PERTURBATION OF SPECTRA IN HILBERT SPACE

By K. O. FRIEDRICHS

LECTURES IN APPLIED MATHEMATICS

Proceedings of the Summer Seminar, Boulder, Colorado, 1960

VOLUME I

LECTURES IN STATISTICAL MECHANICS

By G. Uhlenbeck and G. W. Ford with E. W. Montroll

VOLUME II

MATHEMATICAL PROBLEMS OF RELATIVISTIC PHYSICS

By I. E. Segal with G. W. Mackey

VOLUME III

PERTURBATION OF SPECTRA IN HILBERT SPACE

By K. O. Friedrichs

VOLUME IV

THE GENERAL THEORY OF QUANTIZED FIELDS

By Res Jost

LECTURES IN APPLIED MATHEMATICS

Proceedings of the Summer Seminar, Boulder, Colorado, 1960

VOLUME III

by

KURT O. FRIEDRICHS

Marc Kac, *Editor*
The Rockefeller Institute

Perturbation of Spectra in Hilbert Space

by

KURT O. FRIEDRICHS

COURANT INSTITUTE OF MATHEMATICAL SCIENCES,
NEW YORK UNIVERSITY

1965
AMERICAN MATHEMATICAL SOCIETY
PROVIDENCE, RHODE ISLAND

The Summer Seminar was conducted, and the proceedings prepared in part, by the American Mathematical Society under the following contracts and grants:

Grant NSF-G12432 from the National Science Foundation.

Contract No. AT(30-1)-2482 with the United States Atomic Energy Commission.

Contract Nonr-3081(00) with the Office of Naval Research.

Contract DA-19-020-ORD-5086 with the Office of Ordnance Research.

Library of Congress Catalog Card Number 60-12712

Printed in the United States of America

Foreword

This is the third of a series of four volumes which are to contain the Proceedings of the Summer Seminar on Applied Mathematics, arranged by the American Mathematical Society and held at the University of Colorado for the period July 24 through August 19, 1960. The Seminar was under the sponsorship of the National Science Foundation, Office of Naval Research, Atomic Energy Commission, and the Office of Ordnance Research.

For many years there was an increasing barrier between mathematics and modern physics. The separation of these two fields was regrettable from the point of view of each—physical theories were largely isolated from the newer advances in mathematics, and mathematics itself lacked contact with one of the most stimulating intellectual developments of our times. During recent years, however, mathematicians and physicists have displayed alacrity for mutual exchange. This Seminar was designed to enlarge the much-needed contact which has begun to develop.

The purpose of the Seminar was primarily instructional, with emphasis on basic courses in classical quantum theory, quantum theory of fields and elementary particles, and statistical physics, supplemented by lectures specially planned to complement them. The publication of these volumes is intended to extend the same information presented at the Seminar to a much wider public than was privileged to actually attend, while at the same time serving as a permanent reference for those who did attend.

Following are members of a committee who organized the program of the Seminar:

Kurt O. Friedrichs, Chairman
Mark Kac
Menahem M. Schiffer
George E. Uhlenbeck
Eugene P. Wigner

Local arrangements, including the social and recreational program,

were organized by a committee from the University of Colorado, as follows:

Charles A. Hutchinson
Robert W. Ellingwood

The enduring vitality and enthusiasm of the chairmen, and the cooperation of other members of the university staff, made the stay of the participants extremely pleasant; and the four agencies which supplied financial support, as acknowledged on the copyright page, together with the Admissions Committee, consisting of Bernard Friedman, Wilfred Kaplan, and Kurt O. Friedrichs, Chairman, also contributed immeasurably to the successful execution of the plans for the Seminar.

The Seminar opened with an address given by Professor Mark Kac, Department of Mathematics, Cornell University, on the subject "A Mathematician's Look at Physics: What Sets us Apart and What May Bring us Together." Afternoons were purposely kept free to give participants a chance to engage in informal seminars and discussions among themselves and with the distinguished speakers on the program.

Editorial Committee

V. BARGMANN
G. UHLENBECK
M. KAC, CHAIRMAN

Contents

In Memory
of
Franz Rellich

Preface

The three chapters of this book contain a slightly expanded version of lectures given in 1960 at the Summer Seminar at Boulder. The aim of these lectures was to illuminate some of the puzzling unresolved mathematical questions of the quantum theory of fields, specifically questions arising in the perturbation approach to this theory.

The lectures first gave a short introduction into the well established theory of perturbation of discrete and continuous spectra and the theory of scattering (Chapters I and II). The perturbation problems suggested by field theory were then approached by extending the methods used for continuous spectra (Chapter III).

The presentation of well established perturbation theory in these lectures was rather one-sided and selective. Therefore, appendices are included in the present volume with brief discussions of other approaches and additional material such as Rellich's theory of discrete spectra, Kato's theory of the wave operator, and multiple channel scattering. The third appendix supplements the specific approach to the perturbation problems described in Chapter III.

The treatment of perturbation theory in Chapters I and II and their appendices is essentially rigorous except for some sections in the appendices. The treatment of field problems (Chapter III and Appendix AIII) is essentially formal, though partly described in rigorous terms; but the main problem here is in no way solved. Some explanations seem in order to explain why such a tentative approach to a mathematical problem is presented.

In earlier approaches to the quantum theory of fields the Hamiltonian energy operator was split into a free and an interaction operator and the basic questions were reduced to the problem of finding a unitary transformation of the total (disturbed) into the free (undisturbed) Hamiltonian. This problem differs from that of the theory of single

particles in many ways, in particular in the nature of the spectrum of the energy operator, which allows for the presence of an indefinite number of particles; nevertheless, this problem has many features in common with that of classical theory. It was natural for workers on quantum theory of fields to try to carry over to field problems perturbation methods which were effective for single particle problems; however, severe difficulties appeared. Eventually, it was recognized that these difficulties were partly due to the fact that the disturbed and undisturbed operator did not have the same spectrum; also it was recognized that the latter defect could be remedied by properly adjusting or "renormalizing" certain constants entering free and interaction operators, namely an additive energy constant and the masses of the single particles associated with the field. Moreover, it was seen that by adjusting additional constants some—in quantum electrodynamics even all—of the divergences of the terms of the perturbation series could be made to disappear. The question whether or not the adjusted perturbation series converge was left open and investigation of this question led to strong indications that these series do not converge—except in a few relatively trivial cases. At the same time it became clear that splitting the Hamiltonian in free and interaction operators was merely a mathematical device without direct physical significance and, since these two operators are mathematically not well defined anyway, it was natural to abandon the perturbation approach for the investigation of fundamental matters. Nevertheless, this approach was retained for the treatment of special models and for computational purposes. The definite computational success of the perturbation approach in essential cases strongly suggests that in one way or other perturbation expansions will have an asymptotic validity for the future correct theory.

There are many different features of field theory that produce difficulties in the perturbation approach. We confine ourselves to selecting some of these features while disregarding others. We shall assume that the kernel of the interaction Hamiltonian is a rather smooth function of the pertinent variables; in making this assumption we eliminate the divergences of the terms of the perturbation expansion, but at the same time we forego the possibility of working with a local Lorentz invariant interaction. The interaction kernel shows a certain singularity if the interaction is translation invariant. At first we shall give up even this property and work with what we call totally smooth kernels; but we retain as the basic property of field quantities that they are built up in terms of annihilation and creation operators.

It is due to this property that only one adjustment is needed to align the spectra of disturbed and undisturbed energy operators: it consists in adding a constant to one of these operators. This constant is given by the Goldstone formula. In case of translation invariance two other quantities should be renormalized: the energy of a single particle as a function of its momentum (or simply its mass in a Lorentz invariant theory) and the wave amplitude. Our approach to these problems is based on an explicit elimination of what may be called contributions from disconnected graphs.

Our treatment is strongly influenced by the work of van Hove [**44**] and, more specifically, by that of Hugenholtz [**45**]. Van Hove does not restrict himself to annihilation-creation operators. Our somewhat more explicit results are in principle covered by his. It should be mentioned that also J. Schwartz [**47**] does not restrict himself to annihilation-creation operators and therefore is able to obtain rigorous results.

It would be possible to transform our approach into one that corresponds to Dyson's expansion and renormalization procedure; if this were done, it would appear that our procedure does not lend itself well to describing interaction kernel (or charge) renormalization.

It was not our intention to make any claims as to the suitability of our procedure in handling actual problems in the quantum theory of fields. Our intention was only to show how the perturbation problems of this theory appear when they are approached with the same tools that were effective for classical perturbation of continuous spectra, thus possibly contributing to an understanding of the character of some of the strange features that becloud mathematical field theory. It is hoped that the mathematical considerations advanced in this book, even if they are inconclusive, will nevertheless serve this purpose.

My thanks are due to P. Rejto for his assistance and his many valuable suggestions.

This book is dedicated to my late friend Franz Rellich, who initiated the mathematical theory of perturbations in Hilbert space.

CHAPTER I

The Perturbation Problem. Perturbation of Discrete Spectra

1. Basic notions of spectral theory in Hilbert space. Let me begin these lectures by recalling some of the basic notions of the theory of operators in Hilbert space [1; 6; 8; 9]. The Hilbert space $\mathfrak{H}$ is a linear space whose elements will be called "vectors" and denoted by capital Greek letters. We assume this space to be "complex", so that for any pair of complex numbers c, c_1 the linear combination $c\Phi + c_1\Phi^{(1)}$ of any of its vectors belongs to it.

The inner product (Ψ, Φ) of two of its vectors will be assumed linear in Φ, the second vector; it is then conjugate linear in the first vector since the relation

$$(\Phi, \Psi) = \overline{(\Psi, \Phi)}$$

holds. The unit form (Φ, Φ) is positive unless $\Phi = 0$ and the norm $\|\Phi\| = (\Phi, \Phi)^{1/2}$ satisfies the triangle inequality $\|\Phi + \Psi\| \leqq \|\Phi\| + \|\Psi\|$. The space $\mathfrak{H}$ is complete with respect to this norm.

A (linear) operator A, which assigns to vectors Φ in $\mathfrak{H}$ a vector $A\Phi$ in $\mathfrak{H}$, is called "bounded" if the inequality $\|A\Phi\| \leqq a\|\Phi\|$ holds for all vectors Φ with an appropriate number a. Such a bounded operator is here always assumed to be defined in the whole Hilbert space $\mathfrak{H}$. If the operator A is unbounded, however, it will not be defined in the whole Hilbert space, but only in a dense subspace $\mathfrak{H}_A$, its "domain".

The operator A is called Hermitean if the relation

$$(A\Psi, \Phi) = (\Psi, A\Phi)$$

holds for all vectors Φ, Ψ for which it is defined.

An operator P satisfying the relation

$$P^2 = P$$

is called a "projector" since it projects every vector into a subspace, such that every vector $P\Phi$ of that subspace is projected into itself.

If the projector is Hermitean the projection is "orthogonal" and satisfies the inequality

$$\|P\Phi\| \leqq \|\Phi\|.$$

The major aim of the spectral analysis of an operator A will be achieved if to every function $f(\alpha)$ (of an appropriate class) an operator $f(A)$ has been so assigned that the sum and product of two such operators correspond to the sum and product of the corresponding functions. Thus, the operator A^2 corresponds to the function $f(\alpha) = \alpha^2$ and the operator $\sum_{\nu=0}^{n} c_\nu A^\nu$ corresponds to the polynomial $f(\alpha) = \sum_{\nu=0}^{n} c_\nu \alpha^\nu$.

Among the functions $f(\alpha)$ for which the operator $f(A)$ is defined there should be the unit step function $\eta_{\mathcal{I}}(\alpha)$ of any interval $\mathcal{I}$ on the real α-axis; that is the function defined by

$$\eta_{\mathcal{I}}(\alpha) = \begin{cases} 1 & \text{for } \alpha \text{ in } \mathcal{I}, \\ 0 & \text{for } \alpha \text{ not in } \mathcal{I}. \end{cases}$$

Evidently, this function satisfies the relation $\eta_{\mathcal{I}}^2(\alpha) = \eta_{\mathcal{I}}(\alpha)$. Consequently, the corresponding operator $\eta_{\mathcal{I}}(A)$ is a projector, the "spectral" projector of the operator A. The subspace into which this projector projects will be called the "eigenspace" of A associated with the interval $\mathcal{I}$.

The assignment of spectral projectors to an interval constitutes the "spectral resolution" of the operator provided the projector assigned to the full α-axis is the identity. This notion covers the notions of point-eigenvalue and point-eigenvector. Let the interval $\mathcal{I}$ consist of just one point α_0 and denote the corresponding "step function" by $\eta_0(\alpha)$. Then, the relation

$$(\alpha - \alpha_0)\eta_0(\alpha) \equiv 0$$

holds and hence also the relation $(A - \alpha_0)\eta_0(A)\Phi = 0$ for every vector Φ. In other words, any vector $\Phi_0 = \eta_0(A)\Phi$ in the associated eigenspace satisfies the relation

$$A\Phi_0 = \alpha_0\Phi_0.$$

If there is such a vector $\Phi_0 \neq 0$ it is called a (point)-eigenvector and α_0 is called a (point)-eigenvalue of A.

In general, of course, the spectral analysis of an operator A is not simply completed by enumerating all point-eigenvalues; the spectrum of A may contain more. We say an open interval $\mathcal{I}$ lies outside of the spectrum of A if the spectral projector $\eta_{\mathcal{I}}(A)$ is zero. The "spectrum" of A may then be defined as the complement of its open exterior

formed by all such open sets. This spectrum may, for example, consist of single point-eigenvalues and their limit points; it may contain segments without point-eigenvalues; it may consist just of the infinite α-axis for that matter.

The notion of spectral projector of an operator in Hilbert space allows us to formulate one of the basic tenets of *quantum theory* in a concise manner. According to quantum theory, the "state" (Φ) of a physical object corresponds to a vector Φ in Hilbert space with norm 1. Any observable (A) corresponds to a Hermitean operator A which possesses a spectral resolution.

Suppose the value of the observable (A) is to be measured while the object is in the state (Φ). Then one may ask, what is the probability that the outcome of this measurement will lie in the interval $\mathcal{J}$. The answer to this question as given by quantum theory is that this probability is

$$\|\eta_{\mathcal{J}}(A)\Phi\|^2,$$

the square of the norm of the projection of the "state vector" Φ into the eigenspace of the interval $\mathcal{J}$. Thus the notion of spectral projector enters.

2. The perturbation problem. There are a number of operators whose spectral resolution can be given more or less explicitly. In general, of course, one must resort to approximation procedures. The most frequently used such procedure is the method of perturbation. This method is applicable if the operator to be analyzed spectrally is, in one sense or another, near to an operator whose spectral resolution is known. This operator, the "perturbed" or "disturbed" one, will be denoted by H; the operator with a known spectral resolution, the "undisturbed" one, will be denoted by H_0, the difference, the "disturbing" operator, by V. The operator H may be imbedded in a set of operators

$$H_\varepsilon = H_0 + \varepsilon V$$

depending on a parameter ε. One then tries to expand the spectral projectors $P_\varepsilon = \eta_{\mathcal{J}}(H_\varepsilon)$ in powers of ε hoping that this expansion converges. This hope will be fulfilled only under favorable circumstances. For, the spectrum of an operator is rather sensitive even to the slightest changes of the operator. Point-eigenvalues may split or coalesce; a continuous part of the spectrum may absorb or shed a point-eigenvalue; a continuous part of the spectrum might even instantaneously become a pure point-spectrum. However, if the disturbance V is sufficiently "gentle" the changes of the spectrum induced by it are also mild and can be described.

We shall first formulate the simple classical problem of disturbing a single isolated point-eigenvalue and describe the classical power series expansion of disturbed eigenvalues and eigenvectors. The validity of this formal procedure was first established by Rellich in a series of fundamental papers [2]. We shall justify this procedure in different ways, one of which involves the more general problem of the perturbation of an isolated segment of the spectrum. We shall show that such a segment and its projector vary even analytically for a while before the other parts of the spectrum start to interfere.

In Chapter II we shall first treat the perturbation of a pure continuous spectrum and describe the process of scattering associated with it. Here a new machinery must be developed, based on the notion of spectral representation.

Next we shall consider a case in which the spectrum of the undisturbed operator consists of a continuous section and point-eigenvalue. This case has the same mathematical structure as the "Lee model" allowing one to describe some of the peculiarities observed in the quantum theory of fields, such as the "cloud" phenomenon and mass renormalization.

This problem will lead right over to perturbation problems of the type occurring in the quantum theory of fields, treated in Chapter III.

3. Perturbation of an isolated point-eigenvalue. Let us first describe the classical procedure for the perturbation of a single point-eigenvalue. Accordingly, we assume that the undisturbed operator H_0 is Hermitean, possesses a spectral resolution, and has a single eigenvalue ω_0 with an eigenvector $X_0 \neq 0$. We assume this eigenvalue to be isolated so that the equation $(H_0 - \omega_0)X = \Psi$ has a solution X whenever the given right member Ψ is orthogonal to X_0.

Suppose the disturbed operator $H = H_0 + \varepsilon V$ has a point-eigenvalue ω_ε which, together with an eigenvector X_ε, depends analytically on ε. Then expansions

$$\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots,$$
$$X_\varepsilon = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots$$

are valid for sufficiently small values of ε. Insertion into the equation

$$(H_0 + \varepsilon V)X_\varepsilon = \omega_\varepsilon X_\varepsilon$$

yields the sequence of equations

$$(*) \qquad \begin{aligned} (H_0 - \omega_0)X_0 &= 0, \\ (H_0 - \omega_0)X_1 &= -(V - \omega_1)X_0, \\ (H_0 - \omega_0)X_2 &= -(V - \omega_1)X_1 + \omega_2 X_0, \\ &\cdots. \end{aligned}$$

Of course, the vector X_ε is not uniquely determined by these equations since it may be multiplied by any function of ε. Since $X_0 \neq 0$ this factor could be so chosen that the norm $\|X_\varepsilon\|$ of X_ε is 1 for sufficiently small ε. Instead, we prefer to impose the linear condition

$$(X_0, X_\varepsilon) = 1,$$

which implies

$$(**) \qquad (X_0, X_1) = (X_0, X_2) = \cdots = 0.$$

The sequence of equations (*) can be solved successively for $X_0, X_1, \cdots$ (the first one is satisfied by definition) provided the right-hand sides are orthogonal to X_0. This latter condition together with (**) yields the values

$$\omega_1 = (X_0, VX_0), \quad \omega_2 = (X_0, VX_1),$$

of the expansion coefficients of the eigenvalue ω_ε. The solutions $X_1, X_2, \cdots$ are made unique by the requirements $(X_0, X_1) = (X_0, X_2) = \cdots = 0$. This is essentially the procedure of Schrödinger.

The question arises naturally: Do the series obtained in this manner converge? This convergence could be proved directly under simple assumptions on H_0 and V; but it can more effectively be established in an indirect manner as follows.

One first establishes the existence of the solution $X_\varepsilon, \omega_\varepsilon$ for sufficiently small $|\varepsilon|$ and then proves the continuous differentiability of these quantities with respect to ε. This should be done for complex values of ε and it should be shown that the derivatives are independent of the direction in which they are taken. Then it follows that X_ε and ω_ε are analytic functions of ε and hence these functions possess convergent power series expansions.

Note that we cannot require the disturbing operator εV to be Hermitean if we allow ε to be complex. In fact, the Hermitean character of the operator V will not be used in most of our considerations. On occasion, though, we shall point out specific consequences that result if V is Hermitean.

These remarks are made to support the attitude we are going to take, namely, that establishing the existence of the perturbed eigenvectors and eigenvalues is more urgent than proving the convergence of series expansions for them.

We shall show how to establish the existence of perturbed eigenvalues in two different ways.

The first method, called the "explicit" method, is based on the relation

$$\omega_\varepsilon - \omega_0 = \varepsilon(X_0, VX_\varepsilon)$$

which follows from the relation $(\omega_\varepsilon - \omega_0)X_\varepsilon = (H_0 - \omega_0)X_\varepsilon + \varepsilon V X_\varepsilon$ together with $(X_0, X_\varepsilon) = 1$ and $(X_0, (H_0 - \omega_0)X_\varepsilon) = 0$. The eigenvalue ω_ε may therefore be eliminated from the equation $\omega_\varepsilon X = H_\varepsilon X$ for $X = X_\varepsilon$, which then attains the form

$$(H_0 - \omega_0)X = -VX + (X_0, VX)X,$$

when we write V in place of εV (here and in the following). Note that this equation, apparently first formulated by C. Bloch, is nonlinear in the unknown vector X. Since ω_0 is an isolated single eigenvalue of H_0, the inverse $(H_0 - \omega_0)^{-1}$ of $H_0 - \omega_0$ can be defined in the subspace of those vectors X which are orthogonal to X_0. To make this inverse unique we stipulate that the vector $(H_0 - \omega_0)^{-1}X$ is also orthogonal to X_0. Now, the right-hand side of our equation has this orthogonality property since $(X_0, X) = 1$ and the same is true of $X - X_0$. Thus we are led to the equation

$$X - X_0 = (H_0 - \omega_0)^{-1}\{-VX + (X_0, VX)X\}.$$

This (nonlinear) equation can be solved by iterations. We shall not carry out the details since they are covered by the treatment of a more general problem discussed in the next section.

While in this explicit method one uses an equation for the eigenvector from which the eigenvalue has been eliminated, one uses the opposite procedure in the "implicit" method: An equation for the eigenvalue is established from which the eigenvector has been eliminated.

The implicit method is similar to a method used by Poincaré for perturbation problems in *Celestial mechanics*.

We introduce the projector P_0 which projects into the eigenspace of the eigenvalue ω_0, that is the space of the multiples of the eigenvector X_0, assumed to have norm 1. P_0 transforms every vector Φ into the vector

$$P_0\Phi = (X_0, \Phi)X_0.$$

Since the eigenvalue ω_0 is assumed to be isolated the operator

$$H_0 - \omega + cP_0$$

possesses an inverse $[H_0 - \omega + cP_0]^{-1}$ for $c \neq \omega - \omega_0$ and $|\omega - \omega_0|$ sufficiently small. If V is bounded and if its bound $\|V\|$ is sufficiently small, also the operator $H - \omega + cP_0$ possesses an inverse for every value of ω sufficiently close to ω_0. For, since the operator $H - \omega + cP_0$ has the same domain as H_0 it can be written as $[H_0 - \omega + cP_0]$

$[1 + (H_0 - \omega + cP_0)^{-1}V]$ and hence its inverse can simply be described as

$$[1 + (H_0 - \omega + cP_0)^{-1}V]^{-1}[H_0 - \omega + cP_0]^{-1},$$

provided $\|V\|$ is small enough. Here, as before, we have suppressed the subscript ε.

Let now X be an eigenvector of H satisfying the condition $(X_0, X) = 1$. Note that

$$P_0X = X_0$$

by virtue of this condition. If ω is the associated eigenvalue, the vector X satisfies the equation

$$(H - \omega + cP_0)X = cX_0$$

and hence, if ω is sufficiently close to ω_0, the equation

$$X = c[H - \omega + cP_0]^{-1}X_0.$$

Multiplying by X_0 we find the relation

$$1 = c(X_0, [H - \omega + cP_0]^{-1}X_0)$$

which contains only the unknown ω and not the unknown X; we regard it as an *equation for the eigenvalue* ω.

It is immediately seen that this equation has a solution for sufficiently small values of ε provided $c \neq 0$. This follows from the implicit function theorem since the derivative of the right-hand side with respect to ω does not vanish for $\varepsilon = 0$. Indeed, this derivative is evidently

$$c(X_0, [H_0 - \omega_0 + cP_0]^{-2}X_0) = c(X_0, c^{-2}X_0) = c^{-1} \neq 0.$$

Having found the eigenvalue ω we find the eigenvector X from the preceding relation and verify that it satisfies the equation $(H - \omega)X = 0$. Using the relation $[H_0 - \omega + cP_0]^{-1}X_0 = [\omega_0 - \omega + c]^{-1}X_0$, the equation for ω can be written as

$$1 = \frac{c}{c - \omega + \omega_0}(X_0, [1 + (H_0 - \omega + cP_0)^{-1}V]^{-1}X_0);$$

it then can be brought into the somewhat more manageable form

$$\omega = \omega_0 + \frac{c}{c - \omega + \omega_0}(X_0, V[1 + (H_0 - \omega + cP_0)^{-1}V]^{-1}X_0),$$

after multiplying it by $c - \omega + \omega_0$ and subtracting

$$c = c(X_0, \{1 + (\omega_0 - \omega + c)^{-1}V\}[1 + (H_0 - \omega + cP_0)^{-1}V]^{-1}X_0)$$

from both sides. In this form, iterations or power series expansion with respect to V could be carried out directly.

In the implicit approach as described the factor c is still arbitrary. For concrete evaluations it is in general opportune to employ the formulas which result when one lets c tend to infinity. From the relations

$$[H_0 - \omega + cP_0]^{-1}P_0 = [\omega_0 - \omega + c]^{-1}P_0$$

and

$$[H_0 - \omega + cP_0]^{-1}(1 - P_0) = [H_0 - \omega]^{-1}(1 - P_0)$$

we deduce the relation

$$[H_0 - \omega + cP_0]^{-1} \to [H_0 - \omega]^{-1}(1 - P_0) \quad \text{as } c \to \infty;$$

hence, we obtain in the limit the equation

$$1 = (X_0, [1 + [H_0 - \omega]^{-1}(1 - P_0)V]^{-1}X_0)$$

or the more manageable form

$$\omega = \omega_0 + (X_0, V[1 + [H_0 - \omega]^{-1}(1 - P_0)V]^{-1}X_0).$$

The last one is the implicit equation of Brillouin and Wigner [**10**; **11**], which is frequently used after expansion with respect to powers of ε (as regards V but not ω).

It may be mentioned (as was observed by B. Zumino) that it is not necessary that the operator P_0 used in the implicit formalism is the projector associated with the first eigenvalue ω_0; it could be any operator of the form

$$P_0\Phi = (\Phi_1, \Phi)\Phi_2,$$

where Φ_1, Φ_2 are any vectors in $\mathfrak{H}$. From the equation

$$(H - \omega + cP_0)X = c(\Phi_1, X)\Phi_2$$

we then conclude

$$X = c(\Phi_1, X)[H - \omega + cP_0]^{-1}\Phi_2,$$

whence, after multiplication by Φ_1,

$$1 = c(\Phi_1, [H - \omega + cP_0]^{-1}\Phi_2).$$

Here we assume $(\Phi_1, X) \neq 0$, which will be satisfied if $(\Phi_1, X_0) \neq 0$ and provided $\omega - \omega_0$ and V are sufficiently small. This modified form of the implicit formalism may sometimes be useful.

4. Perturbation of an isolated part of the spectrum. While in the implicit method one first determines the eigenvalue and then the eigenvector, we proceed in the opposite order in the method to be

described now, which as before, will be called an "explicit" method. We shall describe this method in connection with a problem which is more general than that of the isolated point-eigenvalue, and includes it as a special case.

We assume that the operator H_0 possesses a spectral resolution in the manner described at the beginning and that there is a closed interval $\mathcal{I}_0$ contained in an open interval $\mathcal{I}$ such that the eigenspace $\mathfrak{S}_0$ of $\mathcal{I}$ is the same as that of $\mathcal{I}_0$. In other words, the eigenspaces of the two open intervals composing the difference $\mathcal{I} - \mathcal{I}_0$ should be empty (except for $\Phi = 0$). We may express this condition by $\eta_{\mathcal{I}_0}(H_0) = \eta_{\mathcal{I}}(H_0)$. If it is satisfied we say that $\mathcal{I}_0$ is an isolated segment of the spectrum of H_0.

We want to assign to the operator

$$H_\varepsilon = H_0 + \varepsilon V$$

a subspace $\mathfrak{S}_\varepsilon$ of the Hilbert space which depends continuously on ε and is such that H_ε, when applied to vectors in $\mathfrak{S}_\varepsilon$, produces again vectors in $\mathfrak{S}_\varepsilon$. We maintain that there is such a space $\mathfrak{S}_\varepsilon$ provided V is bounded and $|\varepsilon|$ is small enough; this space $\mathfrak{S}_\varepsilon$, or rather the projector projecting into it, depends analytically on ε and tends to $\mathfrak{S}_0$ as $\varepsilon \to 0$.

We shall find the space $\mathfrak{S} = \mathfrak{S}_\varepsilon$ by finding a projector $P = P_\varepsilon$ that projects into it. (We suppress the subscript ε for the present, assuming it absorbed in V.) The condition that H transforms $\mathfrak{S}$ into $\mathfrak{S}$ can then be expressed by the equation $HP = PHP$, or

$$(*) \qquad (1 - P)HP = 0.$$

We shall establish a solution P of this equation which is near the projector $P_0 = \eta_{\mathcal{I}}(H_0)$ which projects into $\mathfrak{S}_0$, i.e., the projector $P = P_\varepsilon$ will tend to P_0 if ε tends to zero.

Such a projector P can be given by the integral

$$P = \frac{1}{2\pi i} \oint [\lambda - H]^{-1} \, d\lambda,$$

where the loop of integration in the complex plane contains the interval $\mathcal{I}_0$ in its interior and crosses the real axis inside the interval $\mathcal{I}$.

This formula was used by Sz.-Nagy [3] and Kato [4], who derived a perturbation series expansion from it and established the asymptotic character of this series in case the operator H_ε is not assumed to be analytic but possesses a finite or infinite asymptotic expansion with respect to ε.

In this section we shall present a different approach, based on a procedure formulated by C. Bloch a few years ago [7], which is closer than Kato's method to the approach to other perturbation problems discussed in these lectures.

In contrast to the projector given by the integral above, the projector P determined in our approach is not required to be orthogonal; but we do require P_0 to be orthogonal, and thus to be "the" spectral projector of H_0 for the interval $\mathcal{J}_0$. Of P we require that it satisfy the two conditions

$$(**) \qquad PP_0 = P, \qquad P_0P = P_0,$$

the reason being that these conditions are linear, while the condition that P be orthogonal would not be linear. Also, we do not need to impose the (nonlinear) condition $P^2 = P$ explicitly since it is a consequence of the conditions (**): $P^2 = (PP_0)P = P(P_0P) = PP_0 = P$. In other words, the two conditions (**) imply that P is a projector.

We recall that in the case of the perturbation of a simple eigenvalue, treated in the previous section, we did not require the disturbed eigenvector X to have the norm 1. Rather, we required that it satisfy the condition $(X_0, X) = 1$. Introducing the projectors P, P_0 by

$$P_0\Phi = (X_0, \Phi)X_0, \qquad P\Phi = (X_0, \Phi)X,$$

we see that the two conditions (**) are satisfied in that case because of $(X_0, X_0) = (X_0, X) = 1$. We thus realize that the present condition (**) is a natural extension of the previous condition $(X_0, X) = 1$.

