open increasing subsets is an open subbase, then the topology of this space is locally convex.

PROOF. Taking into account that the intersection of a finite number of open decreasing subsets is an open decreasing subset and similarly for open increasing subsets, we see that the hypothesis of the proposition signifies that the set of subsets of the form $V \cap W$, where V is an open decreasing and W an open increasing subset, is an open base. The proposition then results from the observation that each of the subsets $V \cap W$ is open and convex.

§2. Normally ordered spaces

1. In this paragraph we generalize the fundamental results of the theory of normal spaces due to Urysohn. (Bibliographical references are found in the Introduction of this work; for the results of the present paragraph consult also Nachbin, 1.)

A topological space E equipped with a preorder shall be said to be *normally preordered* if, for every two disjoint closed subsets F_0 and F_1 of E, F_0 being decreasing and F_1 increasing, there exist two disjoint open subsets A_0 and A_1 such that A_0 contains F_0 and is decreasing, and A_1 contains F_1 and is increasing. The space shall be said to be *normally ordered* if, in addition, its preorder is an order.

We note that, if the order of E is the discrete order, every subset of E is, at the same time, increasing and decreasing; thus E will be normally ordered if and only if E is a normal space. In other words, the concept of a normally ordered space contains as a special case that of a normal space.

2. Before proving the basic results which we have in mind, we shall introduce some concepts and notation of a merely auxiliary kind and of purely technical interest.

We consider a topological space E equipped with a preorder.

It presents no difficulty to verify that the set of all closed decreasing subsets possesses the three properties which are characteristic of the set of closed subsets of a topological space. The same holds for the set of all closed increasing subsets. (Dual statements hold for open subsets.)

It follows from this that every subset $X \subset E$ determines, in a unique fashion, a subset $D(X) \subset E$ which is defined as the smallest closed decreasing subset containing X. Analogously we define the smallest closed increasing subset $I(X)$ containing X.

If X and Y are subsets of E, we write $X < Y$ in order to indicate that

$$D(X) \cap I(Y) = \emptyset$$

In addition, we use the notation $X \ll Y$ to indicate that two disjoint open subsets A and B exist such that the subset A contains X and is decreasing, and that the subset B contains Y and is increasing.

With these conventions, a topological space E equipped with a preorder is normally preordered if and only if it satisfies one of the following equivalent conditions:

(a) If $X, Y \subset E$ and $X < Y$, then $X \ll Y$.

(b) If $F \subset E$ is a closed decreasing subset, if $V \subset E$ is an open decreasing subset, and if $F \subset V$, then there exists a decreasing open subset $W \subset E$ such that

$$F \subset W , \quad D(W) \subset V$$

Indeed, let us assume first that E is normally preordered. If $X, Y \subset E$ and $X < Y$, then

$$D(X) \cap I(Y) = \emptyset \quad .$$

Since $D(X)$ is a closed decreasing subset which is disjoint from the closed increasing subset $I(Y)$, there exists an open decreasing subset $A \supset D(X)$ which is disjoint from some open increasing subset $B \supset I(Y)$. Since then, $A \supset X$ and $B \supset Y$, we see that $X \ll Y$, that is, (a) is satisfied.

Now let us suppose that (a) is satisfied and that F and V fulfill the requirements of (b). Noting that $X = F$ is a closed decreasing subset which is disjoint from the closed increasing subset $Y = E - V$, hence $X < Y$ and applying (a), we obtain $X \ll Y$; that is, there exists an open decreasing subset $A \supset X$ which is disjoint from some open increasing subset $B \supset Y$. We set $W = A$. Then W is an open decreasing subset and, since $W \subset E - B$, and $E - B$ is a closed decreasing subset, it follows that

$$D(W) \subset D(E-B) = E - B \subset E - Y = V$$

Thus (b) is satisfied.

Finally, we assume that (b) is satisfied. Let F_0 be a closed decreasing subset which is disjoint from some closed increasing subset F_1. Setting $F = F_0$ and $V = E - F_1$, we can apply (b) and so obtain an open decreasing subset W such that

$$F \subset W , \quad D(W) \subset V$$

We set

$$A_0 = W , \quad A_1 = E - D(W)$$

Then A_0 and A_1 are disjoint open subsets such that A_0 contains F_0 and is decreasing and that A_1 contains F_1 and is increasing. Thus E is normally preordered.

3. We can now establish the first basic result of this chapter. In the case in which the preorder considered is the discrete order, Theorem 1 reduces to Urysohn's theorem on the separation of disjoint closed subsets of a normal space by a continuous function. The proof given below is a generalization of that due to Urysohn.

THEOREM 1. In order that a topological space E equipped with a preorder be normally preordered, it is necessary and sufficient that, for any two disjoint closed subsets $F_0, F_1 \subset E$ where F_0 is decreasing and F_1 is increasing, there exist on E a continuous increasing real-valued function f such that $f(x) = 0$ for $x \in F_0$, $f(x) = 1$ for $x \in F_1$, and $0 \leq f(x) \leq 1$ for $x \in E$.

PROOF. Let us assume first that such a function f exists. We set:

$$A_0 = [x;\ x \in E,\ f(x) < \tfrac{1}{2}] \quad ,$$
$$A_1 = [x;\ x \in E,\ f(x) > \tfrac{1}{2}] \quad .$$

From the hypotheses made concerning f it follows that A_0 contains F_0 and is an open decreasing subset, that A_1 contains F_1 and is an open increasing subset, and that $A_0 \cap A_1 = \emptyset$. Thus the topological space E is normally preordered.

Conversely, let us assume that E is a normally preordered space. We consider the two sets F_0 and F_1 referred to in the theorem.

Every number ξ, $0 \leq \xi \leq 1$, which is a dyadic fraction (that is, a number of the form $k/2^n$ where k and n are integers such that $0 \leq k \leq 2^n$ and $n \geq 0$), we define an open decreasing set $V(\xi) \subset E$ in the following manner:

(1) $$V(0) = \emptyset \ , \quad V(1) = E - F_1$$

We consider an integer $n \geq 0$ and assume that we have already defined

[n] $$V(k/2^n) \qquad (k = 0, 1, \ldots, 2^n)$$

in such a fashion that the following conditions are satisfied:

(1_n) $V(k/2^n) \subset E$ is an open decreasing subset $(k = 0, 1, \ldots, 2^n)$

(2_n) $D[V(k/2^n)] \subset V[(k+1)/2^n]$ $(k = 0, \ldots, 2^n - 1)$.

(3_n) $F_0 \subset V(k/2^n)$ $(k = 1, 2, \ldots, 2^n)$.

(4_n) $V(0) = \emptyset$.

Formulas (1) show that for $n = 0$ these conditions are, indeed, satisfied. We now consider the case of an arbitrary integer $n \geq 0$. We define

$$[n+1] \qquad V(k/2^{n+1}) \qquad (k = 0, 1, \ldots, 2^{n+1})$$

as follows.

If k is even, that is, if $k = 2p$, where $p = 0, \ldots, 2^n$, we set:

$$V(k/2^{n+1}) = V(p/2^n) \quad . \tag{2}$$

If $k \neq 1$ and odd, that is, if $k = 2p + 1$ where $p = 1, \ldots, 2^n - 1$, we note that, by virtue of (2_n),

$$D[V(p/2^n)] \subset V[(p+1)/2^n] \tag{3}$$

Now the first member of (3) is a closed decreasing subset and the second member is an open decreasing subset. Making use of the preordered normality of the space, we can determine an open decreasing set W_0 such that

$$D[V(p/2^n)] \subset W_0 , \quad D(W_0) \subset V[(p+1)/2^n] \tag{4}$$

We then set:

$$V(k/2^{n+1}) = W_0$$

If $k = 1$, we note that, by virtue of (3_n),

$$F_0 \subset V(1/2^n)$$

Making use once more of the preordered normality of the space, we can determine an open decreasing set W_1 such that

$$F_0 \subset W_1, \quad D(W_1) \subset V(1/2^n) \tag{5}$$

Then we set:

$$V(1/2^{n+1}) = W_1$$

The sequence $[n+1]$ is, hence, defined and the condition (1_{n+1}) is immediately seen to be satisfied. The condition (2_{n+1}) is also satisfied; if $k = 0$, this follows from (4_n); if $k > 0$ and even, it follows from the first part of (4); if $k = 1$, it follows from the second part of (5); and finally, if $k > 1$ and odd, it suffices to note the second part of (4). In an identical manner, we find that (3_{n+1}) and (4_{n+1}) are also fulfilled. Moreover, (2) shows that the sequence [n+1] is an extension of the sequence [n].

In consequence of what has just been shown, we can by induction, define an open decreasing set $V(\xi) \subset E$ for every dyadic number ξ, $0 \leq \xi \leq 1$ in such a way that

a) $\zeta < \zeta'$ implies $D[V(\zeta)] \subset V(\zeta')$;

b) $V(\zeta) \supset F_0$ if $\zeta > 0$;

c) $V(0) = \emptyset$, $V(1) = E - F_1$.

We now note that if $x \in E$, then $x \in E - V(0)$, and we define a real-valued function f on E in the following manner: $f(x)$ shall be the supremum of the dyadic numbers ζ, $0 \leq \zeta \leq 1$, such that $x \in E - V(\zeta)$. Obviously,

$$0 \leq f(x) \leq 1 \quad .$$

If $x \in F_0$, then b) shows that $x \in V(\zeta)$ for every dyadic number $\zeta > 0$. From this it follows at once that $f(x) = 0$.

If $x \in F_1$, then c) shows that x $E - V(1)$ so that $f(x) = 1$.

Suppose that $x, y \in E$ and $x \leq y$. If $x \in E - V(\zeta)$, the fact that $V(\zeta)$ is decreasing and thus, $E - V(\zeta)$ is increasing implies that $y \in E - V(\zeta)$ so that $\zeta \leq f(y)$. Making use of the arbitrariness of ζ, we conclude that

$$f(x) \leq f(y) \quad \text{if} \quad x \leq y \quad ;$$

that is, f is an increasing function on E.

Finally, noting that

$$\overline{V(\zeta)} \subset D[V(\zeta)] \quad ,$$

we obtain as a consequence of a), the following condition:

a') $\zeta < \zeta'$ implies $\overline{V(\zeta)} \subset V(\zeta')$,

and by the same reasoning as that used in the proof of Urysohn's separation theorem, we prove that the function f is continuous (see the references for Urysohn's theorem in the Introduction). Theorem 1 is thus established.

4. Before turning to the second fundamental theorem of this paragraph, we shall establish a preliminary basic result stated in the test as a lemma (see Lemma 2). We begin with the following observation:

> LEMMA 1. Let E be a topological space equipped with a preorder and let $X, Y, V, \ldots$ be subsets of E. Then a) if $X \ll Y$, there exists an open decreasing subset $V \supset X$ such that $V \ll Y$;
> b) if $X_i \ll Y$ $(i = 1, \ldots, n)$ then $\cup_i X_i \ll Y$.
> (Analogous statements are true for Y.)

PROOF. Suppose that $X \ll Y$. By definition, there exist two disjoint open sets V and W such that V contains X and is decreasing, and that W contains Y and is increasing. From $V \subset V$ and $Y \subset W$, we then deduce that $V \ll Y$ whereby a) is established.

Now suppose that $X_i \ll Y$ $(i = 1, \ldots, n)$. This signifies that we can determine disjoint open sets V_i and W_i such that V_i contains X_i and is decreasing, and that W_i contains Y and is increasing where $i = 1, \ldots, n$. We set:

$$V = \cup_i V_i, \quad W = \cap_i W_i$$

Clearly V is an open decreasing set and W an open increasing set, and

$$V \supset \cup_i X_i, \quad W \supset Y$$

Furthermore,

$$V \cap W = \cup_i (V_i \cap W) \subset \cup_i (V_i \cap W_i) = \emptyset \quad ;$$

that is, V and W are disjoint. Thus $\cup_i X_i \ll Y$ which prove b).

We now proceed to prove the following result:

> LEMMA 2. Let E be a topological space equipped with a preorder. If
> $$X = \cup_i X_i, \quad Y = \cup_j Y_j \quad ,$$
> $$X \subset E, \quad Y \subset E \quad , \qquad (i, j = 1, \ldots)$$
> $$X \ll Y_j, \quad X_i \ll Y,$$
> then $X \ll Y$.

PROOF. Since $X_1 \ll Y$, we can determine an open decreasing set V' such that

$$V' \supset X_1, \quad V' \ll Y \quad ,$$

(Lemma 1, a). Analogously, $X \ll Y_1$ implies the existence of an open increasing set W' such that

$$W' \supset Y_1 , \quad X \ll W'$$

Finally, since $X_1 \ll Y_1$ (we have, for example, $X_1 \subset X$ and $X \ll Y_1$) we can determine an open decreasing set V'' and an increasing open set W'' such that

$$V'' \supset X_1 , \; W'' \supset Y_1 , \quad V'' \cap W'' = \emptyset$$

Setting

$$V_1 = V' \cap V'' , \quad W_1 = W' \cap W'' ,$$

we obtain an open decreasing set V_1 and an open increasing set W_1 such that

$$V_1 \supset X_1 , \qquad V_1 \ll Y ,$$
$$W_1 \supset Y_1 , \qquad X \ll W_1 ,$$
$$V_1 \cap W_1 = \emptyset .$$

Suppose the open decreasing sets V_i and the open increasing sets W_j $(i, j = 1, \ldots, n)$ have already been defined in such a way that

$$V_i \supset X_i , \qquad V_i \ll Y ,$$
$$W_j \supset Y_j , \qquad X \ll W_j ,$$
$$V_i \cap W_j = \emptyset \qquad (i, j = 1, \ldots, n)$$

We have already indicated how to achieve this situation for $n = 1$. We shall now pass from the case [n] to the case [n+1].

Since $X_{n+1} \ll Y$, we can determine an open decreasing set V' such that

$$V' \supset X_{n+1} , \quad V' \ll Y ,$$

(Lemma 1 a). In an identical way, $X \ll Y_{n+1}$ proves the existence of an open increasing set W' such that

$$W' \supset Y_{n+1} , \quad X \ll W'$$

Furthermore, since $X_{n+1} \ll Y_{n+1}$ (we have, for example, $X_{n+1} \subset X$ and $X \ll Y_{n+1}$), we can determine an open decreasing subset V'' and an open increasing subset W'' such that

$$V'' \supset X_{n+1} , \quad W'' \supset Y_{n+1} , \quad V'' \cap W'' = \emptyset$$

Finally, it results from $X_{n+1} \subset X$ and from one of the conditions [n] (namely, the second part of the second line) that

$$X_{n+1} \ll W_j \quad (j = 1, \ldots, n)$$

whence

$$X_{n+1} \ll \bigcup_{j=1}^{n} W_j$$

(Lemma 1, b). We can, therefore, determine an open decreasing set V''' such that

$$V''' \supset X_{n+1}, \quad V''' \ll \bigcup_{j=1}^{n} W_j$$

In a strictly corresponding fashion, making use of $Y_{n+1} \subset Y$ and of one of the conditions [n], we obtain an open increasing set W''' with the following properties:

$$W''' \supset Y_{n+1}, \quad \bigcup_{i=1}^{n} V_i \ll W'''$$

We set

$$V_{n+1} = V' \cap V'' \cap V''' ,$$
$$W_{n+1} = W' \cap W'' \cap W''' .$$

Clearly the set V_{n+1} is an open decreasing set and W_{n+1} is an open increasing set. Furthermore, by virtue of the various precautions taken, we see that

$$V_{n+1} \supset X_{n+1}, \qquad W_{n+1} \supset Y_{n+1},$$
$$V_{n+1} \ll Y, \qquad X \ll W_{n+1},$$
$$V_{n+1} \cap W_j = \emptyset, \qquad V_i \cap W_{n+1} = \emptyset,$$
$$(i, j = 1, \ldots, n+1)$$

Combining these conditions with [n], we obtain the case [n+1].

Thus we can, by induction, construct two sequences of sets V_i and W_j which are pairwise disjoint, that is, for which

$$V_i \cap W_j = \emptyset \quad (i, j = 1, 2, \ldots)$$

where $V_i \supset X_i$ and is an open decreasing set, and where $W_j \supset Y_j$ and is an open increasing set.

We now define

$$V = \bigcup_{i=1}^{\infty} V_i, \quad W = \bigcup_{j=1}^{\infty} W_j$$

It is clear that the sets V and W are disjoint, that V contains X and is an open decreasing set, and that W contains Y and is an open increasing set. Thus $X \ll Y$ as we wished to establish.

5. We are now fully prepared to prove the second fundamental result which we have in mind, namely, that relative to the extension of continuous increasing real-valued functions. In the case in which the preorder considered is the discrete order, this result furnishes the theorem concerning the extension of continuous functions due to Urysohn.

> THEOREM 2. Let E be a normally preordered space, let $F \subset E$ be a closed subset of E, and let f be a bounded real-valued function which is continuous and increasing on F. We shall indicate by $A(\xi)$ the set of all points $x \in F$ such that $f(x) \leq \xi$ and by $B(\xi)$ the set of all points $x \in F$ such that $f(x) \geq \xi$, where ξ is a real number.
>
> In order that the function f may be extended to E in such a way as to become a continuous bounded increasing real-valued function of E, it is necessary and sufficient that
>
> $$\xi < \xi' \text{ imply } A(\xi) < B(\xi')$$

PROOF. Let us first assume that f has an extension f^*. We designate by $A^*(\xi)$ the set of all points $x \in E$ such that $f^*(x) \leq \xi$ and by $B^*(\xi)$ the set of all points $x \in E$ such that $f^*(x) \geq \xi$. Observing that $A^*(\xi)$ is a closed decreasing set which contains $A(\xi)$, we obtain

$$D[A(\xi)] \subset A^*(\xi)$$

Similarly,

$$I[B(\xi)] \subset B^*(\xi)$$

Thus $\xi < \xi'$ implies that

$$D[A(\xi)] \cap I[B(\xi)] \subset A^*(\xi) \cap B^*(\xi\ ) = \emptyset ,$$

furnishing $A(\xi) < B(\xi')$.

Conversely, let us suppose that the function f defined on the set F has the properties enumerated in the statement of the theorem. Since f is bounded, we may assume, without loss of generality, that $0 \leq f \leq 1$. And, indeed, if we select two numbers p and q such that $p < q$ and that

$$p \leq f(x) \leq q \quad \text{for every} \quad x \in F \quad ,$$

we can also write

$$0 \leq \frac{f(x) - p}{q - p} \leq 1 \qquad (x \in F) \quad ,$$

and it obviously suffices to replace f by the function f' as defined by

$$f'(x) = \frac{f(x) - p}{q - p} \qquad (x \in F) \quad ,$$

in order to obtain in f' a function of the desired property, $0 \leq f' \leq 1$.

For every dyadic number ζ, $0 \leq \zeta \leq 1$, we now define an open decreasing set $V(\zeta) \subset E$ in the following manner.

We first define

$$V(0) = \emptyset \ , \quad V(1) = E$$

We then consider an integer $n \geq 0$ and assume that we have already defined the sets

[n] $$V(k/2^n) \quad (k = 0, 1, \ldots, 2^n)$$

in such a way that the following conditions are satisfied:

(1_n) $V(k/2^n) \subset E$ is an open decreasing subset $(k = 0, 1, \ldots, 2^n)$,

(2_n) $D[V(k/2^n)] \subset V[(k+1)/2^n] \quad (k = 0, \ldots, 2^n - 1)$,

(3_n) $D[V(k/2^n)] << I[B(\xi)] \quad (k = 0, 1, \ldots, 2^n;\ \xi > k/2^n)$,

(4_n) $D[A(\xi)] << E - V(k/2^n) \quad (k = 0, 1, \ldots, 2^n;\ \xi < k/2^n)$,

where ξ is a real number. These conditions are actually, satisfied for $n = 0$. We now define

[n+1] $$V(k/2^{n+1}) \quad (k = 0, 1, \ldots, 2^{n+1})$$

by means of the following construction.

If k is even, that is, if $k = 2p$ where $p = 0, \ldots, 2^n$, we set

(1) $$V(k/2^{n+1}) = V(p/2^n)$$

If k is odd, that is, if $k = 2p + 1$, where $p = 0, \ldots, 2^n - 1$, we set temporarily consider the following series:

$$X_0 = D[V(p/2^n)] \ , \qquad X_i = D[A(k/2^{n+1} - 1/i)] \ ,$$
$$Y_0 = E - V[(p+1)/2^n] \ , \qquad Y_j = I[B(k/2^{n+1} + 1/j)] \ ,$$

where $i, j = 1, 2, \ldots$.