We insert $H = H_0 + V$ in our equation (*) and simplify the expression $(1 - P)H_0P$. Using the two conditions (**) and also the fact that P_0, as a spectral projector of H_0, commutes with H_0, we find

$$PH_0P = PP_0H_0P = PH_0P_0P = PH_0P_0 = PP_0H_0 = PH_0.$$

Hence $(1 - P)H_0P = H_0P - PH_0 = H_0(P - P_0) - (P - P_0)H_0$. We therefore may write out equation (*) in the form

$$H_0(P - P_0) = (P - P_0)H_0 - (1 - P)VP.$$

We cannot introduce an inverse of the operator H_0, but we can introduce an operator Z_0 such that

$$Z_0H_0 = H_0Z_0 = 1 - P_0.$$

To this end we may employ the function $\zeta(\omega)$, given by $\zeta(\omega) = \omega^{-1}$ for ω outside $\mathcal{J}$, $\zeta(\omega) = 0$ for ω in $\mathcal{J}$, or simply,

$$\zeta(\omega) = \omega^{-1}[1 - \eta_{\mathcal{J}}(\omega)],$$

assuming, without restriction, that the point $\omega = 0$ lies in the interior of the open interval $\mathcal{J}$. Then we need only set

$$Z_0 = \zeta(H_0).$$

Since $P_0 = \eta_{\mathcal{J}}(H_0)$, and since $\omega\zeta(\omega) = 1 - \eta_{\mathcal{J}}(\omega)$, this operator Z_0 has the desired property. Multiplying our equation for $P - P_0$ by Z_0 we obtain the equation

$$P = P_0 + Z_0(P - P_0)H_0 - Z_0(1 - P)VP = f(P).$$

This equation can be attacked by iterations

$$P_1 = f(P_0), \quad P_2 = f(P_1), \quad \cdots.$$

As will be shown in Appendix A4, these iterations converge to a solution and this solution is unique, provided the operator V (measured by ε) is sufficiently small. At the same time it will be shown that each of the operators $P = P_1, P_2, \cdots$ satisfies the two conditions (**); the same is then true of the limit operator which is thus seen to be a projector. Using these facts, one may verify that the solution P of the present equation satisfies the original equation $(1 - P)HP = 0$. At the same time one may prove that the operator $P = P_\varepsilon$ resulting from $V_\varepsilon = \varepsilon V$ tends to P_0 as $\varepsilon \to 0$. Also one may prove that P_ε has a continuous derivative with respect to ε, independently of the direction; in other words, that P_ε is analytic in ε, regular at $\varepsilon = 0$. We shall not carry out details.

The space $\mathfrak{S}_\varepsilon$ is now defined as the range of the projector P_ε and these projectors P_ε may be used to set up a transformation of the space $\mathfrak{S}_0$ into the space $\mathfrak{S}_\varepsilon$ and vice versa. In fact, the operators

$$U^{\pm} = 1 \mp (P_\varepsilon - P_0)$$

have this property,

$$U_\varepsilon^+ \mathfrak{S}_\varepsilon = \mathfrak{S}_0, \qquad U_\varepsilon^- \mathfrak{S}_0 = \mathfrak{S}_\varepsilon,$$

as immediately verified from conditions (**). The existence of such a pair of transformations evidently implies that *the dimension of* $\mathfrak{S}_\varepsilon$ *is the same as that of* $\mathfrak{S}_0$. Thus it is seen that the space $\mathfrak{S}_\varepsilon$ has this property as we had required.

With the aid of the transformations $U_\varepsilon^{\pm}$ we may form the operator

$$\tilde{H}_\varepsilon = U_\varepsilon^+ H_\varepsilon U_\varepsilon^-$$

which transforms the space $\mathfrak{S}_0$ into itself. Suppose the operator H_ε has a spectral resolution, then the relation

$$f(\tilde{H}_\varepsilon) = U_\varepsilon^+ f(H_\varepsilon) U_\varepsilon^-$$

holds for every function f for which $f(H_\varepsilon)$ is defined (since it holds for polynomials because of $U_\varepsilon^- U_\varepsilon^+ = 1$); thus in particular, the relation

$$\eta_{\mathcal{J}}(\tilde{H}_\varepsilon) = U_\varepsilon^+ \eta_{\mathcal{J}}(H_\varepsilon) U_\varepsilon^-$$

for the spectral projectors of H_ε and $\tilde{H}_\varepsilon$ holds. It provides a one-to-one correspondence of the spectral resolutions of the operators H_ε and $\tilde{H}_\varepsilon$ in the spaces $\mathfrak{S}_\varepsilon$ and $\mathfrak{S}_0$.

In this way the problem of the perturbation of the eigenspace of the segment $\mathcal{J}_0$ for the operator H_0 is reduced to a perturbation problem which takes place solely in the eigenspace $\mathfrak{S}_0$ of this segment for the operator H_0. *What happens to the spectrum of H_ε in $\mathfrak{S}_\varepsilon$ when ε varies is thus determined by finding out what happens with the spectrum of $\tilde{H}_\varepsilon$ in $\mathfrak{S}_0$.*

Suppose the segment $\mathcal{J}_0$ consists of just one point ω_0 and suppose that this eigenvalue of H_0 is simple so that the eigenspace associated with it is one-dimensional, consisting just of the multiples of an eigenvector X_0. Then our statement implies that the space $\mathfrak{S}_\varepsilon$ is also one-dimensional, consisting of the multiples of a vector X_ε. Since H_ε transforms the space $\mathfrak{S}_\varepsilon$ into itself it transforms X_ε into a multiple of X_ε; in other words, X_ε is an eigenvector of H_ε. Thus the existence statement of the preceding section is implied by the present statement.

Suppose the segment $\mathcal{J}_0$ consists again of just one point, but that the eigenspace $\mathfrak{S}_0$ has a finite dimension $r > 1$, so that the eigenvalue is not simple. Then our statement implies that the space $\mathfrak{S}_\varepsilon$ also has the dimension r. The operator H_ε in $\mathfrak{S}_\varepsilon$ or, what is equivalent, the operator $\tilde{H}_\varepsilon$ in $\mathfrak{S}_0$, is an operator acting in a finite-dimensional space. The problem of the perturbation of a multiple eigenvalue is thus reduced to the corresponding problem in a finite-dimensional space.

This perturbation problem, even in a finite-dimensional space, is far from trivial. Its complete mathematical solution was given only as late as 1936 by F. Rellich [2, aI], who proved that the multiple eigenvalue splits into r analytic branches if the Hermitean operator H_ε depends analytically on ε, and that this need not at all be the case if H_ε is not analytic in ε. An account of this theory for the finite-dimensional case is given in Appendix A2.

We shall not describe Rellich's treatment of the perturbation of a multiple eigenvalue in an infinite-dimensional space. Instead we simply observe that this problem can be reduced to that of a perturbation in a finite-dimensional space. Such a reduction, in fact, was established in the present section. It is true that the operator $\tilde{H}_\varepsilon$, into which we have transformed the operator H_ε, is not Hermitean in

general, but this does not matter. Since $\tilde{H}_\varepsilon$ is equivalent to the Hermitean operator H_ε, its eigenvalues are real, and that is sufficient for the validity of Rellich's statement in the finite-dimensional case.

Finally, attention should be called to the fact that perturbations of point spectra have also been treated for operators acting in a Banach space. See [**8**, Vol. I] and references given there.

CHAPTER II

Perturbations of Operators having Continuous Spectra

5. Spectral representation. The perturbation of a continuous spectrum cannot be handled in the same way as the perturbation of a point spectrum, by singling out particular points or particular segments in it for investigation; all parts must be treated simultaneously. This requires a new machinery. Before we can describe such a new machinery it is necessary to formulate the notion of "spectral representation", which will enable us to exhibit spectral resolutions concretely and which will serve as our tool in the treatment of the perturbation of continuous spectra.

A "functional representation" of the vectors Φ in Hilbert space consists in a one-to-one linear assignment of these vectors to functions $\phi(\alpha)$ of a real variable α,

$$\Phi \Leftrightarrow \phi(\alpha),$$

such that the inner product of two vectors Φ, Ψ is given by an integral of the form

$$(\Psi, \Phi) = \int \overline{\psi(\alpha)}\phi(\alpha)\, dm(\alpha)$$

involving a measure function $m(\alpha)$, or by an integral expression of a more general type not described here. The real variable α may run from $-\infty$ to $+\infty$. The functions $\phi(\alpha)$ need not be defined in intervals of m-measure zero; in any case, the values of ϕ in such intervals may be disregarded. More generally, we consider two functions $\phi(\alpha)$ as identical if they differ at most in a set of m-measure zero. Except for this qualification, every vector in the Hilbert space $\mathfrak{H}$ should correspond to exactly one m-measurable function $\phi(\alpha)$ for which

$$\int |\phi(\alpha)|^2\, dm(\alpha) < \infty.$$

Vice versa, every such function should correspond to exactly one vector in $\mathfrak{H}$.

The representation of a vector by a sequence of coordinates

$$\Phi \quad \leftrightarrow \quad \phi_n, \qquad n = 1, 2, 3, \cdots,$$

with

$$(\Psi, \Phi) = \sum_{n=1}^{\infty} \bar{\psi}_n \phi_n,$$

is clearly an example of such a functional representation. Here n takes the place of the variable α and the measure function is piecewise constant with the jumps 1 at the positive integers.

The values of the functions ϕ may be complex numbers, or they may themselves be complex-valued functions of "accessory" variables or, in fact, vectors of an "accessory" Hilbert space. The product $\bar{\psi}\phi$ of the values of two such functions is then itself to be given as the sum or integral with respect to these accessory variables or, in fact, as the inner product in the accessory Hilbert space. If the dimension of the accessory space is greater than 1, we speak of a "multiple" representation; otherwise, i.e., if the value of Φ is just a complex number, we call the representation "simple".

A functional representation is said to give a "spectral representation" of an operator A if for every vector Φ on which A is applicable the vector $A\Phi$ is represented by the function $\alpha(\sigma)\phi(\sigma)$,

$$A\Phi \quad \leftrightarrow \quad \alpha(\sigma)\phi(\sigma),$$

with the aid of a "spectral function" $\alpha(\sigma)$. The representation is called "direct" if $\alpha(\sigma) \equiv \sigma$. In that case we use α instead of σ as spectral variable. Otherwise, the representation will be called "indirect". At present we shall consider only direct spectral representations. If A possesses a simple direct spectral representation we call this operator "simple". A transformation which transforms a representation into a spectral one will be called a "spectral transformation".

Assuming that the unit form is given by the integral

$$(\Phi, \Phi) = \int |\phi(\alpha)|^2 \, dm(\alpha),$$

we can describe the spectrum of the operator A as the support of the measure function $m(\alpha)$; i.e., as the complement of the union of all open intervals in which $m(\alpha)$ is constant.

With the aid of a spectral representation the function $f(A)$ of the operator A is easily described when $f(\alpha)$ is an m-measurable function; it is simply characterized by the correspondence

$$f(A)\Phi \quad \leftrightarrow \quad f(\alpha)\phi(\alpha).$$

This operator $f(A)$ is applicable on those vectors Φ for which $f(\alpha)\phi(\alpha)$ is square m-integrable.

Of particular importance is the unit step function $\eta_{\mathcal{I}}(\alpha)$ for an interval $\mathcal{I}$ on the real α-axis. Since application of the operator $\eta_{\mathcal{I}}(A)$ on the vector Φ corresponds to multiplication of $\phi(\alpha)$ by $\eta_{\mathcal{I}}(\alpha)$ we may write

$$\eta_{\mathcal{I}}(A)\Phi \quad \Leftrightarrow \quad \begin{cases} \phi(\alpha) & \text{for } \alpha \text{ in } \mathcal{I}, \\ 0 & \text{for } \alpha \text{ not in } \mathcal{I}. \end{cases}$$

In other words, application of this operator is simply represented by the process of reducing to zero the values of the representing function $\phi(\alpha)$ outside of the interval $\mathcal{I}$.

The operators $\eta_{\mathcal{I}}(A)$ are the *spectral projectors* of the operator A which we have described in §1. In giving these projectors in terms of a spectral representation *we have given the spectral resolution* of this operator.

6. Perturbation of simple continuous spectra. In treating the perturbation of an operator with a continuous spectrum we assume from the outset that the undisturbed operator H_0 is given in a spectral representation. Specifically, we assume that every vector Φ is represented directly by a function $\psi(\omega)$ of a real variable ω:

$$\Phi \underset{0}{\Leftrightarrow} \psi(\omega),$$

such that application of the operator H_0 is represented through multiplication by ω:

$$H_0\Phi \underset{0}{\Leftrightarrow} \omega\psi(\omega).$$

The functions $\psi(\omega)$ should be defined on the whole ω-axis or a closed subset $\mathcal{I}$ of it. The unit form of Φ is supposed to be given by the integral

$$(\Phi, \Phi) = \int_{\mathcal{I}} |\psi(\omega)|^2 \, d\omega$$

extended[1] over the spectrum $\mathcal{I}$ of H_0.

The values ψ themselves may be functions of accessory variables or vectors of an "accessory" Hilbert space and the statements made in

[1] In place of the Lebesgue measure on $\mathcal{I}$ here implied we could have chosen another measure function $m(\omega)$ with the set $\mathcal{I}$ as support. If this measure function is absolutely continuous we could reintroduce $d\omega$ in place of $dm(\omega)$ simply by taking $(dm/d\omega)^{1/2}\psi(\omega)$ as representer of Φ in place of $\psi(\omega)$. In that case, thus, requiring $dm(\omega)$ to be $d\omega$ on $\mathcal{I}$ is no restriction. But the requirement that $m(\omega)$ be absolutely continuous is an essential restriction. See also Appendix A9.

this section hold under these general circumstances. Still, in giving our arguments we shall not make this fact explicit; rather, we shall use a language which may appear to imply that the representation is simple, that is to say, that the values of the representers ψ are just complex numbers.

We have put the subscript 0 under the representation sign in order to distinguish the spectral representation of H_0 from that of the disturbed operator

$$H = H_0 + V$$

which we intend to determine. Accordingly, we shall speak of the H_0- and the H-representation.

Our aim is to show that for appropriate disturbances V the disturbed operator H admits a spectral representation of the same kind as H_0:

$$\Phi \Leftrightarrow \phi(\omega),$$
$$H\Phi \Leftrightarrow \omega\phi(\omega),$$

involving representers $\phi(\omega)$ which form the same class of functions as the representers $\psi(\omega)$. Clearly, if this is possible, the spectrum of H is the same as that of H_0.

Moreover, in case H is Hermitean, we want the H-representation to be such that the unit form is given by the integral

$$(\Phi, \Phi) = \int_{\mathfrak{s}} |\phi(\omega)|^2 \, d\omega.$$

Our aim will be achieved if we are able to describe a transformation of the H_0-representers ψ into H-representers ϕ and vice versa.

We shall try to effect such a description with the aid of a pair of operators $U^{\pm}$ to be so chosen that the H-representer ϕ of the vector Φ is at the same time the H_0-representer of the vector $U^{+}\Phi$, while the H_0-representer ψ of the vector Φ is the H-representer of $U^{-}\Phi$.

That is, we should like to find an H-representation together with a pair of operators $U^{\pm}$ which should be such that the representation

(+) $\quad U^{+}\Phi \underset{0}{\Leftrightarrow} \phi(\omega)$ is equivalent with $\Phi \Leftrightarrow \phi(\omega)$

and

(−) $\quad U^{-}\Phi \Leftrightarrow \psi(\omega)$ is equivalent with $\Phi \underset{0}{\Leftrightarrow} \psi(\omega)$.

The requirement that the H_0-representation of $U^{+}\Phi$ gives a spectral representation of H evidently means that the H_0-representer $U^{+}H\Phi$ should be the H-representer of $H\Phi$; i.e., it should be the function $\omega\phi(\omega)$:

$$U^{+}H\Phi \underset{0}{\Leftrightarrow} \omega\phi(\omega).$$

On the other hand, the requirement that $\phi(\omega)$ be the H-representer of $U^+\Phi$ implies that the function $\omega\phi(\omega)$ is the H_0-representer of the vector $H_0U^+\Phi$,

$$H_0U^+\Phi \underset{0}{\leftrightarrow} \omega\phi(\omega).$$

Comparison of these two formulas yields the equation

$$(1)^+ \qquad U^+H = H_0U^+.$$

In a similar way one derives from formula $(-)$ the equation

$$(1)^- \qquad HU^- = U^-H_0.$$

Operators $U^\pm$ with this property are said to "intertwine" with H and H_0.

Combining the equivalences $(+)$ and $(-)$ we obtain the relations

$$(2)^+ \qquad U^-U^+ = 1$$

and

$$(2)^- \qquad U^+U^- = 1,$$

which express the fact that U^+ is the inverse of U^- and vice versa.

We now proceed in the reverse order: First we find a pair of operators $U^\pm$ which satisfy equations (1) and (2); then we define the representation $\Phi \leftrightarrow \phi(\omega)$ through formulas $(\pm)$, and finally verify that this *H-representation is a spectral representation of the operator* H.

In order to find operators $U^\pm$ with the desired properties we write equations $(1)^\pm$ in the form

$$[H_0, U^+] = H_0U^+ - U^+H_0 = U^+V,$$
$$[H_0, U^-] = H_0U^- - U^-H_0 = -VU^-,$$

where the notation $[A, B]$ is used for the commutator $AB - BA$ of the two operators A and B.

This form of the equations for $U^\pm$ suggests [**10**; **12**] that we should first solve the equation

$$[H_0, Z] = H_0Z - ZH_0 = R$$

for Z when R is a given operator. To be sure, the solution of this equation is an essential ingredient of the theory of perturbation of continuous spectra to be presented here.

How one can solve this equation for Z will be described later on. For operators R of a certain class we shall introduce a definite solution

$$Z = \Gamma R,$$

which is linear homogeneous in R; any other solution then differs from ΓR by the addition of an operator commuting with H_0, such as a function of H_0.

The operators $U^{\pm}$, being solutions of the equations $[H_0, U^{\pm}] = \pm R^{\pm}$ with

$$(\times) \qquad R^{+} = U^{+}V, \qquad R^{-} = VU^{-},$$

will not be of the form $\pm \Gamma R^{\pm}$, but of the form $1 \pm \Gamma R^{\pm}$, which appears natural since the operators U should reduce to the identity for $V = 0$.

Inserting

$$(3)^{\pm} \qquad U^{\pm} = 1 \pm \Gamma R^{\pm}$$

into the definitions of the operators $R^{\pm}$ just given we find for these operators the equations

$$(4)^{\pm} \qquad R^{+} = (1 + \Gamma R^{+})V, \qquad R^{-} = V(1 - \Gamma R^{-}).$$

Our procedure [**10**; **12**] now consists in first solving these equations $(4)^{\pm}$ and then defining the operators in $U^{\pm}$ by relations $(3)^{\pm}$. Clearly, these operators $U^{\pm}$ will satisfy equations $(1)^{\pm}$. Later on we shall show how to achieve that they also satisfy conditions $(2)^{\pm}$.

Note that equations (4) are linear while those of the perturbation of point-eigenvalues were nonlinear. This is connected with the fact that there is no shift of spectrum in the present perturbation problem.

An example in which these equations can be solved explicitly will be given in Appendix A6.

In order to find solutions $R^{\pm}$ of equations (4) in general we may either employ iterations or series expansion:

$$R^{-} = V - V\Gamma V + V\Gamma(V\Gamma V) - \cdots,$$
$$R^{+} = V + (\Gamma V)V + \cdots.$$

Clearly, such an expansion is possible only if the terms occurring in this series are in the class of operators R for which the operation Γ is defined. For this it is necessary that whenever R is in this class the products $V\Gamma R$ and $(\Gamma R)V$ are also in this class.

This is the most important condition. It is fortunate that classes of operators R can indeed be defined which satisfy this condition. If the operator V belongs to such a class and, in addition, is small enough in an appropriate sense, the two series do indeed converge to limit operators satisfying equations (4), so that the operators $U^{\pm}$ given by (3) satisfy equation (1).

We should like to describe more specifically the (six) properties that an "admissible" class $\mathfrak{R}$ of operators R should have. Operators belonging to any of these classes will be called "gentle".

1. For any operator R in $\mathfrak{R}$ the equation $[H_0, Z] = R$ has a bounded solution $Z = \Gamma R$, homogeneous linear in R. That is, whenever the vector Φ admits the operator H_0, also $(\Gamma R)\Phi$ admits H_0 and $H_0(\Gamma R)\Phi - (\Gamma R)H_0\Phi = R\Phi$.

2. The products $R_1\Gamma R_2$ and $(\Gamma R_1)R_2$ belong to the class $\mathfrak{R}$ if R_1 and R_2 do.

3. It is possible to introduce a norm $[\![R]\!]$ in the class $\mathfrak{R}$ such that the inequalities

$$[\![R_1\Gamma R_2]\!], [\![(\Gamma R_1)R_2]\!] \leqq [\![R_1]\!]\,[\![R_2]\!]$$

hold.

The next property will first be formulated in a stronger form than necessary; it will be weakened a little later on.

$\bar{4}$. The class $\mathfrak{R}$ is complete with respect to the norm $[\![\ ]\!]$; i.e., if $[\![R^n - R^m]\!] \to 0$ as $n, m \to \infty$ for a sequence $\{R^n\}$ in $\mathfrak{R}$, there is an operator R in $\mathfrak{R}$ such that $[\![R^n - R]\!] \to 0$ as $n \to \infty$.

A fifth property will be formulated later on.

On occasion we shall refer to $[\![\ ]\!]$ as a "gentleness norm".

With reference to an admissible class $\mathfrak{R}$ of gentle operators we can state: *Suppose the operator V belongs to the class $\mathfrak{R}$ and has a norm less than* 1:

$$[\![V]\!] < 1.$$

Then equations $(4)^{\pm}$ *have solutions $R^{\pm}$ belonging to the class $\mathfrak{R}$.*

This is quite obvious since the transformations of R into $V\Gamma R$ and $(\Gamma R)V$ satisfy the inequalities

$$[\![V\Gamma R]\!] \leqq [\![V]\!]\,[\![R]\!], \qquad [\![(\Gamma R)V]\!] \leqq [\![V]\!]\,[\![R]\!],$$

and hence are contracting if $[\![V]\!] < 1$, so that iterations converge.

If the completeness condition $\bar{4}$ is not satisfied one can complete the space $\mathfrak{R}$ with reference to the norm $[\![\ ]\!]$ by introducing ideal elements. The compounds $R_1\Gamma R_2$ and $(\Gamma R_1)R_2$ can then be defined for these elements such that the inequalities 3 hold. Clearly, our equations $(4)^{\pm}$ have a solution in the completion $\overline{\mathfrak{R}}$ of $\mathfrak{R}$. To make sure that the operators $R^{\pm}$ belong to the original class $\mathfrak{R}$ if V does we need only require property 4 described as follows:

4. Whenever R_0 is in the space $\mathfrak{R}$ and R in its completion $\overline{\mathfrak{R}}$, the operators $R_0\Gamma R$ and $(\Gamma R)R_0$ belong to $\mathfrak{R}$. (Note that property 4 implies property 2, which therefore is superseded.)

The statement made above thus holds with reference to a class $\mathfrak{R}$ which is admissible in the extended sense characterized by properties 1, 3, and 4.

It should be mentioned that condition $[\![V]\!] < 1$, imposed so that iterations can converge, is not necessary for the existence of operators $R^{\pm}$. Other conditions on V, not implying smallness, have been given under which equations (4) have a solution with the desired properties; see Appendix A11. Property 2 is the most important one.

In Appendix A7 we shall describe two classes $\mathfrak{R}$ of gentle operators R having the properties 1 to $\bar{4}$, or 4. The operators of these classes are given, or rather "represented", by integral operators,

$$R\Phi \underset{0}{\Leftrightarrow} \int r(\omega; \omega')\phi(\omega')\, d\omega',$$

where the variable ω runs from $-\infty$ to ∞, so that the spectral interval $\mathscr{I}$, referred to at the beginning, is the whole ω-axis. In Appendix A9 we shall remove this restriction on the domain of ω; i.e., on the spectrum of H_0.

The "kernels" $r(\omega; \omega')$ of the first one of our classes will satisfy certain Hölder conditions; the kernels of the second one of our classes are Fourier transforms of absolutely integrable functions.

It is an important property of the kernels of both classes $\mathfrak{R}$ that they are continuous functions of $(\omega_1; \omega_2)$; but they are more restricted than that, although they are not required to be differentiable.

When the operator H_0 is disturbed by an operator V which can be represented by an integral operator with a kernel of any of these two special classes we call this disturbance "gentle" in the special sense.

By way of an abbreviation we write the representation of R by the integral operator with the kernel $r(\omega; \omega')$ in the form

$$R \underset{0}{\Leftrightarrow} r(\omega; \omega')$$

and say that R is represented by the kernel $r(\omega; \omega')$.

We proceed to describe the nature of the operation Γ in terms of the kernels r of the operators R on which it acts. Suppose the equation

$$[H_0, Z] = R$$

has a solution Z which is also represented by an integral operator, with the kernel $z(\omega; \omega')$ say,

$$Z\Phi \underset{0}{\Leftrightarrow} \int z(\omega; \omega')\psi(\omega')\, d\omega';$$

then the commutator $[H_0, Z]$ would be represented through

$$[H_0, Z]\Phi \underset{0}{\Leftrightarrow} \int (\omega - \omega')z(\omega; \omega')\psi(\omega')\, d\omega'.$$

Consequently, we would have

$$r(\omega; \omega') = (\omega - \omega')z(\omega; \omega')$$

or

$$z(\omega; \omega') = (\omega - \omega')^{-1}r(\omega; \omega').$$

Thus the kernel $z(\omega; \omega')$ would be singular for $\omega' = \omega$ (except if $r(\omega; \omega')$ vanishes there at least of first order).

The singular integral operator transforming $\psi(\omega)$ into

$$\int (\omega - \omega')^{-1}r(\omega; \omega')\psi(\omega')\, d\omega'$$

can indeed be defined as Cauchy's principal value in case the kernel $r(\omega; \omega')$ belongs to one of the two classes described in Appendix A7; the operator Z represented by this singular integral operator is indeed a solution of the equation $[H_0, Z] = R$.

Actually we shall not define ΓR as the operator represented by this singular integral operator; we rather define ΓR as the operator represented by

$$\Gamma R\Phi \underset{0}{\Leftrightarrow} \int (\omega - \omega')^{-1}r(\omega; \omega')\psi(\omega')\, d\omega' + i\pi r(\omega; \omega)\psi(\omega),$$

which is just as well a solution of the equation

$$[H_0, \Gamma R] = R.$$

Using the symbolic entity

$$[\omega - \omega']^{-1} = (\omega - \omega')^{-1} + i\pi\delta(\omega - \omega')$$

involving Dirac's delta function, we write symbolically

$$\Gamma R\Phi \underset{0}{\Leftrightarrow} \int [\omega - \omega']^{-1}r(\omega; \omega')\psi(\omega')\, d\omega',$$

or simply

$$\Gamma R \underset{0}{\Leftrightarrow} [\omega - \omega']^{-1}r(\omega; \omega'),$$

and call $[\omega - \omega']^{-1}r(\omega; \omega')$ the symbolic kernel of the symbolic integral operator.

The reason for introducing the operator ΓR in this way is connected with the fact that ΓR can be obtained from the operator $\Gamma_\varepsilon R$, represented for $\varepsilon \neq 0$ by the proper integral operator with the nonsingular kernel

$$(\omega - \omega' - i\varepsilon)^{-1}r(\omega; \omega'),$$

simply by letting ε tend to zero through positive real values.

For, it is known that the analytic function

$$\int (\omega - \omega' - z)^{-1} r(\omega; \omega')\psi(\omega')\, d\omega'$$

of $z = x + iy$, defined in the upper half plane (for integrable $r(\omega; \omega')\psi(\omega')$ of bounded support say), approaches the function

$$\int [\omega - \omega' - x]^{-1} r(\omega; \omega')\psi(\omega')\, d\omega'$$

of x as $y \to 0$ (with $y > 0$).

From this fact one can derive an important relation concerning the product $\Gamma_\varepsilon R_1 \Gamma_\varepsilon R_2$ of two operators $\Gamma_\varepsilon R_1$ and $\Gamma_\varepsilon R_2$. Formally, this product is represented by an integral operator with the kernel

$$\int (\omega - \tilde{\omega} - i\varepsilon)^{-1} r_1(\omega; \tilde{\omega})(\tilde{\omega} - \omega' - i\varepsilon)^{-1} r_2(\tilde{\omega}; \omega')\, d\tilde{\omega},$$

which evidently can be written in the form

$$(\omega - \omega' - 2i\varepsilon)^{-1} \int \{(\tilde{\omega} - \omega' - i\varepsilon)^{-1} + (\omega - \tilde{\omega} - i\varepsilon)^{-1}\} r_1(\omega; \tilde{\omega}) r_2(\tilde{\omega}; \omega)\, d\tilde{\omega},$$

and thus recognized as the kernel of the operator

$$\Gamma_{2\varepsilon}\{R_1 \Gamma_\varepsilon R_2 + (\Gamma_\varepsilon R_2) R_1\}.$$

It can be shown that this operator actually equals the operator $\Gamma_\varepsilon R_1 \Gamma_\varepsilon R_2$ and also that the relation

$$(\#) \qquad \Gamma R_1 \Gamma R_2 = \Gamma\{R_1 \Gamma R_2 + (\Gamma R_2) R_1\}$$

holds, which results formally as $\varepsilon \to 0$. It is in order to attain this important identity that we have added the term $i\pi r(\omega; \omega)\delta(\omega - \omega')$ to the kernel of the operator ΓR.

We add the validity of identity (#) as *property* 5 of our classes $\mathfrak{R}$. Note that this property is not implied by properties 1 to 4, as would be seen if one took another solution of the equation $[H_0, Z] = R$.

As a consequence of identity (#) we shall be able to prove the relation $(2)^-$: $U^+ U^- = 1$.

Clearly, using the identity we can write

$$\begin{aligned} U^+ U^- - 1 &= (1 + \Gamma R^+)(1 - \Gamma R^-) - 1 = \Gamma(R^+ - R^-) - \Gamma R^+ \Gamma R^- \\ &= \Gamma\{R^+ - R^- - R^+ \Gamma R^- - (\Gamma R^+) R^-\} \\ &= \Gamma\{R^+ U^- - U^+ R^-\}. \end{aligned}$$

Now, from relations $(\times)$ we evidently have

$$R^+ U^- = U^+ V U^- = U^+ R^-$$

and hence indeed $U^+ U^- = 1$.

Note that this relation was deduced from the special form of the operation Γ, which guarantees that the class $\mathfrak{R}$ has property 5. The inverse relation $U^- U^+ = 1$ will be deduced from the relation $U^+ U^- = 1$ under two additional conditions. The first is that the class $\mathfrak{R}$ has property 6.

6. There is a number $\gamma_0 > 0$ such that the inequality

$$(\gamma) \qquad \|\Gamma R\| \leqq \gamma_0 \,[\![R]\!] \tag{γ}$$

holds for all operators R in the class $\mathfrak{R}$. Here $\|S\|$ denotes the minimal (or operator) norm $\text{l.u.b.}_{\Phi \neq 0} \|S\Phi\|/\|\Phi\|$ of an operator S.

The second additional condition[2] is that the operator V should have a norm less than $1/(1 + \gamma_0)$:

$$[\![V]\!] < (1 + \gamma_0)^{-1}.$$

Since

$$[\![R^+]\!] \leqq [\![V]\!] + [\![R^+]\!]\,[\![V]\!],$$

we have

$$[\![R^+]\!] \leqq [1 - [\![V]\!]]^{-1}[\![V]\!] < \gamma_0^{-1}$$

and hence

$$\|\Gamma R^+\| \leqq \gamma_0 [\![R^+]\!] < 1.$$

Consequently, the operator $U^+ = 1 \pm \Gamma R^+$ has an inverse and $U^- U^+ = (U^+)^{-1} U^+ U^- U^+ = (U^+)^{-1} U^+ = 1$. Thus we have proved the inverse relation $(2)^+$ of $(2)^-$.

Having established relations $(1)^\pm$ and $(2)^\pm$ we have shown that the operators $U^\pm$ furnish the spectral representation of the operator $H = H_0 + V$.

At the end of this section we make a few *remarks* which are not vital in the context of Chapter II but are of significance for certain important questions raised in Chapter III.

First of all we mention that the relationship between the operators H_0 and H can clearly be written as

$$H = U^- H_0 U^+$$

and that hence every function $f(H)$ can be written as

$$f(H) = U^- f(H_0) U^+.$$

We can go a step further. Suppose an operator W_0 is represented by

[2] It should be mentioned that this condition is not necessary to insure the validity of the relation $U^- U^+ = 1$. Weaker conditions on V have been given to this end; see e.g. [27].

an integral operator (proper or symbolic) with respect to the operator H_0; then the operator

$$W = U^- W_0 U^+$$

is represented by the same integral operator with respect to H. Symbolically:

$$W_0 \underset{0}{\Leftrightarrow} w(\omega; \omega') \quad \text{implies } W \Leftrightarrow w(\omega; \omega')$$

and vice versa. This statement is immediately derived from the fact that the H_0-representer of the vector $\Psi = U^+ \Phi$ is the H-representer of the vector $U^- \Psi = \Phi$.

We now apply this observation to the operators $U^\pm = 1 \pm \Gamma R^\pm$ themselves. So far we have employed the H_0-representation of these operators in which the operators $\Gamma R^\pm$ were represented by integral operators with a singular kernel. We refer to this representation symbolically by the formula

$$U^\pm \underset{0}{\Leftrightarrow} \delta(\omega - \omega') + [\omega - \omega']^{-1} r^\pm(\omega; \omega').$$

Naturally, one may ask for the H-representation of the operators $U^\pm$. We maintain that the answer is given by the formula

$$U^\pm \Leftrightarrow \delta(\omega - \omega') + [\omega - \omega']^{-1} r^\pm(\omega; \omega')$$

which we express by the

INDEPENDENCE REMARK. The H- and H_0-representations of the operators $U^\pm$ are the same.

Once it is made, this remark is immediately derived from the statement made before. We need only observe that, if we take $U^\pm$ as the operator W_0, the transformed operator $W = U^- W_0 U^+$ is given by

$$U^- U^\pm U^+ = U^\pm,$$

so that $W = W_0$. Hence the H-representer of $U^\pm = W$ is the same as the H_0-representer of $U^\pm = W_0$.

Clearly, the independence remark is very helpful if one wants to switch over from the H_0- to the H-representation for some reason or other. In the quantum theory of fields such a switch of attitude is vital in connection with (renormalizing) limiting processes in which the H_0-representation disappears.

7. Scattering. One of the important features of the theory of perturbation of continuous spectra is its role in the analysis of scattering. This process is described by the transition of the distribution of a

"wave amplitude" at an early time to its distribution at a late time, this transition being governed by a differential equation. In the theory of classical wave motion and in quantum mechanics the wave amplitude is taken as a function of the space variables; by Fourier transformation it becomes a function of the momentum components or, what is equivalent, a function of the energy ω and (accessory) angular variables. In the context of our present discussion the wave function $\psi(\omega, t)$ may be regarded as the H_0-representer of a vector $\Phi(t)$, the "Schrödinger state" vector, which depends on the time t. The differential equation governing the wave process is then Schrödinger's equation

$$i\frac{d}{dt}\Phi(t) = H\Phi(t),$$

where $H = H_0 + V$ is the disturbed operator, the Hamiltonian operator of the "total energy". Evidently, this equation has the solution

$$\Phi(t) = e^{-itH}\Phi$$

in terms of the state $\Phi = \Phi(0)$ assigned to the time $t = 0$.

If one lets the time t tend to ∞ or $-\infty$, the wave function $\psi(\omega, t)$ does not attain limits, but, as we shall show, the "adjusted" wave function $\tilde{\psi}(\omega, t) = e^{it\omega}\psi(\omega, t)$ does. This function evidently is the H_0-representer of the "adjusted" Schrödinger state or "interaction state"

$$\tilde{\Phi}(t) = e^{itH_0}e^{-itH}\Phi.$$

Using it we can make our limit statement precise by saying that for each vector Φ there are vectors $\Phi_\pm$ such that $\tilde{\Phi}(t)$ tends strongly to $\Phi_\pm$; i.e.,

$$\tilde{\Phi}(t) \to \Phi_\pm \quad \text{as } t \to \pm\infty.$$

We shall also show that there is an operator S which transforms Φ_- into Φ_+,

$$\Phi_+ = S\Phi_-;$$

this is the "scattering operator" of Wheeler and Heisenberg. (See e.g. [11; 12; 19].)

The H_0-representers $\psi_\pm(\omega)$ of the "limit" states $\Phi_\pm$ may be regarded as describing the "incoming" and the "outgoing" state of the wave motion. The H_0-representer of the scattering operator is a generalized integral operator which transforms the incoming wave function $\psi_-(\omega)$ into the outgoing one, $\psi_+(\omega)$. The kernel of this integral operator is essentially what is called the "S-matrix".

The two limit states $\Phi_\pm$ can be obtained from the state Φ in the form $\Phi_\pm = W_\pm \Phi$ with the aid of two operators $W_\pm$, called "wave operators": The relations $\Phi(t) \to \Phi_\pm$ are equivalent with the limit relations

$$e^{itH_0}e^{-itH} \to W_\pm \quad \text{as } t \to \pm\infty.$$

It is a remarkable fact that these wave operators intertwine with H and H_0; i.e., they satisfy the equation

$$H_0 W_\pm = W_\pm H,$$

the same equation that is satisfied by our operator U^+. In fact, we shall show that $W_- = U^+$ and that $W_+ = U^+$ would be true instead if we had defined $[\omega]^{-1} = (\omega)^{-1} - i\pi\delta(\omega)$ instead of $=(\omega)^{-1} + i\pi\delta(\omega)$. In any case it follows that the limit wave functions $\psi_\pm(\omega)$, being H_0-representers of $W_\pm \Phi$, may serve as H-representers of the state Φ. We may therefore say that the scattering transformation transforms a spectral representation of the operator H associated with the time $t = -\infty$ into a spectral representation of H associated with $t \to \infty$; this is its essential property.

We should mention that the description of scattering given here is somewhat limited in scope; it is not applicable to perturbation problems of more involved types (partly discussed in later sections) in which limits of the operators $e^{itH_0}e^{-itH}$ give only an incomplete description of the scattering process or do not even exist. Nevertheless, in all such cases, scattering will be described by the transformation of one spectral representation of H into another one. Differences in treatment stem only from differences of the way in which "incoming" and "outgoing" representations are defined.

To determine the limits of the operator $e^{itH_0}e^{-itH}$ we employ the transformations $U^\pm$ and write H in the form $H = U^- H_0 U^+$ so that

$$e^{itH_0}e^{-itH} = e^{itH_0}U^- e^{-itH_0}U^+.$$

The operator $e^{itH_0}U^- e^{-itH_0}$ is clearly of the form

$$e^{itH_0}U^- e^{-itH_0} = 1 + \Gamma R^-(t),$$

where the H_0-representer of $\Gamma R^-(t) = e^{itH_0}\Gamma R^- e^{-itH_0}$ is an integral operator with the kernel

$$[\omega - \omega']^{-1} e^{it(\omega-\omega')} r^-(\omega; \omega').$$

Now, it is an important fact that *an integral operator with such a kernel tends strongly to limits* as $t \to \pm\infty$, provided that the operator R is

gentle and belongs to one of the admissible classes $\mathfrak{N}$ described in Appendix A7. Specifically, we state

$$(\times) \qquad \Gamma R(t) \to \begin{cases} \Gamma_\infty R & \text{as } t \to \infty, \\ 0 & \text{as } t \to -\infty, \end{cases}$$

where $\Gamma_\infty R$ is the operator whose H_0-representer has the kernel

$$2\pi i\delta(\omega - \omega')\, r(\omega; \omega').$$

Clearly, the operator $\Gamma_\infty R$ transforms a vector with the H_0-representer $\psi(\omega)$ into the vector with the H_0-representer

$$2\pi i r(\omega; \omega)\psi(\omega).$$

The limit statement made above evidently depends on the presence of the term $i\pi\delta$ in the definition of the operation Γ; if we had taken $-i$ instead of $+i$ there, the roles of $t = +\infty$ and $t = -\infty$ would have been interchanged.

To prove the limit statement one may use the fact that the integral operator representing $\Gamma R(t)$ is the limit of the one with the kernel

$$(\omega - \omega' - i\varepsilon)^{-1} e^{it(\omega - \omega')}\, r(\omega; \omega)$$

as ε tends to zero through positive values. For analytic functions ψ and r one proves the statement after deforming the path of integration; for other ψ and r it then follows from the boundedness of ΓR. Details are given in Appendix A8.

From our limit statement we now can determine the wave operators; i.e., the limits of the operators

$$e^{itH_0} e^{-itH} = [1 + \Gamma R^-(t)] U^+.$$

We introduce the operators

$$S_0^\pm = 1 + \Gamma_\infty R^\pm,$$

and infer from our statements $(\times)$ the relations

$$e^{itH_0} e^{-itH} \to \begin{cases} S_0^- U^+ & \text{as } t \to \infty, \\ U^+ & \text{as } t \to -\infty; \end{cases}$$

they imply that the wave operators are given by

$$W_- = U^+ \quad \text{and} \quad W_+ = SU^+,$$

where *the scattering operator S is given by* $S = S_0^-$.

Repeating our arguments for the inverse

$$e^{itH_0} U^+ e^{-itH_0}$$

of $e^{itH_0}U^-e^{-itH_0}$ we obtain

$$e^{itH}e^{-itH_0} \to \begin{cases} U^-S_0^+ & \text{as } t \to \infty, \\ U^- & \text{as } t \to -\infty, \end{cases}$$

whence, as $t \to \infty$,

$$1 = (e^{itH_0}e^{-itH})(e^{itH}e^{itH_0}) \to S_0^- U^+ U^- S_0^+ = S_0^- S_0^+,$$
$$1 = U^+U^- = U^+(e^{itH}e^{-itH_0})(e^{itH_0}e^{-itH})U^- \to U^+U^-S_0^+S_0^-U^+U^- = S_0^+S_0^-.$$

Thus we are led to the relations

$$S_0^+S_0^- = S_0^-S_0^+ = 1,$$

which show that the transformation of Φ_+ into Φ_- is given by the inverse scattering operator S_0^+. (These relations could be derived directly from $U^-U^+ = U^+U^- = 1$, without employing the limit process, see Appendix A11.)

Since the H_0-representer of the operator $\Gamma_\infty R$ is the kernel

$$2\pi i r(\omega; \omega)\delta(\omega - \omega'),$$

the H_0-representers of the scattering operators $S_0^\pm$ are the kernels

$$[1 \pm 2\pi i r^\pm(\omega; \omega)]\delta(\omega - \omega').$$

If the vector Φ is represented by the function $\psi(\omega)$, the vectors $S_0^\pm\Phi$ are represented as

$$S_0^\pm\Phi \underset{0}{\Leftrightarrow} [1 \pm 2\pi i\, r^\pm(\omega; \omega)]\psi(\omega).$$

It is thus clear that the operators $S_0^\pm$ commute with the undisturbed operator H_0; i.e., $[H_0, S_0^\pm] = 0$. In case the operator H_0 is simple so that the values of the representers ψ are complex numbers, the operators $S_0^\pm$ are functions of H_0, being represented through multiplication by a function of ω. If, however, the values of ψ are vectors of an accessory Hilbert space of dimension >1, multiplication by $r^\pm(\omega; \omega')$ means application of an operator acting in that space, which in general is not just a number. In any case then, *the scattering operators are represented by operators* $1 \pm 2\pi i r^\pm(\omega; \omega)$ *which act in the accessory space.*

If the vectors ψ of the accessory space are represented by sequences of numbers, the operator $1 + 2\pi i r^-(\omega; \omega)$ acting in it is represented by a matrix. This is precisely Heisenberg's S-matrix.

There is another way of describing the scattering process, which is preferable under certain circumstances. In this description one selects

a wave function $\phi(\omega)$ and assigns to it the vector Φ^- whose initial H-representer is ϕ and the vector Φ^+ whose end representer is also ϕ. Scattering is then described as the transformation of Φ^- into Φ^+. Introducing the operators

$$S^{\pm} = U^- S_0^{\pm} U^+$$

one easily verifies that $\Phi^+ = S^- \Phi^-$.

The specific feature of this description is that it involves no explicit reference to the undisturbed operator H_0. Of course, reference to this operator H_0 is involved implicitly, namely, in the definition of the terms "initial" and "end" spectral representation of H.

Still it is frequently advantageous to avoid explicit reference to the operator H_0; to be consistent one then must describe the operators $S^{\pm}$ not in the H_0, but in the H-representation. This is easily done; for, the operators $S^{\pm}$ are related to $S_0^{\pm}$ by the same transformation through which the operator H is related to H_0; hence, it follows from the Independence remark made at the end of §6 that *the H-representers of $S^{\pm}$ are the same as the H_0-representers of $S_0^{\pm}$*, namely, the integral operators with the kernels

$$\delta(\omega - \omega')[1 + 2\pi i\, r^{\pm}(\omega; \omega)].$$

That is to say, the operators $S^{\pm}$ transform a state with the H-representers $\phi(\omega)$ into states with the H-representers

$$[1 + 2\pi i\, r^{\pm}(\omega; \omega)]\phi(\omega).$$

For the actual computation of the scattering transformation it therefore does not matter which of the two attitudes one takes.

8. Perturbation of an operator with point and continuous spectrum. Interesting phenomena may occur if the spectrum of the undisturbed operator H_0 consists partly of a continuous section, such as a finite or half-infinite interval, and partly of a point-spectrum outside of it. For simplicity we assume this point-spectrum to consist of just one point ω_0 and the continuous section to be the half-infinite interval $\omega \geqq \omega_*$ with $\omega_* > \omega_0$.

The H_0-representer of a vector Φ may then be described as consisting of a number ψ_0 and a function $\psi(\omega)$ defined for $\omega \geqq \omega_*$ so that the representation

$$\Phi \underset{0}{\Leftrightarrow} \{\psi_0, \psi(\omega)\}$$

implies

$$H_0\Phi \underset{0}{\Leftrightarrow} \{\omega_0\psi_0, \omega\psi(\omega)\}.$$

Further, we assume that the unit form is given by

$$(\Phi, \Phi) = |\psi_0|^2 + \int |\psi(\omega)|^2 \, d\omega;$$

here, and in the following, the integration extends over $\omega \geqq \omega_*$.

The interaction operator V obviously consists of four parts connecting ψ with ψ, with ψ_0, ψ_0 with ψ, and ψ_0 with ψ_0. For simplicity we assume the term connecting ψ with ψ to be absent. This is no serious restriction since the interaction of ψ with ψ could first be treated separately, in the manner of the theory of §6, with the aid of two operators $U^{\pm}$. The operator $U^{+}(H_0 + V)U^{-}$ then has no ψ-ψ interaction and could hence be taken as operator $H_0 + V$.

Accordingly, we assume that the operator V is given with the aid of two functions $v_{\pm}(\omega)$ and a number v_0 such that $V\Phi$ is represented through

$$V\Phi \underset{0}{\Leftrightarrow} \left\{ v_0\psi_0 + \int v_+(\omega')\psi(\omega') \, d\omega', \, v_-(\omega)\psi_0 \right\},$$

or simply through

$$V \underset{0}{\Leftrightarrow} \begin{pmatrix} v_0 & v_+(\omega') \\ v_-(\omega) & 0 \end{pmatrix}.$$

The functions $v_{\pm}(\omega)$ should be square integrable over $\omega \geqq \omega_*$; moreover, they should belong to one of the two classes of functions described in Appendix A7. Clearly, the operator V is Hermitean if v_- is the complex conjugate of v_+.

The perturbation problem just described has played a considerable role in the discussions about the mathematical character of the quantum theory of fields: for, the simple model introduced by Lee leads just to this problem. (See e.g. [**39**; **44**II].) Various peculiar phenomena met in this theory can be illustrated with this model.

The first question that arises naturally is, what happens with the point-eigenvalue of H_0 under the disturbance V? Suppose the vector $\hat{X}$, represented as $\hat{X} \underset{0}{\Leftrightarrow} \{\hat{\psi}_0, \hat{\psi}(\omega)\}$, is an eigenvector of $H = H_0 + V$ with the eigenvalue $\hat{\omega}$. Then the representers $\hat{\psi}_0$, $\hat{\psi}(\omega)$ satisfy the equations

$$(\omega_0 + v_0 - \hat{\omega})\hat{\psi}_0 + \int v_+(\omega')\hat{\psi}(\omega') \, d\omega' = 0,$$

$$v_-(\omega)\hat{\psi}_0 + (\omega - \hat{\omega})\hat{\psi}(\omega) = 0.$$

From the second equation we find that $\hat{\psi}_0$ and $\hat{\psi}(\omega)$, except for an arbitrary factor, are given by

$$\hat{\psi}_0 = 1,$$

$$\hat{\psi}(\omega) = -(\omega - \hat{\omega})^{-1} v_-(\omega);$$

and from the first one we then derive the equation

$$\hat{\omega} = \omega_0 + v_0 - \int \frac{v_+(\omega')v_-(\omega')}{\omega' - \hat{\omega}}\, d\omega',$$

for the value of $\hat{\omega}$. The latter equation has certainly a solution $\omega < \omega_*$ if the absolute value of the number $v_0 - \int(\omega' - \omega_0)^{-1}|v_+(\omega')v_-(\omega')|\, d\omega'$ is sufficiently small; this solution $\hat{\omega}$ is evidently an eigenvalue of H with the eigenvector given above. In case V is Hermitean, so that $v_+(\omega)v_-(\omega) = |v(\omega)|^2 > 0$, there is only one such eigenvalue.

In addition to this point eigenvalue $\hat{\omega}$, the operator H has the same continuous spectrum as H_0, namely, the half-axis $\omega \geqq \omega_*$. Specifically, we maintain, the operator H admits a spectral representation in which the vectors Φ are represented by functions $\{\phi_0, \phi(\omega)\}$ connected with $\{\psi_0, \psi(\omega)\}$ by explicit formulas which we shall describe below.

Before giving such a description we shall make an adjustment, which at present is just convenient, but which will be vital in the treatment of divergences to be discussed later on.

This adjustment consists in choosing the constant v_0, the "0-term" of the operator V, in such a way that the disturbed eigenvalue $\hat{\omega}$ agrees with the undisturbed one: $\hat{\omega} = \omega_0$. Evidently, this is achieved by setting

$$v_0 = \int (\omega' - \omega_0)^{-1} v_+(\omega')v_-(\omega')\, d\omega'.$$

Once this is done the operators H and H_0 will turn out to have the same spectrum. Moreover, it will be possible to describe the spectral transformation connecting the spectral representations of these operators with the aid of a pair of operators $U^{\pm}$ so chosen that the H-representers χ_0, $\chi(\omega)$ of a state Φ are given as the H_0-representers of the state $U^+\Phi$, while the H_0-representers ψ_0, $\psi(\omega)$ of Φ are the H-representers of $U^-\Phi$. Just as in §6, the operators $U^{\pm}$ will satisfy the equations

$$HU^- = U^-H_0, \qquad U^+H = H_0U^+.$$

The operators $U^{\pm}$ cannot, however, be written in the same form, $1 \pm \Gamma R^{\pm}$, as those of §6. A modification of this form is necessary.

Again we introduce operators R through their H_0-representation, viz.

$$R \underset{0}{\leftrightarrows} \begin{pmatrix} r_0 & r_+(\omega') \\ r_-(\omega) & r(\omega; \omega') \end{pmatrix}.$$

The function $r(\omega; \omega')$ should belong to one of our classes of kernels; and the functions $v_{\pm}(\omega)$ should belong to corresponding classes of functions of one variable.

The essential restriction imposed on the operators R is that their 0-terms should vanish: $r_0 = 0$; only then can we introduce operators ΓR satisfying $[H_0, \Gamma R] = R$. Indeed, the operators given by

$$\Gamma R \underset{0}{\leftrightarrow} \begin{pmatrix} 0 & (\omega_0 - \omega')^{-1} r_+(\omega') \\ (\omega - \omega_0)^{-1} r_-(\omega) & [\omega - \omega']^{-1} r(\omega; \omega') \end{pmatrix},$$

with $[\omega - \omega']^{-1} = (\omega - \omega')^{-1} + i\pi\delta(\omega - \omega')$, have this property.

As in §6, we now can solve the equations

$$R^+ = (1 + \Gamma R^+)V \quad \text{and} \quad R^- = V(1 - \Gamma R^-),$$

provided the 0-term v_0 of V is chosen properly. In the present problem the solutions $R^\pm$ can be given explicitly, namely, through the representations

$$R^+ \underset{0}{\leftrightarrow} \begin{pmatrix} 0 & v_+(\omega') \\ \dfrac{v_-(\omega)}{h^-(\omega)} & \dfrac{v_-(\omega)v_+(\omega')}{(\omega - \omega_0)h^-(\omega)} \end{pmatrix},$$

$$R^- \underset{0}{\leftrightarrow} \begin{pmatrix} 0 & \dfrac{v_+(\omega')}{h^+(\omega)} \\ v_-(\omega) & \dfrac{v_-(\omega)v_+(\omega')}{(\omega - \omega_0)h^+(\omega')} \end{pmatrix},$$

with functions $h^\pm(\omega)$ given by

$$(\omega - \omega_0)h^\pm(\omega) = \omega - \omega_0 - v_0 \pm \int [\pm\omega \mp \tilde{\omega}]^{-1} v_+(\tilde{\omega}) v_-(\tilde{\omega})\, d\tilde{\omega}.$$

One readily verifies that these operators $R^\pm$ satisfy the two equations provided the 0-term v_0 of V is given the same value that was derived above from the condition that the point-eigenvalues should not shift.

Although the operators $1 \pm \Gamma R^\pm$ satisfy the intertwining relations $(1 + \Gamma R^+)H = H_0(1 + \Gamma R^+)$ and $H(1 - \Gamma R^-) = (1 - \Gamma R^-)H_0$, we shall not take these operators as $U^\pm$ since they do not satisfy the relations $U^+U^- = U^-U^+ = 1$.

To show this we shall evaluate the product $(1 + \Gamma R^+)(1 - \Gamma R^-)$, and to this end we shall first evaluate the difference $\Gamma R^+\Gamma R^- - \Gamma(R^+\Gamma R^- + (\Gamma R^+)R^-)$ (for any pair of operators $R^\pm$, not just the solutions of the equations above). The operator Γ can be applied on the operator $R^+\Gamma R^- + (\Gamma R^+)R^-$ since the 0-term of this operator vanishes, as readily verified. On the other hand, the 0-term of $\Gamma R^+\Gamma R^+$, and hence of the above difference, is not zero; a simple computation shows that it is given by $(\Gamma R^+\Gamma R^-)_0 = -\sigma_0$ with

$$\sigma_0 = \int (\tilde{\omega} - \omega_0)^{-2} r_+^+(\tilde{\omega}) r_-^-(\tilde{\omega})\, d\tilde{\omega}$$

Aside from the 0-term, however, the terms of the matrix representing the above difference do vanish, as readily verified. Introducing the operator σ by the representation

$$\sigma \underset{0}{\Leftrightarrow} \begin{pmatrix} \sigma_0 & 0 \\ 0 & 0 \end{pmatrix},$$

we may formulate this result as the formula

$$\Gamma R^+ \Gamma R^- = \Gamma(R^+ \Gamma R^- + (\Gamma R^+)R^-) - \sigma.$$

As in §6 we can derive the relation $R^+(1 - \Gamma R^-) = (1 + \Gamma R^+)R^-$, and using it we arrive at the identity

$$(1 + \Gamma R^+)(1 - \Gamma R^-) = 1 + \sigma.$$

If V is Hermitean, the functions $v_\pm$ are complex conjugate to each other so that σ_0 is real positive. In general we shall assume that the deviation of V from a Hermitean operator is sufficiently small to insure that $1 + \sigma_0 \neq 0$. Then we can introduce the number

$$\tau_0 = (1 + \sigma_0)^{1/2}$$

and the operator τ represented by

$$\tau \underset{0}{\Leftrightarrow} \begin{pmatrix} \tau_0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Once our identity is written in the form

$$(1 + \Gamma R^+)(1 - \Gamma R^-) = \tau^2$$

it becomes clear how to define the operators $U^\pm$. We simply set

$$U^+ = \tau^{-1}(1 + \Gamma R^+), \qquad U^- = (1 - \Gamma R^-)\tau^{-1}.$$

For then, evidently, the relation

$$U^+ U^- = 1$$

holds and, if V is sufficiently small, the same is true of

$$U^- U^+ = 1.$$

The intertwining relations $U^+ H = H_0 U^+$ and $HU^- = U^- H_0$ then follow from the fact that τ commutes with H_0.

In the problem of the quantum theory of fields which leads to the perturbation problem just solved, the quantity ω_0 corresponds to the energy of a certain type of particle. The shift of the eigenvalue from the value ω_0 to the value $\hat{\omega}$ corresponds to a shift of this energy under the influence of the perturbation.

Since only the eigenstates of the operator H and not those of H_0 are stationary, the "disturbed" particles, having the energy $\hat{\omega}$, are the physically real ones, while the "undisturbed" ones, having the energy ω_0, are regarded as artificial entities.

Since actually only the combination $\mathring{\omega} = \omega_0 + v_0$ enters the description of the operator H, it is quite arbitrary which fraction of this sum is assigned to the undisturbed particle as its energy ω_0 and which fraction v_0 is assigned to the perturbation. How this is done makes a difference inasmuch as the transition formulas are expressed in terms of ω_0.

The original formulation of the quantum theory of fields corresponds to taking $v_0 = 0$ so that $\omega_0 = \mathring{\omega}$; then the disturbed energy $\hat{\omega}$ differs from the undisturbed one. This is the attitude "before renormalization". The attitude "after renormalization" corresponds to taking the physically real energy as the undisturbed one, $\omega_0 = \hat{\omega}$, as we have done here; the deviation $\mathring{\omega} - \hat{\omega}$ then appears as part of the disturbance, $\mathring{\omega} - \hat{\omega}_0 = v_0$. Accordingly, one refers to the real energy $\hat{\omega}$ as the "renormalized" one, and to $\mathring{\omega}$ as the "unrenormalized" energy. This question will also be discussed later on in a related context; see §17.

Actually, one speaks in general of renormalized and unrenormalized masses rather than energies; this is appropriate in Lorentz invariant theories since there the energy of a particle is determined by its mass.

In the problem that we have considered so far, it does not make any real difference which of the two attitudes one takes, but it does make a difference in problems involving divergences. For, if in these problems limiting processes are employed it does make a difference whether the quantity $\hat{\omega}$ or the quantity $\mathring{\omega}$ is kept fixed in such a process.

Divergences arise from the fact that in the physically realistic case the functions $v_\pm(\omega)$ are not square integrable; in fact, they are given by $v_\pm(\omega) = g\omega^{1/2}$, with an appropriate "coupling" constant g. Instead of immediately discussing the implication of this form of $v_\pm$ we shall consider the effect of divergences in stages. To this end we introduce the quantities

$$Y_\rho = \int_{\omega_*}^{\infty} \frac{|v(\omega)|^2}{\omega^\rho}\, d\omega, \qquad \rho = 0, 1, 2, 3,$$

assuming $\omega_* > 0$ and, for simplicity, $v_- = v$, $v_+ = \bar{v}$, $|v|^2 = v_+ v_-$.

We require the function $v(\omega)$ to be such that the functions $\zeta(\omega)v(\omega)$ belong to one of the two classes described in Appendix A7 provided $\zeta(\omega)$, the "form factor", is infinitely differentiable and of bounded support.

While we have assumed so far (Case 0) that the number Y_0 is finite, we now consider the possibility that this quantity is infinite.

In case $Y_0 = \infty$ the operator V is not defined for vectors Φ with $\psi_0 \neq 0$; for, otherwise $(V\Phi, V\Phi) \geqq \int |v(\omega)|^2 \, d\omega |\psi_0|^2$ would be infinite. Although the operator V is thus not defined in a dense subspace of the Hilbert space, the operator $H_0 + V$ can be defined in a dense subspace provided the numbers Y_1, and hence Y_2 are finite (Case 1). This is quite obvious since, by virtue of $Y_2 < \infty$, the quantities τ_0 and $h^{\pm}(\omega)$ and hence the transformation operators $U^{\pm}$ are defined. These operators allow one to transform the dense subspace of vectors Φ for which H_0 is defined through $\Psi = U^{+}\Phi$ into a dense set of vectors where H can simply be defined by

$$H\Psi = U^{+}H_0U^{-}\Psi = U^{+}H_0\Phi.$$

Thus both operators H and H_0 are defined in dense subspaces, but not the difference V.

This fact may appear strange but there is nothing wrong with it. After all, unbounded operators are never defined in the full Hilbert space, but only in dense subspaces, and it can happen that the intersection of two dense subspaces is not dense (or even empty).

While the operator V is not defined in a dense subspace, the associated bilinear form $(\Phi, V\Phi^{(1)})$ can be defined for a dense subset; for example, for the set of all vectors Φ, $\Phi^{(1)}$ whose representers $\psi(\omega)$ and $\psi^{(1)}(\omega)$ have bounded support. It is, of course, well known that there exist properly defined bilinear forms which cannot be associated with an operator defined in a dense set. One might perhaps think one could dispose of the difficulty encountered by going over to bilinear forms, or "matrix elements", instead of using operators. That is not sufficient, however, as will become clear from the discussion of the next case.

In Case 2, characterized by $Y_1 = \infty$, $Y_2 < \infty$, we find that the term v_0, given by the expression $\int (\omega' - \omega)^{-1} |v(\omega')|^2 \, d\omega'$, is infinite. Now, this term is the matrix element $v_0 = (\Phi_0, V\Phi_0)$ where $\Phi_0 \leftrightarrow_0 \{1, 0\}$. Thus in this case not even the bilinear form of V is defined in a dense subspace.