By virtue of (2_n) and of the preordered normalcy of the space, we have

(2) $$X_0 \ll Y_0$$

Furthermore, (3_n) shows that

$$X_0 \ll I[B(k/2^{n+1})]$$

Observing that

(3) $$I[B(k/2^{n+1})] \supset Y_j \quad (j = 1, 2, \ldots)$$

and forming the union of the second members, we obtain

(4) $$X_0 \ll \bigcup_1^\infty Y_j$$

From (2), (4), and Lemma 1, b) we deduce that

(5) $$X_0 \ll \bigcup_0^\infty Y_j$$

Analogously, by (4_n)

(6) $$X_i \ll Y_0 \quad (i = 1, 2, \ldots)$$

Since, by virtue of the hypothesis of the theorem,

$$X_i < I[B(k/2^{n+1})] \quad (i = 1, 2, \ldots) \quad ,$$

we see that, in turn, by virtue of the preordered normality of the space E,

$$X_i \ll I[B(k/2^{n+1})] \quad (i = 1, 2, \ldots)$$

Taking into account once more the inclusion relation (3), we obtain

(7) $$X_i \ll \bigcup_1^\infty Y_j \quad (i = 1, 2, \ldots)$$

From (6) and (7)

(8) $$X_i \ll \bigcup_0^\infty Y_j \quad (i = 1, 2, \ldots)$$

We can now condense (5) and (8) into

(9) $$X_i \ll \bigcup_0^\infty Y_j \quad (i = 0, 1, \ldots)$$

On the other hand, (4_n) shows that

$$D[A(k/2^{n+1})] \ll Y_0$$

Observing that

(10) $$D[A(k/2^{n+1})] \supset X_i \quad (i = 1, 2, \ldots)$$

and forming the union of the second members, we have

(11) $$\bigcup_1^\infty X_i \ll Y_0 \quad .$$

By (2) and (11),

(12) $$\bigcup_0^\infty X_i \ll Y_0$$

Analogously,

(13) $$X_0 \ll Y_j \quad (j = 1, 2, \ldots)$$

on the basis of (3_n). Since, by virtue of the hypothesis of the theorem,

$$D[A(k/2^{n+1})] < Y_j \quad (j = 1, 2, \ldots) \quad ,$$

we see that, in turn, by virtue of the preordered normality of the space E

$$D[A(k/2^{n+1})] \ll Y_j \quad (j = 1, 2, \ldots)$$

Combining this fact with the inclusion relation (10), we conclude that

(14) $$\bigcup_1^\infty X_i \ll Y_j \quad (j = 1, 2, \ldots)$$

Combining into (13) and (14)

(15) $$\bigcup_0^\infty X_i \ll Y_j \quad (j = 1, 2, \ldots) \quad ,$$

and then condensing (12) and (15), we obtain

(16) $$\bigcup_0^\infty X_i \ll Y_j \quad (j = 0, 1, \ldots)$$

We now fix our attention on (9) and (16) and recall Lemma 2. The conclusion in the statement of this lemma permits us to guarantee that

$$\bigcup_0^\infty X_i \ll \bigcup_0^\infty Y_j$$

In other words, we can determine an open decreasing subset $V \subset E$ and an open increasing subset $W \subset E$ such that

(17) $$V \supset X_i \ , \qquad W \supset Y_j \quad (i, j = 0, 1, \ldots)$$
$$V \cap W = \emptyset.$$

Having defined V and W in this manner, we define

$$V(k/2^{n+1}) = V \quad .$$

The definition of the sequence [n+1] is, thus, complete.

Condition (1_{n+1}) is, clearly, satisfied.

Condition (2_{n+1}) is also satisfied; if k is even, this is a consequence of the fact that $X_0 \subset V$ (on the basis of the first part of (17)); and if k is odd, it results from the relation

$$D(V) \subset E - W \subset E - Y_0$$

(on the basis of the fact that $E - W$ is a closed decreasing subset and of the third and second part of (17)).

Condition (3_{n+1}) is an immediate corollary of (3_n) if k is even. In the case in which k is odd, that is, $k = 2p + 1$, we can determine a sufficiently large integer $j \geq 1$ in such a manner that we have

$$k/2^{n+1} + 1/j \leq \xi \quad ,$$

and, the, it suffices to observe that

$$D(V) \subset E - W \subset E - Y_j \subset \dot{E} - I[B(\xi)]$$

in order to be able to conclude (on the basis of the preordered normality of the space) that

$$D(V) \ll I[B(\xi)] \quad .$$

Finally, (4_{n+1}) results from (4_n) if k is even. If k is odd, that is, if $k = 2p + 1$, we select a sufficiently large integer $i \geq 1$ in such a way that the inequality

$$\xi \leq k/2^{n+1} - 1/i$$

is satisfied and note that

$$D[A(\xi)] \subset X_i \subset V \quad .$$

Then (once more by virtue of the preordered normality of the space)

$$D[A(\xi)] \ll E - V \quad ,$$

as we desired.

Formula (1) shows that the sequence [n+1] is an extension of the sequence [n]. This justifies the inductive construction of an open decreasing set $V(\zeta) \subset E$ for every dyadic number ζ, $0 \leq \zeta \leq 1$, in a manner which causes the following conditions to be satisfied:

a) $\zeta < \zeta'$ implies $D[V(\zeta)] \subset V(\zeta')$;

b) $\zeta < \xi$ implies $V(\zeta) \cap B(\xi) = \emptyset$;

c) $\zeta > \xi$ implies $A(\xi) \subset V(\zeta)$;

where ζ and ζ' are dyadic numbers such that $0 \leq \zeta,\ \zeta' \leq 1$, and ξ is a real number.

From b) we obtain

$$V(\zeta) \cap \bigcup_{\xi > \zeta} B(\xi) = \emptyset ,$$

that is,

b') $V(\zeta) \cap F \subset A(\zeta)$.

Analogously, we obtain from c)

$$\bigcup_{\xi < \zeta} A(\xi) \subset V(\zeta) ,$$

that is,

c') $F - B(\zeta) \subset V(\zeta)$.

Having constructed $V(\zeta)$ so as to possess the properties enumerated above, we now define a real-valued function f^* on the space E by means of the two following formulas:

$$f^*(x) = \sup\ [\zeta;\ x \in E - V(\zeta)]$$
$$= \inf\ [\zeta;\ x \in V(\zeta)] .$$

In order to justify the first definition of f^*, we note that $x \in E - V(0)$, to justify the second that $x \in V(1)$.

Clearly, $0 \leq f^*(x) \leq 1$.

Before establishing the properties which f^* is supposed to have, we must prove that the two expressions for f^* agree. As a matter of fact, if

$$x \in E - V(\zeta) \quad \text{and} \quad x \in V(\zeta') ,$$

then $\zeta < \zeta'$ (since, by property a), $\zeta \geq \zeta'$ would imply $V(\zeta) \supset V(\zeta')$). Thus

$$\sup\ [\zeta;\ x \in E - V(\zeta)] \leq \inf\ [\zeta';\ x \in V(\zeta')] .$$

We now set

$$I = \inf\ [\zeta';\ x \in V(\zeta')]$$

If $I = 0$, we note that

$$\sup\ [\zeta;\ x \in E - V(\zeta)] \geq 0 = I$$

If $I > 0$, we consider an arbitrary dyadic number $\bar{\xi}$ such that $0 < \bar{\xi} < I$. Then $x \in E - V(\bar{\xi})$ (since $x \in V(\bar{\xi})$ would imply $I \leq \bar{\xi}$), whence

$$\sup [\xi ; x \in E - V(\xi)] \geq \bar{\xi} \quad ,$$

and, letting $\bar{\xi}$ tend to I, we obtain

$$\sup [\xi ; x \in E - V(\xi)] \geq I \quad .$$

The proof of the equality

$$\sup [\xi ; x \in E - V(\xi)] = \inf [\xi ; X \in V(\xi)]$$

is, thus, completed.

We now verify that the function f^* is an extension of f. If

$$x \in F, \quad x \in E - V(\xi) \quad ,$$

we note that, by virtue of c'),

$$E - V(\xi) \subset B(\xi) \cup (E-F)$$

whence $x \in B(\xi)$; this means that, by the definition of $B(\xi)$, $f(x) \geq \xi$. Taking into account that ξ is arbitrary as long as the condition $x \in E - V(\xi)$ is fulfilled and using the first definition of f^*, we obtain

$$f(x) \geq f^*(x) \quad .$$

On the other hand, if

$$x \in F, \quad x \in V(\xi) \quad ,$$

then, by b'), $x \in A(\xi)$; that is, in accordance with the definition of $A(\xi)$, $f(x) \leq \xi$. Noting again that ξ is arbitrary as long as the condition $x \in V(\xi)$ is fulfilled, and using the second definition of f^*, we can write

$$f(x) \leq f^*(x) \quad .$$

Comparing the two inequalities obtained, we see that $f(x) = f^*(x)$ for every point $x \in F$. Thus f^* is, indeed, an extension of f to the entire space.

In order to verify that the function f^* is increasing and continuous, one proceeds exactly as in the end of the proof of the preceding theorem with respect to the analogous verifications. We therefore omit the details. The theorem is, thus, proved.

In the case of a normal space, every continuous function defined on a closed subset of the space can be extended to the entire space without the loss of its continuity. Now the preceding theorem does not maintain that *every* continuous increasing function defined on a closed subset of a normally preordered space can be extended to the entire space and still remain continuous and increasing; the theorem guarantees this only for bounded functions which satisfy the necessary and sufficient condition indicated in the statement of the theorem. From the point of view which interests us at the present moment, the fact that the functions are assumed to be bounded represents a requirement of little importance.

As regards the necessary and sufficient condition referred to, it may be automatically satisfied in some important cases; this happens, for example, in the case of the following theorem as well as that of Theorem 6 which will be proved later.

> THEOREM 3. Let E be a normally preordered space. We consider a closed subset $F \subset E$ with the following property: if $X, Y \subset F$ and $X < Y$ in the space F (equipped with the topology and the preorder induced by E), then $X < Y$ in E. Every continuous increasing bounded real-valued function defined on F can be extended to the entire space E without the loss of these properties.

PROOF. We retain the notation used in Theorem 2. $A(\xi)$ is a closed decreasing set in the space F and f is a continuous and increasing function. By the same token, $B(\xi)$ is a closed increasing set in the space F. Now $\xi < \xi'$ implies that

$$A(\xi) \cap B(\xi') = \emptyset$$

whence

$$A(\xi) < B(\xi')$$

in the space F. Using the hypothesis of the theorem, we see that

$$A(\xi) < B(\xi')$$

in the space E. The present theorem is, thus, an easily derived consesequence of the preceding one.

§3. Compact ordered spaces

1. Among the topological spaces which render the most important services to general topology, are the compact Hausdorff spaces. This fact is due, principally, to two properties which are characteristic of compact Hausdorff spaces: in the first place, although these spaces may be infinite, they admit approximation by finite sets (in the sense expressed in precise terms by the covering property of Borel-Lebesgue); and, in the second place, they are complete spaces in the sense of the theory of uniform spaces due to Weil.

For this reason, a theory of topological ordered spaces must necessarily include a study of the concept of a compact ordered space and an analysis of its principal properties.

A *compact ordered space* is a compact space equipped with a closed order (see §1). Proposition 2, §1, shows that such a space is, indeed, a compact Hausdorff space. If, in addition, we take into account that the diagonal of the square of a Hausdorff space is a closed subset of this square, we see that a compact space equipped with the discrete order is a compact ordered space if and only if it is a compact Hausdorff space. In other words, the concept of a compact ordered space contains, as a particular case, the concept of a compact Hausdorff space.

2. We start by proving the following result.

> PROPOSITION 4. Let E be a topological space equipped with a closed order. If $K \subset E$ designates a compact subset of E, then the decreasing subset $d(K)$ and the increasing subset $i(K)$ generated by K are closed.

PROOF. We prove that the subset $d(K)$ is closed. For this purpose we consider an arbitrary point $a \in E - d(K)$. Then $a \leq x$ is false for every point $x \in K$. Applying Proposition 1, §1, we see that, corresponding to every point $x \in K$, we can determine an increasing neighborhood V_x of a and a decreasing neighborhood W_x of x which are disjoint. By the compactness of the set K there exist points

$$x_1, x_2, \ldots, x_n \in K \quad ,$$

finite in number, such that the decreasing neighborhoods corresponding to them cover K, that is,

$$K \subset W_{x_1} \cup \ldots \cup W_{x_n}$$

We define the set V by

$$V = V_{x_1} \cap \ldots \cap V_{x_n}$$

Clearly, the set V thus determined constitutes an increasing neighborhood of the point a. Furthermore,

$$V \cap K \subset V \cap (W_{x_1} \cup \ldots \cup W_{x_n}) = (V \cap W_{x_1}) \cup \ldots \cup (V \cap W_{x_n})$$
$$\subset (V_{x_1} \cap W_{x_1}) \cup \ldots \cup (V_{x_n} \cap W_{x_n}) = \emptyset ,$$

that is, $K \subset E - V$. Noting that the set $E - V$ is decreasing, we obtain

$$d(K) \subset E - V ,$$

that is,

$$V \cap d(K) = \emptyset .$$

Thus every point of the complement of $d(K)$ possesses a neighborhood which is disjoint from $d(K)$; that is, $d(K)$ is closed. Similar reasoning holds for the subset $i(K)$. The proposition is, thus, proved.

3. An immediate application of the preceding proposition furnishes the following result:

> PROPOSITION 5. Let us consider a compact ordered space E, a decreasing subset $F \subset V$ and a neighborhood V of F. Then there exists an open decreasing neighborhood W of F contained in V.

PROOF. We set

$$W = E - i(\overline{E-V}) .$$

Since $\overline{E - V}$ is closed and thus, compact by the compactness of E, and since the order of E is closed by hypothesis, we can apply the preceding proposition and conclude that $i(\overline{E-V})$ is a closed subset; then W is an open subset. As the complement of an increasing subset, W is decreasing. Thus

$$i(\overline{E-V}) \supset \overline{E - V} \supset E - V$$

whence follows $W \subset V$. Finally, $F \subset W$, that is,

$$F \cap i(\overline{E-V}) = \emptyset \quad .$$

Let us now suppose that a point t exists such that

$$t \in F \,, \quad t \in i(\overline{E-V}) \quad .$$

The second condition signifies that there exists a point $x \in \overline{E - V}$ such that $t \geq x$. From $t \in F$, $t \geq x$ follows that $x \in F$ since F is decreasing; this contradicts $x \in \overline{E - V}$ since V is a neighborhood of the set F. The proposition is, thus, established.

4. We now derive the theorem on which rests a great part of the interest pertaining to the concept of a compact ordered space.

> THEOREM 4. We consider a compact ordered space E. If the subsets $F_0, F_1 \subset E$ are closed sets such that $x_0 \geq x_1$ is false for any two points $x_0 \in F_0$ and $x_1 \in F_1$, then it is possible to determine two disjoint open subsets V_0 and V_1 of E such that V_0 contains F_0 and is decreasing, and that V_1 contains F_1 and is increasing.

PROOF. We start out by considering two points $a, b \in E$ such that $a \geq b$ is false. We then maintain that it is possible to determine two disjoint open sets V and W such that V contains a and is decreasing, and that W contains b and is increasing. Now, $d(a)$ and $i(b)$ are disjoint and closed (see Proposition 1, §1). By virtue of the fact that every compact Hausdorff space is normal (and by Proposition 2, §1), we can determine two disjoint open sets V' and W' such that $V' \supset d(a)$ and $W' \supset i(b)$. We note that $d(a)$ is a decreasing set and apply the preceding proposition; there exists, then, an open decreasing set V such that

$$d(a) \subset V \subset V' \tag{1}$$

In a strictly corresponding manner, we see that there exists an open increasing set W having the following property:

$$i(b) \subset W \subset W' \quad . \tag{2}$$

From (1) and (2) and from the fact that $V' \cap W' = \emptyset$, we see at once that

$$a \in V \,, \quad b \in W \,, \quad V \cap W = \emptyset$$

as we wished to show.

We now go on to consider a point $a \in E$ and a closed set $F \subset E$ such that $a \geq x$ is false for every point $x \in F$. Basing ourselves on the preceding case, we shall show that it is possible to determine two disjoint open sets V and W such that V contains a and is decreasing, and that W contains F and is increasing. Indeed, we know that for every point $x \in F$ the relation $a \geq x$ is false; thus, two disjoint open sets V_x and W_x may be determined such that V_x contains a and is decreasing, and W_x contains F and is increasing. Now F is also a compact set; consequently, there exist points

$$x_1, x_2, \ldots, x_n \in F \quad ,$$

finite in number, such that the open increasing sets corresponding to them cover the set F, that is,

$$F \subset W_{x_1} \cup W_{x_2} \cup \ldots \cup W_{x_n} \tag{3}$$

We define two sets V and W as follows:

$$V = V_{x_1} \cap V_{x_2} \cap \ldots \cap V_{x_n} \quad ,$$
$$W = W_{x_1} \cup W_{x_2} \cup \ldots \cup W_{x_n}$$

Clearly, V is an open decreasing set and W is an open increasing set. Furthermore, V contains a and W contains F as the inclusion relation (3) shows. Finally, by means of a computation already employed earlier in analogous situations, we verify that V and W are disjoint as we desired.

As a last step, we consider the two sets F_0 and F_1 appearing in the statement of the theorem. For every point $x \in F_0$, this point x and the set F_1 are in exactly the same relation as a and F in the preceding discussion. Therefore we can determine two disjoint open sets V_x and W_x such that V_x contains x and is decreasing, and that W_x contains F_1 and is increasing. Now F_0 is also a compact set; hence there exist points

$$x_1, x_2, \ldots, x_n \in F_0 \quad ,$$

finite in number, such that

$$F_0 \subset V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_n}$$

We define two sets V and W by the equations

$$V = V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_n} ,$$

$$W = W_{x_1} \cap W_{x_2} \cap \ldots \cap W_{x_n}$$

As in the preceding case, V is an open decreasing set and W an open increasing set. Furthermore, $V \supset F_0$ and $W \supset F_1$. Finally, V and W are also disjoint. It then suffices to set $V_0 = V$ and $V_1 = W$ in order to establish the theorem.

An immediate consequence of Theorem 4 and of the definition of a normally preordered space is the following:

THEOREM 4. [COROLLARY] Every compact ordered space is a normally ordered space.

We may, therefore, apply to compact ordered spaces the results relative to the separation of closed sets by continuous increasing functions, and to the extension of continuous increasing functions, defined only for a closed subset of the space, to the entire space.

There follows another interesting consequence of Theorem 4.

THEOREM 5. In every compact ordered space E, the set consisting of the open decreasing subsets and the open increasing subsets is an open subbase.

PROOF. We designate the set in question by H. We then maintain that H is separating in the following sense: if $a, b \in E$ are two distinct points, there exist sets $V, W \in H$ such that

$$a \in V , \quad b \in W , \quad V \cap W = \emptyset$$

Indeed, since $a \neq b$, one of the two inequalities $a \geq b$, $b \geq a$ must be false. In any case, it is sufficient to apply the first part of the proof of Theorem 4 in order to ascertain that H actually is separating. Now in a compact space, every separating set consisting of open subsets is necessarily an open subbase (cf. Bourbaki, 2, Corollary 2, p. 62). Hence the theorem is proved.

5. We now complete our study of the extension of increasing continuous functions to the entire space with the following result (as regards the case of a discrete order, cf. Stone, 1):

THEOREM 6. We consider a normally ordered space E equipped with a closed order and we let $F \subset E$ be a compact subset of E. Then every continuous increasing real-valued function on F can be extended to the entire space in such a way as to remain continuous and increasing.

PROOF. Let f be the given function. The compactness of F implies that f is bounded. Furthermore, let $X, Y \subset F$ be two sets such that $X < Y$ in the space F. The space E is a Hausdorff space (see Proposition 2, §1) and the subset F is compact. Consequently, F is closed in E. It follows therefore, that $\bar{X}, \bar{Y} \subset F$ where the bars indicate the closures of the respective sets in the space E or the space F. Now $\bar{X}$ is contained in the smallest closed decreasing set of the space F which contains X, and $\bar{Y}$ is contained in the smallest closed increasing set of F which contains Y. Making use of the fact that $X < Y$ in the space F, we conclude that $x \geq y$ is false for any two points $x \in \bar{X}$ and $y \in \bar{Y}$; in other words, we have, in the space E,

$$d(\bar{X}) \cap i(\bar{Y}) = \emptyset \quad . \tag{1}$$

By virtue of Proposition 4 and of the fact that $\bar{X}$ and $\bar{Y}$ are compact sets, we see that $d(\bar{X})$ and $i(\bar{Y})$ are closed sets in E. Hence

$$d(\bar{X}) = D(\bar{X}) \ , \quad i(\bar{Y}) = I(\bar{Y})$$

so that $\bar{X} < \bar{Y}$ in the space E and, thus $X < Y$ in the space E. It remains to apply Theorem 3 of §2.

CHAPTER II

UNIFORM ORDERED SPACES

Summary.

The present chapter is devoted to the analysis of the relations between uniform structure (in the sense of A. Weil) and order. Almost all topological spaces occurring in the applications of general topology are spaces which satisfy, at least, the condition of being completely regular. This circumstance is due to the fact, observed by Tychonoff, that every topological subspace of a compact Hausdorff space and every metric space are completely regular, and to the theorem established by Pontrjagin that every topological group is also a completely regular space. For this reason it is natural to expect that a conveniently chosen concept of a completely regular ordered space should play a distinguished role in a general theory of topological ordered spaces. After the introduction of this concept of a completely regular ordered space (§1), we define the so-called semi-uniform structures (which differ from the uniform structures of A. Weil only by the omission of the axiom of symmetry) and we show that from every semi-uniform structure originates, in a natural manner, an associated uniform structure and preorder, thus giving rise to a uniform preordered space (§2). Later we establish the theorem on the metrization of uniform preordered spaces (Theorem 8, §2), which is due to A. Weil for the case of a discrete order, and we determine the class of topological preordered spaces that may be equipped with a uniform structure compatible with their topology so as to become uniform preordered spaces (Theorem 9, §2). In the case of a discrete order, this result was established for the first time by A. Weil as a generalization of Pontrjagin's fundamental theorem on topological groups mentioned above. Finally, we present a simple and satisfactory criterion which permits us to recognize whether a uniform space

which is, at the same time, a preordered space is also a uniform preordered space (Theorem 10, §2). We then apply this criterion to two cases which, practically, embrace all those occurring with any frequency in the applications, namely, the case of uniform lattices and of topological ordered groups.