Nevertheless, the transformation operators $U^{\pm}$ can be defined. At first sight this might appear not to be so since the integral involved in the definition of $h^{\pm}(\omega)$ is not finite now; moreover, the infinite term v_0 enters this definition. However, it is just the latter fact which saves the situation; for the infinity of v_0 just balances the infinity of the integral referred to.

To see this we go back to Case 1 or 0 where both this integral and the number r_0 are finite. Substituting for the latter number its expression as an integral and combining it with the integrals entering the definition of $(\omega - \omega_0)h^{\pm}(\omega)$, we find the expression

$$h^{\pm}(\omega) = 1 \mp \int [\pm\tilde{\omega} \mp \omega]^{-1}(\omega_0 - \tilde{\omega})^{-1} v^{+}(\tilde{\omega}) v^{-}(\tilde{\omega})\, d\tilde{\omega}$$

for $h^{\pm}(\omega)$, which is evidently defined even in Case 2. Here then we meet the process of "removing a divergence" with the aid of an infinite term resulting from renormalization.

Adopting the new expression for $h^{\pm}$ to begin with, we see that the transformations $U^{\pm}$ are defined in Case 2. Since, as can be easily verified, these operators satisfy the relation $U^{+}U^{-} = U^{-}U^{+} = 1$ for sufficiently small Y_2, they transform any dense subspace of the Hilbert space into a dense subspace and hence both operators H and H_0 are defined in such dense subspaces, while the operator V cannot be so defined.

To be sure, it is unnatural to define the operator H via the transformations $U^{\pm}$. It is more natural to define H by a limiting process from a sequence of operators $H_\nu = H_0 + V_\nu$ belonging to Case 0. To this end one may employ a form factor $\zeta_\nu(\omega)$ which approaches 1 as $\nu \to \infty$. Instead of defining H directly as the limit of the sequence of H_ν, one should first define appropriate bounded functions $f(H)$ as limits of $f(H_\nu)$ and then define H with the aid of such functions $f(H)$. In carrying out this limiting process it is indeed vital that the eigenvalue $\hat{\omega}$ $(= \omega_0)$ of the Hamiltonian H_ν is kept fixed or forced to tend to a limit and that the quantity $\omega_0 + r_0$, the unrenormalized energy of the undisturbed particle, is allowed to tend to infinity.

In Case 3, in which $Y_2 = \infty$, and which comprises the physically realistic case with $v(\omega) = g\omega^{1/2}$, it seems impossible to salvage the transformation operators $U^{\pm}$.

One has attempted to allow the coupling constant g_ν in $v_{\pm}(\omega) = g_\nu v^0(\omega)$ to tend to zero in an appropriate manner while the form factor $\zeta_\nu(\omega)$ tends to 1. After replacement of $v(\omega)$ by $g_\nu\zeta_\nu(\omega)v^0(\omega)$, the last term in the matrix ΓR^{+}, for example, is written in the form

$$(\omega - \omega_0)^{-1}[\omega - \omega']^{-1}\zeta_\nu v^0(\omega)\zeta_\nu v^0(\omega') \cdot \Big\{ g_\nu^{-2} + \int (\tilde{\omega} - \omega_0)^{-2} |\zeta_\nu v^0(\tilde{\omega})|^2\, d\tilde{\omega} - (\omega_0 - \omega') \int (\tilde{\omega} - \omega_0)^2 [\tilde{\omega} - \omega']^{-1} |\zeta_\nu v^0(\tilde{\omega})|^2\, d\omega \Big\}^{-1},$$

so that the infinity is concentrated in the first integral in the denominator. Evidently, this integral becomes positively infinite as $\zeta_v \to 1$. If now the coupling constant g_v could be chosen imaginary, it could be so chosen that the term $g_v^{-2} \to -\infty$ in such a way that it cancels the positive infinity resulting from the first integral. This procedure, however, would imply that the operator H is not Hermitean, in violation of one of the basic principles of quantum theory. Such attempts have therefore been abandoned by most physicists.

In our presentation we have not required the operator V to be Hermitean; therefore our theory is applicable to the case of imaginary $g_v \neq 0$. In the limiting case, however, we have $\tau_0 = 0$ and it is no longer possible to form the operators $U^{\pm}$ from $1 \pm \Gamma R^{\pm}$. The significance of this occurrence will be discussed in Chapter III in a wider context.

A new phenomenon occurs in the description of *scattering* in connection with the present example.

We first describe the scattering process by connecting the limit of the adjusted Schrödinger state $\tilde{\Phi}(t) = e^{itH_0}e^{-itH}\Phi$ as $t \to \infty$ with the limits of this state for $t \to -\infty$. Because of $HU^- = U^-H_0$ we have $e^{-itH}U^- = U^-e^{-itH_0}$ and hence we may write

$$\tilde{\Phi}(t) = e^{itH_0}U^-e^{-itH_0}\Phi_-$$

with $\Phi_- = U^+\Phi$. This would not have been possible without our renormalization, $\hat{\omega} = \omega_0$, whereby H and H_0 have acquired the same spectrum.

We denote the H_0-representer of Φ_- by $\{\phi_0, \phi(\omega)\}$. The H_0-representer $\{\psi_0(t), \psi(\omega, t)\}$ of the state $\tilde{\Phi}(t)$ is then given by

$$\psi_0(t) = \tau_0^{-1}\phi_0 - \int e^{it(\omega_0 - \omega')} \frac{p_+(\omega')}{\omega_0 - \omega'} \phi(\omega')\, d\omega',$$

$$\psi(\omega, t) = -\tau_0^{-1}e^{it(\omega - \omega_0)} \frac{v_-(\omega)}{\omega - \omega_0} \phi_0 + \phi(\omega)$$

$$+ v_-(\omega) \int e^{it(\omega - \omega')} \frac{p_+(\omega')}{[\omega - \omega'](\omega_0 - \omega')} \phi(\omega')\, d\omega',$$

with $p_{\pm}(\omega) = v_{\pm}(\omega)/h^{\pm}(\omega)$.

As $t \to \pm\infty$ the function $\psi_0(t)$ tends to a limit:

$$\psi_0(t) \to \tau_0^{-1}\phi_0 \quad \text{as } t \to \pm\infty$$

by virtue of the Riemann-Lebesgue Lemma. The last two contributions to $\psi(\omega, t)$ also converge, but not the first one; all we can assert is

that the asymptotic behavior of the state $\tilde{\Phi}(t)$ as $t \sim -\infty$ is described by

$$\psi_0(t) \sim \tau_0^{-1}\phi_0,$$

$$\psi(\omega, t) \sim -\tau_0^{-1}e^{it(\omega-\omega_0)}\frac{v_-(\omega)}{\omega-\omega_0}\phi_0 + \phi(\omega),$$

while as $t \sim +\infty$

$$\psi_0(t) \sim \tau_0^{-1}\phi_0,$$

$$\psi(\omega, t) \sim -\tau_0^{-1}e^{it(\omega-\omega_0)}\frac{v_-(\omega)}{\omega-\omega_0}\phi_0 + \left[1 + 2\pi i\, v_-(\omega)\frac{p_+(\omega)}{\omega_0-\omega}\right]\phi(\omega).$$

Introducing the operators U_{as}^- and S_0^- by their H_0-representations

$$U_{as}^- \underset{0}{\Leftrightarrow} \begin{pmatrix} \tau_0^{-1} & 0 \\ \tau_0^{-1}\dfrac{v_-(\omega)}{\omega-\omega_0} & 1 \end{pmatrix}$$

and

$$S_0^- \underset{0}{\Leftrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 + 2\pi i\, v_-(\omega)\dfrac{p_+(\omega)}{\omega_0-\omega} \end{pmatrix}$$

we may write these descriptions in the form

$$\tilde{\Phi}(t) \sim e^{itH_0}U_{as}^-e^{-itH_0}\Phi_\pm \quad \text{as } t \sim \pm\infty$$

with $\Phi_+ = S_0^-\Phi_-$. The state $U_{as}^-\Phi_-$ is the *asymptotic state* introduced by van Hove [**44**] in a very general context.

The term $-e^{it(\omega-\omega_0)}v_-(\omega)(\omega-\omega_0)^{-1}\phi_0$ is frequently referred to as "cloud term". The fact that it does not converge to zero as $t \to \pm\infty$ implies that the state it represents does not converge to zero in the strong sense. Still, this state converges to zero in the weak sense by virtue of the Riemann-Lebesgue Lemma, that is, by virtue of the oscillatory character of the "cloud factor" $e^{it(\omega-\omega_0)}$. Thus the operator $e^{itH_0}e^{-itH}$ has weak limits as $t \to \pm\infty$ and the same is true of the inverse operator $e^{itH}e^{-itH_0}$; but the weak limits of these operators are not inverse to each other. This is consistent with the fact that the product of the weak limits need not be the weak limit of the product.

However, if the factor τ_0^{-1} is omitted from the weak limit formulas the resulting operators are inverse to each other.

The operators thus resulting for $t \to \infty$ should be regarded as the scattering operators $S_0^\pm$; for $t \to -\infty$ they reduce to the identity. The operator S_0^+ is given by

$$S_0^+ \underset{0}{\Leftrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 - 2\pi i\dfrac{p_-(\omega)}{\omega-\omega_0}v_+(\omega) \end{pmatrix},$$

while S_0^- was already given above. They satisfy the relations $S_0^- S_0^+ = S_0^+ S_0^- = 1$, as could readily be verified.

The scattering operator S_0^- can also be obtained directly by a limiting process, namely, by

$$e^{itH_0}(U_{as}^-)^{-1}e^{-itH} \to S_0^- \quad \text{as } t \to \infty$$

while for $t \to -\infty$ the limit of this operator is the identity; this is evident from the asymptotic description of the state $\tilde{\Phi}(t)$ given above. Thus it is seen that indeed the states $\Phi_\pm$ can be extracted by a limit process from an adjusted Schrödinger state; but the adjustment consists in multiplying $e^{-itH}\Phi$ by $e^{itH_0}(U_{as}^-)$ rather than simply by e^{itH_0}. Different types of modification of the adjustment will be met in other problems; see Appendix A14.

CHAPTER III

Perturbation by Annihilation-Creation Operators

9. Particle representation. The perturbation problems of the quantum theory of fields differ in many respects from the perturbation problems we have treated in the first two chapters. The disturbing operator in particular, has very special properties: It is expressible in terms of "annihilation and creation operators" (these terms will be explained later in this section) and it is restricted by the requirement that the interaction it represents is "local" and "Lorentz-invariant". It is the latter two properties which produce the divergences that beset the quantum theory of fields.

We shall give up the restriction to Lorentz-invariant local interaction and choose disturbing operators for which, in general, no divergences occur. We shall, however, retain the requirement that the disturbing operator be built up in terms of annihilation and creation operators. In treating the perturbation problem for such operators we shall develop a procedure which is a natural extension of the formalism used for the perturbation of continuous spectra described in Chapter II and which is related to the formalism used in the quantum theory of fields. We shall not, however, be able to prove the validity of this procedure.

The undisturbed operator H_0 will be described with reference to a particular representation of the Hilbert space, the "Fock" representation, also called "particle representation" for reasons explained later on.

In this representation every vector Φ in the Hilbert space $\mathfrak{H}$ is represented by a sequence of functions of 0, 1, 2, $\cdots$ variables,

$$\Phi \underset{0}{\Leftrightarrow} \{\psi_0, \psi_1(\omega_1), \psi_2(\omega_1, \omega_2), \cdots\},$$

where ψ_0 is just a complex number. The real variables $\omega_1, \omega_2, \cdots$ all run over the same interval. In most specific cases this will be the interval $\omega_* \leqq \omega < \infty$, where ω_* is some positive number; but actually we

shall rarely make use of this restriction. The functions $\psi_n(\omega_1, \cdots, \omega_n)$ are to be either symmetric, i.e., invariant under permutation of their variables, or antisymmetric, i.e., change sign if the permutation is odd, and not if it is even. (Accordingly, we speak of the cases of symmetry and antisymmetry.) Also, these functions should be square integrable and the sum of these squares, divided by factorials, should be finite. In fact, this sum should give the unit form. Using the abbreviations:

$$\psi_n(\omega_1, \cdots, \omega_n) = \psi_n((\omega)_n),$$

$$\int \cdots \int \cdots d\omega_1 \cdots d\omega_n = \int \cdots d(\omega)_n$$

and

$$n! = \frac{1}{n!},$$

we may write these requirements in the form

$$(\Phi, \Phi) = |\psi_0|^2 + \sum_{n=1}^{\infty} n! \int |\psi_n(\omega)_n|^2 \, d(\omega)_n < \infty.$$

The operator H_0 will be defined by describing the particle representers of the vector $H_0\Phi$ in terms of those of the vector Φ; the nth "component" of $H_0\Phi$ is obtained from the nth component $\psi_n(\omega_1, \cdots, \omega_n)$ of Φ simply by multiplying it by the sum $\omega_1 + \cdots + \omega_n$ provided $n > 0$. The zeroth component of H_0 is to be zero. Thus H_0 is defined by the representation

$$H_0\Phi \underset{0}{\Leftrightarrow} \{0, \omega_1\psi_1(\omega_1), (\omega_1 + \omega_2)\psi_2(\omega_1, \omega_2), \cdots\}$$

or, shortly, by

$$H_0\Phi \underset{0}{\Leftrightarrow} \left\{\sum^n \omega\psi_n(\omega)_n\right\},$$

where the abbreviation

$$\sum^n \omega = \omega_1 + \cdots + \omega_n,$$

is used. The operator H_0 is evidently defined in a dense subspace of $\mathfrak{H}$.

The representation employed here is not a spectral representation for H_0 in the strict sense; but a proper spectral representation could easily be derived from it. In doing so one would introduce the number n together with $n - 1$ variables such as $\omega_1, \cdots, \omega_{n-1}$ as accessory variables and the sum $\omega = \sum^n \omega$ as spectral variable. In any case it is clear that H_0 has the point-eigenvalue 0 with the eigenvector $\Phi_0 \underset{0}{\Leftrightarrow} \{1, 0, 0, \cdots\}$ and a continuous spectrum in the interval $\omega_* \leqq \omega$ with infinite multiplicity for $2\omega_* \leqq \omega$.

Representations of "state vectors" Φ by sequences of functions in the manner described play a role in the quantum theory of fields inasmuch as a field may be regarded as consisting of an indefinite number of particles, called "bosons" in case of symmetry, "fermions" in case of antisymmetry. The probability that there are exactly n particles present in the field when it is in the state Φ with $(\Phi, \Phi) = 1$ is then given by the nth term in the series for (Φ, Φ). Furthermore, the value of $\psi_1(\omega_1)$ is the amplitude of the probability that only one particle is present in the field and that its energy is equal to ω_1. Quite generally, the value $\psi_n(\omega)_n$ is the amplitude of the probability that exactly n particles are present in the field with the energies $\omega_1, \cdots, \omega_n$. The total energy present in the field would then be the sum $\omega_1 + \cdots + \omega_n$.

From our description of the operator H_0 it is now clear that this operator corresponds to the energy of the totality of particles present in the field or simply to the *energy of the field.*

The interpretation of the representation described justifies the term "particle representation". In accordance with it we shall call a state vector Ψ_m an m-particle state if its components ψ_n are zero for all $n \neq m$. The zero particle state Ψ_0 will also be called "vacuum state".

The description of the particle representation as we have given it is too restricted for the purpose of field theory. The state of a single particle in this theory is not just characterized by referring to its energy; other observables are needed for a complete description of this state such as the direction of the momentum of the particle, possibly spin components, and quantities that indicate which kind of particle it is. Collecting such "accessory" variables in a quantity σ we may say that the pair (ω, σ) stands for a complete set of observables associated with a single particle. The functions ψ_n should then depend on the n pairs of variables $(\omega_1, \sigma_1), (\omega_2, \sigma_2), \cdots, (\omega_n, \sigma_n)$.

Among the variables which form the quantity σ one may include one, ι say, to indicate whether the particle is a boson, $\iota = 0$, or a fermion, $\iota = 1$. To distinguish these two cases one should stipulate that any permutation of the n pairs of variables $(\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n)$ does, or does not, change the sign of ψ_n according as it does, or does not, induce an odd permutation of those pairs (ω, σ) for which $\iota = 1$. Pairs of variables with $\iota = 0$ and $\iota = 1$ will be referred to as "boson" and "fermion variables". If both occur we shall speak of the case of partial antisymmetry, or of a boson-fermion mixture.

In the following we shall, in general, suppress the accessory variables σ, but we keep in mind that an expression such as $|\psi_n|^2 = \bar{\psi}_n \psi_n$ should

stand for a sum with respect to the discrete variables covered by σ and an integral with respect to the continuous such variables. On occasion we shall refer to the suppressed variables σ as "silent".

Our next task is to describe the interaction energy operators that we shall adopt as disturbing operators V. These operators will belong to a class of general operators K which transform the representers of a state Φ into the representers of the state $K\Phi$ with the aid of integration processes. Such an operator K should be of the form

$$K = \sum_{l=0,\, m}^{\infty} K_{lm},$$

where for any non-negative integer r the operator K_{lm} transforms an $(r + m)$-state Ψ_{r+m} into an $(r + l)$-state with the aid of m integrations; for $r < 0$ we should have $K_{lm}\Psi_{r+m} = 0$. Specifically, the operator K_{lm} should be "generated" by a sequence of functions $k^r_{lm}((\omega)_l; (\omega')_m)$, called "kernels", which have the same symmetry property in the l variables $(\omega)_l$ and in the m variables $(\omega')_m$ that was required of the components ψ_l and ψ_m of Φ. We assume $k^r_{lm} = 0$ for $r < 0$. The "values" of the function k^r_{lm} are to be operators acting on functions of the accessory variables σ. A vector ψ_{r+m} with the representer $\psi_{r+m}(\omega)_{r+m}$ should then be transformed by the operator K_{lm} into the $(r + l)$-vector $K_{lm}\Psi_{r+m}$ with the representer

$$r! \sum \text{Perm} \pm \int \cdots \int k^r_{lm}(\omega_{r+1}, \cdots, \omega_{r+l}; \omega'_{r+1}, \cdots, \omega'_{r+m}) \cdot \psi_{r+m}(\omega_1, \cdots, \omega_r, \omega'_{r+1}, \cdots, \omega'_{r+m})\, d\omega'_{r+1} \cdots d\omega'_{r+m}.$$

The sign "Perm" is to indicate any permutation of the variables $\omega_1, \cdots, \omega_{r+l}$, that is to say, of the variables not involved in an integration. The plus or minus sign is to be taken according as the permutation is even or odd with respect to all fermion variables. The sign "$\sum$ Perm" is to indicate that the sum of the results of all indicated permutations is to be taken. It is necessary to add the results of permutations in order to achieve that the resulting function of $\omega_1, \cdots, \omega_{r+l}$ has the required symmetry property. Since each of these $(r + l)!$ functions has this property already with respect to the first r arguments of ψ_{r+m}, permutation of these variables would not be necessary. Division by $r!$ is equivalent to restricting oneself to permutations in which the order of these r variables is not changed.

It should also be said that in case $m = 0$ the variables ω' and the symbols $\int \cdots d\omega'$ are to be omitted, so that actually no integration is involved in applying an operator K_{l0}.

Owing to the symmetrization entering the definition of the operators K, these operators differ essentially from the gentle operators treated in Chapter II no matter how smooth the functions k^r_{lm} may be chosen. This will be explained in Appendix A16.

The operators of the quantum theory of fields and those of the "theory of many particles" belong to a particular class of operators K, namely, the class of operators $G = \sum_{l,m=0}^{\infty} G_{lm}$ whose kernels are independent of r,

$$g^r_{lm} = g^0_{lm} = g_{lm} \quad \text{for } r = 0, 1, \cdots.$$

Thus, for operators of this class the way the operator G_{lm} transforms an m-state into an l-state determines the way it transforms an $(m + r)$-state into an $(l + r)$-state.

The function $g_{lm}((\omega)_l; (\omega')_m)$ will be assumed to be the kernel of an integral transformation which transforms any square integrable function of m variables into such a function of l variables, so that G_{lm} is applicable on any m-state Ψ_m. It is readily verified then that G_{lm} is also applicable on any n-state Ψ_n for arbitrary n. Each operator G_{lm} is evidently applicable on vectors Φ having only a finite number of components $\psi_n \neq 0$ and produces vectors $G_{lm}\Phi$ with this property. Note that the space of these vectors, which will be denoted by $\mathfrak{H}$, is dense in $\mathfrak{H}$. Also any finite sum $\sum_{l,m} G_{lm}$ is applicable in $\mathfrak{H}$ and produces vectors in $\mathfrak{H}$. Special provisions, however, must be made for infinite sums $\sum_{l,m} G_{lm}$.

There is another, equivalent, description of the operators of the type G which, because of its extreme conciseness, is invaluable for the treatment of quantum problems of fields or many particles. This is the description of the operator G in terms of "annihilation and creation operators".

The *annihilation operator* $A^-(\omega)$ depends on a real variable ω (and possibly on a silent variable σ); its effect on a vector Φ can be described by the formula

$$A^-(\omega)\Phi \underset{0}{\Leftrightarrow} \{\psi_1(\omega), \psi_2(\omega_1, \omega), \psi_3(\omega_1, \omega_2, \omega), \cdots\}.$$

Thus, by "freezing" the last variable of each representer ψ_n, the operator $A^-(\omega)$ transforms an n-state into an $(n - 1)$-state provided $n > 0$; the vacuum state Φ_0 is completely annihilated

$$A^-(\omega)\Phi_0 = 0.$$

In general, the functions of $(\omega)_n$ resulting from application of the operator $A^-(\omega)$ will not be square integrable for every ω; accordingly,

the operator $A^-(\omega)$ has only a symbolic significance. A proper operator can be defined with the aid of a "test function" $g(\omega)$. It is convenient to write this operator symbolically as $\int g(\omega)A^-(\omega)\,d\omega$ and define it by describing its effect on a vector Φ in terms of the representation of Φ:

$$\int g(\omega)A^-(\omega)\,d\omega\Phi$$
$$\underset{0}{\Leftrightarrow} \left\{\int g(\omega)\psi_1(\omega)\,d\omega, \int g(\omega)\psi_2(\omega_1, \omega)\,d\omega, \int g(\omega)\psi_3(\omega_1, \omega_2, \omega)\,d\omega, \cdots\right\}.$$

The symbolic *creation operator* $A^+(\omega)$ will be characterized by defining the proper operator $\int A^+(\omega)g(\omega)\,d\omega$ which depends on a test function g. Specifically, we define this operator by the representation

$$\int A^+(\omega)g(\omega)\,d\omega\Phi \underset{0}{\Leftrightarrow} \{0, \psi_0 g(\omega_1), \psi_1(\omega_1)g(\omega_2) + \psi_1(\omega_2)g(\omega_1), \psi_2(\omega_1, \omega_2)g(\omega_3) + \psi_2(\omega_1, \omega_3)g(\omega_2) + \psi_2(\omega_2, \omega_3)g(\omega_1), \cdots\}.$$

It is clear how the nth component on the right should be formed: One should add the n functions which result from $\psi_{n-1}(\omega_1, \cdots, \omega_{n-1})g(\omega_n)$ by interchanging ω_n successively with $\omega_1, \cdots, \omega_{n-1}$ after supplying these functions with a + or − sign according as the permutation is even or odd with respect to the fermion variables.

The most important property of the annihilation and creation operators is that they obey the *canonical commutation law*. In case of pure symmetry this law reads

$$\left[\int g'(\omega')A^-(\omega')\,d\omega', \int A^+(\omega)g(\omega)\,d\omega\right] = \int g'(\omega)g(\omega)\,d\omega,$$

where

$$[A, B] = AB - BA$$

denotes the commutator of the operators A and B and where g and g' are any test functions.

This relation can also be described without employing test functions, simply by the symbolic relation

$$[A^-(\omega'), A^+(\omega)] = \delta(\omega' - \omega),$$

where δ signifies Dirac's delta function. In either form the commutation relation can be readily verified.

In case of antisymmetry the commutator is to be replaced by the

"anticommutator" $\{A, B\} = AB + BA$. In case of partial antisymmetry the relation takes the form

$$A^-(\omega', \sigma')A^+(\omega, \sigma) - (-1)^{\iota\iota'}A^+(\omega, \sigma)A^-(\omega', \sigma') = \delta(\omega' - \omega)\,\delta(\sigma' - \sigma),$$

with $\iota = \iota(\sigma)$ and $\iota' = \iota(\sigma')$. (We recall that $\iota = 0$ for a boson state, $\iota = 1$ for a fermion state.) The term $\delta(\sigma' - \sigma)$ is a product of the Dirac and Kronecker delta functions of all continuous and discrete variables which compose $\sigma' - \sigma$. We shall refer to this general case only on a few occasions.

The annihilation and creation operators should commute among each other,

$$[A^-(s'), A^-(s)] = [A^+(s'), A^+(s)] = 0,$$

in case of symmetry; in case of antisymmetry the factor $(-1)^{\iota\iota'}$ should be supplied.

Another important, and easily verified, property of the annihilation and creation operators should be mentioned although we shall refer to it only incidentally. This is the property that the operators $\int \bar{g}(\omega')A^-(\omega')\,d\omega'$ and $\int A^+(\omega)g(\omega)\,d\omega$ are formally adjoint to each other.

Instead of employing just one annihilation or creation operator and a test function of one variable, one can form proper operators by employing several annihilation and creation operators and a function of several variables in place of a test function. We take a function $g_{lm}((\omega)_l; (\omega')_m)$ of l variables ω and m variables ω' which has the same symmetry property in $(\omega)_l$ and in $(\omega')_m$ that was required of the components ψ_l and ψ_m. Then we form the operator

$$G_{lm} = \int\int A^+(\omega_l)\cdots A^+(\omega_1)g_{lm}((\omega)_l; (\omega')_m)A^-(\omega'_1)\cdots A^-(\omega'_m)\,d(\omega)_l\,d(\omega')_m.$$

To make the meaning of this symbolic description clear we specify how to apply this operator G_{lm} on an n-state Ψ_n. To carry out the m annihilation processes one should rename the last m variables of the component $\psi_n((\omega)_n)$ of Ψ_n, calling them $\omega'_1, \cdots, \omega'_m$. Next one should multiply $\psi_n((\omega)_{n-m}; (\omega')_m)$ by $g_{lm}((\omega)_l; (\omega')_m)$ and integrate the product with respect to $\omega'_1, \cdots, \omega'_m$. The result considered as a function of the $n - m$ first variables $(\omega)_{n-m}$ of ψ_n is the $(n - m)$-component of the vector

$$\int g_{lm}((\omega)_l; (\omega')_m)A^-(\omega'_1)\cdots A^-(\omega'_m)\,d(\omega')_m\Psi_n.$$

To apply the l creation operators one should rename the variables $(\omega)_l$, calling them $\omega_{n-m+1}, \cdots, \omega_{n-m+l}$, and then perform all permutations of the $n - m + l$ variables $(\omega)_{n-m+l}$ of the resulting function, disregarding permutations of the first $n - m$ arguments of the original component ψ_n. The resulting functions, after having been supplied with the proper sign, should then be added. The function thus obtained is the $(n - m + l)$th component of the $(n - m + l)$-state $G_{lm}\Psi_n$.

Thus we see: When acting on an m-state Ψ_m the operator G_{lm} produces an l-state. The lth component of the state $G_{lm}\Psi_m$ is evidently obtained by applying an integral operator with a kernel given by g_{lm} on the mth component ψ_m of Ψ_m. When G_{lm} is applied on a state Ψ_n with $n > m$, it also acts as an integral operator, but only m of the n variables of the component ψ_n of Ψ_n are affected by the integration. If G_{lm} acts on a state Ψ_n with $n < m$ it produces 0.

One readily verifies that the operators denoted by G_{lm} earlier are operators of exactly the type described now.

Operators G_{lm} of this type will be called "basic annihilation-creation operators". If we want to indicate that the operator G_{lm} is generated by the integral operator with the kernel g_{lm} we shall write

$$G_{lm} \underset{0}{\leftrightarrows} g_{lm}((\omega)_l; (\omega')_m).$$

Operators of the form

$$G = \sum_{l,m=0} G_{lm}$$

will be called "annihilation-creation operators" quite generally.

We now stipulate that the disturbing operator V should be such an annihilation-creation operator.[1]

As a matter of fact, even the undisturbed energy operator H_0 is of the annihilation-creation form inasmuch as it can symbolically be written as

$$H_0 = \int A^+(\omega)\omega A^-(\omega)\, d\omega,$$

as is easily verified. Obviously this operator is of the type G_{11}, but its kernel, $g_{11}(\omega; \omega') = \omega\delta(\omega - \omega')$, is not a proper function.

[1] In this restriction our approach differs from that of van Hove and Jack Schwartz [44; 47], who admit more general operators of type K. By restricting ourselves to operators of class (G) we can describe a formalism which is simpler than, though in principle covered by, the formalism of van Hove. For certain types of operators of class (K) which are not of class (G), J. Schwartz obtains rigorous results.

In our discussion of the perturbation of continuous spectra we have emphasized the importance of the smoothness of the kernel of the disturbing integral operator. To remain mathematically as realistic as possible, we shall require a high degree of smoothness of the kernels $v_{lm}((\omega)_l; (\omega')_m)$ of our present disturbing operator V. We shall deal with two types of smoothness: "total" and "conservation" smoothness.

We call an operator of type G *totally smooth* if the kernels g_{lm} are continuous functions of the $l + m$ variables $(\omega)_l$ and $(\omega')_m$.

In the case of "conservation smoothness" the kernels g_{lm} of the operators G_{lm} possess a certain delta function as a factor. This factor insures that a state Ψ_n in which n particles with a definite total momentum are present is transformed by application of the operator G_{lm} into a state with the same value of the total momentum. The disturbing operators in all realistic and unrealistic models in the quantum theory of fields that are treated by physicists are endowed with this conservation character provided this disturbing operator corresponds to the interaction of one field with another (or with itself) but not with an external source. The interaction of a field with a fixed external source may be approximately represented by a totally smooth disturbing operator V.

At first we shall restrict ourselves to totally smooth disturbances; the case of conservation smoothness (with the restriction $V_{0m} = V_{l0} = 0$) will to a certain degree be treated in §§17, 18 and Appendix A19.

Since we want to take our disturbing operators from the class of annihilation-creation operators it is important to know whether or not such operators are bounded. First we state that *in case of symmetry the operators* G_{lm}, *except for* $l = m = 0$, *are unbounded.* For the operators G_{01} and G_{10} this follows immediately from the commutation relation; cf. e.g. [**41**]. For general operators G_{lm} this fact is implied by the results of Galindo mentioned below. The same is true for a finite sum for convenience written as $\sum_{l+m>0} G_{lm}$, but not necessarily for an infinite sum of operators G_{lm}.

If the disturbing operator V is unbounded the question arises whether or not the operator $H_0 + V$, defined in $\mathfrak{H}$ can be extended to a strictly selfadjoint operator in case V is formally selfadjoint. This is well known to be true if V is positive-definite or at least semi-bounded; under very wide conditions it was shown by Galindo [**51**] to be true for operators given as a finite sum $V = \sum_{l+m>0} V_{lm}$.

On the other hand, by exploiting the unboundedness of such operators, again in case of symmetry, Galindo has established that *the operator* $H_0 + V$ *has no lower bound* if V can be written in the

form $\sum_{0<l+m\leqq s} V_{lm}$ (with $s < \infty$ being the smallest such subscript) provided V_{ss} is not positive-definite. As a consequence *the spectrum of this operator reaches down to* $-\infty$. Certainly then, such an operator $H_0 + V$ is not equivalent with the operator H_0. We shall refer to this fact in the next section.

Similar facts can be expected to obtain in the case of partial antisymmetry; but the situation is quite different in the case of pure antisymmetry. For, in case of antisymmetry the basic operators $\int A^+(\omega)g(\omega)\,d\omega$ and $\int \bar{g}(\omega')A^-(\omega')\,d\omega'$ are bounded if g is square integrable,

$$\int \overline{g(\omega)}g(\omega)\,d\omega < \infty.$$

This follows, by virtue of the commutation law, from the identity

$$\begin{aligned}\|A^+\cdot g\Phi\|^2 + \|\bar{g}\cdot A^-\Phi\|^2 &= (\Phi, \{\bar{g}\cdot A^-A^+\cdot g + A^+\cdot g\bar{g}\cdot A^-\}\Phi)\\ &= (\bar{g}\cdot g)(\Phi, \Phi),\end{aligned}$$

see [**41**], in which we have used the abbreviation

$$f\cdot g = \int f(\omega)g(\omega)\,d\omega,$$

where the function $f(\omega)$ may also stand for an operator such as $A^{\pm}(\omega)$.

Using this fact one can show that the operators V_{lm} are bounded provided the kernels v_{lm} belong to an appropriate class of smooth functions. If the operator V consists of a finite sum of such operators V_{lm}, the operator $H_0 + V$ is defined whenever H_0 is. In case of antisymmetry, therefore, it is not a priori excluded that the disturbed operator $H_0 + V$ is equivalent with the undisturbed operator H_0.

Incidentally, a similar remark applies to the case in which V is linear in the annihilation-creation operators of boson variables, but nonlinear in those of fermion variables.

10. Perturbation by totally smooth annihilation-creation operators. Our aim is to find a spectral representation of the operator

$$H = H_0 + V.$$

To this end, we should like to find a pair of bounded operators $U^{\pm}$, inverse to each other, which transform H_0 into H according to the relation

$$H = U^-H_0U^+.$$

If such operators $U^{\pm}$ existed, the operators H and H_0 would be equivalent and hence would have the same spectrum. But we certainly

have no right to expect that such operators $U^{\pm}$ exist unless the disturbing operator V obeys severe restrictions; as a matter of fact, we have no right to expect that total smoothness of the operator V be sufficient for this existence.

There are several reasons for this. First of all the operator $H_0 + V$ might be unbounded below (this can happen, as was mentioned at the end of the last section) so that the spectrum of $H_0 + V$ reaches to $-\infty$. Next, if $H_0 + V$ is bounded below, the spectrum of this operator might be quite different from that of H_0; in particular, it need not be associated with a particle representation (see e.g. §16). But even if this were the case, the spectrum of $H_0 + V$ might be shifted relative to that of H_0. In fact, this should be expected to occur in general.

Note that the undisturbed operator H_0 has a point eigenvalue, the eigenvalue zero with the vacuum state Ψ_0 as eigenvector. In general, one must expect that this eigenvalue will change under the perturbation provided terms V_{0m} or V_{l0} are present among the components V_{lm} of V. At the same time even the continuous spectra $n\omega_* \leqq \omega$ associated with the n-particle state must be expected to move. It is most remarkable that these changes will be of a very simple type if, as we have assumed, the operator V is an annihilation-creation operator. The eigenvalue zero and the lower ends $n\omega_*$ of the spectra $n\omega_* \leqq \omega$ will simply move by the same amount. This remarkable feature is indicated by the formalism we are going to develop.

In a situation as described it is not difficult to eliminate the shifts, and thus to achieve that the operator H has the same spectrum as H_0, provided one is willing to tamper slightly with the disturbing operator V. In fact, one single adjustment of V is sufficient: the proper choice of the term V_{00} of V. Discussion of this adjustment will be one of the major items of our exposition.

We mention incidentally that in the case of conservation smoothness an additional adjustment is needed, namely, that of the term V_{11}, or, what can be shown to be equivalent, the adjustment of the energy function $\omega(k)$ which gives the energy of a single particle in terms of its momentum k.

In accordance with what we have said above we shall keep the value of the 00-term V_{00} of V in abeyance when we try to find a pair of operators $U^{\pm}$.

The operators $U^{\pm}$ will be given in terms of another pair of operators $T^{\pm}$ which also intertwine with H and H_0; i.e., satisfy the same "transformation equations"

$$T^{+}H = H_0T^{+}, \qquad HT^{-} = T^{-}H_0$$

as $U^{\pm}$, but which are not required to be inverse to each other. Instead, they are required to be of the form

$$T^{\pm} = 1 \pm \Gamma R^{\pm},$$

where the operation Γ is so chosen that the operators $\Gamma R^{\pm}$ are solutions of the equation

$$[H_0, \Gamma R^{\pm}] = R^{\pm}.$$

In contrast with what is the case for the simple perturbation problem treated in Chapter II, being of the form $1 \pm \Gamma R^{\pm}$ will not necessarily make the two operators $T^{\pm}$ inverse to each other. But we shall be able to derive from the operators $T^{\pm}$ other operators $U^{\pm}$ which have this property.

If the transformation equations are satisfied by $T^{\pm} = 1 \pm \Gamma R^{\pm}$, the relations

$$[H_0, \Gamma R^{\pm}] = T^{+}V, \qquad [H_0, \Gamma R^{-}] = VT^{-}$$

hold for the operators $\Gamma R^{\pm}$; consequently, the operators $R^{\pm}$ satisfy the equations

$$R^{+} = (1 + \Gamma R^{+})V, \qquad R^{-} = V(1 - \Gamma R^{-}).$$

We shall try to find annihilation-creation operators $R^{\pm}$ which solve these equations. Clearly, operators $T^{\pm} = 1 \pm \Gamma R^{\pm}$ formed with such $R^{\pm}$ satisfy the transformation equations.

This set-up is exactly the same as that for the perturbation of continuous spectra treated in §6. The difference will become visible when, in trying to mimic the procedure of §6, we shall investigate the smoothness properties of the kernels of the integral operators generating the operators $R^{\pm}$.

First of all let us discuss the operation Γ. Let G_{lm} be an operator generated by a kernel g_{lm} and hence given by

$$G_{lm} = \int A^{+}(\omega_1) \cdots A^{+}(\omega_l) g_{lm}((\omega)_l; (\omega')_m) A^{-}(\omega'_1) \cdots A^{-}(\omega'_m)\, d(\omega)_l\, d(\omega')_m.$$

Then the commutator $[H_0, G_{lm}]$ is also an annihilation-creation operator. Applying the operators $H_0 G_{lm}$ and $G_{lm} H_0$ on an $(r + m)$-state, one will observe in subtracting the results that the r unaffected ω's cancel out and thus find

$$[H_0, G_{lm}] = \int A^{+}(\omega_1) \cdots A^{+}(\omega_l)(\omega - \omega')_{lm} g_{lm}((\omega)_l; (\omega')_m) A^{-}(\omega'_1) \cdots A^{-}(\omega'_m)\, d(\omega)_l\, d(\omega')_m$$

with

$$(\omega - \omega')_{lm} = \omega_1 + \cdots + \omega_l - \omega'_1 - \cdots - \omega'_m.$$

In view of this formula one should expect that a solution G_{lm} of the equation $[H_0, G_{lm}] = Q_{lm}$ will be given by the operator generated by the kernel $[\omega - \omega']^{-1}_{lm} q_{lm}$, where $q_{lm}((\omega)_l; (\omega')_m)$ is the kernel generating the annihilation-creation operator Q_{lm}, and $[\omega - \omega']^{-1}_{lm}$ is a particular inverse of $(\omega - \omega')_{lm}$.

In analogy with the choice made in §6 we define

$$[\omega - \omega']^{-1}_{lm} = (\omega - \omega')^{-1}_{lm} + i\pi\delta(\omega - \omega')_{lm}$$

except for $lm = 0$. For $l = 0$ and for $m = 0$ we omit the delta function. For sufficiently smooth kernels q_{lm} we then shall define ΓQ_{lm} as the operator with the kernel

$$[\omega - \omega']^{-1}_{lm} q_{lm}((s)_l; (s)_m).$$

The operation Γ is also applicable to linear combinations $Q = \sum_{lm} Q_{lm}$ of operators Q_{lm} provided no 00-term Q_{00} occurs among them. Clearly, if the requirement

$$Q_{00} = 0$$

is violated, the operation Γ cannot be defined. For, the obvious relation $(\omega - \omega')_{00} = 0$ implies $[H_0, G_{00}] = 0$, so that the equation $[H_0, G_{00}] = Q_{00}$ has no solution unless $Q_{00} = 0$. This condition $Q_{00} = 0$ will play a major role in the subsequent analysis.

In particular, this condition must be observed when we try to solve the equations for the operators $R^{\pm}$ formulated above. Since the operation Γ is to be applicable to the operators $R^{\pm}$, we are forced to require these operators to have no 00-terms. Here then, we meet our *first obstacle*: Suppose we want to solve the equations for $R = R^{\pm}$ by iterations

$$R^{(0)} = V, \qquad R^{(n+1)} = (1 + \Gamma R^{(n)})V,$$

and

$$R^{(0)} = V, \qquad R^{(n+1)} = V(1 - \Gamma R^{(n)}).$$

In trying to do that we see that this process cannot be continued unless the operators $R^{(n+1)}$ satisfy the condition $R^{(n+1)}_{00} = 0$ if the $R^{(n)}$ do; but this will in general not be the case. The same difficulty arises if we try to solve these equations by a series expansion

$$R^{+} = V + (\Gamma V)V + (\Gamma(\Gamma V)V) + \cdots,$$
$$R^{-} = V - V\Gamma V - V\Gamma(V\Gamma V) - \cdots.$$

This obstacle can be overcome. But aside from it, other obstacles will be met when one tries to introduce a class of operators, to which

V should be restricted, such that $P\Gamma Q$ and $(\Gamma P)Q$ belong to this class whenever P and Q do. To analyze these circumstances it is necessary to describe in detail the nature of the product of two annihilation-creation operators.

11. Products and graphs. At first we shall consider two general basic annihilation-creation operators F_{jk} and G_{lm} and rewrite the product $F_{jk}G_{lm}$, given by

$$F_{jk}G_{lm} = \int\int\int\int A^+(v_1)\cdots A^+(v_j)f_{jk}A^-(v'_1)\cdots A^-(v'_k)$$
$$\cdot A^+(\omega_1)\cdots A^+(\omega_l)g_{lm}A^-(\omega'_1)\cdots A^-(\omega'_m)\,d(\omega)_l\,d(v)_l\,d(\omega')_m\,d(v')_m$$

in a different, "ordered", form. To this end we apply the basic commutation law (see §9),

$$A^-(v')A^+(\omega) = (-1)^{\iota'}A^+(\omega)A^-(v') + \delta(v' - \omega),$$

successively until there is no longer any annihilation operator ahead of a creation operator. The result is an expression of the product $F_{jk}G_{lm}$ as a sum of basic annihilation-creation operators, referred to as "contributions" to the product.

One of these contributions is simply obtained by placing all annihilation operators to the right, all creation operators to the left and supplying the proper sign; it is called the "Wick product" and indicated by double dots :

$$:F_{jk}G_{lm}: = \int\int\int\int A^+(v_1)\cdots A^+(\omega_l)f_{jk}(-1)^{\Sigma^k\iota\cdot\Sigma^l\iota}g_{lm}A^-(v'_1)\cdots$$
$$A^-(\omega'_m)\,d(\omega)_l\,d(v)_j\,d(v'_l)_k\,d(\omega')_m,$$

where $\sum^k\iota$, $\sum^l\iota = \iota(\tau_1) + \cdots + \iota(\tau_k)$ and $\sum^k\iota$, $\sum^l\iota = \iota(\sigma_1) + \cdots + \iota(\sigma_l)$. As we recall, if the value of σ associated with a particle is such that $\iota(\sigma) = 0$ the particle is a boson, and if $\iota(\sigma) = 1$ it is a fermion.

All other contributions to the product $F_{jk}G_{lm}$ are formed with the aid of at least one delta function $\delta(v' - \omega)$ where v' stands for one of the variables $v'_1, \cdots, v'_k$ and ω for one of the variables $\omega_1, \cdots, \omega_l$. This delta function can be made to disappear by substituting $v' = \omega$ in g_{lm} and then integrating with respect to the variable v'. The integration with respect to ω remains. We say that the variable v' was "contracted" with ω.

The sum of all contributions to $F_{jk}G_{lm}$ which involve contractions will be called the "attached" product of F_{jk} and G_{lm} and denoted by $F_{jk} \multimap G_{lm}$. Thus we may write

$$F_{jk}G_{lm} = F_{jk} \multimap G_{lm} + :F_{jk}G_{lm}:.$$

The Wick product has the important property, under circumstances which we are going to describe, that the order of its factors may be interchanged

$$:G_1G_2: = :G_2G_1:.$$

We then say that the "Wick factors commute". Certainly they do in the case of pure symmetry since in that case annihilation operators A^- as well as creation operators A^+ commute among themselves. In case of pure antisymmetry the Wick factors do not commute if the operators G_1 and G_2 are odd functions, but do if these operators are even functions of the $A^\pm$. To cover the case of partial antisymmetry we say that an operator G_{lm} is "even in fermion annihilation and creation operators" if the kernel

$$g_{lm}((\omega_1, \sigma_1), \cdots, (\omega_l, \sigma_l); (\omega'_1, \sigma'_1), \cdots, (\omega'_m, \sigma'_m))$$

vanishes for those values of the arguments $(\omega_1, \sigma_1), \cdots, (\omega'_m, \sigma'_m)$ which involve an odd number of fermion variables; that is, variables (ω, σ) with $\iota(\sigma) = 1$. If both operators F_{jk} and G_{lm} have this property the factors in their Wick product commute.

Eventually, in §12, we shall assume that our interaction operator V has this property of being even in fermion annihilators and creators.

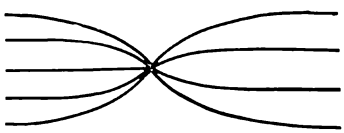

FIGURE 1. GRAPH OF BASIC OPERATOR G_{54}

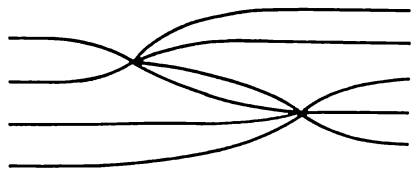

FIGURE 2. GRAPH OF CONTRIBUTION TO PRODUCT $F_{24}G_{43}$ WITH TWO CONTRACTIONS

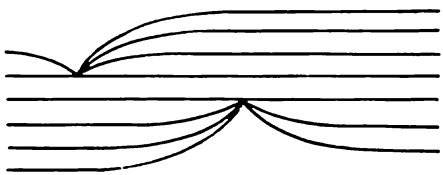

FIGURE 3. GRAPH OF WICK PRODUCT: $F_{24}G_{43}$

It is perhaps worthwhile to describe a graphical representation of operators G_{lm} and their products.

To an lm-operator G_{lm} we assign as its graph a point to which $l + m$ segments, called "prongs", are attached, l pointing to the left, m to the right, all extending the same horizontal distance from the vertex.

If two operators F_{jk}, G_{lm} are to be multiplied we place the graph of F_{jk} to the left of that of G_{lm}. To a contribution to the product $F_{jk}G_{lm}$ which involves r contractions we assign a graph obtained by connecting r annihilation prongs of F_{jk} with r creation prongs of G_{lm} and extending the unaffected annihilation prongs of F_{jk} to the end level of those of G_{lm} and the unaffected creation prongs to the end level of those of F_{jk}.

The graph of a Wick product, not involving any contractions, is called "disconnected".

Each contribution involving r contractions will be denoted by

$$F_{jk} \overset{\circ}{\underset{r}{-}} G_{lm};$$

all these contributions are equal, as follows immediately from the commutation laws and the required symmetry property of the kernels. The number of these contributions involving r contractions is

$$N_{kl}^{r} = C_{k,r}r!C_{l,r} = k!l!(k-r)!(l-r)!r!,$$

where $C_{ab} = a!b!(a-b)!$ is a binomial coefficient; for, these contributions are obtained by making all $C_{l,r}$ selections of r creation prongs of G_{lm}, all $C_{k,r}$ selections of r annihilation prongs of F_{jk}, and then making all $r!$ contractions between them. The total attached product can therefore be written as

$$F_{jk} \overset{\circ}{-} G_{lm} = \sum_{r>0} N_{kl}^{r}(F_{jk} \overset{\circ}{\underset{r}{-}} G_{lm}).$$

Let us return to the investigation of products of the type $P_{jk}\Gamma Q_{lm}$ and $(\Gamma P_{jk})Q_{lm}$. Suppose then that G_{lm} is an operator of the form $G_{lm} = \Gamma Q_{lm}$ and that Q_{lm} and P_{jk} having kernels sufficiently smooth to admit the operation Γ. Then the kernel of each contribution to the attached product of these two operators involves an integration with respect to at least one of the variables ω; this integration smoothes out the singularity $[\omega - \omega']^{-1}$ which is present because of the presence of the operation Γ, just in the same way as this was the case for the single particle interaction treated in §6. Because of this smoothing effect one should expect that the attached products $P_{jk} \overset{\circ}{\underset{r}{-}} \Gamma Q_{lm}$ and $P_{jk} \overset{\circ}{\underset{r}{-}} Q_{m}$ again are operators of type Q admitting the operation Γ (unless they are 00-terms) provided the smoothness requirements on the kernels are properly chosen.

We should like to formalize these matters somewhat. We assume that the operators P_{jk} and Q_{lm} belong to a linear manifold $\mathfrak{Q}$ of operators having various properties of which two will be formulated presently, while the others will be formulated when they are used for

the first time. Classes $\mathfrak{Q}$ of operators Q having these properties will be described in Appendix A17.

We assume that every operator in $\mathfrak{Q}$ is the linear combination $\sum_{lm} Q_{lm}$ of operators Q_{lm} in $\mathfrak{Q}$. The subclass of the operators $\sum_{l+m>0} G_{lm}$, that is of those for which $Q_{00} = 0$, will be denoted by $\mathfrak{Q}'$. Then we require (I) that the operation Γ should be defined for operators Q in $\mathfrak{Q}'$ and there should exist a dense subspace $\ddot{\mathfrak{H}}$ of $\mathfrak{H}$ such that the operators ΓQ for Q in $\mathfrak{Q}'$ are applicable in $\ddot{\mathfrak{H}}$. The class of operators ΓQ with Q in $\mathfrak{Q}'$ will also be referred to as class $\Gamma\mathfrak{Q}'$. The important property (II) now is that the attached products

$$P \multimap \Gamma Q \quad \text{and} \quad \Gamma P \multimap Q$$

should belong to the class $\mathfrak{Q}$ provided P and Q belong to $\mathfrak{Q}'$.

The situation is quite different as regards the Wick products $:P\Gamma Q:$ and $:(\Gamma P)Q:$; their kernels are formed from those of P and Q without integration; the singularity of $[\omega - \omega']^{-1}$ is not smoothed out. These Wick products, therefore, are not totally smooth and hence do not belong to the class $\mathfrak{Q}$. This is the cause of our *second obstacle*: The products $P\Gamma Q$, and $(\Gamma P)Q$ do not belong to the class $\mathfrak{Q}$ if P and Q belong to $\mathfrak{Q}'$ since the Wick contribution is included in these products.

We shall overcome this "second obstacle" by replacing the basic equations for the operators $R^{\pm}$ by other, equivalent, equations which involve attached products instead of ordinary products; then the trouble caused by the Wick products will be eliminated.

Our subsequent developments will largely be based on the analogue of the identity

$$\Gamma(P\Gamma Q + (\Gamma P)Q) = \Gamma P \Gamma Q$$

which was formulated in §6 for operators which now would be called Q_{11} and P_{11} restricted to act on states Ψ_1. Carrying over the arguments of §6 one will find that this identity holds even for the attached products formed with any annihilation-creation operators Q, P in $\mathfrak{Q}'$ provided the attached product $\Gamma P \multimap \Gamma Q$ has no 00-component. In this connection one should observe that the operator $P_{0k} \multimap \Gamma Q_{l0} + \Gamma P_{0k} \multimap Q_0$ never has a 00-term since the factors $[\tilde{\omega}]_0^{-1}$ and $[-\tilde{\omega}]_{0k}^{-1}$ supplied by the Γ-operation cancel out. Here $\tilde{\omega}$ refers to the variables involved in the contraction. The same, obviously, applies to $P \multimap \Gamma Q + \Gamma P \multimap Q$. If $(\Gamma P \multimap \Gamma Q)_{00} = 0$ is not assumed, the identity takes the form

$$\Gamma(P \multimap \Gamma Q + \Gamma P \multimap Q) = \Gamma P \multimap \Gamma Q - (\Gamma P \multimap \Gamma Q)_{00}.$$

This identity in fact holds for each single contribution to $P \multimap \Gamma Q$

$+ \Gamma P \multimap Q$ provided that corresponding contributions of $P \multimap \Gamma Q$ and $\Gamma P \multimap Q$, i.e., contributions having the same graph, are combined.

Although the Wick products $G = :P\Gamma Q:$ and $G = :(\Gamma P)Q:$ with P and Q in $\mathfrak{Q}'$ do not belong to the class $\mathfrak{Q}$, operators ΓG can be defined for them as being applicable to vectors in the space $\ddot{\mathfrak{H}}$ described a little earlier, so that the relation $[H_0, \Gamma Q] = Q$ holds; at the same time the identity

$$\Gamma:P\Gamma Q + (\Gamma P)Q: = :(\Gamma P)\Gamma Q:$$

can be verified. Note that neither of the Wick products occurring here has a 00-term since P and Q do not.

We do not want to go into details, but accept the last formulas as the basis of our subsequent analysis.

The operation Γ can similarly be extended to Wick products of several factors and one readily verifies the identity

$$\Gamma:Q_1\Gamma Q_2 \cdots \Gamma Q_n + (\Gamma Q_1)Q_2\Gamma Q_3 \cdots \Gamma Q_n + \cdots + (\Gamma Q_1 \cdots \Gamma Q_{n-1})Q: = :\Gamma Q_1 \cdots \Gamma Q_n:.$$

12. Two basic formulas. Basic equations. In case the operators $G_1, \cdots, G_n$ entering the Wick product $:G_1 \cdots G_n:$ are equal (we then set $G_1 = \cdots = G_n = G$), we speak of a "Wick power" $:G^n:$; in case $n - 1$ of these factors are equal, all but the νth factor say, we use the symbol $:G^{\nu-1}G_\nu G^{n-\nu}:$.

At this place we formulate an additional (not numbered) requirement on the operators Q of our class $\mathfrak{Q}$: *These operators Q should be even in the fermion annihilators and creators $A^\pm$.* That is to say each component Q_{lm} should have this property (described in §11). As a consequence, the factors in a Wick product of such operators Q, or ΓQ, commute. In particular, it then follows that the operator $:nQ(\Gamma Q)^{n-1}:$ can be written in the form

$$:nQ(\Gamma Q)^{n-1}: = :Q(\Gamma Q)^{n-1} + (\Gamma Q)Q(\Gamma Q)^{n-2} + \cdots + (\Gamma Q)^{n-1}Q:.$$

The right number here is evidently the commutator of H_0 with $:(\Gamma Q)^n:$, so that we may write

$$[H_0, :(\Gamma Q)^n:] = :nQ(\Gamma Q)^{n-1}:.$$

Applying the operator Γ on both sides we obtain the identity

$$:nQ(\Gamma Q)^{n-1}: = :(\Gamma Q)^n:,$$

which agrees with the last identity of §11 for $Q_1 = \cdots = Q_n = Q$.

With the aid of the operators $:G^n:$ we form the "Wick exponential function" defined by

$$:e^G: = 1 + \sum_{n=1}^{\infty} :n!\, G^n:,$$

actually used only for G in $\Gamma\mathfrak{Q}'$; with Q in $\mathfrak{Q}'$; with the aid of the operators $:nQ(\Gamma Q)^n:$ we form the series

$$:Qe^{\Gamma Q}: = \sum_{n=1}^{\infty} :(n-1)!\, Q(\Gamma Q)^{n-1}:.$$

As an additional property of our class $\mathfrak{Q}$ we now require that

(III) *These series converge to operators applicable in the subspace* $\ddot{\mathfrak{H}}$ *of* $\mathfrak{H}$ *if the operator* Q *is of the class* $\mathfrak{Q}'$. The class $\mathfrak{Q}$ described in Appendix A17 will have this property.

Applying the operation Γ on each term of the series for $:Qe^{\Gamma Q}:$ we obtain the relation

$$\Gamma:Q\sum_{n=1}^{\infty}(n-1)!(\Gamma Q)^{n-1}: = :\sum_{n=1}^{\infty} n!(\Gamma Q)^n:,$$

which can simply be written as

$$1 + \Gamma:Qe^{\Gamma Q}: = :e^{\Gamma Q}:. \tag{I}$$

This is our *first basic formula.*

This formula is quite suggestive. If we recall the formalism of the perturbation of continuous spectra, reviewed at the beginning of §10, we are clearly led to expect that *the desired operators* $T^{\pm}$ *are Wick exponential functions*

$$T^{\pm} = :e^{\pm\Gamma Q^{\pm}}:,$$

involving two operators $Q^{\pm}$ belonging to the class $\mathfrak{Q}'$ if this class is chosen sufficiently wide. For, according to the basic formula (I) these operators would then be of the desired form

$$T^{\pm} = 1 \pm \Gamma R^{\pm}$$

with operators $R^{\pm}$ given as

$$R^{\pm} = :Q^{\pm}e^{\pm\Gamma Q^{\pm}}:.$$

Equations

$$R^{+} = T^{+}V, \qquad R^{-} = VT^{-},$$

which are equivalent with

$$[H_0, T^{+}] = T^{+}V, \qquad [H_0, T^{-}] = -VT^{-},$$

and hence with

$$T^+(H_0 + V) = H_0T^+, \qquad (H_0 + V)T^- = T^-H_0,$$

then assume the form

$$:Q^+e^{rQ^+}: = :e^{rQ^+}:V, \qquad :Q^-e^{rQ^-}: = V:e^{rQ^-}:.$$

These equations, which should be solved for $Q^{\pm}$, are not yet in a manageable form. It is possible to put them in an accessible form since it is possible quite generally to *write the product of any operator, such as V, with a Wick exponential function as the Wick product of another operator with the Wick exponential function.*

To describe how this can be done we introduce, at first in a purely formal manner, a particular kind of product of an operator with a Wick power and a Wick exponential function.

We call "(left) connected product" of an operator W of type G with a Wick power $:G^n:$ of an operator G with $G_{00} = 0$ the sum of all those contributions to the product $W:G^n:$ in which W is contracted with each of the n factors G. (The graphs of two connected contributions to the product $W_{24}:G_{21}^3:$ is shown in Figure 4.) We denote the left connected product of W and $:G^n:$ by

$$W \prec :G^n: ;$$

FIGURE 4. GRAPHS OF TWO CONNECTED CONTRIBUTIONS TO THE PRODUCT $W_{24}:G_{21}^3:$

this product is to be zero if W has fewer than n annihilation prongs. We now define, at first purely formally, the left connected product of the operator W with a Wick exponential function by the series

$$W \prec :e^G: = \sum_{n=0}^{\infty} W \prec :\tfrac{1}{n!}\, G^n:,$$

setting

$$W \prec :1: = W, \qquad W \prec :G: = W \circ G.$$

In a similar manner we define the right connected product

$$:G^n: \succ W \quad \text{and} \quad :e^G: \succ W.$$

In terms of these connected products our *second basic formulas* read

$$\text{(II)} \qquad \begin{aligned} W{:}e^G{:} &= {:}(W \prec {:}e^G{:}){:}e^G{:}, \\ {:}e^G{:}W &= {:}({:}e^G{:} \succ W){:}e^G{:}. \end{aligned}$$

Evidently, these formulas express the ordinary product with a Wick exponential function as a Wick product.

The validity of these formulas could be established rigorously for operators W in $\mathfrak{Q}$ and G in $\Gamma\mathfrak{Q}'$, when $\mathfrak{Q}$ is the class described in Appendix A17. Here we shall give only a formal derivation. Therefore, we regard *the validity of formulas* (II) *as requirement* (IV) *on our class* $\mathfrak{Q}$.

The (formal) proof of identities (II) is quite simple. We first write the product $V{:}G^n{:}$ as the sum of those contributions in which V is contracted with μ of the factors of G^n, letting μ run from 1 to n. Evidently, the contribution from μ factors of G^n is

$$\binom{n}{\mu}{:}(V \prec {:}G^\mu{:})G^{n-\mu}{:},$$

since there are $C_{n,\mu}$ choices of μ factors. Summing over μ from 0 to n, we obtain the relation

$$\begin{aligned} V{:}G^n{:} &= \sum_{\mu=0}^{n} \binom{n}{\mu}{:}(V \prec {:}G^\mu{:})G^{n-\mu}{:} \\ &= {:}VG^n{:} + n{:}(V \prec G)G^{n-1}{:} + \binom{n}{2}{:}(V \prec {:}G^2{:})G^{n-2}{:} \\ &\quad + \cdots + V \prec {:}G^n{:}. \end{aligned}$$

Hence

$$\begin{aligned} V{:}e^G{:} &= \sum_{n=0}^{\infty} n! \, V{:}G^n{:} = \sum_{n=0}^{\infty} n! \sum_{\mu=0}^{n} \binom{n}{\mu}{:}(V \prec {:}G^\mu{:})G^{n-\mu}{:} \\ &= \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \nu! \, \mu! \, {:}(V \prec {:}G^\mu{:})G^\nu{:} = {:}(V \prec {:}e^G{:}){:}e^G{:}. \end{aligned}$$

Confronting the formulas (II) with the equations for $Q^\pm$ to be solved we see that solutions of them are simply given by solutions of the equations

$$Q^+ = {:}e^{\Gamma Q^+}{:} \succ V, \qquad Q^- = V \prec {:}e^{-\Gamma Q^-}{:}.$$

These *basic equations* take the place of the equations

$$R^+ = (1 + \Gamma R^+)V \quad \text{and} \quad R^- = V(1 - \Gamma R^-)$$

described in §10.