§1. Completely regular ordered spaces

1. The concept of a completely regular space was introduced into general topology by Tychonoff, especially in connection with the problem of the compactification of a topological space, that is, the problem of embedding a given topological space into a compact space as a dense topological subspace. This connection between complete regularity and compactification is expressed by the fact that a topological space admits a Hausdorff compactification if and only if it is a completely regular space. Later, after A. Weil's work on the theory of uniform spaces, this concept of complete regularity appeared on the scene once more in an essential role in as much as a Hausdorff space can be equipped with a uniform structure compatible with its topology if and only if it is a completely regular space. From then on, this concept became popular in general topology; this circumstance is due to the fact that completely regular spaces are the first in the hierarchy of topological spaces relative to which one can successfully make use of an interdependence between the topology of the space and the system of continuous numerical functions on the space. These spaces may be generalized in the following manner.

A *uniformizable preordered space* is, by definition, a topological space E equipped with a preorder which satisfies the two following conditions:

1) if $a \in E$ and if V designates a neighborhood of a, there exist two continuous real-valued functions f and g on E, where f is increasing and g is decreasing, such that

$$0 \leq f \leq 1, \quad 0 \leq g \leq 1 \quad ,$$
$$f(a) = 1 \ , \quad g(a) = 1 \quad , \quad .$$
$$\inf [f(x), g(x)] = 0 \quad \text{if} \quad x \in E - V$$

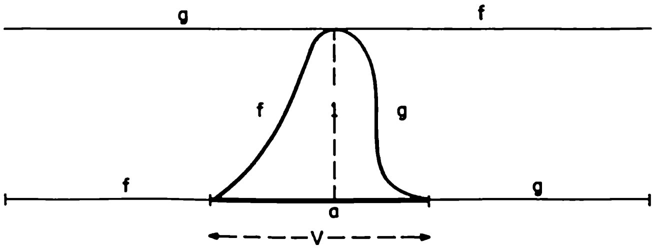

Fig. 16

2) if $a, b \in E$ and $a \leq b$ is false, there exists a continuous increasing, real-valued function f on E such that

$$f(a) > f(b) \quad .$$

The use of the term *uniformizable* in the above definition will be justified later when we establish the necessary and sufficient condition which a topological space equipped with a preorder has to satisfy in order that there exist a uniform structure compatible with the topology of the space under which it becomes a uniform preordered space (Theorem 9, §2). Compare with the end of Section 4, §3 of the Introduction.

We begin by establishing two simple properties of a uniformizable preordered space.

PROPOSITION 6. Every uniformizable preordered space is a uniformizable space. The preorder is closed, and the set formed of the open decreasing and the open increasing subsets is an open subbase.

PROOF. Let E be the space under consideration. We denote by $a \in E$ a point of this space and by V a neighborhood of this point. We determine the functions f and g in accordance with condition 1) of the preceding definition. Then we introduce a function h defined by the equation

$$h(x) = \sup [0, f(x) + g(x) - 1] \quad \text{where} \quad x \in E$$

Clearly, the function h is continuous and $h \geq 0$. Furthermore,

$$f(x) + g(x) - 1 \leq 1 + 1 - 1 = 1$$

whence $h \leq 1$, and

$$f(a) + g(a) - 1 = 1 + 1 - 1 = 1 \quad ,$$

that is, $h(a) = 1$. Finally, if $x \in E - V$, the last part of condition 1) shows that either $f(x) = 0$ or $g(x) = 0$. We consider the first case (the second is analogous). Then

$$f(x) + g(x) - 1 = g(x) - 1 \leq 0$$

whence $h(x) = 0$. We can, therefore, conclude that E is a uniformizable space.

We now consider two points $a, b \in E$ such that $a \leq b$ is false. We refer to condition 2) and determine the function f there indicated. Let ξ be a real number such that $f(a) > \xi > f(b)$. We define two sets V and W by the following equations:

$$V = [x;\ x \in E,\ f(x) > \xi] \quad ;$$
$$W = [x;\ x \in E,\ f(x) < \xi] \quad .$$

Making use of the fact that f is continuous and increasing, we verify that V is an increasing neighborhood of a and W a decreasing neighborhood of b. Moreover, V and W are, obviously, disjoint. Applying Proposition 1, §1, Chapter I, we see that the preorder is closed.

Finally, let a point $a \in E$ and a neighborhood V of a be given. We once more apply the preceding condition 1) and let f and g be the corresponding functions. We define two sets W_1 and W_2 be means of the equations

$$W_1 = [x;\ x \in E,\ f(x) > 0] \quad ,$$
$$W_2 = [x;\ x \in E,\ g(x) > 0] \quad .$$

Clearly, W_1 is an open increasing subset and W_2 is an open decreasing subset. Furthermore, condition 1) shows that

$$a \in W_1 \cap W_2 \subset V \ ;$$

this justifies the last assertion made in the proposition.

2. A *completely regular ordered space* is a uniformizable preordered space which has the following equivalent properties:

a) the Hausdorff axiom is satisfied;

b) the preorder of the space is an order.

We must, first of all, show that conditions a) and b) are equivalent. Suppose that a) is satisfied. We then consider two distinct points $a, b \in E$. By a) we can determine a neighborhood V of a which does not contain b. We next apply condition 1) of the definition of a uniformizable preordered space and let f and g be the two corresponding functions. The last part of 1) shows that either $f(b) = 0$ or $g(b) = 0$; in the first case, we see that $a \leq b$ is false since $a \leq b$ would imply

$$1 = f(a) \leq f(b) = 0$$

which is impossible; and in the second that $a \geq b$ is false for a similar reason. Consequently, b) is satisfied. If we, conversely, assume that the space has property b), it suffices to apply Proposition 6 proved above and to refer to Proposition 2, §1, Chapter I to obtain a).

We note that, by virtue of condition b), we can from now on use the expression *uniformizable ordered space* synonymously with *completely regular ordered space*.

In view of the definitions of a topological subspace and preordered subspace, we easily obtain the following result:

> PROPOSITION 7. Every topological and preordered subspace of a uniformizable preordered space or of a completely regular ordered space is a space with the same properties.

3. We conclude the present paragraph with a last result which in the case of a discrete order was first obtained by Tychonoff.

> THEOREM 7. Every topological and ordered subspace of a compact ordered space is a completely regular ordered space.

PROOF. Let E be the given space. By virtue of Proposition 7, it suffices to establish that this compact ordered space itself is a completely regular ordered space. For this purpose, we consider a point $a \in E$ and a neighborhood V of a. On the basis of Theorem 5, §3, Chapter I and of an observation made in the course of the proof of Proposition 3, §1, Chapter I, there exist two open subsets W_1 and W_2 of E, of which the first is decreasing and the second increasing, such that

$$a \in W_1 \cap W_2 \subset V \tag{1}$$

We note that $d(a)$ is a decreasing and closed subset (Proposition 1, §1, Chapter I) which, because of $a \in W_1$, is disjoint from the increasing and closed subset $E - W_1$. We determine a continuous increasing real-valued function g' on E such that

$$0 \leq g' \leq 1 \quad ,$$
$$g'(x) = 0 \quad \text{if} \quad x \in d(a) \quad ,$$
$$g'(x) = 1 \quad \text{if} \quad x \in E - W_1 \quad .$$

From the second condition follows, in particular, that $g'(a) = 0$. We recall that such a function g' exists by virtue of Theorem 4 [Corollary], §3 and of Theorem 1, §2, Chapter I. We now set $g = 1 - g'$. Clearly, then

$$\begin{aligned} &0 \leq g \leq 1 \quad , \\ (2) \qquad &g(a) = 1 \quad , \\ &g(x) = 0 \quad \text{if} \quad x \in E - W_1 \quad , \end{aligned}$$

and g is a continuous decreasing function.

In a corresponding fashion, observing that $i(a)$ is a closed increasing subset that is disjoint from the closed decreasing subset $E - W_2$, we can determine a continuous increasing real-valued function f on E such that

$$\begin{aligned} &0 \leq f \leq 1 \quad , \\ (3) \qquad &f(a) = 1 \quad , \\ &f(x) = 0 \quad \text{if} \quad x \in E - W_2 \quad . \end{aligned}$$

These two functions f and g show that condition 1) of the definition of a uniformizable preordered space is satisfied. Indeed, it suffices to take into account (2) and (3) and to note that (1) implies

$$E - V \subset (E-W_1) \cup (E-W_2) \quad ,$$

and from this the third part of 1), obviously, follows.

We now consider two points $a, b \in E$ such that $a \leq b$ is false. Then

$$i(a) \cap d(b) = \emptyset$$

Applying, once more, the propositions and theorems mentioned above, we obtain a continuous increasing real-valued function f such that

$$f(x) = 0 \quad \text{if} \quad x \in d(b) \quad ,$$
$$f(x) = 1 \quad \text{if} \quad x \in i(a) \quad .$$

In particular, $f(a) = 1$ and $f(b) = 0$, that is, $f(a) > f(b)$ as we desired.

Finally, we note that E is an ordered (and not only a preordered) space. The theorem is, thus, proved.

§2. Uniform ordered structures

1. The concept of a uniform structure or of uniformity was introduced into general topology by A. Weil in order to make possible a sufficiently general formulation of the concept of a uniformly continuous function, in the same way in which the concept of a topological structure or a topology furnishes the terms necessary for the formulation of the concept of a continuous function. With this theory of uniform spaces, Weil succeeded in harmoniously uniting into one system several aspects common to the study of metric spaces, compact spaces, and topological groups. In the present paragraph, it is our objective to analyse the concept of a uniform ordered space. This generalization of a uniform space is obtained by omitting one of the axioms governing uniform structures and by appropriately interpreting the mathematical structure so resulting. (Bibliographical references are found in the Introduction; concerning the material of the present paragraph, consult also, Nachbin, 2 and 3.)

A *semi-uniform structure* on a set E is a filter F of subsets of the square E^2 of the set E which has the following properties:

1) if $V \in F$, then $\Delta \subset V$ where Δ designates the diagonal of E^2;

2) if $V \in F$, then there exists a $W \in F$ such that $W \circ W \subset V$.

Every uniform structure is, obviously, a semi-uniform structure, but the converse is not necessarily true.

If F is a semi-uniform structure, the set F^{-1} of subsets of E^2 of the form V^{-1}, where $V \in F$, is also, as is easily verified, a semi-uniform structure and is called the *dual* of F.

We consider a semi-uniform structure F on E. We designate by F^* the set of the subsets of E^2 of the form $V \cap W^{-1}$ where $V, W \in F$. We assert that F^* is a uniform structure on E. Indeed, it is not difficult to prove that F^* is a semi-uniform structure. Furthermore, if $V \cap W^{-1}$

designates an arbitrary member of F^*, we note that

$$(V \cap W^{-1})^{-1} = W \cap V^{-1}$$

so that the first member of this inequality belongs to F^*. This proves that F^* satisfies the axiom of symmetry and is, therefore, a uniform structure. We shall call F^* the uniform structure *generated* by, or *associated* with, the given semi-uniform structure F.

The topology determined by the uniform structure F^* (cf. Weil, 1 and Bourbaki, 2, Definition 1, p. 92) will be called the topology *generated* by, or *associated* with, the given semi-uniform structure F.

We now represent by G the intersection of all the sets $V \in F$ and we assert that G is the graph of a preorder on E. If $x, y \in E$, the relation $x \leq y$ shall be defined as the relation $(x, y) \in G$ and shall be shown to have the properties required by the definition of a preorder. And, indeed, if $x \in E$, then

$$(x, x) \in \Delta \subset V$$

for every $V \in F$ whence follows $(x, x) \in G$, that is, $x \leq x$. Now consider three points $x, y, z \in E$ and assume that

$$x \leq y \ , \quad y \leq z \ ,$$

that is,

$$(x, y) \in G \ , \quad (y, z) \in G \ .$$

Given an arbitrary member $V \in F$, we determine $W \in F$ is such a way that $W \circ W \subset V$ (by condition 2) of the definition of F). Now

$$(x, y) \in G \subset W \ , \quad (y, z) \in G \subset W \ ,$$

whence we conclude that

$$(x, z) \in W \circ W \subset V$$

for every $V \in F$. Thus $(x, z) \in G$ which signifies that $x \leq z$ and completes the proof of the assertion that G is the graph of a preorder on the set E. We shall call this preorder *generated* by, or *associated* with the semi-uniform structure F.

A *uniform preordered space* is, by definition, a uniform space which is, at the same time, a preordered set in such a way that there exists on it at least one semi-uniform structure which generates the uniform structure and the preorder given on the space.

A *uniform ordered space* is a uniform preordered space which satisfies the two following equivalent conditions:

a) the uniform structure of the space is a Hausdorff uniform structure;

b) the preorder of the space is an order.

These conditions are equivalent since, by definition, they are respectively expressed by the equalities

$$\Delta = \bigcap (V \cap W^{-1}) \ , \quad \Delta = G \cap G^{-1}$$

which are, in turn, equivalent since G is the intersection of all the sets $V \in F$.

2. We shall now establish two simple properties of uniform preordered structures.

PROPOSITION 8. The preorder of every uniform preordered space is closed.

PROOF. Let E be the space. We consider two points $a, b \in E$ such that $a \leq b$ is false, that is, $(a, b) \in E^2 - G$; then there exists a set $V \in F$ such that $(a, b) \in E^2 - V$ (where G and F have the same meaning as above). We choose $V_1 \in F$ such that $V_1 \circ V_1 \subset V$, and then, $W \in F$ such that $W \circ W \subset V_1$. We define the sets A and B by

$$A = i[W(a)] \ , \quad B = d[W^{-1}(b)] \ .$$

From $A \supset W(a)$ follows that A is an increasing neighborhood of a; similarly, $B \supset W^{-1}(b)$ shows that B is a decreasing neighborhood of b. By virtue of Proposition 1, §1, Chapter I, it suffices to show that the sets A and B are disjoint. Let us suppose, now, that there exists a point

$$z \in A \cap B \ .$$

Since $z \in A$, there exists a point $x \in W(a)$ such that $x \leq z$. Then

$$(a, x) \in W \ , \quad (x, z) \in G \subset W$$

whence

$$(a, z) \in W \circ W \subset V_1 \tag{1}$$

Similarly, since $z \in B$, there exists a point $y \in W^{-1}(b)$ such that $z \leq y$. Then

$$(z, y) \in G \subset W \ , \quad (b, y) \in W^{-1} \quad \text{or} \quad (y, b) \in W$$

which furnishes

$$(z, b) \in W \circ W \subset V_1 \quad . \tag{2}$$

Combining (1) and (2), we obtain

$$(a, b) \in V_1 \circ V_1 \subset V$$

which contradicts $(a, b) \in E^2 - V$. The proposition is, thus, proved.

PROPOSITION 9. The topology of every uniform preordered space is locally convex.

PROOF. Let E, F, F^*, and G have the same meanings as above. Let $a \in E$ be a point of E and A a neighborhood of a. By virtue of the definition of the topology associated with a uniform structure, we can determine $W \in F^*$ such that $A = W(a)$. Recalling the definition of F^*, we easily see that there exists a set $V \in F$ such that

$$V \cap V^{-1} \subset W \ . \tag{1}$$

We then determine $V_1 \in F$ is such a manner that $V_1 \circ V_1 \subset V$. We set

$$W_1 = V_1 \cap V_1^{-1} \ , \quad B = k[W_1(a)]$$

(where k indicates the convex hull of the corresponding set). Since

$$B \supset W_1(a) \ , \quad W_1 \in F^* \ ,$$

we see that B is a convex neighborhood of a. The proposition will be proved if we show that $B \subset A$. For this purpose, we consider a point $x \in B$. There exist, then, two points $x', x'' \in W_1(a)$ such that

$$x' \leq x \leq x'' \quad .$$

Now

$$(a, x') \in W_1 \subset V_1 \ , \quad (x', x) \in G \subset V_1$$

show that

$$(a, x) \in V_1 \circ V_1 \subset V \quad . \tag{2}$$

Similarly,

$$(a, x'') \in W_1 \subset V_1^{-1} \quad \text{or} \quad (x'', a) \in V_1, \quad (x, x'') \in G \subset V_1$$

show that

(3) $$(x, a) \in V_1 \circ V_1 \subset V \quad \text{or} \quad (a, x) \in V^{-1} \quad .$$

Thus, by virtue of (1), (2), and (3).

$$(a, x) \in V \cap V^{-1} \subset W \quad .$$

that is, $x \in W(a) = A$ as we wished to show.

3. We consider a set E. A *semi-metric* on E is a real-valued function m defined on the square E^2 such that

1) if $x, y \in E$, then $m(x, y) \geq 0$;
2) $m(x, x) = 0$;
3) $m(x, z) \leq m(x, y) + m(y, z)$.

These three conditions are included in the axioms defining the concept of a metric due to Fréchet. We note, however, that m is not required to satisfy the symmetry condition $m(x, y) = m(y, x)$. We give just one example of a semi-metric. If E designates the set of real numbers, the function m defined by

$$m(x, y) = \sup(0, x-y)$$

is a semi-metric on E.

Every semi-metric m on E determines a semi-uniform structure F on E as follows. We designate by m_ε the set of points (x, y) of E^2 such that

$$m(x, y) \leq \varepsilon$$

where $\varepsilon > 0$. Let F be the set of the subsets of E^2 which contain at least one subset of the form m_ε where $\varepsilon > 0$. Clearly, F is a filter on E^2. We now verify that F is a semi-uniform structure. If $x \in E$, then

$$m(x, x) = 0 \leq \varepsilon \ , \quad \text{that is,} \quad (x, x) \in m_\varepsilon$$

for every $\varepsilon > 0$. Thus $(x, x) \in V$ for every $V \in F$, that is, $\Delta \subset V$. We now consider any set $V \in F$; by definition, there exists an $\varepsilon > 0$ such that $m_\varepsilon \subset V$. We set $W = m_{\varepsilon/2}$. Then $W \in F$ and $W \circ W \subset V$, since if

$$(x, y) \in W \quad \text{or} \quad m(x, y) \leq \varepsilon/2 \quad ,$$
$$(y, z) \in W \quad \text{or} \quad m(y, z) \leq \varepsilon/2 \quad ,$$

then

$$m(x, z) \leq m(x, y) + m(y, z) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad ,$$

This means that $(x, z) \in V$ which proves the inclusion relation $W \circ W \subset V$. Thus F is a semi-uniform structure.

The topology and the preorder generated by the semi-uniform structure F described above shall be called the topology and the preorder associated with m.

There follows a simple result, which we shall use a little later.

PROPOSITION 10. Let m be a semi-metric on E. For every point $b \in E$, the function $m(x, b)$ of x is continuous and increasing according to the topology and the preorder associated with m. Similarly, for every $a \in E$, the function $m(a, y)$ is continuous and decreasing.

PROOF. We shall, in the first place, establish the continuity of the function $m(x, b)$. Given a point $\xi \in E$ at which we want to establish the continuity of this function, we note that

$$m(x, b) \leq m(x, \xi) + m(\xi, b) \quad ,$$
$$m(\xi, b) \leq m(\xi, x) + m(x, b) \quad .$$

It follows from this that

(1) $$-m(\xi, x) \leq m(x, b) - m(\xi, b) \leq m(x, \xi) \quad .$$

Let us consider an arbitrary number $\varepsilon > 0$ and define a set A by

(2) $$A = [x; x \in E, m(x, \xi) \leq \varepsilon , \; m(\xi, x) \leq \varepsilon] \quad .$$

In view of the definition of the topology associated with a semi-metric, we see that A is a neighborhood of ξ according to the topology associated with m. Now, (1) and (2) show that

$$|m(x, b) - m(\xi, b)| \leq \varepsilon$$

provided that $x \in A$, and this establishes the continuity of $m(x, b)$ relative to the variable x.

We now prove that $m(x, b)$ is an increasing function of x. For this purpose, we consider two points $x', x'' \in E$ such that $x' \leq x''$, that is, such that the point (x', x'') belongs to the intersection of all the sets $V \in F$. By the definition of F, we see that

$$(x', x'') \in m_\varepsilon \quad \text{or} \quad m(x', x'') \leq \varepsilon \quad ,$$

for every $\varepsilon > 0$. Consequently $m(x', x'') = 0$. It follows that

$$m(x', b) \leq m(x', x'') + m(x'', b) = m(x'', b) \quad ,$$

as we wished to show.

The proof of that part of the statement relating to the function $m(a, y)$ is analogous.