The advantage of the present equations over the former ones lies in the fact that (V) *for Q in $\mathfrak{Q}'$ and W in $\mathfrak{Q}$ the connected products* $W \prec :e^{\Gamma Q}:$ and $:e^{\Gamma Q}: \succ W$ are well defined and *belong to* $\mathfrak{Q}$, provided this class is properly chosen. The class $\mathfrak{Q}$ described in Appendix A17 has that property. The reason that this class can be so chosen is that every factor $\Gamma\mathfrak{Q}$ of the Wick powers $:(\Gamma Q)^n:$ which form the Wick exponential function, is contracted at least once with W, so that every one of the n factors $[\omega - \omega']^{-1}$ in $:(\Gamma Q)^n:$ is smoothed out. Thus we see that *we have overcome our second obstacle*: We would now be able to carry out the iteration steps for our new basic equations if only we were sure that the operators $V \prec :e^{\Gamma Q}:$ and $:e^{\Gamma Q}: \succ V$, appearing in these steps, had no 00-terms. That need not be the case, however. That is to say, we have not yet overcome our first obstacle.

This first obstacle can easily be overcome if one is willing to adjust the value of the 00-term V_{00} of the disturbing operator V instead of prescribing it.

We set

$$V = V' + V_{00},$$

so that V' is in $\mathfrak{Q}'$, and write our second basic equation (which we prefer to the first one when we give details) in the form

$$Q^- = V' \prec :e^{-\Gamma Q^-}: + V_{00},$$

observing that the term V_{00} does not contract with $:e^{-\Gamma Q^-} - 1:$. Now we need only require that V_{00} be given by

$$V_{00} = -(V' \prec :e^{-\Gamma Q^-}:)_{00}$$

and write our second basic equation in the "modified" form

$$Q^- = V' \prec :e^{-\Gamma Q^-}: - (V' \prec :e^{-\Gamma Q^-}:)_{00}.$$

Clearly, the right-hand side of this modified equation has no 00-term; thus *the first obstacle is eliminated.* A solution Q^- of the modified equation is evidently a solution of the unmodified one provided one chooses for the term V_{00} the value given by the preceding relation.[2]

This relation expresses V_{00} in terms of V' and Q^-; inasmuch as Q^- is determined by V' also V_{00} is given in terms of V'. The above expression for the term V_{00} is the "Goldstone formula" in its basic form; see [**45**; **46**].

[2] The fact that adjustment of a single term is sufficient is due to the annihilation-creation character of V. This becomes quite clear from the work of J. Schwartz [47] in which annihilation-creation operators are regarded as a special case of more general operators K which are shown to require, in general, a host of adjustments.

Instead of working with the second basic equation we could just as well have worked with the first basic equation by setting

$$V_{00} = -(:e^{\Gamma Q^+}: \succ V')_{00}$$

and writing this second equation in the "modified" form

$$Q^+ = :e^{\Gamma Q^+}: \succ V' - (:e^{\Gamma Q^+}: \succ V')_{00}.$$

In this way we would be led to two adjustments of the constant term V_{00}. Fortunately, as we shall show in §13, these two adjusted values of V_{00} are the same. Thus we can achieve by a single adjustment of the value of V_{00} that the solutions $Q^{\pm}$ of the modified equations also satisfy the unmodified equations.

As was remarked earlier, the process of iterations employed in trying to solve the two modified equations can be carried out since the operator V was assumed to belong to the class $\mathfrak{Q}'$. It is quite another question whether or not the iteration process converges. Although for a few problems, namely, those that can be solved explicitly, see, e.g., §16, this convergence could be established, it is not at all clear that there is convergence, even in the cases in which the operators $H_0 + V$ and H_0 are equivalent to each other. Also, if there is convergence at all, the limit operators $Q^{\pm}$ cannot be expected to belong to the class $\mathfrak{Q}'$ unless this class has been chosen wide enough. Certainly the class chosen in Appendix A17 is not wide enough for this purpose. Reasons why it is not possible to enlarge this class by employing the same arguments that were used in §6 will be given in Appendix A17. Here then we have met a *third obstacle*; it is the major difficulty in the perturbation problems discussed, and has not been overcome.

Thus we are not able to proceed rigorously; nevertheless, we proceed formally, hoping that the formalism developed will some day prove to be the appropriate asymptotic description of the solution of perturbation problems of the type here considered. Accordingly, we proceed under the assumption that our modified equations have solutions $Q^{\pm}$ in these classes so that the operators

$$T^{\pm} = :e^{\Gamma Q^{\pm}}:$$

satisfy the equations

$$(H_0 + V)T^- = T^- H_0, \qquad T^+(H_0 + V) = H_0 T^+.$$

We must try to construct operators $U^{\pm}$ from these operators $T^{\pm}$ which satisfy the same equation but, in addition, are inverse to each other.

13. Inverse relations. In order to find operators $U^{\pm}$ in place of $T^{\pm}$ we shall first evaluate the product $T^{+}T^{-}$. At the same time we shall show that the two adjusted 00-components $V_{00}^{\pm}$ of V are the same:

$$V_{00}^{+} = V_{00}^{-}.$$

These considerations will be based on the assumption that the 00-component of the operator $T^{+}T^{-}$ is positive,

$$(T^{+}T^{-})_{00} > 0.$$

Since $T^{+}T^{-} = 1$ for $V = 0$ this will be true if V is sufficiently small, provided the $T^{\pm}$ depend continuously on V.

We employ the operators

$$R^{\pm} = :Q^{\pm}e^{\pm\Gamma Q^{\pm}}:,$$

where $Q^{\pm}$ are solutions of our basic equations. By virtue of the second basic formula of §12, the operators $R^{\pm}$ can be written in the form $R^{+} = T^{+}(V' + V_{00}^{+})$, $R^{-} = (V' + V_{00}^{-})T^{-}$. Multiplying R^{+} and R^{-} by T^{-} and T^{+} on the right and on the left we obtain the relation

$$R^{+}T^{-} - T^{+}R^{-} = T^{+}(V_{00}^{+} - V_{00}^{-})T^{-} = (V_{00}^{+} - V_{00}^{-})T^{+}T^{-}.$$

We maintain that the 00-component of the left-hand side vanishes. To show this we use the relations $T^{\pm} = 1 \pm \Gamma R^{\pm}$ and write

$$(R^{+}T^{-} - T^{+}R^{-})_{00} = (R^{+} - R^{-})_{00} - (R^{+}\Gamma R^{-} + (\Gamma R^{+})R^{-})_{00}.$$

Now we have $R_{00}^{\pm} = (:Q^{\pm}e^{\Gamma Q^{\pm}}:)_{00} = 0$ since $Q_{00}^{\pm} = 0$. Furthermore, we have

$$(R^{+}\Gamma R^{-} + (\Gamma R^{+})R^{-})_{00} = 0$$

since the Γ-factors $[-\omega']_n^{-1}$ and $[\omega]_n^{-1}$ of R^{-} and R^{+} cancel. Consequently, we have $(R^{+}T^{-} - T^{+}R^{-})_{00} = 0$ and hence

$$(V_{00}^{+} - V_{00}^{-})(T^{+}T^{-})_{00} = 0.$$

In view of the assumption $(T^{+}T^{-})_{00} \neq 0$ we conclude that $V_{00}^{+} = V_{00}^{-}$, so that indeed the adjustment of the constant V_{00} is the same whether determined from the first or the second basic equation. At the same time we obtain the relation

$$R^{+}T^{-} - T^{+}R^{-} = 0$$

and, hence, the relation

$$R^{+} - R^{-} - R^{+}\Gamma R^{-} - (\Gamma R^{+})R^{-} = 0,$$

which we shall use in evaluating the product $T^+T^- = (1 + \Gamma R^+) \cdot (1 - \Gamma R^-)$.

Following the procedure of §6 we shall express $\Gamma R^+ \Gamma R^-$ in terms of $\Gamma(R^+\Gamma R^- - (\Gamma R^+)R^-)$. We recall the basic identity

$$\Gamma(P\Gamma Q + (\Gamma P)Q) \equiv \Gamma P \Gamma Q - (\Gamma P \Gamma Q)_{00},$$

for operators in $\mathfrak{Q}$, which was formulated in §11 for Wick and attached products, and which depended on the fact that $(P\Gamma Q + (\Gamma P)Q)_{00} = 0$. A corresponding identity holds for a more general class of operators, in particular, for operators of the form $R^\pm = :Q^\pm e^{\pm \Gamma Q^\pm}:$. That is to say, the "extended basic identity"

$$\Gamma(R^+\Gamma R^- + (\Gamma R^+)R^-) = \Gamma R^+ \Gamma R^- - (\Gamma R^+ \Gamma R^-)_{00}$$

holds. It is readily proved formally if one uses the relation $(R^+\Gamma R^- + (\Gamma R^+)R^-)_{00} = 0$ derived above; we refrain from giving a rigorous proof.

Since $R^+\Gamma R^- + (\Gamma R^+)R^- = R^+ - R^-$, as was shown above, the extended basic identity leads to the relation

$$\Gamma(R^+ - R^-) = \Gamma R^+ \Gamma R^- - (\Gamma R^+ \Gamma R^-)_{00}.$$

Using this relation in $T^+T^- = (1 + \Gamma R^+)(1 - \Gamma R^-)$ we find

$$T^+T^- = 1 - (\Gamma R^+ \Gamma R^-)_{00}.$$

Thus we see that T^+T^- differs from the identity by an additional 00-term. The right-hand side evidently equals the 00-term, $(T^+T^-)_{00}$, of the left-hand side. Since this term was assumed to be positive we can introduce the number

$$\tau = [1 - (\Gamma R^+ \Gamma R^-)_{00}]^{1/2}$$

and write our result as

$$T^+T^- = \tau^2.$$

We now define the operators $U^\pm$ by

$$U^+ = \tau^{-1}T^+, \qquad U^- = T^-\tau^{-1}.$$

Clearly, these operators satisfy the relation

$$U^+U^- = 1;$$

at the same time these operators intertwine with H and H_0, i.e., the relations $U^+H = H_0U^-$, $HU^- = U^-H_0$ hold. If the operators $U^\pm - 1$ were sufficiently small with respect to an appropriate norm, the inverse relation

$$U^-U^+ = 1$$

would follow from $U^+U^- = 1$. From now on we shall assume this inverse relation to hold.

While describing operators $U^\pm$ was our major aim, other items should be discussed, in particular, items referring to the scattering process. Before doing this we must exploit our knowledge of the operators $U^\pm$ for a "change of attitude".

14. Change of attitude. With the aid of the operators $U^\pm$ we may introduce new annihilation and creation operators which play the same role for the complete energy operator H as the operators $A^\pm$ play for the undisturbed operator H_0. We need only set

$$B^\pm(\omega) = U^- A^\pm(\omega) U^+$$

and verify that these operators $B^\pm$ satisfy the same commutation laws as the $A^\pm$. We then may associate a particle representation with the operators $B^\pm$ by stipulating that the B-particle representer of a state Φ should be the A-particle representer of the state $\Psi = U^+\Phi$. The B-vacuum state Φ_0 is then related to the A-vacuum state Ψ_0 by the relation $\Phi_0 = U^-\Psi_0$ and, more generally, if Ψ_n is an n-state with respect to $A^\pm$, the transform

$$\Phi_n = U^-\Psi_n$$

is an n-state with respect to the B-particles. A definite physical interpretation of these B-particles will be given in §15.

We also may introduce annihilation-creation operators in terms of the operators $B^\pm$. Denoting such an operator expressed with the aid of the $A^\pm$ by $G(A)$, we may describe the corresponding B-operator by

$$G(B) = U^- G(A) U^+.$$

In particular, writing

$$H_0(A) = \int A^+(\omega)\omega A^-(\omega)\, d\omega$$

for the undisturbed energy operator, we find the expression

$$H = H_0(B) = \int B^+(\omega)\omega B^-(\omega)\, d\omega$$

for the disturbed energy operator $H = H_0 + V = U^- H_0 U^+$. Any relation between the operators $A^\pm$ and H_0 clearly leads to a corresponding relation between $B^\pm$ and H. · For example, the relation

$$e^{-itH_0} A^\pm(\omega) e^{itH_0} = e^{\mp it} A^\pm(\omega),$$

which can immediately be read off from the A-particle representation, leads to the relation

$$e^{-itH}B^{\pm}(\omega)e^{itH} = e^{\mp it\omega}B^{\pm}(\omega),$$

which we shall use in §15.

The transformation operators $U^{\pm}$, built up in terms of A-annihilation-creation operators, should be regarded as functionals of the $A^{\pm}$, and accordingly, we should write $U^{\pm} = U^{\pm}(A)$. We then can introduce operators $U^{\pm}(B)$ which depend on the $B^{\pm}$ in the same way as the $U^{\pm}(A)$ depend on the $A^{\pm}$. Actually, however, the $U^{\pm}$ are the same no matter whether they are expressed in terms of the A or the B. This fact is formulated as the

INDEPENDENCE REMARK.

$$U^{\pm}(B) = U^{\pm}(A).$$

Its proof is immediate; for, by virtue of $U^{+}U^{-} = U^{-}U^{+} = 1$, we have

$$U^{\pm}(B) = U^{-}U^{\pm}(A)U^{+} = U^{-}U^{\pm}U^{+} = U^{\pm}.$$

This remark, which corresponds to that made in Chapter II, §6 (and is implied in the literature), is very helpful in connection with the following considerations.

In quantum theory the B-particles are regarded as physically real particles (the "incoming" particles, as we shall show in §15) while the A-particles have only a preliminary significance. If in a renormalization procedure a nonsmooth interaction is approximated by a smooth one, it is appropriate to keep the B-particles fixed and, if necessary, to sacrifice the A-particles in the limit. In any such procedure it is necessary to express all operators, such as functions $f(A)$ of the $A^{\pm}$, in terms of the $B^{\pm}$. When this is done we speak of a "change of attitude". For us the change of attitude will be important for the analysis of scattering.

To carry out this change of attitude we first define the operator $f(A)$ as

$$f(A) = U^{+}f(B)U^{-}$$

and then express the operators $U^{\pm}$ in terms of B. By virtue of our independence remark it is good enough for this purpose to determine the $U^{\pm}$ in terms of the A, for their dependence on the $B^{\pm}$ is then also determined, so that we can write

$$f(A) = U^{+}(B)f(B)U^{-}(B).$$

In the following discussion we shall employ the B-particle representation, i.e., a representation of vectors Φ by sequences of functions $\phi_n(\omega_n)$.

The variable ω then stands for the energy of a single B-particle. Note that the A-kernels $g_{lm}((\omega)_l; (\omega')_m)$ of an operator $G = f(A)$ are at the same time the B-kernels of the operator $f(B)$. The operator $H = H_0(B)$, rather than $H_0 = H_0(A)$, is now represented by multiplying ϕ_n by $[\omega]_n$ and the notation Γ will now be used for the Γ operation associated with H.

We know that the operators $U^+ = \tau^{-1}T^+$ and $U^- = T^-\tau^{-1}$ are the same whether expressed in terms of $A^\pm$ or $B^\pm$; we maintain that that is true separately for τ and $T^\pm$. The reason is that the number τ_B is given as the same functional of the representers of $Q^\pm(B)$ that gives $\tau = \tau_A$ in terms of the representers of $Q^\pm(A)$, and these representers are the same.

Denoting the operators $Q^\pm(B)$, rather than $Q^\pm(A)$, by $Q^\pm$ we have $T^\pm = T^\pm(B) = :e^{\pm\Gamma Q^\pm}:$.

Working with functions of the $B^\pm$ rather than with functions of the $A^\pm$ results in what has been referred to as a "change of attitude". It is desirable to adopt this change of attitude in quantum theory since there the B-particles, rather than the A-particles, are regarded as physically real particles. The A-particles are fictitious entities introduced to facilitate the mathematical description of the problem. Therefore, also the annihilation and creation operators $A^\pm(\omega)$ have no direct physical significance. Still, they do have an indirect physical significance inasmuch as they are connected with the basic "field quantity". In the case of a pure boson field this connection is given by the formulas

$$A^\pm(\omega) = (\omega/2)^{1/2}[\phi(\mp k) \pm (i\omega)^{-1}\dot{\phi}(\mp k)],$$

where $k = (k_1, k_2, k_3)$ is the momentum of the particle and $\omega = |\,|k|^2 + \mu^2|^{1/2}$ its energy, while $\phi(k)$ and $\dot{\phi}(k)$ are the Fourier transforms of the field quantity $\phi(x, t)$, and its time derivative $\partial\phi(x, t)/\partial t$ at the time $t = 0$.

All physically significant processes, such as scattering, should be described in terms of the B-particles. Also, if in a process of removal of divergences a local interaction is approximated by a smooth, though not local one, it is appropriate to keep the B-particle annihilation-creation operators fixed, and, if necessary, to sacrifice the existence of A-particles in the limit.

15. Scattering. Naturally, one will try to carry over to the present problem the treatment of scattering given in §7 for single particle

interaction. We shall show that this leads to incongruities, while a second approach, which was hinted at in §7 and which is physically more realistic anyway, leads to a description of scattering which is quite appropriate.

The *first approach to* the analysis of *scattering* refers to the asymptotic behavior of the "adjusted Schrödinger operator"

$$e^{itH_0}e^{-itH}$$

as $\pm t$ tends to infinity.

This operator transforms the state Φ, the Schrödinger state of the field at the time $t = 0$, into the adjusted Schrödinger state

$$\tilde{\Phi}(t) = e^{itH_0}\Phi(t) = e^{itH_0}e^{-itH}\Phi$$

at the time t. In describing its asymptotic behavior we may use the A-language or, changing attitude, we may use the B-language. We choose the latter.

Accordingly, we write the above operator in the form

$$U^+e^{itH}U^-e^{-itH}$$

and describe the operator

$$U^-(t) = e^{itH}U^-e^{-itH}$$

asymptotically as $t \to \pm\infty$. Setting $Q = Q(B)$ we introduce the operators $Q^-_{lm}(t)$ with the kernels

$$e^{it[\omega-\omega']_{lm}}q^{\pm}((\omega)_l;(\omega')_m),$$

where $[\omega - \omega']_{lm} = [\omega]_l - [\omega']_m = \omega_1 + \cdots + \omega_l - \omega'_1 - \cdots - \omega'_m$, and set

$$Q^-(t) = \sum_{lm} Q^-_{lm}(t);$$

then we may write the operators $U^-(t)$ in the form

$$U^-(t) = e^{itH}U^-e^{itH} = :e^{\Gamma Q^-(t)}:_T{}^{-1}.$$

It is necessary to separate the terms which involve the subscript 0 from the others. Accordingly, we write

$$Q = Q_{0\cdot} + Q_{\cdot\cdot} + Q_{\cdot 0}$$

with

$$Q_{0\cdot} = \sum_{m=1}^{\infty} Q_{0m}, \qquad Q_{\cdot 0} = \sum_{l=1}^{\infty} Q_{l0},$$

$$Q_{\cdot\cdot} = \sum_{l,m=1}^{\infty} Q_{lm}.$$

Since $Q_{0\cdot}$ does not contract on the left and $Q_{\cdot 0}$ does not contract on the right we may write

$$:e^{\Gamma Q(t)}: = e^{\Gamma Q_{\cdot 0}(t)}:e^{\Gamma Q_{\cdot\cdot}(t)}:e^{\Gamma Q_{0\cdot}(t)}.$$

As we have done in §6 for the operator $\Gamma Q_{11}(t)$, one can derive for the operator $\Gamma Q_{\cdot\cdot}(t)$ the relation

$$\Gamma Q_{\cdot\cdot}(t) \to \begin{cases} 1 & \text{as } t \to -\infty, \\ \Gamma_{00} Q_{\cdot\cdot} & \text{as } t \to \infty, \end{cases}$$

where $\Gamma_{00} Q_{lm}$ is the operator with the kernel

$$2\pi i \delta[\omega - \omega']_{lm} q_{lm}((\omega)_l; (\omega')_m)$$

(in the B-representation).

Convergence of an operator $G(t)$ to an operator G_∞ is to mean, here and in the following, that the relation

$$G(t)\Phi \to G_\infty \Phi$$

should hold for all states Φ in $\mathring{\mathfrak{H}}$, i.e., for all states Φ with a finite number of nonvanishing components. We shall refer to this kind of convergence, somewhat improperly, as "strong" convergence.

The limit behavior of the operators $\Gamma Q_{0m}(t)$ and $\Gamma Q_{l0}(t)$ is different from that of $\Gamma Q_{lm}(t)$ (for $l \neq 0$, $m \neq 0$) since the denominators $\sum^m (-\omega')$ and $\sum^l \omega$ in their kernels are not singular. By virtue of this fact we can deduce from the Riemann-Lebesgue Lemma the relation

$$\Gamma Q_{0\cdot}(t) \to 0.$$

(For details see Appendix A18.) No such relation holds for $\Gamma Q_{\cdot 0}(t)$, at least not in the strong sense, since no integration is involved in forming the representers of $\Gamma Q_{\cdot 0}(t)\Phi$. Consequently, our asymptotic description of the adjusted Schrödinger operator is

$$e^{itH_0}e^{-itH} \sim \begin{cases} U^+ e^{\Gamma Q_{\cdot 0}(t)} \tau^{-1} & \text{as } t \sim -\infty, \\ U^+ e^{\Gamma Q_{\cdot 0}(t)} S^+ \tau^{-1} & \text{as } t \sim \infty, \end{cases}$$

where

$$S^\pm = :e^{\Gamma_\infty Q_{\cdot\cdot}^\pm}:.$$

Thus we see that, in general, our adjusted Schrödinger operator $e^{itH_0} \cdot e^{-itH}$ does not converge as $t \to \pm\infty$ in the "strong" sense. The states

$$U^+ e^{\Gamma Q_{\cdot 0}(t)} \tau^{-1}\Phi(0) \quad \text{and} \quad U^+ e^{\Gamma Q_{\cdot 0}(t)} S^+ \tau^{-1}\Phi(0),$$

describing asymptotically the adjusted Schrödinger states $\tilde{\Phi}(t)$, are

essentially van Hove's "asymptotic states", cf. [44]. The presence of the term $e^{\Gamma Q_{\vec{0}}^{\pm}(t)\tau^{-1}}$ expresses the fact that asymptotically the interaction is not wiped out and that a "cloud" of those particles that produce the interaction persists. The factors $e^{\Gamma Q_{\vec{0}}^{\pm}(t)\tau^{-1}}$ have therefore been called "cloud factors".

Let us turn to the *second approach to scattering*. There one asks for the asymptotic behavior of adjusted operators $F(t)$ corresponding to observables F whose values are to be measured or have been measured at the time t.

We restrict ourselves to considering the operators $A^{\pm}(\omega)$, which are connected with the field quantities $\phi(x, t)$ in the manner indicated at the end of §14, instead of a general operator F. We introduce the time dependent operators

$$A^{\pm}(\omega, t) = e^{itH}A^{\pm}(\omega)e^{-itH}$$

and the "adjusted" time dependent operators

$$\tilde{A}^{\pm}(\omega, t) = e^{\mp it\omega}A^{\pm}(\omega, t),$$

which, because of $e^{\mp it\omega}B^{\pm}(\omega) = e^{-itH}B^{\pm}(\omega)e^{itH}$, see §14, can be written in the form

$$\tilde{A}^{\pm}(\omega, t) = U^{+}(t)B^{\pm}(\omega)U^{-}(t)$$

with

$$U^{\pm}(t) = e^{itH}U^{\pm}e^{-itH}.$$

From the second expression for $\tilde{A}^{\pm}(t) = \tilde{A}^{\pm}(\omega, t)$ we find for these operators the asymptotic expression

$$\tilde{A}^{\pm}(t) \sim \tau^{-1}e^{\Gamma Q_{\vec{0}}^{\pm}(t)}S^{+}B^{\pm}e^{-\Gamma Q_{\vec{0}}^{-}(t)}S^{-}\tau^{-1},$$

valid for $t \sim \infty$, with $S^{\pm}$ replaced by 1 for $t \sim -\infty$.

It is a remarkable fact that the two cloud factors appearing here cancel out in the limit, as can be shown by a more detailed analysis given in Appendix A18. Thus, we arrive at the limit statements (in the "strong" sense)

$$\tilde{A}^{\pm}(t) \rightarrow \begin{cases} B^{\pm} & \text{as } t \rightarrow -\infty, \\ S^{+}B^{\pm}S^{-} & \text{as } t \rightarrow \infty. \end{cases}$$

It is by virtue of this result that we may interpret the "modified" annihilation and creation operators $B^{\pm}$ as those of the "incoming" particles. The annihilation and creation operators of the "outgoing" particles, $S^{+}B^{\pm}S^{-}$, are related to those of the incoming particles by the scattering operator S^{-} and its inverse S^{+}.

16. Remarks about divergences in the case of smooth interaction. In §8 we have shown in connection with a simple example that it may happen that the disturbed as well as the undisturbed operators H and H_0 are defined while the interaction operator V is not defined.

A similar situation may arise in the general case of totally smooth interaction. There are some differences, however, since the problem of §8 is not a special case of such an interaction; it may be regarded as a degenerate case of the problem of conservation interaction treated later on in §A20.

We shall discuss the nature of the improperly defined operators V in connection with a very simple special problem in which the basic equations for $Q^{\pm}$ (see §13) have a solution, in fact, an explicit solution. This is the problem in which the disturbing operator

$$V = V_{10} + V_{01} + V_{00},$$

with

$$V_{10} = A^{+} \cdot v_{10} = \int A^{+}(\omega) v_{10}(\omega)\, d\omega,$$

$$V_{01} = v_{01} \cdot A^{-} = \int v_{01}(\omega') A^{-}(\omega_1)\, d\omega',$$

is linear in $A^{\pm}$. The solution $Q^{\pm}$ is simply

$$Q^{\pm} = V' = V_{10} + V_{01},$$

as one easily verifies. The operator $\Gamma Q^{\pm} = \Gamma V'$ is then generated by the kernels $\gamma v_{10}(\omega) = v_{10}(\omega)/\omega$ and $\gamma v_{01}(\omega') = -v_{01}(\omega')/\omega'$, and one readily verifies the validity of the expressions

$$V_{00} = \int \frac{v_{01}(\omega) v_{10}(\omega)}{\omega}\, d\omega,$$

$$\tau^2 = \exp\left\{ \int \frac{v_{01}(\omega) v_{10}(\omega)}{\omega^2}\, d\omega \right\}$$

for the constants V_{00} and τ^2. The operators $T^{\pm}$ are given by

$$T^{\pm} = e^{\pm A^{+} \cdot \gamma v_{10}} e^{\pm \gamma v_{01} \cdot A^{-}}.$$

From the identities

$$A^{\pm}(\omega) e^{A^{\pm} \cdot q} = e^{A^{\pm} \cdot q} (A^{\pm}(\omega) \pm q(\omega)),$$

one readily verifies (see e.g. [**41**]) that the operators $B^{\pm}$ are simply given as

$$B^{+}(\omega) = A^{+}(\omega) - \gamma v_{01}(\omega), \qquad B^{-}(\omega) = A^{-}(\omega) + \gamma v_{10}(\omega),$$

so that indeed the disturbed operator can be written as

$$H = \int B^{+}(\omega) \omega B^{-}(\omega)\, d\omega.$$

The operator $V = V' + V_{00}$ can evidently be defined in an appropriate subspace of $\mathfrak{H}$ provided that V_{00} is finite and the functions $v_{01}(\omega)$ and $v_{10}(\omega)$ are square integrable (Case 0). If the last condition is not satisfied (we then speak of Case 1 if $|V_{00}| < \infty$), the operator V_{10} is not applicable on the vacuum state, although inner products or "matrix elements" $(\Psi_n, V\Psi_m)$ can be defined for an appropriate class of states Ψ_m and Ψ_n. If $V_{00} = \infty$, even this is not possible. Nevertheless, the operators $U^{\pm}$ may just as well be defined provided only the functions $\gamma v_{10}(\omega)$ and $\gamma v_{01}(\omega)$ are square integrable so that $|\tau|^2 < \infty$. In that case (Case 2) the operator H may be defined as $H = U^{-}HU^{+}$ although the operator V is not defined.

In case $\tau^2 = \infty$, however, (Case 3) the operator H can no longer be defined in this manner, but it is possible to define the operator

$$H_{\mathcal{I}} = \int_{\mathcal{I}} B^{+}(\omega)\omega B^{-}(\omega)\, d\omega$$

corresponding to the total energy of particles having an energy ω in the finite interval $\mathcal{I}$. Similarly, one can introduce the operator corresponding to the total number of particles with ω in $\mathcal{I}$,

$$N_{\mathcal{I}} = \int_{\mathcal{I}} B^{+}(\omega)B^{-}(\omega)\, d\omega.$$

In Case 2 these operators are defined even if $\mathcal{I}$ covers the whole ω-ray $\omega \geqq \omega_*$ so that $N_{\mathcal{I}} = N$ and $H_{\mathcal{I}} = H$; in Case 3 this is not true of either of these two operators. In this case, thus, no number operator N and hence no particle representation is associated with the operators $B^{\pm}$; nevertheless, these operators satisfy the canonical commutation laws $[B^{-}(\omega'), B^{+}(\omega)] = \delta(\omega' - \omega)$.

This fact indicates that the operators $A^{\pm}$ and $B^{\pm}$ are not equivalent. If $v_{01} = \bar{v}_{10}$, so that the operator V is Hermitean if it is defined, these operators are unitarily equivalent in Cases 0, 1, and 2, but not in Case 3.

Here we have assumed $\omega_* > 0$. In case $\omega_* = 0$, a different case distinction can be made even if $\int |V_{01}(\omega)|^2 d\omega$ and $\int |V_{10}(\omega)|^2 d\omega$ are finite. It may then happen that $V_{00} < \infty$ but $\tau^2 = \infty$. In this case the operator H is defined directly as $H = H_0 + V$, but the number operator N and the transformations $U^{\pm}$ are not defined. Again the particle representation is associated with the operators $B^{\pm}$ although these operators $B^{\pm}$ satisfy the canonical commutation laws.[3]

[3] It was in connection with this example that the author first observed the possibility, independently observed by others, that the canonical commutation laws can be realized by operators $B^{\pm}$ with which no particle representation can be associated; see [41].

In the case with $\omega_* = 0$ just described one speaks of an "infrared catastrophe", since the trouble arises from the behavior of $v_{10}(\omega)$ and $v_{01}(\omega)$ as $\omega \to 0$. In Case 3 discussed before, in which the trouble arises from the behavior as $\omega \to \infty$, one speaks of an "ultraviolet catastrophe".

Instead of taking the attitude that the operators $A^{\pm}$ are associated with a particle representation but possibly not the $B^{\pm}$, we may just as well take the attitude that the $B^{\pm}$ are associated with such a representation and the $A^{\pm}$ possibly not. This is the attitude taken in the quantum theory of fields, where the $B^{\pm}$ refer to the field of incoming particles with the energy operator H while the "free particles" associated with the $A^{\pm}$ are considered just fictitious entities so that the "free total energy" H_0 need not be defined.

There is still another possibility, namely, the attitude taken by Segal in some of his work; see [54]. With any set of operators that satisfy the canonical commutation laws (for bosons), such as the $A^{\pm}$ or $B^{\pm}$, Segal associates a C^*-algebra of bounded operators of which the selfadjoint ones correspond to bounded physical observables. Then he proves that any two such algebras are algebraically isomorphic in a unique way. This remarkable fact makes it unnecessary to stipulate whether or not the $A^{\pm}$ or the $B^{\pm}$ are associated with a number representation if one restricts oneself to observables that correspond to operators of this algebra.