4. We now proceed to establish one of the basic results of the theory of uniform ordered spaces. In the case in which the order considered is discrete, this theorem was established for the first time by A. Weil. He obtained it as a generalization of already known results on the metrization of topological spaces and, in particular, of topological groups (cf. Kakutani, 1; Weil, 1; and Bourbaki, 4, Proposition 1, p. 35, Proposition 2, p. 7). The proof which we shall adopt rests on the observation that the axiom of symmetry plays only a secondary role in the case of Weil's theorem. Rather than omit this proof because it follows the same line of argument as that by Bourbaki, we prefer to include it in this text so as not to disturb the continuity of the exposition.

THEOREM 8. Every semi-uniform structure associated with a semi-metric is a filter which has a countable base. Conversely, every semi-uniform structure which is a filter with a countable base is associated with at least one semi-metric.

PROOF. We consider a semi-metric m on E and the semi-uniform structure F associated with m. We set $V_n = m_{1/n}$ $(n = 1, 2, \ldots)$. Then the sequence $V_1, \ldots, V_n, \ldots$ is a countable base for F. Indeed, if $V \in F$, there exists an $\varepsilon > 0$ such that $m_\varepsilon \subset V$. We select an integer $n \geq 1$ such that $n \geq 1/\varepsilon$. Then

$$V_n \subset m_\varepsilon \subset V$$

which proves the first assertion.

Conversely, let the semi-uniform structure F on E be a filter that has a countable base $V_1, \ldots, V_n, \ldots$. Without loss of generality, we can assume that $V_1 \supset \ldots \supset V_n \supset \ldots$.

We note, to begin with, that if $V \in F$, there exists a set $W \in F$

such that $W \circ W \circ W \subset V$. Indeed, there exists a set $U \in F$ such that $U \circ U \subset V$. Furthermore, there exists a set $W \in F$ such that $W \circ W \subset U$. It follows that

$$W \circ W \circ W = \Delta \circ W \circ W \circ W \subset$$
$$\subset W \circ W \circ W \circ W \subset U \circ U \subset V$$

as we desired.

Having noted this fact, we define $W_1 = V_1$. We assume the set $W_n \in F$ already to have been defined. We then determine $W_{n+1} \in F$ in such a manner that

$$W_{n+1} \circ W_{n+1} \circ W_{n+1} \subset W_n , \quad W_{n+1} \subset V_{n+1} .$$

In this fashion, we obtain a sequence $W_n \in F$ $(n = 1, 2, \ldots)$, such that

(1) $$W_{n+1} \circ W_{n+1} \circ W_{n+1} \subset W_n , \quad W_n \subset V_n \quad (n = 1, 2, \ldots) .$$

By means of this sequence, we proceed to define a real-valued function g on the square E^2 as follows:

1) $g(x, y) = 0$ if $(x, y) \in W_n$ for every $n = 1, 2, \ldots,$
2) $g(x, y) = 1/2^k$ if $(x, y) \in W_k$ and $(x, y) \in E^2 - W_{k+1}$,
3) $g(x, y) = 1$ if $(x, y) \in E^2 - W_1$.

We note that the first part of (1) implies that

$$W_1 \supset W_2 \supset \ldots \supset W_n \supset \ldots \quad ;$$

this proves that the function g is fully defined by the conditions 1), 2), 3) above.

Clearly, $0 \leq g \leq 1$ and $g(x, x) = 0$ for every point $x \in E$.

From the definition of the function g follows at once that

2') $$g(x, y) \leq 1/2^k \text{ if } (x, y) \in W_k ,$$

and that

(2") $$g(x, y) > 1/2^{k-1} \text{ if } (x, y) \in E^2 - W_k$$

We define a semi-metric m in the following way. Given $x, y \in E$ we consider an arbitrary sequence

(2) $$x = z_0, z_1, \ldots, z_p = y \quad (p \geq 1)$$

of points of E and we set

$$m(x, y) = \inf [g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)] ,$$

where the infimum is taken with respect to all the sequences of the type (2). We must prove that m is actually a semi-metric and that, furthermore,

(3) $$\tfrac{1}{2}\, g(x, y) \leq m(x, y) \leq g(x, y) \quad .$$

The second part of inequality (3) is obvious since it suffices to set $p = 1$, $z_0 = x$ and $z_1 = y$ in (2). On the other hand, the second part of (3) proves that $m(x, x) = 0$, and on the basis of the definition of m, it is clear that

$$m(x, z) \leq m(x, y) + m(y, z) \ , \quad m \geq 0 \quad ;$$

that is, m is a semi-metric.

The first part of (3) is equivalent to

(4) $$\tfrac{1}{2}\, g(x, y) \leq g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)$$

for every sequence (2). We establish (4) by induction on the integer p. For $p = 1$, (4) is obvious. Assume that $p \geq 2$ and that (4) is ture for $1, \ldots, p-1$. We set

$$a = g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)$$

In the case $a \geq \tfrac{1}{2}$, inequality (4) clearly holds since $g(x, y) \leq 1$. Let us, then, assume that $a < \tfrac{1}{2}$.

We examine the case $a = 0$. Making use of the induction hypothesis, we have

$$\begin{aligned} 0 = a &= [g(z_0, z_1) + \cdots + g(z_{p-2}, z_{p-1})] + g(z_{p-1}, z_p) \\ &\geq \tfrac{1}{2}\, g(x, z_{p-1}) + g(z_{p-1}, y) \end{aligned}$$

whence

$$g(x, z_{p-1}) = g(z_{p-1}, y) = 0 \quad ,$$

that is,

$$(x, z_{p-1}) \in W_k, \quad (z_{p-1}, y) \in W_k \qquad (k = 1, 2, \ldots)$$

Replacing k by $k+1$, we obtain by (1),

$$(x, y) \in W_{k+1} \circ W_{k+1} \subset W_k \qquad (k = 1, 2, \ldots)$$

It follows that $g(x, y) = 0$ which proves (4) for the case $a = 0$.

Now suppose that $a > 0$.

We designate by h the smallest integer such that $1 \leq h \leq p$ and that

(5) $$g(z_0, z_1) + \cdots + g(z_{h-1}, z_h) > a/2$$

We note that (5) is satisfied if $h = p$ (since $a > 0$ implies $a > a/2$) so that such a minimal h exists. We assert that

(6) $$g(x, z_{h-1}) \leq a , \quad g(z_{h-1}, z_h) \leq a , \quad g(z_h, y) \leq a \quad .$$

Indeed, the first of the inequalities (6) is, clearly, satisfied if $h = 1$. Let us assume that $h \geq 2$. Since h is the smallest integer for which $1 \leq h \leq p$ and which satisfies (5), we see that

$$g(z_0, z_1) + \cdots + g(z_{h-2}, z_{h-1}) \leq a/2 \quad .$$

Now, $1 + (h-2) = h - 1 \leq p - 1$. Using in addition the induction hypothesis, we obtain

$$\tfrac{1}{2}g(x, z_{h-1}) \leq g(z_0, z_1) + \cdots + g(z_{h-2}, z_{h-1}) \leq \tfrac{1}{2}a$$

This completes the proof of the first inequality (6).

The second inequality is obvious on the basis of the definition of a.

If $h = p$, the last inequality is, clearly, satisfied. Let us assume that $h \leq p - 1$. Applying (5) and the definition of a, we have

$$g(z_h, z_{h+1}) + \cdots + g(z_{p-1}, z_p) < a/2 \quad .$$

Now, $1 + (p-1) - h = p - h \leq p - 1$. Using the induction hypothesis once more, we obtain

$$\tfrac{1}{2}g(z_h, y) \leq g(z_h, z_{h+1}) + \cdots + g(z_{p-1}, z_p) < \tfrac{1}{2}a$$

This completes the proof of the last inequality (6).

Having established these inequalities, we recall that $a > 0$ and consider the integer k defined by

(7) $$1/2^k \leq a < 1/2^{k-1}$$

Since $a < 1/2$, we have $k \geq 2$. Furthermore,

$$(x, z_{h-1}) \in W_k , \quad (z_{h-1}, z_h) \in W_k , \quad (z_h, y) \in W_k \quad ;$$

if, namely, we had for example, $(x, z_{h-1}) \in E^2 - W_k$, we should also have by 2")

$$g(x, z_{h-1}) > 1/2^{k-1}$$

which contradicts (6) and (7). Thus

$$(x, y) \in W_k \circ W_k \circ W_k \subset W_{k-1} \quad ,$$

and consequently, by 2') and (7),

$$g(x, y) \leq 1/2^{k-1} \leq 2a \quad .$$

Thus (4) is proved and, consequently, (3) is satisfied.

It remains to prove that m determines the given semi-uniform structure F. If $V \in F$, there exists an integer $k \geq 1$ such that $V_k \subset V$. Now $W_k \subset V_k$. The first part of (3) shows that

$$[(x, y) ; m(x, y) \leq 1/2^{k+1}] \subset [(x, y) ; g(x, y) \leq 1/2^k] \quad .$$

By 2') and 2"), the last member of this inclusion relation is W_k. Therefore $m_\varepsilon \subset V$ where $\varepsilon = 1/2^{k+1}$. Conversely, let us assume that $V \subset E^2$ and that $V \supset m_\varepsilon$ for a certain $\varepsilon > 0$. We determine an integer $k \geq 1$ in such a way that $\varepsilon \geq 1/2^k$. By the second part of (3), we see that $W_k \subset m_{1/2^k} \subset V$, and since $W_k \in F$, we have $V \in F$. The proof of the theorem is, thus, complete.

5. There follows now, the second basic result of the theory of uniform ordered spaces, which, in the case of the discrete is due to Weil and which extends a theorem of Pontrjagin on topological groups.

> THEOREM 9. Every uniform preordered space is a uniformizable preordered space. Conversely, every uniformizable preordered space can be equipped with a uniform structure in such a way that it becomes a uniform preordered space.

PROOF. Let E be a uniform preordered space. We designate by F a semi-uniform structure which generates the uniform structure and the preorder of E. We then consider a point $a \in E$ and a neighborhood V of a according to the topology of E. By the definition of the topology generated by a semi-uniform structure, there exists a set $W \in F$ such that

$$W(a) \cap W^{-1}(a) \subset V$$

We set $W_1 = W$ and assume $W_n \in F$ already to have been defined. We then determine $W_{n+1} \in F$ in such a way that

$$W_{n+1} \circ W_{n+1} \subset W_n \quad ,$$

and indicate by F' the filter of subsets of E^2 which admits $W_1, \ldots,$

W_n, ... as a base; it is clear that the sets W_n can be taken as the base of a filter and that the filter F' so obtained is a semi-uniform structure on E which admits a countable base.

We apply Theorem 8 and construct a semi-metric m on E which defines the semi-uniform structure F'. We then introduce the real-valued functions f' and g' on E defined as follows:

$$f'(x) = m(x, a) , \quad g'(x) = m(a, x) .$$

Noting that every member of the filter F' is also a member of the filter F and applying Proposition 10, we see that f' is a continuous increasing function and g' a continuous decreasing function on E.

Now $W \in F'$ and m defines F'; thus there is an $\varepsilon > 0$ such that $m_\varepsilon \subset W$. We next define two real-valued functions, f'' and g'' on E, in the following manner:

$$f''(x) = \sup[0, 1 - m(a, x)/\varepsilon] ,$$
$$g''(x) = \sup[0, 1 - m(x, a)/\varepsilon] .$$

It is clear that f'' is continuous and increasing and g'' continuous and decreasing. Furthermore, we have

(1) $$f'' \geq 0, \quad g'' \geq 0, \quad f''(a) = 1, \quad g''(a) = 1 .$$

If $x \in E - V$, then

(2) $$\inf [f''(x), g''(x)] = 0$$

Indeed, if we had $f''(x) > 0$ and $g''(x) > 0$, we should by the definition of f'' and g'', have

$$1 - m(a, x)/\varepsilon > 0 , \quad 1 - m(x, a)/\varepsilon > 0 ,$$

that is,

$$m(a, x) < \varepsilon , \quad m(x, a) < \varepsilon$$

Consequently,

$$(a, x) \in m_\varepsilon \subset W ,$$
$$(x, a) \in m_\varepsilon \subset W \quad \text{or} \quad (a, x) \in W^{-1} ,$$

so that

$$x \in W(a) \cap W^{-1}(a) \subset V ,$$

which contradicts the hypothesis that $x \in E - V$, so (2) is proved.

If we now define the functions f and g by

$$f(x) = \inf [1, f''(x)] \quad ,$$
$$g(x) = \inf [1, g''(x)] \quad ,$$

and take into account relations (1) and (2) above, we see that these functions f and g have all the properties indicated in the first condition of the definition of a uniformizable preordered space.

We now go on to consider two points $a, b \in E$ such that $a \leq b$ is false. By the definition of the preorder associated with a semi-uniform structure, there exists a set $W \in F$ such that $(a, b) \in E^2 - W$.

We repeat the construction used in the previous case, setting $W_1 = W$ and determining W_n $(n = 1, 2, \ldots)$, F', m, and ε in the manner there indicated. If we had $m(a, b) = 0$, we should have $m(a, b) \leq \varepsilon$, whence would result

$$(a, b) \in m_\varepsilon \subset W \quad ,$$

and this would contradict the choice of W. Therefore,

$$m(a, b) > 0$$

We introduce the continuous increasing real-valued function f defined by

$$f(x) = m(x, b) \quad .$$

Since

$$f(a) = m(a, b) > 0 = m(b, b) = f(b) \quad ,$$

we see that the second condition of the definition of a uniformizable preordered space is satisfied. The proof of the first part of the theorem is, thus, complete.

Conversely, we consider a uniformizable preordered space E. We indicate by f an arbitrary continuous increasing real-valued function on E and introduce the set $W_f \subset E^2$ defined by

$$W_f = [(x, y);\ f(x) - f(y) < 1] \quad .$$

Clearly, $W_f \supset \Delta$. The collection of all sets of the form W_f can, therefore, be taken as the subbase of a filter F on E^2. It is a question of routine to verify that the filter F so obtained is a semi-uniform structure, and this detail will, therefore, be omitted. It remains to be shown that F generates precisely the topology and the preorder given.

Now since for every point $a \in E$ and for every continuous increasing real-valued function f defined on E (the continuity referring to the given topology) the sets

$$[y;\ (a, y) \in W_f] \quad ,$$
$$[x;\ (x, a) \in W_f] \quad ,$$

are (according to the given topology) neighborhoods of a, we see that every subset which is open according to the topology generated by F is also open according to the given topology. Conversely, we consider a point $a \in E$ and one of its neighborhoods V according to the given topology. Making use of the fact that E is a uniformizable preordered space, we can determine two continuous real-valued functions f and g, where f is increasing and g decreasing, such that

$$0 \leq f \leq 1 \quad , \quad 0 \leq g \leq 1 \quad ,$$
$$f(a) = 1 \quad , \quad g(a) = 1 \quad ,$$
$$\inf\,[f(x),\ g(x)] = 0 \quad \text{if} \quad x \in E - V$$

We assert that

(3) $$W_f(a) \cap W_{1-g}^{-1}(a) \subset V \quad ;$$

this is true since, if

$$x \in W_f(a) \cap W_{1-g}^{-1}(a) \quad ,$$

then $x \in W_f(a)$ furnishes $(a, x) \in W_f$ or $f(a) - f(x) < 1$ whence follows $f(x) > 0$ as $f(a) = 1$; furthermore, $x \in W_{1-g}^{-1}(a)$ signifies that $(a, x) \in W_{1-g}^{-1}$, that is, $(x, a) \in W_{1-g}$ or

$$[1 - g(x)] - [1 - g(a)] < 1$$

whence follows that $g(x) > 0$ since $g(a) = 1$. We can, thus, assert that

$$\inf[f(x),\ g(x)] > 0$$

and this, by the third property of f and g, requires that $x \in V$ as we wished to show.

Now the inclusion relation (3) implies that V is a neighborhood of a according to the topology generated by the semi-uniform structure F. Consequently, every subset that is open according to the given topology is also open according to the topology generated by F. Combining this fact with the converse observation made earlier, we conclude that the two topologies are identical.

As a last step, we prove that the preorder generated by F is identical with the given preorder. If $a \leq b$ were $a, b \in E$, then $f(a) \leq f(b)$ whence

$$f(a) - f(b) \leq 0 < 1 \quad ;$$

this shows us that $(a, b) \in W_f$ for every continuous real-valued function on E which is increasing according to the given preorder; but, then, $(a, b) \in W$ for every $W \in F$; that is, $a \leq b$ according to the preorder determined by F. Furthermore, if $a \leq b$ is false according to the given preorder, there exists a continuous real-valued function which is increasing according to that preorder such that $f(a) > f(b)$. Without loss of generality, we can assume that $f(a) - f(b) = 1$ since, in the contrary case, it suffices to substitute for f the function defined by the expression

$$\frac{f(x) - f(b)}{f(a) - f(b)}$$

Clearly, then, $(a, b) \in E^2 - W_f$ and thus, $a \leq b$ is false according to the preorder determined by F. Again combining this fact with the converse observation made earlier, we see that the two preorders are identical. The theorem is, therefore, proved.

6. As soon as the definition of a uniform ordered space is formulated, the following problem arises.

Let a space E be given which is, at the same time, a uniform space and a preordered space. Under what conditions of interdependence between the uniform structure and the preorder is E a uniform preordered space?

In agreement with the definition adopted, this means that there exists a semi-uniform structure on E which generates precisely the uniform structure and the preorder given on E. It is, therefore, desirable to obtain sufficiently simple conditions for the existence of such a semi-uniform structure, conditions that are easily verified in some concrete important cases.

An interesting result in this direction, of which we shall make two applications later, is the following:

THEOREM 10. Let E be a uniform space which is, at the same time, a preordered space. Let F^* stand for the filter of subsets of E^2 which define the uniform structure of E and G for the graph of the preorder of E in E^2. In order that E be a uniform preordered space it is sufficient that

a) given $V \in F^*$, there exists a set $W \in F^*$ such that

$$W \circ G \subset G \circ V \quad ;$$

b) given $V \in F^*$, there exists a set $W \in F^*$ such that

$$(G \circ W) \cap (W \circ G^{-1}) \subset V \quad ;$$

c) for every $a \in E$, the set $i(a) = \{x; x \in E, x \geq a\}$ be closed.

PROOF. Suppose that a), b), and c) are satisfied. Since

$$G \circ V \supset \Delta \qquad (\text{since } G, V \supset \Delta) \quad ,$$
$$(G \circ V_1) \cap (G \circ V_2) \supset G \circ (V_1 \cap V_2) \quad ,$$

the sets of the form $G \circ V$, where $V \in F^*$, can be taken as the base of a filter F on E^2. It is our objective to establish that F is a semi-uniform structure which determines the given uniform structure and preorder.

From $G \circ V \supset \Delta$, where $V \in F^*$, we see that every member of F contains Δ. In order to complete the proof that F is a semi-uniform structure, it therefore sufficies to show that, if $V \in F^*$, there exists a set $W \in F^*$ such that

(1) $$(G \circ W) \circ (G \circ W) \subset G \circ V$$

Now in terms of its graph G, the transitive property of a preorder signifies that $G \circ G \subset G$. We determine a set $V' \in F^*$ in such a manner that

$$V' \circ V' \subset V \ .$$

Then, making use of the condition a) as stated in the theorem, we determine a set $V'' \in F^*$ such that

$$V'' \circ G \subset G \circ V' \quad .$$

Setting $W = V' \cap V''$, it is clear that $W \in F^*$ and that

$$G \circ W \circ G \circ W \subset G \circ V'' \circ G \circ V' \subset G \circ G \circ V' \circ V' \subset G \circ V$$

This proves (1). So F is a semi-uniform structure.

We now go on to show that the uniform structure associated with F is identical with the given uniform structure. For this purpose, we establish two facts: in the first place, that

$$G \circ V \in F^*$$

for every $V \in F^*$; and, in the second place, that corresponding to every set $V \in F^*$, there exists a set $W \in F^*$ such that

(2) $$(G \circ W) \cap (G \circ W)^{-1} \subset V$$

The first fact results in a simple manner from

$$G \circ V \supset \Delta \circ V = V$$

and from one of the properties of filters. In order to establish the inclusion relation (2), we make use of condition b) in the statement of the theorem and, once $V \in F^*$ is given, we determine $W' \in F^*$ in such a way that

$$(G \circ W') \cap (W' \circ G^{-1}) \subset V$$

Setting

$$W = W' \cap W'^{-1} \quad ,$$

it follows that $W \in F^*$ and the inclusion relation (2) is obviously verified.

Finally, we prove that the preorder determined by F is identical with the given preorder, or, in equivalent terms that

(3) $$G = \bigcap_{V \in F^*} G \circ V$$

And, indeed, we note that

$$G \circ V \supset G \circ \Delta = G$$

and thus, relation (3) is valid provided that we replace the $=$ sign by $\subset$. We next assume that

$$(a, b) \in E^2 - G \quad ,$$

that is, that $a \leq b$ is false. By hypothesis, the set $i(a)$ is closed (condition c) of the statement of the theorem) and $b \in E - i(a)$ so that there exists a neighborhood B of b such that

$$B \cap i(a) = \emptyset .$$

We now determine a set $V \in F^*$ in such a way that $V^{-1}(b) = B$. We assert that

$$(a, b) \in E^2 - G \circ V ,$$

since, in the contrary case, there would exist a point $x \in E$ such that

$$(a, x) \in G , \quad (x, b) \in V ,$$

whence would follow

$$a \leq x \text{ or } x \in i(a) , \qquad (b, x) \in V^{-1} \text{ or } x \in V^{-1}(b) ,$$

in contradiction to the hypothesis that B and $i(a)$ are disjoint. Thus equality (3) is proved.