While in the simple example presented in this section one is not forced to take this intrinsic attitude, it might well be necessary to do so in connection with less simple problems.

Incidentally, it appears likely that in the general problem treated in this chapter it depends just on the nature of the kernels v_{l1} and v_{1m}, and not on the kernels v_{lm} with $l > 0$ and $m > 0$, whether or not the $B^{\pm}$ are equivalent with the $A^{\pm}$.

17. Perturbation obeying a conservation law. If a physical entity such as a field is not subjected to external forces, its total momentum will not be changed by the internal interaction described by the disturbing interaction energy operator $V = \sum_{lm} V_{lm}$. This conservation of total momentum is reflected by the presence of the delta function

$$\delta(k_1 + \cdots + k_l - k'_1 - \cdots - k'_m)$$

as a factor of the kernel v_{lm} generating the operator V_{lm}, as was mentioned at the end of §9. Here k is the three-dimensional momentum vector of a particle; each of its three components is to range from $-\infty$ to $+\infty$. The presence of this delta factor in the

kernels of V implies a loss of smoothness of the kernel, and produces shifts of the spectrum of $H = H_0 + V$, in addition to the constant shift described in the previous sections.

Among the (discrete) accessory variables, which, together with k, characterize the state of a single particle, there may be some for which also a conservation law holds; such a variable may, for example, be a quantity which has the value 0 if the particle is a photon and the values ± 1 if it is a negaton or positon. We denote such variables by σ and set $(k, \sigma) = s$. We shall in general pay no attention to the remaining accessory variables, referred to as "silent". The delta factor expressing the conservation of the total values of k and σ (the "conservation factor", for short) will be denoted by

$$\delta(k_1 + \cdots - k'_m)\,\delta(\sigma_1 + \cdots - \sigma'_m) = \delta(s_1 + \cdots - s'_m)$$

or

$$\delta[k - k']_{lm}\,\delta[\sigma - \sigma']_{lm} = \delta[s - s']_{lm}$$

for short. Writing $\int \cdots ds$ in place of $\sum_\sigma \int^3 \cdots d^3k$ we may introduce operators

$$G_{lm} = \int\int A^+(s_1) \cdots A^+(s_l)\,\delta[s - s']_{lm} g^0((s)_l;(s')_m) \cdot A^-(s'_1) \cdots A^-(s'_m)\,d(s)_l\,d(s')_m$$

in terms of a "reduced kernel" g^0 assumed to be smooth. We observe that any (attached or Wick) product of two such operators is again an operator of this type, as readily verified by using the formula $\delta(a)\delta(b) = \delta(a)\delta(a + b)$.

The energy ω of a single particle will be regarded as a function of s; it *is required to be independent of the silent accessory variables.* We may just as well assume

$$\omega(s) = (|k|^2 + \mu^2(\sigma))^{1/2},$$

where $\mu(\sigma) \geqq \omega_*$ is the mass of the kind of particle which is characterized by the value of σ. An important requirement will be imposed on the function $\omega(s)$, namely, that it be *strictly subadditive*; that is, the inequality

$$\omega(s_1 + \cdots + s_n) < \omega(s_1) + \cdots + \omega(s_n) \quad \text{for } n > 1$$

should hold. Clearly, if the function $\mu(\sigma)$ has this property, the same is true of the function $\omega = (\mu^2 + |k|^2)^{1/2}$, since then

$$\begin{aligned}[\omega_1 + \cdots + \omega_n]^2 - \omega^2(s_1 + \cdots + s_n) &\geqq [|k_1| + \cdots + |k_n|]^2 - |k_1 + \cdots + k_n|^2 \\ &\quad + [\mu(\sigma_1) + \cdots + \mu(\sigma_n)]^2 - \mu^2(\sigma_1 + \cdots + \sigma_n) > 0.\end{aligned}$$

Instead of ω we shall adopt s as the visible variable of our representers ψ. Thus we represent the state Φ by

$$\Phi \quad \leftrightarrow \quad \{\psi_n(s)_n\}$$

with

$$(\Phi, \Phi) = |\psi_0|^2 + \sum_{n=1} n! \, |\psi_n(s)_n|^2 \, d(s)_n.$$

Note that any state with a definite total value $s^0 = (k^0, \sigma^0)$ of s is improper since its representers are of the form

$$\psi_n(s)_n = \delta\left(\sum^n s - s^0\right)\psi_n^0(s)_n.$$

Clearly, a conservation smooth operator G transforms such an improper state into another improper state with the same total value of s. Actually, we shall not work with such improper states.

As before, the undisturbed energy operator H_0 is described by

$$H_0\Phi \quad \leftrightarrow \quad \left\{\sum^n \omega\psi_n(s)_n\right\}$$

with $\sum^n\omega = \omega(s_1) + \cdots + \omega(s_n)$ for $n \geqq 1$, and $\sum^0\omega = 0$.

The interaction operator V is again assumed to be of the form

$$V = \sum_{lm} V_{lm};$$

but the operators V_{lm} are now represented through

$$V_{lm} \quad \leftrightarrow \quad \delta[s - s']_{lm} v_{lm}^0((s)_l; (s')_m)$$

by kernels consisting of a smooth function $v_{lm}^0((s)_l; (s')_m)$, the *reduced kernel*, multiplied by the conservation factor.

We shall assume that pure annihilation and creation components, or "vacuum interaction terms", are absent from V, i.e., we assume

$$V_{0m} = 0, \qquad V_{l0} = 0 \quad \text{for all } l, m.$$

It should be mentioned that this simplifying assumption is never satisfied in field theory. For, there the operator V is of the form

$$V = \sum_n \int^n \delta[s]_n v_n^0((s)_n) : (A^+(-s_1) + A^-(s_1)) \cdots (A^+(-s_n) + A^-(s_n)) : d(s)_n$$

and therefore always comprises pure creation and annihilation terms:

$$V_{n0} = \int^n \delta[s]_n v_n^0((-s)_n) A^+(s_1) \cdots A^+(s_n) \, d(s)_n$$

and

$$V_{0n} = \int^{n} \delta[s']_n v_n^0((s')_n) A^-(s_1') \cdots A^-(s_n')\, d(s')_n.$$

Because of the delta singularity of the kernels of these operators, the difficulty indicated at the end of the preceding section is expected to occur. (Haag's theorem is concerned with a related situation; cf. e.g. [54].)

Our reason for making this simplifying assumption was not so much to avoid this difficulty but to separate the phenomena caused by the presence of the δ-factor from those caused by the vacuum interaction terms.

As before, our aim is to give the spectral analysis of the operator

$$H = H_0 + V$$

after having adjusted the disturbance V such that H and H_0 have similar spectral representations.

Again we shall seek operators $T^{\pm}$ such that

$$HT^- = T^- H_0, \qquad T^+ H = H_0 T^+,$$

and we shall again assume that there are such operators having the form

$$T^{\pm} = :e^{\pm \Gamma Q^{\pm}}:$$

with operators $Q^{\pm}$ satisfying the equations

$$Q^+ = :e^{\Gamma Q^+}: \succ V \quad \text{and} \quad Q^- = V \prec :e^{-\Gamma Q^-}:.$$

The operator V should belong to an appropriate class $\mathfrak{Q}$ of operators Q whose kernels carry the conservation delta factor and for which $Q_{10} = Q_{0m} = 0$; moreover, the operators $Q^{\pm}$ should belong to a subclass $\mathfrak{Q}''$ of $\mathfrak{Q}$ consisting of operators for which the operation Γ is so defined that

$$[H_0, \Gamma Q] = Q.$$

Here an obstacle arises which is the analogue of the first obstacle met in the case of totally smooth interaction. While in that case the first obstacle concerns 00-operators, such as V_{00}, in the present case the 11-operators, such as V_{11}, are critical.

A 11-operator G_{11} is of the form

$$G_{11} = \int A^+(s_1)\delta(s_1 - s_1')g^0(s_1; s_1')A^-(s_1')\, ds_1'\, ds_1$$

$$= \int A^+(s)g(s)A^-(s)\, ds,$$

where $g(s) = g^0(s; s)$ is a matrix as regards the silent variables. (Somewhat improperly, we shall on occasion call $g(s)$, rather than $g^0(s_1; s_1')$, the reduced kernel of G_{11}.) Note that the undisturbed energy operator H_0 is also of the form G_{11} with $g(s) = \omega(s)$. Since $\omega(s)$ is independent of the silent variables it commutes with any other $g(s)$, and hence H_0 commutes with every operator G_{11}:

$$[H_0, G_{11}] = 0.$$

Since therefore the 11-component Q_{11} of Q commutes with H_0, the operation Γ cannot be defined for this component unless it vanishes. We therefore must require of the operators Q in $\mathfrak{Q}''$ that they have no 11-term,

$$Q_{11} = 0.$$

It then follows that the right-hand sides of the equations for $Q^{\pm}$ have no 11-terms either, provided these equations have solutions. We want to express this condition in terms of the 11-terms of V. Accordingly, we set

$$V = V'' + V_{11}$$

and note that the 11-term of $V_{11} \prec :e^{-\Gamma Q^-}:$ is $V_{11} - V_{11} \multimap \Gamma Q^-$, so that our first equation becomes

$$Q^- = V'' \prec :e^{-\Gamma Q^-}: - V_{11} - V_{11} \multimap \Gamma Q^-.$$

The last term here has no 11-term since $Q_{11}^- = 0$; the 11-term of $V'' \prec :e^{-\Gamma Q^-}:$ is simply $-(V'' \multimap \Gamma Q^-)_{11}$. Thus our condition becomes

$$V_{11} = (V'' \multimap \Gamma Q^-)_{11}.$$

In a similar manner we may write the equation for Q^+ in the form

$$Q^+ = :e^{\Gamma Q^+}: \prec V'' + V_{11} + \Gamma Q^+ \multimap V_{11};$$

the condition on V_{11} being

$$V_{11} = -(\Gamma Q^+ \multimap V'')_{11}.$$

In discussing the case of total smoothness we have observed that in forming the connected product $V \prec :(\Gamma Q)^n:$ the singularities are smoothed out by integration and thereby our second obstacle was overcome. In the present case of conservation smoothness this argument applies only to those contributions to the product in which each factor ΓQ undergoes at least two contractions since one integration is lost because of the presence of the conservation factor. Thus a new form of the second obstacle appears.

We have not succeeded to characterize a class $\mathfrak{Q}$ of operators such that the operators $V \prec :e^{rQ}:$ and $:e^{rQ}: \succ V$ belong to it if V'' and Q belong to $\mathfrak{Q}''$, and anyway, we are not able to establish a class of operators V for which our equations for $Q^{\pm}$ have solutions. We therefore are forced to assume that there is such a class $\mathfrak{Q}''$ and that there are solutions of our equations belonging to it. Instead of specifying this assumption in detail we prefer to proceed formally.

Suppose then that there are two solutions $Q^{\pm}$ of our two equations in which the components V_{11} are to be expressed in terms of V'' and $Q^{\pm}$ as indicated. The first question that arises is whether or not the two operators V_{11} determined from these two solutions are the same. This is indeed the case as can be shown by arguments similar to those employed in §13 for the case of smooth interaction. Having said this we proceed to discuss the significance of the adjustment of the component V_{11}.

The operator V_{11} is evidently of the form

$$V_{11} = \int A^{+}(s)v(s)A^{-}(s)\,ds,$$

where the quantity v is a matrix as regards the silent accessory variables; it will be referred to as the "single particle self-energy".

Adding the term V_{11} to H_0, rather than to V'', we may write $H = H_0^0 + V''$ and consider $H_0^0 = H_0 + V_{11}$ as a new undisturbed energy operator; it corresponds to the "unrenormalized" one in the perturbation procedure of field theory. The operator H_0^0 may be written in the form

$$H_0^0 = \int A^{+}(s)\omega^0(s)A^{-}(s)\,ds,$$

where

$$\omega^0(s) = \omega(s) + v(s)$$

is the sum of the energy of a real incoming particle and the "self-energy"; it is the "unrenormalized" energy while $\omega(s)$ is the "renormalized" one.

If $v(s)$ does not depend on the silent variables, as $\omega(s)$ was assumed to, we may regard $\omega^0(s)$ as the energy of a fictitious single particle, formed under the influence of the self-interaction of the field. In case of a Lorentz invariant theory, the function $\omega^0(s)$ will be of the form

$$\omega^0(s) = (|k|^2 + (\mu^0)^2(\sigma))^{1/2}$$

and thus be generated by an "unrenormalized" mass μ^0.

The remarkable fact is that a single adjustment of the term V_{11} is sufficient to make the spectral representation of H similar to that of H_0 provided that our equations have solutions. If one were to compare the spectrum of the disturbed operator $H = H_0^0 + V''$ with that of the unrenormalized undisturbed operator H_0^0, one would find that it differs from it in the following manner: In the H-vacuum state $U^-\Psi_0$ the operator H has the eigenvalue 0, as H_0^0 has in the state Ψ_0; for, $U^-\Psi_0 = \Psi_0$, because of $Q_{0m} = Q_{l0} = 0$. In a state $U^-\Psi_1$ the eigenvalue of H runs over the range of the function $\omega^0(s) - v(s)$; the spectrum of H in an n-particle state $U^-\Psi_n$ is the range of the function $\omega^0(s_1) - v(s_1) + \cdots + \omega^0(s_n) - v(s_n)$. Thus it is seen that the shift of the n-particle spectrum is determined by the shift of the one-particle spectrum. The shift is eliminated if one compares the spectrum of H with that of the renormalized undisturbed energy operator H_0, rather than with that of the unrenormalized one, H_0^0.

The next item is the evaluation of the product T^+T^-. This evaluation can be carried out in a manner similar to that in which the corresponding evaluation for smooth interaction was performed. The result is that the operator

$$T^+T^- = \tau^2$$

is an exponential function

$$\tau^2 = \exp\left[2\int A^+(s)\theta(s)A^-(s)\,ds\right]$$

of a 11-operator, generated by a reduced kernel $\theta(s)$, which is a matrix as regards the silent variables. Specifically, the function $e^{2\theta(s)} - 1$ is given as the reduced kernel of the 11-operator

$$A^+(s)(e^{2\theta(s)} - 1)A^-(s)\,ds = \Gamma Q_{1\cdot}^+ \times \Gamma Q_{\cdot 1}^-,$$

where the symbol $\times$ is to indicate that those terms of the product are to be taken in which all annihilation prongs of $Q_{1\cdot}^+$ and all creation prongs of $Q_{\cdot 1}^-$ are contracted; see Appendix A19.

Using the operator τ we may introduce the operators

$$U^+ = \tau^{-1}T^+, \qquad U^- = T^-\tau^{-1},$$

which then satisfy the relation

$$U^+U^- = 1.$$

We assume the inverse relation

$$U^-U^+ = 1$$

to hold.

With the aid of these transformation operators $U^{\pm}$, we can then introduce the transformed annihilation operators $B^{\pm}(s) = U^{-}A^{\pm}(s)U^{+}$. Changing our attitude we write the disturbed energy H in the form

$$H = \int B^{+}(s)\omega(s)B^{-}(s)\,ds.$$

Also we write $U^{+} = \tau^{-1}{:}e^{\Gamma Q^{+}}{:}$ and $U^{-} = {:}e^{-\Gamma Q^{-}}{:}\tau^{-1}$, where now $Q^{\pm}$ and

$$\tau = \exp\left[\int B^{+}(s)\theta(s)B^{-}(s)\,ds\right]$$

are functionals of the $B^{\pm}$ rather than the $A^{\pm}$.

Next we should discuss *scattering*. The adjusted time dependent operators

$$\begin{aligned}\tilde{A}^{\pm}(s, t) &= e^{\mp it\omega(s)}A^{\pm}(s, t)\\ &= U^{+}(t)B^{\pm}(s)U^{-}(t)\end{aligned}$$

(see §15), or rather the operators $\int A^{\pm}(s, t)f(s)\,ds$, do not converge strongly to limits as $t \to \pm\infty$; cloud terms appear. These cloud terms are weaker, however, than those occurring in the adjusted Schrödinger operator $e^{itH_0}e^{-itH}$. In any case, these adjusted operators converge weakly to limits, given (see Appendix A20) by the relations

$$\left.\begin{matrix}e^{-it\omega(s)}A^{+}(s, t)e^{\theta(s)}\\ e^{+it\omega(s)}e^{\theta(s)}A^{-}(s, t)\end{matrix}\right\} \to \begin{cases}B^{\pm}(s) & \text{as } t \to -\infty,\\ S^{+}B^{\pm}(s)S^{-} & \text{as } t \to \infty.\end{cases}$$

Here the operators $S^{\pm}$ are given by

$$S^{+} = \tau^{-1}{:}e^{\Gamma Q_{\cdot\cdot}^{+}}{:}\tau, \qquad S^{-} = \tau{:}e^{-\Gamma Q_{\cdot\cdot}^{-}}{:}\tau^{-1},$$

where

$$Q_{\cdot\cdot}^{\pm} = \sum_{l>1,\,m>1} Q_{lm}^{\pm}$$

consists of those contributions to $Q = \sum_{lm} Q_{lm}$ which do not involve Q_{1m} and Q_{l1}.

The asymptotic formulas show that the operators $B^{\pm} = B^{\pm}{}_{\mp\infty}$ are associated with the "incoming" particles while the operators $S^{+}B^{\pm}S^{-} = B^{\pm}{}_{\pm\infty}$ are associated with the "outgoing" particles.

Clearly, S^{-} is the *scattering operator*. It commutes with H since $\Gamma_{\infty}Q_{\cdot\cdot}$ and τ do and it transforms a state Φ with the $B^{(\mathrm{in})}$-representers $\phi_n^{(\mathrm{in})}$ into a state $S^{-}\Phi$ whose $B^{(\mathrm{in})}$-representers $\phi_n^{(\mathrm{out})}$ are the $B^{(\mathrm{out})}$-representers of Φ.

Note that a single incoming particle state $\Phi_1 = U^{-}\Psi_1 \leftrightarrow_B \{0, \phi(s_1), 0, \cdots\}$ is not changed by applying the operator: $e^{-\Gamma Q_{\cdot\cdot}^{-}}$: and

hence not by applying S^-. Consequently, *a single incoming particle state is at the same time a single outgoing particle state.* That this be so is one of the basic postulates of renormalization theory due to Källén. In our set-up this property results from the fact that the operator $Q_{..}^-$ has no $l1$-term.

The operators $B_{\text{in}}^{\pm}$ and $B_{\text{out}}^{\pm}$ are the limits, after adjustment with $e^{\pm it\omega}$, not of the operators $A^{\pm}(s, t)$, but of the "renormalized" operators

$$\hat{A}^+(s, t) = \tilde{A}^+(s, t)e^{\theta(s)}, \qquad \hat{A}^-(s, t) = e^{\theta(s)}\tilde{A}^-(s, t).$$

We add a remark about the role of this "amplitude renormalization" in field theory.

In field theory the annihilation and creation operators, as well as the associated particles, have no direct physical significance, but are connected with physically significant quantities, the "field amplitude" ϕ and its time derivative $\dot{\phi}$, which in case of a boson field are given, possibly except for a factor, by the expressions

$$\phi(s) = \frac{1}{(2\omega(s))^{1/2}}\,[A^+(-s) + A^-(s)],$$

$$\dot{\phi}(s) = i\left(\frac{\omega(s)}{2}\right)^{1/2}[A^+(-s) - A^-(s)].$$

We assume that the disturbance V is Hermitean, which is the case in field theory, so that the operators Q^+, ΓQ^+, and A^+ are adjoint to Q^-, $-\Gamma Q^-$, and A^-, while ϕ and $\dot{\phi}$ are selfadjoint. Also, we assume θ to be a multiple of the unit matrix with respect to the silent variables; moreover, we assume the relation $\theta(-s) = \theta(s)$ to hold.

The time dependent quantities

$$\phi(s, t) = \frac{1}{(2\omega(s))^{1/2}}\,[A^+(-s, t) + A^-(s, t)],$$

$$\dot{\phi}(s, t) = i\left(\frac{\omega(s)}{2}\right)^{1/2}[A^+(-s, t) - A^-(s, t)]$$

are expected to behave asymptotically as the outgoing and incoming quantities

$$\phi_{\pm}(s, t) = \frac{1}{(2\omega(s))^{1/2}}\,[e^{it\omega(s)}B_{\pm\infty}^+(-s) + e^{-it\omega(s)}B_{\pm\infty}^-(s)],$$

$$\dot{\phi}_{\pm}(s, t) = \left(\frac{\omega(s)}{2}\right)^{1/2}[e^{it\omega(s)}B_{\pm\infty}^+(-s) + e^{-it\omega(s)}B_{\pm\infty}^-(s)].$$

The asymptotic formulas derived before show that this will not be

the case unless $\theta(s) = 0$, and that it will be the case if in the definition of the quantities ϕ and $\dot{\phi}$ the operators $A^{\pm}$ are replaced by the "renormalized" operators

$$\hat{A}^{\pm}(s) = e^{\theta(s)}A^{\pm}(s).$$

It is for this reason that the quantities ϕ and $\dot{\phi}$ so renormalized are regarded as those corresponding to the observable field amplitude and its time derivative, in spite of the fact that then the commutation law for these quantities,

$$[\phi(s), \dot{\phi}(s')] = ie^{2\theta(s)}\,\delta(s - s'),$$

differs from the canonical one in the presence of the factor $e^{2\theta(s)}$. The renormalization here involved is referred to as "wavefunction" or "amplitude renormalization". It is usually described for Lorentz invariant theories, where θ is constant.

Note that this amplitude renormalization is here introduced without reference to possible divergences; actually, it is optional if θ is finite, but necessary if $\theta = \infty$.

A remark may be added concerning another type of renormalization related to "charge renormalization". It comes about by absorbing the factors τ and τ^{-1} which occur in the definition of the operators $S^{\pm}$. Setting

$$\hat{Q}^{+} = \tau^{-1}Q^{+}\tau \quad \text{and} \quad \hat{Q}^{-} = \tau Q^{-}\tau^{-1},$$

one may write

$$S^{\pm} = :e^{\pm\Gamma_{\infty}\hat{Q}^{\pm}}:.$$

Note that the factor τ cannot be absorbed in the operators $U^{\pm}$ in this manner.

To determine the reduced kernels $\hat{q}^{0\pm}_{lm}((s)_l; (s')_m)$ of the operators $\hat{Q}^{\pm} = \hat{Q}^{\pm}(B)$ one need only observe that

$$\tau^{\mp 1}B^{+}(s)\tau^{\pm 1} = B^{+}(s)e^{\mp\theta(s)} \quad \text{and} \quad \tau^{\mp 1}B^{-}(s)\tau^{\pm 1} = e^{\pm\theta(s)}B^{-}(s).$$

One then finds

$$\hat{q}^{0+}_{lm}((s)_l; (s')_m) = e^{-[\theta]_l}\, q^{0+}_{lm}((s_l); (s')_m)e^{[\theta]_m},$$
$$\hat{q}^{0-}_{lm}((s)_l; (s')_m) = e^{[\theta]_l}\, q^{0-}_{lm}((s')_l; (s')_m)e^{-[\theta]_m}.$$

Introducing operators $\hat{V}^{+} = \tau^{-1}V\tau$ and $\hat{V}^{-} = \tau V\tau^{-1}$ with reduced kernels $\hat{V}^{0\pm}_{lm}$ one may write the basic equations for the operators $Q^{\pm}$ in the form

$$\hat{Q}^{+} = (:e^{\Gamma\hat{Q}^{+}}: \succ \hat{V}^{+})'', \qquad \hat{Q}^{-} = (\hat{V}^{-} \prec :e^{-\Gamma\hat{Q}^{-}}:)'',$$

where $(\ )''$ means $(\ ) - (\ )_{11}$.

Actually, one will introduce a single "renormalized" interaction operator instead of $\hat{V}^+$ and $\hat{V}^-$, namely, the operator $\hat{V}(B)$ with the reduced kernels

$$\hat{V}^0_{lm}((s)_l; (s')_m) = e^{-[\theta]_l} v^0_{lm}((s)_l; (s')_m) e^{-[\theta]_m}.$$

Note that the interaction operator $V(A)$ can then be written as

$$V(A) = \hat{V}(\hat{A})$$

in terms of the renormalized kernels and the renormalized field operators.

In field theory, where θ is constant, this renormalization of V is referred to as "coupling constant" or "charge" renormalization, since the operator V carries the charge or some other coupling constant as a factor. (In electrodynamics, actually only one out of two contributions to the charge renormalization is covered in this way.)

The operator $\hat{V}$ plays a role in a different procedure of absorbing the factor τ employed for the removal of a certain divergence, namely, the divergence of the leading integral involved in the definition of e^θ. This will be indicated at the end of §18.

18. Remarks about removal of divergences. In this last section we shall describe in general terms what happens to the operator Q^- if the kernel $v(s)$ of the self-energy operator V_{11} is infinite and also if, in addition, the same is true of the factor $e^{\theta(s)}$, which enters the description of the operator U^-. The fate of the operator Q^+ is similar to that of Q^- in an obvious manner and will not be discussed.

The operator V_{11} was chosen as

$$V_{11} = (V' \multimap \Gamma Q^-)_{11},$$

in order that the basic equation

$$Q^- = V' \multimap :e^{-\Gamma Q^-}: - V_{11} \multimap \Gamma Q^- + V_{11}$$

admit iterations or formal expansion. We shall show that certain terms contributed by $V' \multimap :e^{-\Gamma Q^-}:$ to this formal expansion can be combined with certain terms of the expansion of $V_{11} \multimap \Gamma Q^-$ in such a way that the divergence of the integral which causes the infinity of $v(s)$ is reduced, if not removed. Also we shall show, to a certain degree, that if e^θ is infinite, other infinite terms occur in the formation of Q^- which can be combined with e^θ so as to reduce (or remove) the divergence, provided the term e^θ is made to enter the formation of Q^- through interaction energy renormalization.

In this section we shall not give a complete derivation of the stated facts, not even in the formal sense. We shall be satisfied with describing some of the essential ideas involved under simplified circumstances.

We imagine the operator $Q = Q^-$ to be developed in a series

$$Q = \sum_{n=0}^{\infty} \sum_{l,m} Q_{lm}^{(n)}$$

in terms of operators $Q_{lm}^{(n)}$ of degree n in V'.

Out of the operator $Q_{lm}^{(n)}$ we shall construct particular contributions to the operator $Q_{lm}^{(n+r)}$ for any $r > 0$, assigned to the contributions

$$(V_{11})_r = (-1)^r V_{lm_r} \times \Gamma(V_{l_{r-1}m_{r-1}} \times \cdots \times \Gamma V_{l_1 1})$$

to the component $V_{11}^{(r)}$ of the operator V_{11}. The cross $\times$ is to indicate that all annihilation prongs of $V_{l_{r-1}m_{r-1}}, \cdots, V_{l_1 1}$ and all creation prongs of $V_{l_1 m_1}, \cdots, V_{l_2 m_2}$ are contracted with each other. The contribution to $Q_{lm}^{(n+r)}$ assigned to $(V_{11})_r$, denoted by $(V_{11})_r \overset{a}{\multimap} \Gamma Q_{lm}^{(n)}$, will then be given as

$$(V_{11})_r \overset{a}{\multimap} \Gamma Q_{lm}^{(n)} = (-1)^r V_{lm_r} \times \Gamma V_{l_{r-1}m_{r-1}} \times \cdots \times \Gamma(V_{l_1 1} \multimap \Gamma Q_{lm}^{(n)}) \cdots).$$

Note that only one prong of $Q_{lm}^{(n)}$ is contracted so that the resulting contribution to $Q^{(n+r)}$ has indeed l creation and m annihilation prongs just as $Q_{lm}^{(n)}$.

Now we observe that the operator $-(V_{11})_r \multimap \Gamma Q_{lm}^{(n)}$ is a contribution to the second term $-V_{11} \multimap \Gamma Q$ of our basic equation and hence to $Q_{lm}^{(n+r)}$. Consequently, the sum $(V_{11})_r \overset{b}{\multimap} \Gamma Q_{lm}^{(n)} = (V_{11})_r \overset{a}{\multimap} \Gamma Q_{lm}^{(n)} - (V_{11})_r \multimap \Gamma Q_{lm}^{(n)}$ contributes to $Q_{lm}^{(n+r)}$; we shall show that the terms of this sum can be so combined that the leading singularities in them cancel out, so that the divergence of these terms will be reduced if not removed.

We first consider the leading contributions

$$V_{1b} \times \Gamma V_{b1},$$

to the operator $V_{11}^{(2)}$, where the cross $\times$ is to indicate that both creation prongs of V_{b1} are to be contracted. For simplicity we assume $b = 2$. Moreover, we shall denote the contribution $V_{12} \times \Gamma V_{21}$ simply by $(V_{11})_2$; the formulas we shall derive for this contribution will be similar to those that hold for the whole operator $V_{11}^{(2)}$.

Let

$$v_{12}(s; s_1', s_2') = \delta(s - s_1' - s_2') v_{12}^0(s_1', s_2')$$

and

$$v_{21}(s_1, s_2; s') = \delta(s_1 + s_2 - s')v_{21}^0(s_1, s_2)$$

be the kernels of the operators V_{12} and V_{21}; then the reduced kernel of the operator $(V_{11})_2 = V_{12} \times \Gamma V_{21}$ is evidently given by the function

$$\int v_{12}^0(\tilde{s}, s - \tilde{s})[\omega(\tilde{s}) + \omega(s - \tilde{s}) - \omega(s)]^{-1}v_{21}^0(\tilde{s}, s - \tilde{s})\, d\tilde{s}.$$

We next observe that in the series for Q together with every term $Q_{lm}^{(n)}$ also the term

$$V_{12} \times \Gamma(V_{21} \multimap \Gamma Q_{lm}^{(n)})$$

occurs as a contribution to $Q_{lm}^{(n+2)}$. For, the expression $-V_{21} \multimap \Gamma Q_{lm}^{(n)}$ is clearly a contribution to $V' \multimap :e^{-\Gamma Q}:$ and hence to Q, and the same is therefore true of $V_{12} \multimap \Gamma(V_{21} \multimap \Gamma Q_{lm}^{(n)})$. We shall employ the notation

$$V_{12} \times \Gamma(V_{21} \multimap \Gamma Q_{lm}^{(n)}) = (V_{11})_2 \overset{a}{\multimap} \Gamma Q_{lm}^{(n)}$$

for this term to indicate that the operators V_{12} and V_{21} involved are the same that occur in V_{11}.