7. A first application of the preceding theorem furnishes the following result.

PROPOSITION 11. Every Hausdorff uniform space E which is, at the same time, a sup-lattice such that $x \vee y$ is a uniformly continuous function of (x, y), is a uniform ordered space.

PROOF. Our procedure will simply be to verify that conditions a), b), and c) of Theorem 10 are satisfied in the case in question.

We indicate by F^* the filter on E^2 which defines the uniform structure on E and by G the graph of the order of E.

In order to establish a), we consider an arbitrary set $V \in F^*$ and determine $W \in F^*$ in such a way that, if

$$(x', x'') \in W , \quad (y', y'') \in W , \tag{1}$$

then

$$(x' \vee y', x'' \vee y'') \in V , \tag{2}$$

this being possible by the uniform continuity of the supremum with respect to its two arguments. We assert that

$$W \circ G \subset G \circ V$$

Indeed, if (x, y) is a point belonging to the first member of this inclusion relation, then there exists a point $t \in E$ such that

$$(x, t) \in W , \quad (t, y) \in G \text{ or } t \leq y$$

Noting that

$$(x, t) \in W , \quad (y, y) \in W \quad ,$$

and taking into account (1) and (2), we obtain

$$(x \vee y, t \vee y) \in V \quad ,$$

or

$$(x \vee y, y) \in V \tag{3}$$

since $t \vee y = y$. On the other hand, we note that $x \leq x \vee y$, that is,

$$(x, x \vee y) \in G \quad . \tag{4}$$

Combining (3) and (4), we conclude that the point (x, y) also belongs to $G \circ V$, and this completes the proof of a).

In order to establish b), we consider a set $V \in F^*$ and, then determine $W_1 \in F^*$ in such a way that

$$W_1 \circ W_1^{-1} \circ W_1 \subset V$$

Making use once more of the uniform continuity of the supremum, we select a set $W_2 \in F^*$ such that

$$(x' \vee y', x'' \vee y'') \in W_1 \tag{5}$$

whenever

$$(x', x'') \in W_2 , \quad (y', y'') \in W_2 \tag{6}$$

Setting

$$W = W_1 \cap W_2$$

we assert that

$$(G \circ W) \cap (W \circ G^{-1}) \subset V$$

And indeed, if (x, y) designates a point belonging to the first member of this inclusion relation, then there follows directly from $(x, y) \in G \circ W$ the existence of a point $u \in E$ such that

$$(x, u) \in G \text{ or } x \leq u , \qquad (u, y) \in W \quad ,$$

and similarly, there follows from $(x, y) \in W \circ G^{-1}$ that there is a point

$v \in E$ such that

$$(x, v) \in W, \quad (v, y) \in G^{-1} \quad \text{or} \quad y \leq v$$

Now the relations

$$(x, v) \in W \subset W_2, \quad (u, y) \in W \subset W_2 \quad ,$$

together with (5) and (6), imply that

$$(x \vee u, v \vee y) \in W_1, \quad \text{that is} \quad (u, v) \in W_1$$

since $x \vee u = u$ and $v \vee y = v$. Combining the relations

$$(x, v) \in W \subset W_1, \quad (v, u) \in W_1^{-1}, \quad (u, y) \in W \subset W_1 \quad ,$$

we have

$$(x, y) \in W_1 \circ W_1^{-1} \circ W_1 \subset V$$

whereby the proof of b) is completed.

Finally, we note that $x \vee y$ is a uniformly continuous function of the two variables x and y simultaneously and, thus, a continuous function of each variable separately. It follows from this that, for every point $a \in E$, the set

$$i(a) = [x; x \vee a = x]$$

is closed since E is a Hausdorff space (and therefore, the diagonal of E^2 is closed). If follows that condition c) is satisfied and the theorem is proved.

8. Another important application of Theorem 10 is given in

PROPOSITION 12. In order that a topological Abelian group E which is, at the same time, a preordered group, be a uniform preordered space, it is necessary and sufficient that

1) for every neighborhood A of the element 0 of E there exist another neighborhood B of 0 such that

$$0 \leq x \leq y \in B \text{ implies } x \in A \quad ;$$

2) the set P of the positive elements of E be closed.

PROOF. The conditions are necessary. In fact, 1) is an immediate consequence of Proposition 9, §2, Chapter II. As regards 2), it suffices to combine Proposition 8, §2, Chapter II with Proposition 1, §1, Chapter I.

Conversely, suppose that conditions 1) and 2) as stated in the theorem are satisfied. We shall show that conditions a), b), and c) of Theorem 10 are also satisfied.

For this purpose, we indicate by F^* and G the filter on E^2 which defines the uniform structure of E and the graph of the preorder of E, respectively. We note that, if A is an arbitrary neighborhood of 0 in E and if we define

$$A^* = [(x, y); x, y \in E, x - y \in A] \quad ,$$

then $A^* \in F^*$ and the collection of all sets A^* constitutes a base of the filter F^* (by the definition of the uniform structure of a topological group). We assert that

$$A^* \circ G = G \circ A^* \quad .$$

Indeed, if the point (x, y) belongs to the first member of this equlaity, there exists a point $u \in E$ such that

$$(x, u) \in A^* \ , \quad \text{that is} \quad x - u \in A \ ,$$
$$(u, y) \in G \ , \quad \text{that is} \quad u \leq y \quad .$$

Setting

$$v = x + (y-u) \quad ,$$

we have

$$x \leq v \ , \qquad \text{that is} \quad (x, v) \in G \ ,$$
$$v - y = x - u \in A \ , \quad \text{that is} \quad (v, y) \in A^* \quad .$$

This shows that the point (x, y) also belongs to the second member of the equality to be established, and so proved the inclusion in one sense; the inclusion in the opposite sense is proved in an analogous manner.

Having recognized this, we consider an arbitrary set $V \in F^*$. We determine a neighborhood A of 0 in such a way that $A^* \subset V$. Then

$$G \circ V \supset G \circ A^* = A^* \circ G \quad ,$$

and it suffices to take $W = A^*$ in order to be able to conclude that condition a) of Theorem 10 is satisfied.

In order to establish b), we consider a set $V \in F^*$ and determine a neighborhood A of 0 such that $A^* \subset V$. By the fact that E is a topological group, we can determine a neighborhood B of 0 such that

$B + B \subset A$. In agreement with condition 1) of the present proposition, we can determine a neighborhood C of o such that

$$o \leq x \leq y \in C \text{ implies } x \in B$$

Then, again by the fact that E is a topological group, we can determine a neighborhood D of o, where $D \subset B$, such that $D - D \subset C$ (the minus sign, naturally, being interpreted in the sense of the theory of Abelian groups). Setting $W = D^*$, we have $W \in F^*$ and

$$(G \circ W) \cap (W \circ G^{-1}) \subset V \quad .$$

Indeed, let (x, y) be a point belonging to the first member of this inclusion. From the fact that $(x, y) \in G \circ W$, it follows that there exists a point $u \in E$ such that

$$(x, u) \in G, \text{ that is, } x \leq u \ ,$$
$$(u, y) \in W, \text{ that is, } u - y \in D \ .$$

Similarly, it follows from $(x, y) \in W \circ G^{-1}$ that there exists a point $v \in E$ such that

$$(x, v) \in W, \text{ that is, } x - v \in D \ ,$$
$$(v, y) \in G^{-1}, \text{ that is, } y \leq v$$

We set

$$s = v - y \ , \quad t = (u-y) - (x-v) \quad .$$

Then

$$t = (u-y) - (x-v) \in D - D \subset C \quad .$$

Consequently,

$$o \leq s \leq t \in C \text{ implies } s \in B$$

Finally, it follows from

$$x - v \in D \subset B \ , \quad v - y = s \in B$$

that

$$x - y = (x-v) + (v-y) \in B + B \subset A \quad ,$$

that is

$$(x, y) \in A^* \subset V \quad ,$$

and hereby condition b) of Theorem 10 is established.

In order to obtain condition c) of Theorem 10, it suffices to take into account condition 2) of the present proposition and to note that the transformation $x \to a + x$ is a homeomorphism in every topological Abelian group. Our proposition is, thus proved.

We shall conclude this paragraph with the study of an important category of uniform ordered spaces to which Theorem 10 does not apply since condition a) of this theorem is not necessarily fulfilled.

PROPOSITION 13. Every compact ordered space is a uniform ordered space.

PROOF. Consider a compact ordered space E and let G be the graph of its order. The general theory of uniform spaces teaches us (Bourbaki, 2, Theorem 1, p. 107) that the filter F^* of the neighborhoods of the diagonal Δ of E^2 is a uniform structure on E which is compatible with the topology of E and which, more precisely, is the only uniform structure with this property. We consider the filter F of the neighborhoods of G in E^2. We shall show that F is a semi-uniform structure which generates the uniform structure F^* and the order G.

We first show that F is a semi-uniform structure. If $V \in F$, then $V \supset \Delta$ since $V \supset G$. Moreover, given $V \in F$, there exists a set $W \in F$ such that $W \circ W \subset V$, and it suffices to establish this fact under the assumption that V is open. Let us suppose, for a moment, that it is impossible to determine $W \in F$ in such a way that $W \circ W \subset V$. In other words, given any set $W \in F$, there exist points $x, y \in E$ such that

$$(x, y) \in E^2 - V, \quad (x, y) \in W \circ W,$$

that is, there exists a point $t \in E$ such that

$$(x, t) \in W, \quad (t, y) \in W.$$

We designate by V' the subset of the cube $E^3 = E \times E \times E$ formed of all the points (x, t, y) such that $(x, y) \in E^2 - V$ and $t \in E$. We note that V' is compact since $E^2 - V$ and E are compact.

For every $W \in F$, we designate by $\widetilde{W}$ the set of all the points $(x, t, y) \in E^3$ such that

$$(x, y) \in E^2 - V, \quad (x, t) \in W, \quad (t, y) \in W$$

Clearly, $\tilde{W} \subset V'$; and, as seen above, the assumption that $W \circ W \subset V$ is false signifies that $\tilde{W}$ is not empty. It follows that the collection of sets $\tilde{W}$, where $W \in F$, can be taken as the base of a filter f on V'. Making use of the compactness of V', we see that the filter f has at least one accumulation point; let (a, h, b) be such a point.

We assert that $a \leq h$. Indeed, suppose $a \leq h$ is false. Since E is normally ordered (by Theorem 4, Corollary, §3, Chapter I), we can determine an open increasing set $P \ni a$ and an open decreasing set $Q \ni h$ which are disjoint. The topological space E is, moreover, normal and P is a neighborhood of the closed set $i(a)$. We can, therefore, determine a closed neighborhood P' of $i(a)$ such that $P' \subset P$. We note that P' is then a neighborhood of a. Similarly, we can determine a closed neighborhood Q' of h such that $Q' \subset Q$. The set

$$W = E^2 - P' \times Q'$$

is open and contains G (since, if there existed a point (x, y) common to G and $P' \times Q'$, we should have

$$x \leq y , \quad x \in P' \subset P , \quad y \in Q' \subset Q \quad ,$$

and this would contradict the fact that P is increasing and disjoint from Q). This means that $W \in F$ and, thus, $\tilde{W} \in f$. Furthermore, $P' \times Q' \times E$ is a neighborhood of (a, h, b) in E^3 which is disjoint from $\tilde{W}$ for, if (x, t, y) were a point belonging to this neighborhood, we would have

$$(x, t) \in E^2 - W \quad \text{whence} \quad (x, t, y) \in E^3 - \tilde{W} \quad .$$

Now this fact contradicts the property of (a, h, b) to be an accumulation point of the filter f. Consequently, $a \leq h$.

The relation $h \leq b$ is established in a corresponding manner. Combining the two inequalities obtained, we see that $a \leq b$ so that

$$(a, b) \in G \subset V$$

and this contradicts the fact that $(a, h, b) \in V'$ (we recall that f is a filter on V'). This contradiction shows that, given $V \in F$, there exists $W \in F$ such that $W \circ W \subset F$. Thus F is a semi-uniform structure.

We now prove that the semi-uniform structure F determines the order whose graph is G. This is true since E^2 is a Hausdorff space (Proposition 2, §1, Chapter I) and since in every Hausdorff space every set (and, in particular the set G in the space E^2) is identical with the intersection of its neighborhoods.

As a last step, we prove that the semi-uniform structure F determines the uniform structure F^*. By virtue of the very definitions of F and F^*, it is clear that every subset which is open according to the topology defined by F is also open according to the topology defined by F^*, that is, according to the topology originally given on E. As was already seen, F determines an order and not only a preorder. Thus the topology defined by F is a Hausdorff topology. On the basis of a property of compact spaces which we have already used (see Bourbaki, 2, Corollary 2, p. 62), we conclude that the two topologies defined above are identical. The uniqueness of the uniform structure compatible with the topology of a compact Hausdorff space then implies that the uniform structure determined by F is identical with that determined by F^*. The result is, thus, established.

Clearly, $\widetilde{W} \subset V'$; and, as seen above, the assumption that $W \circ W \subset V$ is false signifies that $\widetilde{W}$ is not empty. It follows that the collection of sets $\widetilde{W}$, where $W \in F$, can be taken as the base of a filter f on V'. Making use of the compactness of V', we see that the filter f has at least one accumulation point; let (a, h, b) be such a point.

We assert that $a \leq h$. Indeed, suppose $a \leq h$ is false. Since E is normally ordered (by Theorem 4, Corollary, §3, Chapter I), we can determine an open increasing set $P \ni a$ and an open decreasing set $Q \ni h$ which are disjoint. The topological space E is, moreover, normal and P is a neighborhood of the closed set $i(a)$. We can, therefore, determine a closed neighborhood P' of $i(a)$ such that $P' \subset P$. We note that P' is then a neighborhood of a. Similarly, we can determine a closed neighborhood Q' of h such that $Q' \subset Q$. The set

$$W = E^2 - P' \times Q'$$

is open and contains G (since, if there existed a point (x, y) common to G and $P' \times Q'$, we should have

$$x \leq y \ , \quad x \in P' \subset P \ , \quad y \in Q' \subset Q \quad ,$$

and this would contradict the fact that P is increasing and disjoint from Q). This means that $W \in F$ and, thus, $\widetilde{W} \in f$. Furthermore, $P' \times Q' \times E$ is a neighborhood of (a, h, b) in E^3 which is disjoint from $\widetilde{W}$ for, if (x, t, y) were a point belonging to this neighborhood, we would have

$$(x, t) \in E^2 - W \quad \text{whence} \quad (x, t, y) \in E^3 - \widetilde{W} \quad .$$

Now this fact contradicts the property of (a, h, b) to be an accumulation point of the filter f. Consequently, $a \leq h$.

The relation $h \leq b$ is established in a corresponding manner. Combining the two inequalities obtained, we see that $a \leq b$ so that

$$(a, b) \in G \subset V$$

and this contradicts the fact that $(a, h, b) \in V'$ (we recall that f is a filter on V'). This contradiction shows that, given $V \in F$, there exists $W \in F$ such that $W \circ W \subset F$. Thus F is a semi-uniform structure.

We now prove that the semi-uniform structure F determines the order whose graph is G. This is true since E^2 is a Hausdorff space (Proposition 2, §1, Chapter I) and since in every Hausdorff space every set (and, in particular the set G in the space E^2) is identical with the intersection of its neighborhoods.

As a last step, we prove that the semi-uniform structure F determines the uniform structure F^*. By virtue of the very definitions of F and F^*, it is clear that every subset which is open according to the topology defined by F is also open according to the topology defined by F^*, that is, according to the topology originally given on E. As was already seen, F determines an order and not only a preorder. Thus the topology defined by F is a Hausdorff topology. On the basis of a property of compact spaces which we have already used (see Bourbaki, 2, Corollary 2, p. 62), we conclude that the two topologies defined above are identical. The uniqueness of the uniform structure compatible with the topology of a compact Hausdorff space then implies that the uniform structure determined by F is identical with that determined by F^*. The result is, thus, established.

CHAPTER III

LOCALLY CONVEX ORDERED VECTOR SPACES

SUMMARY.

In this chapter we introduce the concepts of a locally convex ordered vector space and of a topological vector lattice and we indicate certain basic properties of these structures. Then after establishing a fundamental result on locally convex directed vector spaces (Theorem 11, §2), we apply it in order to prove the principal theorem of the present chapter; this theorem is concerned with the continuity of positive linear transformations between two locally convex ordered vector spaces (Theorem 12, §2).

§1. Generalities

1. The concept of a locally convex ordered vector space is a synthesis of the concept of a locally convex vector space—which includes Hilbert spaces and Banach spaces—and the concept of an ordered space. From the point of view of the applications of general topology and of the theory of ordered sets to modern function theory, locally convex ordered vector spaces furnish one of the principal examples of a simple mathematical system in which we can successfully combine topological methods and algebraic ideas in order to obtain results that are of interest in other branches of mathematics. An important chapter of mathematics in which the concept of a locally convex ordered vector space frequently occurs is the study of the spectral decomposition of self-adjoint operators and, even more so, the study of algebras of operators in Hilbert space.

A *locally convex preordered vector space* E is a topological vector space which is, at the same time, a preordered vector space in such a way as to satisfy the following conditions:

1) E is a locally convex vector space;

2) given an arbitrary neighborhood A of o in E, it is possible to determine another neighborhood B of o such that

$$o \leq x \leq y \in B \text{ implies } x \in A \quad ;$$

3) the cone P of the positive elements of E is closed.

Proposition 14 below justifies the above terminology.

We are already familiar with conditions 2) and 3) (see Proposition 12, §2, Chapter II). Intuitively speaking, 2) signifies that if x and y are two variable elements of E such that

$$o \leq x \leq y$$

and if y tends to o, then x will tend to o; and 3) is equivalent to the assertion that the limit of every positive and variable element of E is necessarily positive. As regards condition 1), it is concerned with a now classical requirement in the theory of topological vector spaces.

A *locally convex ordered vector space* is a locally convex preordered vector space which satisfies the two following equivalent conditions:

a) the Hausdorff axiom is valid;

b) the preorder of the space is an order.

The equivalence of these two conditions may be established either as a particular case of analogous equivalences previously considered or by repetition, in the case on hand, of the type of reasoning previously employed.

We begin by proving the following proposition:

PROPOSITION 14. In every locally convex preordered vector space, the neighborhoods of every point which are convex both in the sense of the preorder and in the vector sense constitute a base for the neighborhood system of this point.

PROOF. We represent the space by E. We start by showing that, if $X \subset E$ is a convex subset in the vector sense, then its convex hull $Y = k(X)$, in the sense of the preorder, that is, the smallest convex set in the sense of the preorder which contains X, will be convex also in the vector sense. Indeed, let us consider two points $a, b \in Y$ and a number ξ where $0 \leq \xi \leq 1$, and let us set

$$c = \xi a + (1-\xi)b \quad .$$

Since $a \in Y$, there exist two points $a', a'' \in X$ such that

(1) $$a' \leq a \leq a'' \quad .$$

For the same reason, there exist two points $b', b'' \in X$ such that

(2) $$b' \leq b \leq b''$$

Setting

$$c' = \xi a' + (1-\xi)b' \ , \quad c'' = \xi a'' + (1-\xi)b'' \quad .$$

we see that $c', c'' \in X$ because of the convexity of X in the vector sense. Furthermore, (1) and (2) show that

$$c' \leq c \leq c''$$

so that $c \in Y$; thus Y is convex in the vector sense.

We now show that the neighborhoods of every point which are convex in the sense of preorder constitute a base for the neighborhood system of this point. Without loss of generality, we can assume that the point under consideration is the origin o. We designate by A an arbitrary neighborhood of o. We determine a neighborhood B of o such that $B + B \subset A$ which is possible since E is a topological vector space. Then we apply condition 2) of the definition of a locally convex preordered vector space and determine a neighborhood C of o such that

$$o \leq x \leq y \in C \text{ implies } x \in B \quad .$$

Finally, we select a neighborhood D of o such that $D - D \subset C$ (where the minus sign is to be interpreted in the sense of group theory). We define A' as being the convex hull, in the sense of the preorder of $B \cap D$. It follows from

$$A' \supset B \cap D$$

that A' is a convex neighborhood of o in the sense of the preorder. To complete this part of the proof, it suffices to show that $A' \subset A$. For this purpose, we consider a point $x \in A'$ and determine two points x' and x'' such that

$$x', x'' \in B \cap D \ , \quad x' \leq x \leq x''$$

Now

$$x'' - x' \in D - D \subset C$$

and thus,

$$0 \leq x - x' \leq x'' - x' \in C$$

implies that $x - x' \in B$. Hence

$$x = (x-x') + x' \in B + B \subset A$$

whereby the inclusion $A' \subset A$ is proved.

Now, at last, we prove the proposition. As was already stressed, we may assume that the point under consideration is the origin 0. We indicate by A an arbitrary neighborhood of 0. Basing ourselves on an earlier remark, we can determine a neighborhood $B \subset A$ of 0 which is convex in the sense of the preorder of E. Now E is, also, a locally convex vector space and thus, there exists a neighborhood $C \subset B$ of 0 which is convex in the vector sense. We designate by A' the convex hull of C with reference to the preorder. Then, since $A' \supset C$, A' is a convex neighborhood of 0 with reference to the preorder. By the first part of the proof, A' is also convex in the vector sense. Furthermore, it follows from $C \subset B$ that $A' \subset B$ since B is convex with reference to the preorder. The combination of this relation with $B \subset A$ furnishes $A' \subset A$. This concludes the proof.

2. An ordered vector space is said to be a *directed vector space* if every element of the space can be expressed as the difference of two positive elements. We shall now introduce the topological analogue of this algebraic concept.

A *locally convex directed vector space* is a locally convex ordered vector space which satisfies the following condition:

D. For every neighborhood A of 0 the set

$$A \cap P - A \cap P$$

is also a neighborhood of 0.

(P, as before, designates the cone of positive elements: the minus sign must, from now on, be interpreted in the sense of group theory unless there is an indication to the contrary.)

Intuitively speaking, condition D requires that every variable point of E which tends to 0 can be expressed as the difference of two positive variable points which also tend to 0. This results from the second part of the following proposition.

PROPOSITION 15. Every locally convex directed vector space is necessarily a directed vector space. For a locally convex ordered vector space E such that the neighborhood system of every point has a countable base to be a locally convex directed vector space, it is necessary and sufficient that this space E have the following property:

1) if $x_n \in E$ $(n = 1, 2, \ldots)$ and $x_n \to 0$, it is possible to write

$$x_n = u_n - v_n$$

where.

$$u_n, v_n \in E, \quad u_n, v_n \geq 0 \quad (n = 1, 2, \ldots)$$

and where $u_n \to 0$ and $v_n \to 0$.

PROOF. We establish the first part of the proposition. Let E be the space under consideration. Since E is a neighborhood of 0, the definition of a locally convex directed vector space shows that

$$E \cap P - E \cap P = P - P$$

must also be a neighborhood of 0. The only vector subspace of a topological vector space which is a neighborhood of the origin is the space itself. Thus $P - P = E$ which means that E is a directed vector space as we wished to show.

We now prove the second part. We designate by A_n $(n = 1, 2, \ldots)$ a countable base of the neighborhood system of 0. Without loss of generality, we can assume that $A_n \supset A_{n+1}$ $(n \geq 1)$. To start out, we assume that the locally convex ordered vector space E is a locally convex directed vector space. We consider an arbitrary sequence of points $x_n \in E$ $(n = 1, 2, \ldots)$ such that $x_n \to 0$. By induction, we can determine integers $1 \leq n_1 < \ldots < n_i < \ldots$ such that

$$x_n \in A_i \cap P - A_i \cap P \text{ if } n > n_i \quad (i = 1, 2, \ldots) ,$$

since $x_n \to 0$ and $A_i \cap P - A_i \cap P$ is a neighborhood of 0. By virtue of the first part of the present proposition,

$$x_n \in E = P - P \quad \text{if} \quad 1 \leq n \leq n_1$$

so that we can write

$$x_n = u_n - v_n \quad \text{where} \quad u_n, v_n \geq 0 \quad (1 \leq n \leq n_1)$$

Let us assume the points u_n and v_n already to have been defined for $1 \leq n \leq n_i$. Since

$$x_n \in A_i \cap P - A_i \cap P \quad \text{for} \quad n_i < n \leq n_{i+1} \quad ,$$

we can write

$$x_n = u_n - v_n \quad \text{where} \quad u_n, v_n \in A_i \cap P \quad (n_i < n \leq n_{i+1})$$

It is then clear that one can, by induction, construct two sequences u_n, $v_n \in E$ $(n = 1, 2, \ldots)$ such that

$$x_n = u_n - v_n \ , \quad u_n, v_n \geq 0 \quad (n = 1, 2, \ldots)$$

and $u_n, v_n \in A_i$ for $n > n_i$ $(i = 1, 2, \ldots)$; we thus have $u_n \to 0$, $v_n \to 0$, meaning that condition 1) of the proposition is satisfied.

Conversely, let us assume that the locally convex ordered vector space E is not a locally convex directed vector space. Then there exists a neighborhood A' of 0 such that the set $A' \cap P - A' \cap P$ is not a neighborhood of 0. In other words, it is possible to determine a sequence $x_n \in E$ $(n = 1, 2, \ldots)$ such that $x_n \to 0$ and, at the same time,

$$x_n \in E - (A' \cap P - A' \cap P) \quad (n = 1, 2, \ldots) \quad ,$$

where the first minus sign must be interpreted in the sense of set theory. We claim that the sequence $\{x_n\}$ cannot be decomposed in the manner indicated in condition 1) since from $u_n, v_n \to 0$ would result the existence of an integer $N \geq 1$ such that $u_N, v_N \in A'$ from which, in turn, would follow

$$x_N = u_N - v_N \in A' \cap P - A' \cap P \quad ,$$

and this would contradict our hypothesis. The proposition is, thus proved.

3. A *locally convex vector lattice* is a Hausdorff topological vector space E which is, at the same time, a vector lattice satisfying the following conditions:

1) E is a locally convex vector space;

2) $x \vee y$ and $x \wedge y$ are uniformly continuous functions of (x, y).

It would, actually, suffice in condition 2), to assume the uniform continuity of one of the two expressions indicated; for example, the uniform continuity of $x \vee y$ combined with the formula

$$x \wedge y = -[(-x) \vee (-y)]$$

(see Birkhoff, 1, p. 215) entails the uniform continuity of $x \wedge y$.

PROPOSITION 16. Every locally convex vector lattice is a locally convex directed vector space. In order that a locally convex ordered vector space which is, at the same time, a vector lattice, be a locally convex vector lattice, it is necessary and sufficient that the space be a locally convex directed vector space.

PROOF. Let E be a locally convex vector lattice. Since the infimum is uniformly continuous, there exists, for a given neighborhood A of 0, another neighborhood B of 0 such that

$$x' - x'' \in B, \quad y' - y'' \in B$$

imply

$$x' \wedge y' - x'' \wedge y'' \in A$$

It follows from this that

$$0 \leq x \leq y \in B$$

implies

$$x - x = 0 \in B, \quad y - 0 = y \in B, \quad \text{hence} \quad x \wedge y - x \wedge 0 \in A \quad ;$$

since $x \wedge y = x$ and. $x \wedge 0 = 0$, this means that $x \in A$.

On the other hand, since

$$P = [x; x \in E, x \wedge 0 = 0]$$

and since $x \wedge 0$ is a continuous function of x in the Hausdorff space E, we see that P is closed. This completes the proof that E is a locally convex ordered vector space.

Furthermore, E is a locally convex directed vector space. Indeed, let us consider a neighborhood A of 0. Making use of the continuity of $x \vee 0$ for $x = 0$, we can determine a neighborhood B of 0 such that

$x \in B$ implies $x \vee 0 \in A$. We now assert that

$$B \cap (-B) \subset A \cap P - A \cap P$$

If, namely, x belongs to the first member, that is, if $x \in B$ and $-x \in B$, we have $x \vee 0 \in A$ and $(-x) \vee 0 \in A$, and then making use of the familiar formula

$$x = x \vee 0 - [(-x) \vee 0]$$

(see Birkhoff, 1, p. 219) and of the inequalities $x \vee 0 \geq 0$, $(-x) \vee 0 \geq 0$, we obtain

$$x \in A \cap P - A \cap P \quad ,$$

as we claimed. After that, it suffices to observe that $B \cap (-B)$ is a neighborhood of 0 in order to be able to conclude that E is a locally convex directed vector space.

We now establish the second part of the proposition. The necessity of the condition there stated results from the first part. It remains then to prove its sufficiency. From the fact that P is closed, it follows that $P \cap (-P)$ is also closed. Now this intersection reduces to 0 and every topological vector space in which the subset reducing to 0 is closed is necessarily a Hausdorff space. Thus, E is a Hausdorff space.

We now verify that $x_+ = x \vee 0$ is a continuous function of x for $x = 0$. Given a neighborhood A of 0, let B be another such that

$$0 \leq x \leq y \in B \text{ implies } x \in A \quad ;$$

this is possible because of the hypothesis made concerning E. From the fact that E is a locally convex directed vector space it follows that the set C defined by

$$C = B \cap P - B \cap P$$

is also a neighborhood of 0. Now $x \in C$ signifies that x may be written in the form $x = u - v$ where $u, v \geq 0$ and $u, v \in B$. From $u = x + v \geq x$ and $u \geq 0$ it follows that $u \geq x_+$. Thus $0 \leq x_+ \leq u \in B$ furnishes $x_+ \in A$. In short, $x \in C$ implies $x_+ \in A$ which establishes the continuity of x_+ for $x = 0$.

In an analogous manner, we verify that $x_- = (-x) \vee 0$ is also a continuous function of x for $x = 0$. Since

$$|x| = x_+ + x_-$$

(see Birkhoff, 1, p. 220; we call attention to the slight difference between Birkhoff's notation and that adopted by us), $|x|$ is a continuous function of x for $x = 0$.

We finally prove that the supremum is a uniformly continuous function. Let A be an arbitrary neighborhood of 0. We determine a neighborhood B of 0 such that $B + B \subset A$; this is possible because E is a topological vector space. Then, making use of the continuity of $|x|$ for $x = 0$, we determine a neighborhood C of 0 such that $x \in C$ implies $|x| \in B$. If now we have $x' - x'' \in C$ and $y' - y'' \in C$, we also have

$$\begin{aligned} x' \vee y' - x'' \vee y'' &\leq |x' \vee y' - x'' \vee y''| \\ &\leq |x' - x''| + |y' - y''| \in B + B \subset A \end{aligned}$$

(as follows from the repeated application of formula (20) of Birkhoff, 1, p. 220), and this establishes the uniform continuity of the supremum. The proposition is, therefore, proved.

The most important example of a locally convex vector lattice is furnished by a *normed vector lattice*. This is a vector lattice which is, at the same time, a normed vector space such that

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\| \quad .$$

(The proof that a normed vector lattice is a locally convex vector lattice is found in Birkhoff, 1, p. 247.

In particular, normed vector lattices which are complete in the sense of Cauchy are known as *Banach lattices*.

§2. Continuity of positive linear transformations

1. We have already seen that every locally convex directed vector space is also a directed vector space (Proposition 15, §1, Chapter III). Examples are known, however, which prove that the converse of this fact is not necessarily true, that is, a locally convex vector space which is, at the same time, a directed vector space need not be a locally convex directed vector space. In any event, this converse is true in the following important case.

THEOREM 11. For a locally convex ordered vector space E to be a locally convex directed vector space, it is sufficient that the following conditions be satisfied:

1) every point of the space E has a countable base for its neighborhood system;

2) E is complete in the sense of Cauchy;

3) E is a directed vector space.

PROOF. For the convenience of expression, we shall denote the topology given on E as the old topology and shall proceed to construct a new topology in the following manner. For every neighborhood W of 0 in the old topology, we define $\widetilde{W}$ by

$$\widetilde{W} = W \cap P - W \cap P$$

We shall show that the collection of sets of the form $\widetilde{W}$ can be taken as a base for the neighborhoods of the origin in a topology on E relative to which E is a Hausdorff topological vector space; that is, we shall show that this collection is the base of a filter which satisfies the conditions $v1, \ldots, v8$ set down in Nachbin, 4, p. 33. We shall limit ourselves to establishing the condition $v8$ since the examination of the other conditions is nothing but a question of routine.

The verification of $v8$ consists in proving that, for every set of the form $\widetilde{W}$ where W is a neighborhood of 0 according to the old topology and for every point $x \in E$, there exists a number $\bar{\xi} \neq 0$ such that $\bar{\xi}x \in W$. Now, by the third condition of the theorem, the point x may be written in the form $x = u - v$ where $u, v \geq 0$. Since $\xi u \to 0$ when $\xi \to 0$, there exists a number $\xi' > 0$ such that $\xi u \in W$ whenever $|\xi| < \xi'$. In an analogous fashion, we establish the existence of a number $\xi'' > 0$ such that $\xi v \in W$ whenever $|\xi| < \xi''$. If now we select $\bar{\xi}$ is such a way that

$$0 < \bar{\xi} < \xi' , \quad 0 < \bar{\xi} < \xi'' ,$$

we see that $\bar{\xi} \neq 0$ and that

$$\bar{\xi}x = \bar{\xi}u - \bar{\xi}v \in W \cap P - W \cap P = \widetilde{W} ,$$

as we desired. The topology so obtained on E shall be called the new topology.

Let us consider a neighborhood W of 0 according to the old topology. Let us determine another neighborhood V of 0 according to the old topology such that $V - V \subset W$. It follows that

$$\tilde{V} = V \cap P - V \cap P \subset V - V \subset W$$

and thus, W is also a neighborhood of 0 according to the new topology. In other words, every subset of E which is open according to the old topology is also open according to the new topology.

If W_n $(n = 1, 2, \ldots)$ designates a countable base for the neighborhoods of 0 according to the old topology, then $\tilde{W}_n$ $(n = 1, 2, \ldots)$ will constitute a countable base for the neighborhoods of 0 according to the new topology. Thus every point has a countable base for its neighborhoods according to the new topology.

We now prove that E is complete in the sense of Cauchy according to the new topology. For this purpose, we select a base V_n $(n = 1, 2, \ldots)$ for the neighborhoods of 0 relative to the old topology such that

(1) $$-V_n = V_n, \quad V_{n+1} + V_{n+1} \subset V_n \quad (n = 1, 2, \ldots) \;;$$

this can, obviously, be done by means of induction. In particular

(1') $$V_{n+1} \subset V_n, \quad \text{or even} \quad \overline{V_{n+1}} \subset V_n \quad (n = 1, 2, \ldots)$$

(where the bar indicates the closure according to the old topology).

We assume for a moment that we have a sequence $x_n \in E$ $(n = 1, 2, \ldots)$ such that

$$x_{n+1} - x_n \in \tilde{V}_n \quad (n = 1, 2, \ldots)$$

Then we can write

$$x_{n+1} - x_n = u_n - v_n \quad \text{where} \quad u_n, v_n \in V_n \cap P$$

for $n = 1, 2, \ldots$. Now, by virtue of (1),

(2) $$u_n + \ldots + u_p \in V_n + \ldots + V_p \subset V_{n-1} \quad (1 < n \leq p) \quad ,$$

and this, since the space is complete in the sense of Cauchy, proves that the series

$$u_1 + u_2 + \ldots + u_n + \ldots$$

is convergent according to the old topology. In a strictly analogous fashion, we establish the convergence of the series

$$v_1 + v_2 + \ldots + v_n + \ldots$$

We set

$$r_n = u_n + u_{n+1} + \cdots$$
$$s_n = v_n + v_{n+1} + \cdots$$

for $n = 1, 2, \ldots$. The fact that P is closed implies that $r_n, s_n \geq 0$. letting $p \to \infty$ in (2), we obtain, by (1'),

$$r_n \in \overline{V_{n-1}} \subset V_{n-2} \quad ,$$

and, analogously, $s_n \in V_{n-2}$, for $n \geq 3$.

On the other hand, we have

$$x_{n+1} - x_n = u_n - v_n \in V_n - V_n = V_n + V_n \subset V_{n-1}$$

for $n = 2, 3, \ldots$ whence, again by virtue of (1),

$$x_p - x_n = (x_p - x_{p-1}) + \cdots + (x_{n+1} - x_n)$$
$$\in V_{p-2} + \cdots + V_{n-1} \subset V_{n-2} \qquad (3 \leq n < p) \quad ;$$

this establishes the convergence of the series

$$(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{n+1} - x_n) + \cdots \quad ,$$

that is, the convergence of the sequence $x_1, x_2, \ldots, x_n, \ldots$ to a point $x \in E$ according to the old topology. Now

$$x_p - x_n = (x_p - x_{p-1}) + \cdots + (x_{n+1} - x_n)$$
$$= (u_{p-1} - v_{p-1}) + \cdots + (u_n - v_n)$$
$$= (u_n + \cdots + u_{p-1}) - (v_n + \cdots + v_{p-1}) \qquad (1 \leq n < p) \ .$$

Letting $p \to \infty$ in this equality, we obtain

$$x - x_n = r_n - s_n \in V_{n-2} \cap P - V_{n-2} \cap P = \tilde{V}_{n-2} \qquad (n = 3, \ldots)$$

which proves that the sequence $x_1, x_2, \ldots, x_n, \ldots$ converges to the point x according to the new topology.

We now consider an arbitrary Cauchy sequence of points of E relative to the new topology. It is clear, by inductive reasoning, that this sequence contains a subsequence x_n $(n = 1, 2, \ldots)$ such that

$$x_{n+1} - x_n \in \tilde{V}_n \qquad (n = 1, 2, \ldots)$$

By what was shown above, this subsequence converges to a certain point of E according to the new topology. Since the given sequence is a Cauchy sequence, it follows that this sequence also converges to the same point according to the new topology. The proof of the completeness of E according to the new topology is, thus, completed.

Keeping in mind, now, that E is a topological vector space which has a countable base for the neighborhood system of every point, that it is complete in the sense of Cauchy, both according to the old and to the new topology, and that every subset which is open according to the old topology is also open according to the new topology, we can apply a fundamental theorem due to Banach in order to conclude that the two topologies are identical. (Cf. Banach, 1, p. 41, Theorem 6; we call attention to the fact that, in the statement of his theorem, Banach assumes a distance to be given for each of the topologies which, as is well known, is not strictly necessary.) Now the equality of the two topologies precisely signifies that the locally convex ordered vector space E is a locally convex directed vector space as we aimed to prove.

2. We now go on to prove the central result of the present chapter. (With respect to Theorems 11 and 12, consult Nachbin, 5 where Theorem 12 is found stated in another, equivalent, form, as a result relative to the concept of the natural topology of an ordered vector space.)

> THEOREM 12. Given two locally convex ordered vector spaces E and F of which the first is assumed to satisfy the following conditions:
>
> 1) every point of E has a countable base for its neighborhoods;
> 2) E is complete in the sense of Cauchy;
> 3) E is a directed vector space.
>
> Then every positive linear transformation T from E to F is continuous.

PROOF. By virtue of the linearity of the transformation T, it suffices to establish its continuity at the origin of the space E. We reason by reductio ad absurdum and assume that T is not continuous at the origin of E. Then there exists a certain neighborhood W_0 of the origin of F and, by condition 1), a sequence $x_1, x_2, \ldots, x_n, \ldots$ of points of E such that

(1) $$x_n \to 0 , \quad T(x_n) \in F - W_0 \qquad (n = 1, 2, \ldots)$$

(where the minus sign must be interpreted in the sense of set theory). Without loss of generality, we may assume that W_0 is convex in the sense of order; in the contrary case, it would be sufficient to apply Proposition 14, §1, Chapter III and to replace W_0 by another neighborhood of the origin of F which is convex in the sense of order and is contained in W_0.

We select numbers $k_n > 0$ $(n = 1, 2, \ldots)$ such that

$$k_n \to \infty, \quad k_n x_n \to 0 \quad ,$$

which is possible by (1) and by the first condition stated in the theorem (cf. Mazur and Orlicz, 1). Since $k_n x_n \to 0$, we can combine Theorem 11 with Proposition 15, §1, Chapter III to determine two sequences $u_1, u_2, \ldots, u_n, \ldots$ and $v_1, v_2, \ldots, v_n, \ldots$ of points of E such that

$$u_n \to 0, \quad v_n \to 0, \quad k_n x_n = u_n - v_n, \quad u_n \geq 0, \quad v_n \geq 0 \quad (n = 1, 2, \ldots).$$

Then, making use of $u_n \to 0$ and $v_n \to 0$ and taking into account conditions 1) and 2) as stated in the theorem, it is possible to determine a sequence of integers $0 < n_1 < n_2 < \ldots < n_i < \ldots$ such that the series

$$u_{n_1} + u_{n_2} + \ldots + u_{n_i} + \ldots \quad ,$$

$$v_{n_1} + v_{n_2} + \ldots + v_{n_i} + \ldots$$

are convergent. We designate by r, respectively s, the sums of these series. It is clear that

$$k_{n_i} x_{n_i} = u_{n_i} - v_{n_i} \leq u_{n_i} \leq r$$

and, analogously, we obtain

$$k_{n_i} x_{n_i} \geq -s$$

whence

$$-\frac{1}{k_{n_i}} s \leq x_{n_i} \leq \frac{1}{k_{n_i}} r$$

for every $i = 1, 2, \ldots$. From this we obtain

$$-\frac{1}{k_{n_i}} T(s) \leq T(x_{n_i}) \leq \frac{1}{k_{n_i}} T(r) \tag{2}$$

since T is linear and positive (and thus increasing). Now $1/k_{n_i} \to 0$ when $i \to \infty$. If, consequently, we choose a sufficiently large value for the integer i, the first and last members of (2) will belong to W_0. It follows from this that the middle member of (2) also belongs to W_0 in contradiction to the last part of (1).

This contradiction results from the assumption that T is not continuous. The theorem is, thus, proved.

We note that Theorems 11 and 12, which have just been proved, remain true if, instead of assuming that E satisfies the conditions 1) and 2) there stated (that is, assuming that E is an ($\mathfrak{F}$) space according to the terminology adopted by Dieudonné and Schwartz), we assume that E is an ($\mathfrak{L}\,\mathfrak{F}$) space since the proofs given above may, on the basis of a recent paper by Dieudonné and Schwartz (cf. Dieudonné and Schwartz, 2), be generalized to this case.

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APPENDIX

The following text, in some instances, differs slightly in terminology and notation from the preceding monograph.

§1. On topological ordered spaces[1]

1. Let E be a topological space equipped with a preorder.[2] The preorder is said to be: (s-cl) *semiclosed* if, given any $a \in E$, the sets of the elements $x \leq a$ and of the elements $x \geq a$ are closed; (cl) *closed* if its graph, the set of the points (x, y) such that $x \leq y$, is closed in the square E^2. The topology is said to be: (l-c) *locally convex* if the set of convex neighborhoods of each point is a base for the neighborhood system of the point; (w-c) *weakly convex* if the set of open convex subsets is a base of the topology; (c) *convex* if the set consisting of the open decreasing and the open increasing subsets is a system of generators (subbase) of the topology. By means of the concept of Moore-Smith convergence,[3] we can also say that the preorder of E is of the type: (s-cl) if $x_\lambda \to a$, $x_\lambda \leq b$ imply $a \leq b$ and if $x_\lambda \to a$, $x_\lambda \geq b$ imply $a \geq b$; (cl) if $x_\lambda \to a$, $y_\lambda \to b$, and $x_\lambda \leq y_\lambda$ imply $a \leq b$; (l-c) if $x_\lambda \to a$, $z_\lambda \to a$, and $x_\lambda \leq y_\lambda \leq z_\lambda$ imply $y_\lambda \to a$. We have: (cl) $\to$ (s-cl), (c) $\to$ (w-c) $\to$ (l-c). A topological space equipped with a semiclosed (closed) order

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[1] Translated from the Comptes Rendus Acad. Sci. Paris, vol. 226 (1948), p. 381-382.

[2] We use the terminology of "Éléments de Mathématiques" by N. Bourbaki (Act. Sci. Ind. Nos. 846, 858, 916, Paris, 1939-1942). A *preorder* is a *reflexive* and *transitive* relation. A subset P is said to be *decreasing* if $a \leq b \in P \to a \in P$; a corresponding definition holds for *increasing* subsets; P is said to be *convex* if $a \leq b \leq c$, $a \in P$, $c \in P \to b \in P$.

[3] See G. Birkhoff, Moore-Smith convergence in general topology, Ann. of Math., vol. 38 (1937), p. 39-56.

is a Frechet (Hausdorff) space; a topological space which is locally convex with respect to a semiclosed (closed) preorder is a Frechet (Hausdorff) prespace;[4] a preordered space equipped with a locally convex Kolmogoroff topology is ordered. One designates as a compact ordered space a compact space equipped with a closed order.

(1) The topology of a compact ordered space is convex.

(2) If each point of a topological space equipped with a closed order has at least one convex neighborhood the closure of which is compact, then the topology is weakly convex.

(3) In order that the topology of a nondiscrete metric space be locally convex with respect to an arbitrary closed order on the space, it is necessary and sufficient that the space be compact.

2. We consider the following conceivable properties of an ordered set. *Interpolatory property* (i-p): if $a_i \leq b_j$ (i, j = 1, 2), there exists a ζ such that $a_i \leq \zeta \leq b_j$ (i, j = 1, 2); *conditional upper lattice* (c-u-l): every nonempty finite subset which is bounded from above has a supremum; *conditional lower lattice* (c-l-l): the dual of (c-u-l); *ascending chain condition* (a-c-c): every nonempty totally ordered subset (chain) bounded from above has a supremum; *decreasing chain condition* (d-c-c): the dual of (a-c-c); *conditional complete lattice* (c-c-l): every nonempty subset which is bounded from above, respectively from below, has a supremum, respectively an infimum. We have: (i-p) + (d-c-c) → (c-u-l) (and its dual); (c-u-l) + (a-c-c) → (c-c-l) (and its dual); (i-p) + (d-c-c) + (a-c-c) → (c-c-l). Consequently,[5]

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[4] A topological space the topology of which is the inverse image of the topology of a Fréchet (Hausdorff) space is called a *Fréchet (Hausdorff) prespace*.

[5] This theorem contains a result of J. Dieudonné, Sur le théorème de Lebesgue-Nikodym (II), Bull. Soc. Math. France, vol. 72 (1944), p. 193-239 (see the corollary p. 198); and also a result of G. Birkhoff, Lattice Theory, Amer. Math. Soc. Coll. Publ., New York (1940), see the corollary p. 118.

(4) Let E be a topological space equipped with a semiclosed order such that 1) E has the interpolatory property, and 2) every closed totally ordered subset which is bounded (in the sense of the order) is compact. Then E is a conditional complete lattice.

Remark. Condition 2) results from 2*): every segment (understood in the sense of the order) is compact.

3. A topological space E equipped with a preorder (order) is said to be *normally preordered* (*normally ordered*) if, given a closed decreasing subset F_0 and a closed increasing subset F_1 which are disjoint, one can find an open decreasing subset $A_0 \supset F_0$ and an open increasing subset $A_1 \supset F_1$ which are disjoint. Urysohn's results for the case of a discrete order (that is, $x \leq y$ means $x = y$) admit generalization.

(5) E is normally preordered if and only if, given a closed decreasing subset F_0 and a closed increasing subset F_1 which are disjoint, there exists a real-valued function φ which is continuous and increasing on E such that $x \in F_i \rightarrow \varphi(x) = i$ $(i = 0, 1)$. Every compact ordered space is normally ordered.

If $X, Y \subset E$, we write $X < Y$ when the smallest closed decreasing subset containing X and the smallest closed increasing subset containing Y are disjoint. We write g_λ and s_λ for the sets of numbers $\geq \lambda$ and $\leq \lambda$, respectively.

(6) Given a normally preordered space E, let φ be a bounded real-valued function which is continuous and increasing on a closed subset $F \subset E$. In order that there exist a continuous increasing bounded real-valued extension of φ defined on E, it is necessary and sufficient that $\alpha < \beta \rightarrow \varphi^{-1}(s_\alpha) < \varphi^{-1}(g_\beta)$.

(7) Given a normally preordered space E and a subset $F \subset E$, in order that, for every continuous increasing bounded real-valued function on F, there exist a continuous increasing bounded real-valued extension defined on E, it is sufficient that the holding of the relations $X, Y \subset F$ and $X < Y$ in the space F imply the holding of the relation $X < Y$ in the space E. This condition is satisfied when the preorder is a closed order and F is compact.

§2. On uniformizable ordered spaces[6]

A topological space E equipped with a preorder is said to be a *uniformizable preordered space* if 1) the set of the inverse images of the open subsets of the line under the continuous increasing real-valued functions on E is a system of generators (subbase) of the topology; and if 2), whenever $a \leq b$ is false, there exists such a function, say φ, for which $\varphi(a) > \varphi(b)$; thus the preorder of E is closed and its topology is convex. A compact ordered space K is said to be an *ordered compactification* of a topological space E equipped with an order if 1) E is a dense topological ordered subspace of K; and if 2), among the closed order relations on K for which E is an ordered subspace, the relation considered on K is that which has the smallest graph.[7] Two ordered compactifications K_1 and K_2 of E are said to be *equivalent* if there exists a homeomorphism $\varphi\colon K_1 \rightleftarrows K_2$ such that $\varphi(x) = x$ for $x \in E$; it follows that φ will be an order isomorphism. We write $K_1 \geq K_2$ if there exists a continuous function $\varphi\colon K_1 \to K_2$ such that $\varphi(x) = x$ for $x \in E$; it follows that φ is increasing and that

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6 Translated from the Comptes Rendus Acad. Sci. Paris, vol. 226 (1948), p. 547.

7 It suffices that the closure in K^2 of the graph of the order of E be the graph of the order of K; it must be taken into account that, in the general case, the relation on K the graph of which is this closure may not be transitive.

$\varphi(K_1 - E) = K_2 - E$. Notice that $K_1 \geq K_2$ and $K_2 \geq K_1$ if and only if K_1 is equivalent to K_2. Furthermore, every nonempty family $\{K_\lambda\}$ of ordered compactifications of E has a least upper bound ordered compactification of E, $K = \cup_\lambda K_\lambda$, which is unique up to an equivalence and which is such that $K \geq K_\lambda$ for any λ and that, if $K' \geq K_\lambda$ for any λ, then $K' \geq K$.

(1) A topological ordered subspace of a compact ordered space is a uniformizable ordered space.

(2) Every uniformizable ordered space E has an ordered compactification $\beta(E)$ which[8] is unique up to an equivalence and which has the following equivalent properties: a) $\beta(E) \geq K$ for every ordered compactification K of E; b) every continuous increasing bounded real-valued function on E has an extension of the same nature defined on $\beta(E)$; c) every continuous increasing function on E whose values lie in any compact ordered space has an extension of the same nature defined on $\beta(E)$.

In particular, every compact ordered (respectively uniformizable ordered) space is isomorphic to a closed (respectively arbitrary) topological ordered subspace of a product of closed intervals $[0, 1]$.

§3. On uniform ordered spaces[9]

A semiuniform structure on a set E is a filter f of subsets of E^2 such that: 1) if $V \in f$, then $\Delta \subset V$ where Δ is the diagonal of E^2; and 2) if $V \in f$, there exists a $W \in f$ such that $W \circ W \subset V$. The set f^{-1} of the subsets V^{-1} of E^2, where $V \in f$, is such a structure and is called the *dual* of f. The subsets $V(x)$, where $V \in f$, constitute the filter of the neighborhoods of each point $x \in E$ for a topology $\mathcal{T}^-$ which is called the *lower* topology *generated* by f; the lower topology $\mathcal{T}^+$ generated by f^{-1} is called the *upper* topology *generated* by f. The uniform structure $f \cup f^{-1}$ which is the smallest of all uniform structures on E containing the sets $V \in f$, and the topology $\mathcal{T}$ derived from $f \cup f^{-1}$ are said to be *generated*

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[8] For the case of the discrete order see Tychonoff, Math. Annalen, vol. 102 (1930), p. 544-561; E. Čech. Ann. of Math., vol. 38 (1937), p. 823-844; M. H. Stone, Trans. Amer. Math. Soc., vol. 41 (1937), p. 375-481.

[9] Translated from the Comptes Rendus Acad. Sci. Paris, vol. 226 (1948), p. 774-775.

by f; $\mathfrak{J}$ is the supremum of $\mathfrak{J}^-$ and $\mathfrak{J}^+$. The intersection of the sets $V \in f$ is the graph of a preorder on E which is said to be *generated* by f; this preorder becomes an order if $f \cup f^{-1}$ is separated. A *uniform preordered space* is a space equipped with a uniform structure and with a preorder relation, both being generated by at least one semiuniform structure (there always exists one such structure which is greater than all the others). A *pseudosemimetric* on E is a positive real-valued function δ on E^2 such that $\delta(x, x) = 0$, $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$; if in addition, $\delta(x, y) = \delta(y, x) = 0$ implies $x = y$, then δ is called a *semimetric*. Let (δ_λ) be a nonempty family of pseudosemimetrics on E; the filter generated by the subsets of E^2 defined by means of a finite number of inequalities $\delta_\lambda(x, y) < \varepsilon$ is a semiuniform structure on E. Moreover, every semiuniform structure may be obtained in this manner.

(1) A compact ordered space equipped with its natural uniform structure is a uniform ordered space (the generating semiuniform structure is the filter of neighborhoods in E^2 of the graph of the order and, thus, unique).

(2) In order that a topological space E equipped with a preorder may be equipped with a uniform structure compatible with its topology so that it becomes a uniform preordered space, it is necessary and sufficient that E be a uniformizable preordered space.[10]

If E is a uniform space, the set $\mathfrak{F}(E)$ of the closed subsets of E which is equipped with the uniform structure derived from that of E and which is ordered by set inclusion is a uniform ordered space. Now, let E be a uniform space equipped with an order such that the set I(a) of the points $x \leq a$ is closed for any $a \in E$. The correspondence $a \rightleftarrows I(a)$ is an order isomorphism of E with a subset of $\mathfrak{F}(E)$. From this is derived a *sufficient* condition in order that E be a uniform ordered space, namely,

[10] A. Weil, Act. Sci. Ind. No. 531, Paris, 1938, for the discrete case.

the uniform continuity in both senses of $a \rightleftarrows I(a)$ (even in the case of a compact ordered space, $a \to I(a)$ may cease to be continuous); for this uniform continuity to hold, it is necessary and sufficient 1) that, if V is a surrounding on E, that is, an element of the filter in E^2 for the uniform structure of E, there exist a surrounding W such that $x \leq x'$, $y' \in W(x')$ imply the existence of a $y \in V(x)$ with $y \leq y'$; and 2), that for every V there exist a W such that $y \in W(x')$, $y' \in W(x)$, $x \leq x'$, $y \leq y'$ imply $y \in V(x)$. This criterion gives us the following theorem.

(3) In order that a topological group equipped with a group preorder be a uniformizable preordered space (respectively a uniform preordered space with reference to one of the two natural uniform structures of the topological group), it is necessary and sufficient that the set of the elements $x \geq e$ be closed and that the set of the convex neighborhoods of the identity e be a fundamental system for the neighborhoods of e.

The set $C(\Omega)$ of all continuous real-valued functions on a topological space Ω is a uniform ordered space by virtue of the presence of the pseudosemimetrics $\delta_K(f, g) = \sup\{0, f(x) - g(x)\}$ where x varies in K, $K \subset \Omega$ being an arbitrary compact subset.

(4) Every uniform ordered space E is isomorphic to a uniform ordered subspace a) of $C(\Omega)$ for a locally compact Ω (if the filter of the surroundings of E has a countable base, Ω may be assumed compact); and b) of a product of semimetric spaces.

§4. On the continuity of positive linear transformations[11]

A topological ordered vector space is a locally convex real topological vector space which is also an ordered vector space such

[11] Reproduced from the Proceedings of the International Congress of Mathematicians (1950), vol. 1 (1952), Amer. Math. Soc., Providence, R.I. p. 464-465.

that (1) the cone of all positive elements is closed and (2) given any neighborhood V of 0, there is another neighborhood W of 0 such that $0 \leq x \leq y \in W$ implies $x \in V$. Theorem 1: Let $\mathcal{C}$ be a topological ordered vector space such that (1) $\mathcal{C}$ has a countable base of neighborhoods at 0 and is complete in the Cauchy sense, and (2) every element of $\mathcal{C}$ can be expressed as the difference of two positive elements. Then, for any topological ordered vector space $\mathcal{X}$, every positive linear transformation from $\mathcal{C}$ into $\mathcal{X}$ is continuous. Let A be a completely regular space: if A is complete under the weakest uniform structure with respect to which all continuous real-valued functions on A are uniformly continuous, we say that A is saturated. A regular Hausdorff space in which every open covering contains a countable subcovering is normal saturated. Let $\mathcal{C} = \mathcal{C}(A)$ be the topological ordered vector space of all continuous real-valued functions on A. Theorem 2: Given A, then for every topological ordered vector space $\mathcal{X}$ all positive linear transformations from $\mathcal{C}$ into $\mathcal{X}$ are continuous if and only if A is saturated. Corollary: If A is saturated, $\mathcal{X}$ is normed, and $\Phi: \mathcal{C} \to \mathcal{X}$ is positive linear, there is a compact set $K \subset A$ such that $\Phi(f) = 0$ whenever $f \in \mathcal{C}$ and $f(x) = 0$ for every $x \in K$; if A is not saturated, this may be false even for functionals Φ. If $\mathcal{C}$ is an ordered vector space, then among all the topologies under which $\mathcal{C}$ is a topological ordered vector space, there is a strongest one: call it the natural topology of $\mathcal{C}$. Theorems 1 and 2 can then be restated as follows. If $\mathcal{C}$ is as in Theorem 1, then the natural topology of $\mathcal{C}$ as an ordered vector space is the topology already given on $\mathcal{C}$. If $\mathcal{C}$ is as in Theorem 2, then the natural topology of $\mathcal{C}$ as an ordered vector space is the compact open topology if and only if A is saturated.

§5. Linear continuous functionals positive on the increasing continuous functions[12]

Let us consider a compact ordered space E. Call G the graph of the order relation on E and Δ the diagonal of $E \times E$. If ν is a positive Radon measure[13] of finite total mass $\|\nu\|$ on the locally compact space $G - \Delta$, then the Radon measure μ determined on E by

$$\int_E f(t)d\mu(t) = \int_{G-\Delta} [f(y) - f(x)]\, d\nu(x, y)$$

possesses the property that every increasing continuous real-valued function on E has a positive integral with respect to μ. Moreover,

$$\|\nu\| \geq \tfrac{1}{2} \|\mu\|$$

In this article we present a proof that, conversely, given a Radon measure μ on E with the foregoing property, there is a positive Radon measure ν on $G - \Delta$ which determines μ and has total mass

$$\|\nu\| = \tfrac{1}{2} \|\mu\|$$

Let E be a vector space endowed with a weak topology defined by a separating collection of linear functionals on it. Let $X \subset E$ and call Γ the closed convex cone spanned by X. Then Γ is said to be *perfectly spanned* by X if X is a bounded locally compact subset of E such that corresponding to every $\gamma \in \Gamma$ there is a finite positive Radon measure ν on X for which

$$\gamma = \int_X x\, d\nu(x) \quad ,$$

where the vector-valued integral is understood in the weak topology sense.

With this terminology, the result proved here implies that, in the dual $C^*(E)$ of the Banach space C(E) of the continuous real-valued

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[12] Reproduced from Summa Brasiliensis Mathematicae, vol. 2 (1951), p. 135-150.

[13] We follow Bourbaki's terminology. See the articles by Cartan and Godement listed in the Bibliography. See also the book by Halmos.

functions on E and with respect to the weak topology defined on $C^*(E)$ by C(E), the set of functionals φ_{xy}, where $x < y$, defined by

$$\varphi_{xy}(f) = f(y) - f(x) , \quad f \in C(E)$$

spans perfectly the least closed convex cone containing it.

The author wishes to acknowledge here his great indebtedness to Professor J. da Costa Ribeiro, Chairman of the Physics Department, University of Brazil, for the encouraging support to his work.

Compact ordered spaces[14]

1. Consider an ordered set E. The *graph* of its order relation is, by definition, the subset of the Cartesian square $E \times E$ formed by all pairs (x, y) with $x \leq y$. If E, in addition to being an ordered set, is also a topological space, we say that its order relation is *closed* when the graph is a closed subset of the topological space $E \times E$. Notice that a topological space endowed with a closed order relation has to satisfy Hausdorff's separation axiom. Indeed, denoting by G the graph of the order relation and by Δ the diagonal of $E \times E$ we have

$$\Delta = G \cap G^{-1} ,$$

where G^{-1} represents the subset of E^2 which is symmetrical to G with respect to Δ. Therefore Δ is closed, and this is known to be equivalent to the Hausdorff axiom.

By a *compact ordered space* we shall mean a compact space endowed with a closed order relation. When this order relation is taken to be the discrete one, this notion reduces to that of a compact space.

2. Let E be an ordered set. We shall say that a subset $X \subset E$ is *decreasing* if $y \leq x \in X$ implies $y \in X$. In a dual way one defines the notion of an *increasing* subset.

Assume now that E is an ordered set and, at the same time, a topological space. We shall say that E is *normally ordered* if, for any two disjoint closed sets $F_0, F_1 \subset E$, where F_0 is decreasing and F_1 is increasing, there are two disjoint open sets $A_0, A_1 \subset E$, where $A_0 \supset F_0$ is

[14] See the author's note quoted in the Bibliography.

decreasing and $A_1 \supset F_1$ is increasing. This notion reduces to that of a normal space in the case of a discrete order relation.

3. We are now going to show that a compact ordered space is normally ordered. The proof of this fact is slightly less straightforward than the corresponding proof in the discrete order relation case. We shall first establish a few preliminary results.

E being an ordered set and $X \subset E$, we shall denote by $d(X)$ the *decreasing set spanned* by X, that is, the least decreasing set containing X. It is clear that $d(X)$ is the set of all $y \in E$ such that $y \leq x$ for some $x \in X$. Dually one defines the increasing set $i(X)$ spanned by X.

> LEMMA 1. Let E be an ordered set which is also a topological space. The order relation is closed if and only if, for any two points $a, b \in E$ such that $a \leq b$ is false, there is an increasing neighborhood V of a disjoint from some decreasing neighborhood W of b.

PROOF. Assume the order relation closed and let G be its graph. If $a \leq b$ is false, that is, if $(a, b) \in G$ is false, we can find a neighborhood V_1 of a and a neighborhood W_1 of b such that

$$(V_1 \times W_1) \cap G = 0 \quad .$$

In other words, $x \in V_1$ and $y \in W_1$ imply that $x \leq y$ is false. Setting $V = i(V_1)$ and $W = d(W_1)$, we obtain the desired neighborhoods.

Conversely, in case G is not closed, there exists a point (a, b) belonging to the closure of G but not to G itself. It is then clear that

$$(V \times W) \cap G \neq 0$$

from which it follows that $V \cap W \neq 0$ for any increasing neighborhood V of a and any decreasing neighborhood W of b.

> LEMMA 2. Let E be a topological space endowed with a closed order relation. If $K \subset E$ is compact, the decreasing set $d(K)$ spanned by K is closed.

PROOF. Let us consider a point $a \in E - d(K)$. Then $a \leq x$ is false for any point $x \in X$. Using the preceding lemma, we see that, corresponding to any $x \in X$, we can obtain an increasing neighborhood V_x of a

disjoint from some decreasing neighborhood W_x of x. By the compactness of K, a finite number of the neighborhoods W_x, $x \in K$, cover K. Let the set V be defined as the intersection of the corresponding finite number of neighborhoods V_x. It is clear that V is an increasing neighborhood of a disjoint from K and, therefore, from $d(K)$. This completes the proof that the set $d(K)$ is closed.

We notice that there is a dual statement to the one just proved.

LEMMA 3. Let E be a compact ordered space. If $F \subset E$ is a decreasing set and V is a neighborhood of F, there is an open decreasing neighborhood W of F contained in V.

PROOF. Putting

$$W = E - i(\overline{E-V}) \quad ,$$

we obtain a decreasing set. We claim that it is open. This follows immediately from the fact that $\overline{E - V}$ is closed, hence compact, and from the dual to the preceding lemma. Now

$$i(\overline{E-V}) \supset \overline{E - V} \supset E - V$$

and therefore $W \subset V$. We say also that $F \subset W$, that is,

$$F \cap i(\overline{E-V}) = 0$$

Indeed, let us assume the existence of a point.

$$t \in F \ , \quad t \in i(\overline{E-V})$$

The second condition means that there exists some $x \in \overline{E - V}$ such that $x \leq t$. From $t \in F$ and $x \leq t$ we get $x \in F$ which contradicts $x \in \overline{E - V}$ because V is a neighborhood of the set F.

Notice again that there is a dual statement to the one just established.

We have now all that is needed to prove the following result.

THEOREM 1. Every compact ordered space is normally ordered.

PROOF. Call the given space E. To begin with, consider two points $a, b \in E$ such that $a \geq b$ is false. We claim that it is possible to determine two disjoint open sets V and W, where V is decreasing and contains a and W is increasing and contains b. Now the sets $d(a)$ and $i(b)$ are disjoint and closed (by Lemma 2 and its dual). Since E is a normal topological space we can find two disjoint open sets such that

$$V_1 \supset d(a) \ , \quad W_1 \supset i(b)$$

Using the fact that $d(a)$ is decreasing and applying the preceding lemma, we obtain a decreasing open set V such that

$$d(a) \subset V \subset V_1$$

By a dual argument we can get an increasing open set W such that

$$i(b) \subset W \subset W_1 \quad ,$$

and it is then clear that V and W are the desired sets.

Consider now an increasing closed set $F \subset E$ and a point $a \in E$ not belonging to F. Then $a \geq x$ is false for any $x \in F$. By the preceding case, we can find a decreasing open set V_x containing a and disjoint from an increasing open set W_x which contains x. By the compactness of F, a finite number of these sets W_x, $x \in F$, cover F. Call W the union of this finite number of sets W_x and V the intersection of the corresponding finite number of sets U_x. It is then clear that V is a decreasing open set containing a and disjoint from the increasing open set W which contains F.

Finally, if we consider two disjoint closed sets of which one is decreasing and the other increasing and apply the preceding case to the pair formed by an arbitrary point of the decreasing set and the increasing set, a compactness argument leads to the desired ordered normality of every compact ordered space.

4. Our next step consists in observing that Urysohn's separation theorem for normal spaces generalizes to normally ordered spaces.

First of all, we remark that, if E is an ordered set which is also a topological space, then E is normally ordered if and only if, for any decreasing closed set F and any decreasing open set V containing F, there is a decreasing open set V' and a decreasing closed set F' such that

$$F \subset V' \subset F' \subset V$$

We can now prove the following result..

THEOREM 2. Let E be an ordered set which is also a topological space. In order that, for every two disjoint closed subsets $F_0, F_1 \subset E$ where F_0 is decreasing and F_1 is increasing, there should exist an increasing continuous real-valued function f defined on E such that

$$f(x) = i \quad \text{for} \quad x \in F_i \qquad (i = 0, 1) \quad ,$$

it is necessary and sufficient that E be normally ordered.

PROOF. The necessity is trivial. The sufficiency is established along the same lines as in Urysohn's theorem. For this purpose define

$$V(0) = 0 \ , \quad F(0) = F_0 \ , \quad V(1) = E - F_1, \quad F(1) = E \quad .$$

Using ordered normality, we can find a decreasing open set $V(1/2)$ and a decreasing closed set $F(1/2)$ such that

$$V(0) \subset F(0) \subset V(1/2) \subset F(1/2) \subset V(1) \subset F(1) \quad .$$

Again by ordered normality, we can choose two decreasing open sets $V(1/4)$ and $V(3/4)$ and two decreasing closed sets $F(1/4)$ and $F(3/4)$ such that

$$V(0) \subset F(0) \subset V(1/4) \subset F(1/4) \subset V(1/2)$$
$$\subset F(1/2) \subset V(3/4) \subset F(3/4) \subset V(1) \subset F(1)$$

By carrying this interpolation procedure on indefinitely, we succeed in defining a decreasing open set $V(\lambda)$ for every dyadic number λ, $0 \leq \lambda \leq 1$, with the following properties:

1) $\alpha < \beta$ implies $\overline{V(\alpha)} \subset V(\beta)$;

2) $F_0 \subset V(\lambda)$ if $\lambda > 0$;

3) $V(0) = 0$, $V(1) = E - F_1$.

Define f by

$$f(x) = \sup \{\lambda; \ x \in E - V(\lambda)\} \quad .$$

It is well known that f is continuous and has the desired separation property. Moreover f is increasing because each $V(\lambda)$ is decreasing.

All that will be needed in the sequel is the combination of Theorems 1 and 2.

> COROLLARY. If E is a compact ordered space and $a \geq b$ is false, then there is an increasing continuous real-valued function f on E such that $f(a) < f(b)$.

This follows from the fact that $d(a)$ is a decreasing closed set disjoint from the increasing closed set $i(b)$.

5. It is interesting to notice that the study of a compact ordered space is the same thing as the study of a compact space with a "distinguished" class of continuous real-valued functions on it. More exactly

> THEOREM 3. Let E be a compact ordered space. Then the set of all increasing continuous real-valued functions on E is a separating closed semi-vector lattice of continuous real-valued functions on E which contains the constant functions. Conversely, given a compact space E and a set I of continuous real-valued functions on it with the properties just mentioned, there is one and only one way of making E into a compact ordered space so that I becomes the set of all increasing continuous real-valued functions on E.

PROOF. We claim that, if $f \in I$ and $\lambda \geq 0$ is a real number, then $\lambda f \in I$, and $f + g$, $\sup(f, g)$ and $\inf(f, g)$ all belong to I whenever $f, g \in I$. Moreover I is a closed subset of the space of all continuous real-valued functions on E in its usual norm topology. Also there is some $f \in I$ such that $f(a) \neq f(b)$ corresponding to every pair of distinct points $a, b \in E$. Finally all constant functions lie in I. All these statements are clearly true (see the corollary of Theorem 2).

Conversely, assume that I has the indicated properties. From the corollary of Theorem 2 it follows that, if a closed order relation on E is such that the corresponding set of all increasing continuous real-valued functions is equal to I then $a \geq b$ if and only if $f(a) \geq f(b)$ for all $f \in I$. This establishes the uniqueness of the closed order relation on E giving rise to I and suggests the following construction. Define $a \geq b$ for $a, b \in E$ if $f(a) \geq f(b)$ for any $f \in I$. From the fact that I is separating, it follows that we have an order relation on E which is clearly closed. All that remains to be proved is that an increasing continuous real-valued function φ on E belongs to I. For this purpose, in view of the Kakutani-Stone theorem on the structure of closed lattices of continuous functions,[15] we have to exhibit, corresponding to any pair of points $a, b \in E$, a function in I which assumes at these points the same values as φ. If $\varphi(a) = \varphi(b)$ this is clear. Assume $\varphi(a) < \varphi(b)$. Then $a \geq b$ is false; that is, there exists some $f \in I$ such that $f(a) < f(b)$. Choose the real numbers λ and μ so that

$$\lambda f(a) + \mu = \varphi(a) \quad ,$$
$$\lambda f(b) + \mu = \varphi(b) \quad .$$

Since $\lambda > 0$ the function $\lambda f + \mu$ is the member of I with the desired property.

6. We shall now establish two results which, in case the order relation is discrete, reduce to known facts about the interpolation of a continuous function between two semi-continuous functions.[16]

THEOREM 4. Let φ be an upper semi-continuous real-valued function and ψ a lower semi-continuous real-valued function on the compact ordered space E. Assume that

$$\varphi(x) < \psi(x) \quad \text{for} \quad x \in E$$

and that one of these functions is increasing. Then there is an increasing continuous real-valued function f on E such that

$$\varphi(x) < f(x) < \psi(x) \quad \text{for} \quad x \in E \quad .$$

PROOF. Choose a real-valued function m on E such that

$$\varphi(x) < m(x) < \psi(x) \quad \text{for} \quad x \in E \quad .$$

Since ψ is bounded from below on E, we can assume that m is also bounded from below. Let μ be the greatest lower bound of m. Let us consider, for instance, the case in which ψ is increasing. Putting

$$A_x = \{y; \quad \varphi(y) < m(x)\}$$
$$B_x = \{y; \quad \psi(y) > m(x)\} \quad ,$$

[15] See the papers by Kakutani and Stone quoted in the Bibliography.

[16] See the paper by Dieudonné in the Bibliography and, for more complete results, the recent article by Dowker.

we see that B_x is an open increasing set containing x and that A_x is an open set containing x. Since E is a regular space we can, for every x, find a closed neighborhood C_x of x contained in B_x. Call D_x the increasing set spanned by C_x. The dual of Lemma 2 shows that D_x is a closed increasing neighborhood of x contained in B_x. By Theorems 1 and 2 we can choose an increasing continuous real-valued function f_x on E such that

$$f_x(t) \leq m(x) \quad (t \in E) ,$$
$$f_x(t) = \mu \quad (t \in E - B_x) ,$$
$$f_x(t) = m(x) \quad (t \in D_x) .$$

Since $A_x \cap D_x$ is a neighborhood of x, we can select a finite number of points in E whose corresponding neighborhoods cover the space. Define f as the supremum of the finite number of corresponding functions f_x. It is clear that f is continuous and increasing.

Notice that $f_x(t) < \psi(t)$. Indeed, we have

$$f_x(t) = \mu \leq m(t) < \psi(t) \quad (t \in E - B_x) ,$$
$$f_x(t) \leq m(x) < \psi(t) \quad (t \in B_x) .$$

It therefore follows that $f(t) < \psi(t)$ for all $t \in E$.

Moreover, given $t \in E$, if x is chosen so that

$$t \in A_x \cap D_x$$

then

$$f_x(t) = m(x) > \varphi(t) ,$$

showing that $f(t) > \varphi(t)$ for all $t \in E$ as desired.

THEOREM 5. Let φ be an upper semi-continuous real-valued function and ψ a lower semi-continuous real-valued function on the compact ordered space E. Assume that $\varphi \leq \psi$ and that one of these functions is increasing. Then there is an increasing continuous real-valued function f on E such that $\varphi \leq f \leq \psi$.

PROOF. Suppose, for instance, that ψ is increasing. Put

$$\varphi_0 = \varphi - 1 , \quad \psi_0 = \psi + 1$$

and choose an increasing continuous real-valued function f_0 on E so that

$$\varphi_0(t) < f_0(t) < \psi_0(t)$$

for $t \in E$. Assume that we have defined the real-valued functions φ_n, ψ_n and f_n where φ_n is upper semi-continuous, ψ_n is increasing lower semi-continuous, and f_n is increasing continuous, and that

$$\varphi(t) - \frac{1}{2^n} \leq \varphi_n(t) < f_n(t) < \psi_n(t) \leq \psi(t) + \frac{1}{2^n}$$

for any $t \in E$. We then define

$$\varphi_{n+1} = \sup\left(\varphi - \frac{1}{2^{n+1}}, \quad f_n - \frac{1}{2^{n+1}}\right),$$

$$\psi_{n+1} = \inf\left(\psi + \frac{1}{2^{n+1}}, \quad f_n + \frac{1}{2^{n+1}}\right).$$

It is clear that φ_{n+1} is upper semi-continuous, ψ_{n+1} is increasing lower semi-continuous, and

$$\varphi_{n+1}(t) < \psi_{n+1}(t)$$

for $t \in E$. We can therefore find an increasing continuous real-valued function f_{n+1} on E such that

$$\varphi_{n+1}(t) < f_{n+1}(t) < \psi_{n+1}(t)$$

for $t \in E$. Notice also that

$$\varphi(t) - \frac{1}{2^{n+1}} \leq \varphi_{n+1}(t), \quad \psi_{n+1}(t) \leq \psi(t) + \frac{1}{2^{n+1}},$$

for $t \in E$. This shows that we can find a sequence of increasing continuous real-valued functions f_n $(n = 0, 1, \ldots)$ such that

$$\varphi - \frac{1}{2^n} \leq f_n \leq \psi + \frac{1}{2^n}$$

$$\|f_n - f_{n+1}\| \leq \frac{1}{2^{n+1}} \qquad (n = 0, 1, \ldots).$$

Hence $f = \lim f_n$ exists and is an increasing continuous function such that $\varphi \leq f \leq \psi$ as desired.

7. In this section we shall establish a result concerning the approximation of a continuous function by an increasing continuous function. The preceding theorems were considered only to help prove Theorem 6 below. Conversely, it is easy to see how this theorem subsumes the natural corollary to Theorems 1 and 2.

Consider an ordered set E and let f be a real-valued function on it. Then there is on E an increasing real-valued function $F \geq f$ if and only if f is bounded from above on every set $d(x)$, $x \in E$; this condition is satisfied, in particular, if f is bounded from above on E. In this case, among such functions F, there is a least one which we represent by f^* and which is defined by

$$f^*(x) = \sup \{f(y);\ x \geq y\}$$

The dual notion of a greatest increasing function below f is introduced similarly.

LEMMA 4. If E is a compact ordered space and f is an upper semi-continuous real-valued function on E, then f^* exists and is upper semi-continuous.

PROOF. Since f is bounded from above on E, we see that f^* exists. For each real number α, call J_α the set of all real numbers $\lambda \geq \alpha$. A point $x \in E$ satisfies $f^*(x) \geq \alpha$ if and only if, for every $\lambda < \alpha$, there is some $y \in E$ such that $x \geq y$ and $f(y) \geq \lambda$. This establishes

$$f^{*-1}(J_\alpha) = \bigcap_{\lambda<\alpha} i[f^{-1}(J_\lambda)]$$

Since $f^{-1}(J_\lambda)$ is closed, hence compact, the dual to Lemma 2 together with the above equality implies that $f^{*-1}(J_\alpha)$ is closed, as desired.

We can now prove the following theorem.

THEOREM 6. Let E be a compact ordered space, f a continuous real-valued function on E and $\delta \geq 0$ a real number. Then there is an increasing continuous real-valued function F on E such that

$$\|F-f\| \leq \tfrac{1}{2}\delta$$

if and only if $x \leq y$ implies $f(x) \leq f(y) + \delta$.

PROOF. The necessity follows from the fact that

$$\begin{aligned} f(x) &= f(y) + [F(y) - f(y)] + [F(x) - F(y)] + [f(x) - F(x)] \\ &\leq f(y) + \tfrac{1}{2}\delta + 0 + \tfrac{1}{2}\delta = f(y) + \delta \end{aligned}$$

whenever $x \leq y$.

Now we prove the sufficiency. From the assumption that $x \leq y$ implies

$$f(x) \leq f(y) + \delta$$

we see immediately that

$$f^*(y) \leq f(y) + \delta\ , \quad \text{i.e.,} \quad f^* \leq f + \delta\ ,$$

or equivalently,

$$f^* - \tfrac{1}{2}\delta \leq f + \tfrac{1}{2}\delta \quad .$$

Using Lemma 4 and Theorem 5, we can find an increasing continuous real-valued function F on E such that

$$f^* - \tfrac{1}{2}\delta \leq F \leq f + \tfrac{1}{2}\delta \quad ,$$

and since $f \leq f^*$, we conclude that F is the desired function.

Extending positive linear functionals

1. In this section we shall quote two known results concerning the extension of positive linear functionals.

E being an ordered vector space, we shall mean by an ***order unity*** for E an element $e > 0$ of E with the property that corresponding to any $x \in E$ there is a real number $\lambda \geq 0$ such that $-\lambda e \leq x \leq \lambda e$.

THEOREM 7. Let E be an ordered vector space with an order unity e and V a vector subspace of E containing e. Then every positive linear functional on V can be extended to a positive linear functional on E.

COROLLARY. Let E be an ordered vector space with an order unity e and V a vector subspace of E such that $e \leq x$ is false for any $x \in V$ (and, in particular, $e \in V$ is false). Then a positive linear functional μ on V can be extended to a positive linear functional μ on E if and only if, putting

$$\sigma = \sup \{\mu(x); \quad x \in V, \quad x \leq e\} \quad ,$$

we have $\sigma < +\infty$. Moreover, if ρ is a real number, there is such an extension with $\mu(e) = \rho$ if and only if $\rho \geq \sigma$.

Linear continuous functionals positive on the increasing continuous functions

1. We can now prove the following theorem.

THEOREM 8. Let E be a compact ordered space and μ a Radon measure on E with respect to which all increasing continuous real-valued functions on E have a positive integral. Then there is a positive Radon measure ν on the locally compact space $G - \Delta$ (where G is the graph of the order relation and Δ is the diagonal of $E \times E$) of finite total mass $\|\nu\| = \frac{1}{2}\|\mu\|$ such that

$$\int_E f(t)\, d\mu(t) = \int_{G-\Delta} (f(y) - f(x))\, d\nu(x, y)$$

for any continuous real-valued function f on E.

PROOF. Corresponding to every real-valued continuous function f on E let us form the function f^* on G defined by

$$f^*(x, y) = f(y) - f(x) \quad .$$

The mapping $f \to f^*$ is a linear transformation from the space $C(E)$ of all continuous real-valued functions on E onto a vector subspace V of the space $C(G)$ of all continuous real-valued functions on G.

It is clear that $f^* \geq 0$ if and only if f is increasing. In particular, we have $f^* = 0$ if and only if f is both increasing and decreasing, that is, the kernel K of this linear mapping is the collection of all increasing and decreasing real-valued continuous functions on E.

Let μ be the Radon measure on E with the above mentioned property. It follows from this property that $\mu(f) = 0$ whenever $f \in K$, where $\mu(f)$ stands for the integral of f with respect to μ. This implies the existence of a (unique) linear functional μ^* on V which is such that

$$\mu^*(f^*) = \mu(f) \quad \text{for} \quad f \in C(E)$$

and which is positive on V due to the assumtion on μ. In order to apply the corollary of Theorem 7, we note that $1 \leq f^*$ is false for any $f^* \in V$ because every such f^* vanishes on Δ. We proceed now to the evaluation of the number

$$\sigma = \sup\{\mu^*(f^*);\ f^* \in V,\ f^* \leq 1\}$$

First of all, if $f \in C(E)$ and $\|f\| \leq 1$, then

$$f(y) - f(x) \leq 2 \quad ,$$

that is $\frac{1}{2} f^* \leq 1$. It follows that

$$\tfrac{1}{2}\mu(f) = \mu^*\left(\tfrac{1}{2}f^*\right) \leq \sigma \quad ,$$

hence $\frac{1}{2}\|\mu\| \leq \sigma$. On the other hand, let $f^* \in V$ and $f^* \leq 1$. Then $x \leq y$ implies

$$f(y) - f(x) \leq 1 \quad \text{or} \quad (-f)(x) \leq (-f)(y) + 1 \quad .$$

An application of Theorem 6 to the function $-f$ leads to an increasing continuous real-valued function F on E such that

$$\|F + f\| = \|F - (-f)\| \leq \frac{1}{2}$$

and therefore

$$\begin{aligned} \mu^*(f^*) &= \mu(f) = \mu(F+f) - \mu(F) \\ &\leq \mu(F+f) \leq \tfrac{1}{2}\|\mu\| \end{aligned}$$

which proves that $\sigma \leq \frac{1}{2}\|\mu\|$. In conclusion we have

$$\sigma = \tfrac{1}{2}\|\mu\| \quad .$$

By the corollary mentioned, μ^* can be extended to a positive linear functional on $C(G)$, which we again denote by μ^*, such that $\mu^*(1) = \sigma$, and so μ^* becomes a positive Radon measure on the compact space G of total mass σ. Call ν the positive Radon measure induced by μ^* on the locally compact space $G - \Delta$. It is clear (since every f^* vanishes on Δ) that μ is expressed in terms of ν by the equation in the statement of the theorem and so

$$\|\nu\| \geq \tfrac{1}{2}\|\mu\|$$

Moreover it is clear that

$$\|\nu\| \leq \|\mu^*\| = \mu^*(1) = \tfrac{1}{2}\|\mu\| \quad ,$$

and the theorem is proved.

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