Let $q_{lm}^{(n)}((s)_l; (s')_m)$ be the kernel of the operator $Q_{lm}^{(n)}$. The kernel of $V_{11} \overset{a}{\multimap} \Gamma Q_{lm}^{(n)}$ is the sum of l terms corresponding to the l possible contractions; thus it is of the form

$$\sum_{\lambda=1}^{l} w(s_\lambda, \omega^\lambda)[\omega(s_\lambda) - \omega^\lambda]^{-1}q_{lm}^{(n)}((s)_l; (s')_m),$$

where

$$\omega^\lambda = \omega(s_1') + \cdots + \omega(s_m') - \omega(s_1) - \cdots - \omega(s_l) + \omega(s_\lambda)$$

and

$$w(s, \omega^*) = \int v_{12}^0(\tilde{s}, s - \tilde{s})[\omega(\tilde{s}) + \omega(s - \tilde{s}) - \omega^*]^{-1}v_{21}^0(\tilde{s}, s - \tilde{s})\, d\tilde{s}.$$

Now we observe that the kernel of the operator

$$(V_{11})_2 \multimap \Gamma Q_{lm}^{(n)} = (V_{12} \times \Gamma V_{21}) \multimap \Gamma Q_{lm}^{(n)}$$

differs from that of $(V_{11})_2 \overset{a}{\multimap} \Gamma Q_m^{(n)}$ only in the factor w. Specifically, the kernel of this operator can be obtained from that of $V_{11} \overset{a}{\multimap} \Gamma\, Q_{lm}^{(n)}$ simply by replacing the factor $w(s, \omega^\lambda)$ by $w(s_\lambda, \omega(s_\lambda))$. From the identity

$$\{[\omega_1 + \omega_2 - \omega^*]^{-1} - [\omega_1 + \omega_2 - \omega]^{-1}\}[\omega - \omega^*]^{-1}$$
$$= -[\omega_1 + \omega_2 - \omega^*]^{-1}[\omega_1 + \omega_2 - \omega]^{-1}$$

we infer that the kernel of the difference

$$(V_{11})_2 \overset{b}{\multimap} \Gamma Q_{lm}^{(n)} = (V_{11})_2 \overset{a}{\multimap} \Gamma Q_{lm}^{(n)} - (V_{11})_2 \multimap \Gamma Q_{lm}^{(n)}$$

is given by

$$-\sum_{\lambda=1}^{l} y(s_\lambda, \omega^\lambda) q_{lm}^{(n)}((s)_l; (s')_m)$$

with

$$\begin{aligned} y(s, \omega^*) &= \{w(s, \omega(s)) - w(s, \omega_*)\}[\omega(s) - \omega^*]^{-1} \\ &= \int v_{12}^0(s; \tilde{s}, s - \tilde{s})[\omega(\tilde{s}) + \omega(\tilde{s}) + \omega(s - \tilde{s}) - \omega^*]^{-1} \\ &\quad \cdot [\omega(\tilde{s}) + \omega(s - \tilde{s}) - \omega(s)]^{-1} v_{21}^0(\tilde{s}, s - \tilde{s}; s)\, d\tilde{s}. \end{aligned}$$

Whether or not the integrals $w(s, \omega^*)$ and $w(s, \omega)$ converge depends on the behavior of v_{12}^0 and v_{21}^0 as $|\tilde{k}|$ tends to infinity. Since, in case $\omega^2 = \mu^2 + |k|^2$, the terms $\omega(\tilde{s})$ and $\omega(s - \tilde{s})$ behave like $|\tilde{k}|$, their presence in the denominator of the integrals w may be sufficient to produce convergence. If not, the integral $y(s, \omega^*)$ may nevertheless converge since the terms $\omega(\tilde{s})$ and $\omega(s - \tilde{s})$ occur twice in its denominator. If this is the case, the divergence of the terms $w(s, \omega^*)$ and $w(s, \omega)$ has been removed[4] by combining $w(s, \omega^*)[\omega - \omega^*]^{-1}$ and $w(s, \omega)[\omega - \omega^*]^{-1}$ into the one term $y(s, \omega^*)$. We realize that if the divergence of $w(s, \omega^*)$ can be removed in such a case it is just because of the presence of another infinite term, viz., $w(s, \omega^*)$.

We like to free ourselves from the restriction to the contribution $(V_{11})_2 = V_{12} \times \Gamma V_{21}$ to the operator V_{11} and consider a contribution

$$(V_{11})_r = (-1)^r V_{1m_r} \times \Gamma(\cdots \times \Gamma V_{l_1 1})$$

to $V_{11}^{(r)}$ and the corresponding operator

$$(V_{11})_r \overset{a}{\multimap} \Gamma Q = (-1)^r V_{1m_r} \times \Gamma((V_{l_1 1} \multimap \Gamma Q)\cdots),$$

formerly called $'Q_{lm}^{(n+r)}$; we have set $Q_{lm}^{(n)} = Q$ for simplicity.

We denote by ω^ρ the sum of those quantities ω which are associated with the creation prongs of the operator $V_{l_\rho m_\rho} \times \Gamma(\cdots \times \Gamma V_{l_1 1}\cdots)$ and by ω^λ, as before the sum of the ω's associated with the m annihilation prongs minus the sum of the ω's associated with the $l - 1$ not contracted creation prongs of the operator of Q, and finally we let

[4] An example in which the divergence so removed is the only one present will be given in §A20.

$\omega_\lambda = \omega(s_\lambda)$ be associated with the one contracted creation prong of Q. The Γ-factors of $(V_{11})_r \overset{a}{\multimap} \Gamma Q$ can then be written in the form

$$\gamma(\omega^{(n-1)}, \cdots, \omega^{(1)}, \omega^\lambda)[\omega_\lambda - \omega^\lambda]^{-1},$$

where

$$\gamma(\omega^{(r-1)}, \cdots, \omega^{(1)}, \omega^*) = [\omega^{(r-1)} - \omega^*]^{-1} \cdots [\omega^{(1)} - \omega^*]^{-1}.$$

The kernel of the operator $V_{11} \overset{a}{\multimap} \Gamma Q$ may be written in the form

$$\sum_{\lambda=1}^{l} w(s_\lambda, \omega^\lambda)[\omega_\lambda - \omega^\lambda]^{-1} q_{lm}^{(n)}((s)_l; (s')_m)$$

when $w(s, \omega^*)$ is given by

$$w(s, \omega^*) = \int \cdots \gamma(\tilde{\omega}^{(r-1)}, \cdots, \tilde{\omega}^{(1)}, \omega^*) \cdot v_{1m_1}(s; (\tilde{s}')_{m_r}) \cdots v_{l_2 m_2}((\tilde{s})_{l_2}; (\tilde{s}')_{m_2}) v^0_{l_1 1}((\tilde{s})_{l_1})\, d(\tilde{s} \cdots)$$

in terms of the kernels

$$v_{lm}((s)_l; (s')_m) = v^0_{lm}((s)_l; (s')_{m-1})\delta(s_1 + \cdots + s_l - s'_1 - \cdots - s'_m)$$

of the operators V_{lm}. The variables $\tilde{s}$ and $\tilde{s}'$ are to be identified according to the stipulated contractions.

We assume the sequence of operators $V_{1l_1}, \cdots$ and the contractions between them to be such that each term $\omega^{(\rho)}$ is the sum of at least two terms ω, so that the 11-graph of $V_{1m_r} \times \cdots \times \Gamma V_{l_1 1}$ cannot be split into two 11-graphs; else the 11-operator $(V_{11})_r = V_{1m_r} \times \Gamma \cdots V_{l_1 1}$ could not be formed. (This is the condition referred to above.)

We now observe that the kernel of the operator $V_{11} \overset{b}{\multimap} \Gamma Q$ differs from that of the operator $(V_{11})_r \overset{a}{\multimap} \Gamma Q$ in its $r - 1$ first Γ-factors in that $\omega_\lambda = \omega(s_\lambda)$ takes the place of ω^λ in them; that is to say, the product of its Γ-factors is

$$\gamma(\omega^{(r-1)}, \cdots, \omega^{(1)}, \omega_\lambda)[\omega_\lambda - \omega^\lambda]^{-1}.$$

We now combine the operators $(V_{11})_r \overset{a}{\multimap} \Gamma Q$ and $(V_{11})_r \overset{b}{\multimap} \Gamma Q$ simply by subtracting their Γ-factors. The difference of these factors can be written as

$$\{\gamma(\omega^{(r-1)}, \cdots, \omega^\lambda) - \gamma(\omega^{(r-1)}, \cdots, \omega_\lambda)\}[\omega_\lambda - \omega^\lambda]^{-1} = -\gamma'(\omega^{(r-1)}, \cdots, \omega_\lambda, \omega^\lambda),$$

where

$$
\begin{aligned}
&(-1)^r\gamma'(\omega^{(r-1)},\cdots,\omega,\omega^*)\\
&\quad=[\omega^{(r-1)}-\omega^*]^{-1}[\omega^{(r-1)}-\omega]^{-1}[\omega^{(r-2)}-\omega]^{-1}\cdots\\
&\qquad\qquad\cdot[\omega^{(2)}-\omega]^{-1}[\omega^{(1)}-\omega]^{-1}\\
&\qquad+[\omega^{(r-1)}-\omega^*]^{-1}[\omega^{(r-2)}-\omega^*]^{-1}[\omega^{(r-2)}-\omega]^{-1}\cdots\\
&\qquad\qquad\cdot[\omega^{(2)}-\omega]^{-1}[\omega^{(1)}-\omega]^{-1}\\
&\qquad+\cdots+[\omega^{(r-1)}-\omega^*]^{-1}\cdots[\omega^{(1)}-\omega^*]^{-1}[\omega^{(1)}-\omega]^{-1},
\end{aligned}
$$

as verified by using the formula

$$
\begin{aligned}
[a-\omega]^{-1}[\omega-\omega^*]^{-1}&=[a-\omega^*]^{-1}\\
&=[a-\omega^*]^{-1}\{[a-\omega]^{-1}+[\omega-\omega^*]^{-1}\}.
\end{aligned}
$$

Clearly then, the reduced kernel of the difference

$$(V_{11})_r \overset{b}{\circ\!\!-} \Gamma Q = (V_{11})_r \overset{a}{\circ\!\!-} \Gamma Q - (V_{11})_r \circ\!\!- \Gamma Q$$

may be written as

$$-\sum_{r=1}^{l} y(s_\lambda,\omega^\lambda)q_{lm}^{(n)}((s)_l;(s')_m)$$

with

$$
\begin{aligned}
y(s,\omega^*)=\int\cdots\gamma'(\tilde\omega^{(r-1)},\cdots,\omega(s),\omega^*)\\
\cdot v_{1m_r}(s;(\tilde s')_{m_r})\cdots v_{l_2m_2}((\tilde s)_{l_2};(s')_{m_2})v^0_{l_11}((\tilde s)_{l_1})\,d(\tilde s\cdots).
\end{aligned}
$$

Note that in each of the terms composing γ', in addition to the r factors $[\omega^{(r-1)}-\cdots]^{-1}\cdots[\omega^{(1)}-\cdots]^{-1}$, another factor of the form $[\omega^{(n)}-\cdots]^{-1}$ occurs, instead of $[\omega-\omega^*]^{-1}$. It is for this reason that the integrals y may possibly converge even if the integral w does not. That is to say, the operator $(V_{11})_r \overset{b}{\circ\!\!-} \Gamma Q$ may possibly be defined even if the operators $(V_{11})_r \overset{a}{\circ\!\!-} \Gamma Q$ and $(V_{11})_r \circ\!\!- \Gamma Q$ separately are not.

The sum of all integrals $w(s,\omega(s))$ is the self-energy $v(s)$, the reduced kernel of the operator V_{11}. The divergence of $w(s,\omega(s))$ is therefore referred to as self-energy divergence. This term was present as a factor of $q_{lm}^{(n)}$ by virtue of the proper choice of the operator V_{11} or, in other words, by virtue of the proper renormalization of the single particle energy $\omega^0(s)=\omega(s)+v(s)$. One therefore speaks of the removal of a divergence by renormalization of the single particle energy or, in case of a Lorentz invariant theory, by renormalization of the single particle mass.

Appendix to Chapter I

A1. Formal perturbation procedure for multiple eigenvalues. Suppose the eigenvalue ω_0 of the operator H_0 is isolated, as it was assumed in §3, but not simple, having a multiplicity >1. Then one may try again to set up a formal procedure of determining series expansions for the eigenvalues ω_ε and the eigenvectors X_ε of the operator $H_\varepsilon = H_0 + \varepsilon V$; but in doing this one will meet a peculiar difficulty.

Suppose there exists an eigenvalue ω_ε together with an eigenvector X_ε which, in their dependence on the parameter ε, admit series expansions

$$\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \cdots, \qquad X_\varepsilon = X_0 + \varepsilon X_1 + \cdots.$$

Then the coefficients of these expansions satisfy a number of conditions, which we should like to determine.

Insertion of the two expansions into the equation $(H_0 + \varepsilon V - \omega_\varepsilon)X_\varepsilon = 0$ yields the sequence of equations

$$(0) \qquad (H_0 - \omega_0)X_0 = 0,$$

$$(1) \qquad (H_0 - \omega_0)X_1 = (\omega_1 - V)X_0,$$

$$(2) \qquad (H_0 - \omega_0)X_2 = (\omega_1 - V)X_1 + \omega_2 X_0,$$

$$\cdots.$$

First of all we see that the vector X_0 is an eigenvector of the operator H_0 with the eigenvalue ω_0; but the vector X_0 satisfies additional restrictions. Note that X_0 enters the right members of equations (1), (2), $\cdots$; and these right members must have the property that they are perpendicular to all eigenvectors of the operator H_0 with the eigenvalue ω_0; otherwise these equations would have no solutions. In case the eigenvalue ω_0 is simple, as in §3, the relations expressing this property characterize the expansion coefficients $\omega_1, \omega_2, \cdots$ of the eigenvalue ω; but in the case of a multiple eigenvalue ω_0 these relations restrict also the vector X_0.

Let us denote by $\mathfrak{H}_0$ the space of all eigenvectors of H_0 with eigenvalue ω_0, and by P_0 the (ortho-)projector into $\mathfrak{H}_0$. The condition

satisfied by the right member of equation (1) can then be written as $P_0(\omega_1 - V)X_0 = 0$. In view of $X_0 = P_0X_0$ we may rewrite this condition in the form

$$(1)_0 \qquad P_0VP_0X_0 = \omega_1X_0,$$

which may be interpreted as an eigenvalue equation for the operator P_0VP_0 acting in the space $\mathfrak{H}_0$. In case the space $\mathfrak{H}_0$ is one-dimensional this operator consists in just multiplying by a constant, ω_1, and the vector X_0 in $\mathfrak{H}_0$ is not further restricted. If the space $\mathfrak{H}_0$ is multi-dimensional there may be different eigenvalues ω_1 and then X_0 is restricted to be an eigenvector of such an eigenvalue.

Let us denote by $\mathfrak{H}_{01}$ the space of eigenvectors of P_0VP_0 with the eigenvalue ω_1 and by P_{01} the projector into $\mathfrak{H}_{01}$ so that X_0 is restricted by $X_0 = P_{01}X_0$. The vector X_1, being a solution of equation (1), can then be written in the form

$$X_1 = Z_0(\omega_1 - V)X_0 + X_{10},$$

where Z_0 is the operator used in §3 which satisfies the equations

$$(H_0 - \omega_0)Z_0 = 1 - P_0 \quad \text{and} \quad P_0Z_0 = 0,$$

while X_{10} is a vector in $\mathfrak{H}_0$.

We insert the expression found for X_1 into the condition

$$(2)_0 \qquad P_0[(\omega_1 - V)X_1 + \omega_2X_0] = 0$$

on the right member of equation (2). We split this condition in two by applying the projectors P_{01} and $P_0 - P_{01}$ on it. We take the relations $P_{01}P_0 = P_{01}$ and $X_0 = P_{01}X_0$ into account and observe that, by definition of the projector P_{01}, the relation $P_0(\omega_1 - V)P_0P_{01} = 0$ and hence the adjoint relation $P_{01}P_0(\omega_1 - V)P_0 = 0$ holds, so that $P_{02}P_0(\omega_1 - V)X_{10} = 0$. We then find that the first contribution to condition $(2)_0$ can be written in the form

$$(2)'_0 \qquad P_{01}(\omega_1 - V)Z_0(\omega_1 - V)P_{01}X_0 = \omega_2X_0,$$

i.e., in the form of an eigenvalue equation for an operator acting in the space $\mathfrak{H}_{01}$. The second contribution to condition $(2)_0$ will involve a restriction on the vector X_{10}.

We shall not write down this restriction and refrain from carrying out any further steps. It is already obvious at the present stage that continuation of the procedure started will lead to a nested sequence of spaces $\mathfrak{H}_0 \supset \mathfrak{H}_{01} \supset \cdots$, each being an eigenspace of an operator acting in the preceding space, such that the vector X_0 is restricted to lie in all

these spaces. Also, for the projections $X_{10}, X_{20}, \cdots$ of the vectors $X_1, X_2, \cdots$ into $\mathfrak{H}_0$ sequences of restrictions will be found.

Suppose now any vectors $X_0, X_{10}, \cdots$ have been chosen that do satisfy all these restrictions; then one may form the sequence $X_0 + \varepsilon X_1 + \cdots$ and ask whether or not it converges to an eigenvector X_ε of the operator $H_\varepsilon = H_0 + \varepsilon V$. Assuming that this is the case one could use a finite number of terms of such a series for an approximate description of the vector X_ε provided, of course, that these terms can be determined in a finite number of steps. This will be possible only if the spaces in the sequence $\mathfrak{H}_0, \mathfrak{H}_{01}, \cdots$ will not change any more after a finite number of steps.

If the first eigenspace, $\mathfrak{H}_0$, is finite-dimensional, it is clear that the spaces $\mathfrak{H}_0, \mathfrak{H}_{01}, \cdots$ will indeed remain unchanged eventually. This will, in particular, be the case, once a one-dimensional space turns up in this sequence; but that need never happen. There does not seem to be any way of telling beforehand after which step the spaces in the sequence $\mathfrak{H}_0 \supset \mathfrak{H}_{01} \supset \cdots$ do not change any more. Thus, it may happen that in carrying out the procedure described one will never reach a stage at which one knows definitely how to select even the very first term X_0 in the series expansion. This is the difficulty (to which the author's attention was called by S. Parter) referred to above.

A2. Analytic branches of eigenvalues. We shall not try to prove the convergence of the power series formed with terms $X_0, X_1, \cdots, \omega_1, \omega_2, \cdots$ satisfying the restrictions derived from the formal expansion. Just as in the case of the perturbation of a simple eigenvalue we shall prove directly the existence of eigenvalues ω_ε and eigenvectors X_ε which depend analytically on the parameter ε. Assuming that the isolated eigenvalue ω_0 has the finite multiplicity h, we shall show that there are h such analytic "branches" $\omega_\varepsilon, X_\varepsilon$, which reduce to ω_0 and X_0 for $\varepsilon = 0$. (When we speak of an analytic function in the following, we shall always mean to imply that this function is regular at the origin.)

We shall prove here the existence of such analytic branches only for the case that the Hilbert space is finite-dimensional. As is explained at the end of §4, the perturbation problem of an isolated eigenvalue in an infinite-dimensional Hilbert space can be reduced to a corresponding problem in a finite-dimensional space.

When we treated the perturbation of a simple eigenvalue in §3, we first proved the existence of continuous functions $\omega_\varepsilon, X_\varepsilon$ and later on indicated how to establish their analyticity. As to the perturbation of a multiple eigenvalue it was shown by Rellich that even if $H_\varepsilon = H_0 + V_\varepsilon$

depends infinitely differentiably on ε (without being analytic) continuous functions ω_ε, X_ε need not exist and that, even if V_ε depends analytically on ε (without being Hermitean) analytic functions $\omega_\varepsilon, X_\varepsilon$ need not exist. (A counterexample to support the first statement will be given in §A3; the second statement will follow from the considerations below.) One therefore has no right to take it for granted that such analytic functions exist for analytic Hermitean V_ε. Nevertheless, this is the case as Rellich has shown [2, I]. Specifically, we shall prove that *to an isolated eigenvalue ω_0 of the selfadjoint operator H_0 with finite multiplicity h, there exist h eigenvalues ω_ε and eigenvectors X_ε of the operator $H_\varepsilon = H_0 + V_\varepsilon$ depending analytically on ε (near $\varepsilon = 0$) provided the bounded operator V_ε depends analytically on ε and provided the point-eigenvalues of the operator H_ε are real.*

The latter conditions will be satisfied if V_ε, and hence H_ε, is self-adjoint, as Rellich had assumed. On the other hand, Rellich did not restrict himself to bounded disturbances V_ε; we shall do so only for simplicity.

It is sufficient to prove the statement only for operators acting in a finite-dimensional space since the statement for a space of infinite dimension can be reduced to that for a space of finite dimension, in the manner described at the end of §4. At present we shall prove only the existence of analytic branches of eigenvalues of H passing through those of H_0. The full statements will be proved at the end of §5 by using an argument due to Sz.-Nagy.

Accordingly, we consider an n-dimensional space $\mathfrak{H}$ and an operator $H_\varepsilon = H_0 + V_\varepsilon$ acting in it which depends analytically on ε. We assume H_0 to be Hermitean and to have an eigenvalue ω_0 of multiplicity h; of the operator H_ε we assume that all its eigenvalues are real.

This operator H_ε, of course, can be described as an n by n matrix and, therefore, its eigenvalues ω satisfy an equation

$$p_\varepsilon(\omega) = 0,$$

where p_ε is a polynomial of degree n, given as the determinant of the matrix $H_\varepsilon - \omega$. Clearly, the coefficients of p_ε depend analytically on ε and the coefficient of ω^n is 1. Assuming $\omega_0 = 0$, we know that the equation $p_0(\omega) = 0$ has $\omega = 0$ as root of exactly the degree h.

The question arises: Do the roots of a polynomial equation $p_\varepsilon(\omega) = 0$ depend analytically on ε if the coefficients do? The answer is given by the *Puiseux Theorem*. Actually, we do not need the full force of this theorem. It is sufficient to refer to that part of Puiseux's Theorem which states that there exist h analytic functions $\gamma^{(1)}(\zeta), \cdots, \gamma^{(h)}(\zeta)$ of

a complex variable ζ (regular at $\zeta = 0$) and an integer $r \leqq n$ such that the polynomial p_ε can be written as

$$p_\varepsilon(\omega) = [\omega - \gamma^{(1)}(\zeta)] \cdots [\omega - \gamma^{(h)}(\zeta)] q_\varepsilon(\omega)$$

provided ε is taken as $\varepsilon = \zeta^r$; here $q_\varepsilon(\omega)$ is a polynomial in ω of degree $n - h$. Evidently, each function $\gamma^{(\kappa)}(\zeta)$ satisfies the equation

$$p_\varepsilon(\gamma^{(\kappa)}(\zeta)) = 0 \quad \text{with } \varepsilon = \zeta^r.$$

In other words, the roots $\omega_\varepsilon^{(\kappa)} = \gamma^{(\kappa)}(\varepsilon^{1/r})$ are regular-analytic in the variable $\varepsilon^{1/r}$ and not necessarily in ε itself.

For example, the roots $\omega = \pm\sqrt{\varepsilon}$ of the equation $\omega^2 - \varepsilon = 0$ are not regular in ε but are regular in ζ when written as $\omega = \pm\zeta$ with $\varepsilon = \zeta^2$.

Now, Rellich observed that *the roots $\gamma^{(\kappa)}(\zeta)$ are regular functions of $\varepsilon = \zeta^r$ after all, provided they are real for real values of ε.* To show this, let $\gamma(\zeta)$ be one of the functions $\gamma^{(\kappa)}(\zeta)$ so that $\gamma(\zeta)$ is real for real $\varepsilon = \zeta^r$ and in particular for real ζ. Now, the function $\tilde{\gamma}(\zeta) = \gamma(e^{i\pi/r}\zeta)$ is evidently a solution of the equation $p_{\tilde{\varepsilon}}(\tilde{\gamma}) = 0$ with $\tilde{\varepsilon} = (e^{i\pi/r}\zeta)^r = -\zeta^2 = -\varepsilon$; hence, it is one of the functions $p_{-\varepsilon}^{(\kappa)}(\zeta)$ and therefore real for real ε. It follows that the function $\omega_\varepsilon = \gamma(\varepsilon^{1/r})$, defined on the upper ε-half-plane is real for real positive and negative values of ε and hence can be uniquely continued into the lower ε-half-plane by reflection. Since $\gamma(\zeta)$ is regular in ζ at $\zeta = 0$, the function ω_ε is bounded at $\varepsilon = 0$. Consequently, the function ω_ε, defined in the full ε-plane by reflection, is indeed regular in ε at $\varepsilon = 0$.

Applying this observation to the roots of the characteristic equation of the operator $H_\varepsilon = H_0 + V_\varepsilon$, which were assumed real for real ε, we conclude that there are indeed h regular analytic functions ω_ε which are eigenvalues of this operator H_ε. Thus the part of Rellich's Theorem which concerns the eigenvalues is established. The existence of h analytic functions X_ε which are eigenvectors of H_ε with the eigenvalue ω_ε could be derived from this part provided the operator H_ε is assumed to be given equivalent to a Hermitean one for each sufficiently small $|\varepsilon|$.

Another approach to Rellich's Theorem was developed by Sz.-Nagy [3;6] by using the contour integral formula given earlier (p. 9). The same tool was used by Kato [4] to obtain asymptotic expansions of eigenvalues and vectors if the H_ε is only assumed to possess an asymptotic expansion in powers of ε.

A3. Nonanalytic disturbance. It was mentioned in Appendix A2 that even if the operator H_ε depends infinitely differentiably on ε there need not be any eigenvector X_ε which is continuous at $\varepsilon = 0$. Following

Rellich [2, A, I] we give a counterexample. In the two-dimensional space of vectors $\Phi = (\phi_1, \phi_2)$ with $(\Phi, \Phi) = |\phi_1|^2 + |\phi_2|^2$ we let L_ε be the operator given by the diagonal matrix with the components $l_{11}(\varepsilon) = \lambda_\varepsilon^+$, $l_{22}(\varepsilon) = \lambda_\varepsilon^-$, $l_{12}(\varepsilon) = l_{21}(\varepsilon) = 0$, and let R_ε be the rotation through the angle θ_ε, so that $r_{11}(\varepsilon) = r_{22}(\varepsilon) = \cos\theta_\varepsilon, r_{12}(\varepsilon) = -r_{21}(\varepsilon) = \sin\theta_\varepsilon$ are the matrix elements of R_ε. Then we consider the operator

$$H_\varepsilon = V_\varepsilon = R_\varepsilon L_\varepsilon R_\varepsilon^{-1}$$

as our disturbed operator with $H_0 = 0$.

We take $\lambda_\varepsilon^\pm$ as infinitely differentiable and vanishing with all derivatives at $\varepsilon = 0$, such as $\lambda_\varepsilon^\pm = \pm e^{-c/\varepsilon^2}$; and finally we take $\theta_\varepsilon = c_0/\varepsilon$ so that $\theta_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Then the coefficients of the matrix V_ε are infinitely differentiable and vanish with all derivatives for $\varepsilon = 0$. Nevertheless, the eigenvectors

$$R_\varepsilon(1, 0) = (\cos\theta_\varepsilon, -\sin\theta_\varepsilon),$$
$$R_\varepsilon(0, 1) = (\sin\theta_\varepsilon, \cos\theta_\varepsilon)$$

of H_ε do not tend to any limit as ε tends to zero. Thus it is seen that there are no eigenvectors X_ε continuous at $\varepsilon = 0$.

A second observation of Rellich [2, I] was that an operator depending analytically on two parameters $\varepsilon_1, \varepsilon_2$ need not possess an eigenvalue which is regular-analytic for $\varepsilon_1 = \varepsilon_2 = 0$.

Indeed, the operator given by the matrix

$$H_{\varepsilon_1\varepsilon_2} = V_{\varepsilon_1\varepsilon_2} = \begin{pmatrix} 2\varepsilon_1 & \sqrt{2}(\varepsilon_1 + \varepsilon_2) \\ \sqrt{2}(\varepsilon_1 + \varepsilon_2) & 2\varepsilon_2 \end{pmatrix}$$

acting on vectors (ϕ_1, ϕ_2) has the eigenvalues

$$\lambda = \varepsilon_1 + \varepsilon_2 + \sqrt{2}\,(\varepsilon_1^2 + \varepsilon_2^2)^{1/2},$$

which are not regular in $\varepsilon_1, \varepsilon_2$ at $\varepsilon_1 = \varepsilon_2 = 0$.

A4. The question of uniform perturbation of the spectrum. So far we have described the perturbation of a single point-eigenvalue or of an isolated segment of the spectrum by establishing the associated projector P_ε for an ε-interval $|\varepsilon| \leqq \varepsilon_0$. This interval depended essentially on the undisturbed eigenvalue and on its distance from its neighbors; it may shrink to zero when the undisturbed eigenvalue approaches an accumulation point.

The question may be asked under which conditions does that not happen. Specifically, with reference to an operator with a pure point-spectrum, one may ask when perturbed analytic point-eigenvalues exist

in a uniform interval for the parameter ε. Moreover, if this is so, one may ask when the eigenvalues thus produced by the perturbation form the complete spectrum of the disturbed operator.

Rellich has derived a sufficient condition for this last property in connection with certain differential operators [2, IV]. Significant results on these questions were also obtained by E. Heinz [5]. Here we shall be satisfied with showing, in connection with two examples due to Rellich, that the last property may or may not obtain.

Both examples refer to the space of functions $f(x)$ defined for $0 \leqq x \leqq 1$ for which the integral

$$\|f\|^2 = \int_0^1 |f(x)|^2 \, dx$$

is finite. The undisturbed and disturbed operators will be denoted by L_0 and $L_\varepsilon = L_0 + \varepsilon L_1$ since we want to reserve the notations H_0 and $H_\varepsilon = H_0 + \varepsilon V$ for their inverses.

The undisturbed operator of our *first example* is given by

$$L_0 f(x) = \int_0^x x' f(x') \, dx' + x \int_x^1 f(x') \, dx',$$

while the disturbance is

$$\varepsilon L_1 f(x) = \varepsilon x \int_0^1 x' f(x') \, dx'.$$

In our *second example* we take

$$L_0 f(x) = (1 - x) \int_0^x x' f(x') \, dx' + x \int_x^1 (1 - x') f(x') \, dx'$$

and the same disturbance εL_1.

Since the operators L_0 and $L_0 + \varepsilon L_1$ are integral operators with continuous kernels they possess an essentially discrete spectrum, i.e., the eigenspaces of every closed interval not containing the origin are finite-dimensional. In fact, as we shall see the spectrum of both operators L_0 consists of a sequence of positive eigenvalues tending to zero.

The same is true of the first operator $L_\varepsilon = L_0 + \varepsilon L_1$ for $\varepsilon > -1$; the eigenvalues of L_ε and appropriate eigenfunctions depend analytically on ε for $\varepsilon < 1$. The second operator L_ε also possesses eigenvalues and eigenvectors which depend analytically on ε, for all ε; but these eigenvalues do not form the whole spectrum for negative ε. For such values of ε there is a single additional negative eigenvalue which tends to zero as $\varepsilon \to 0$.

The second example thus shows that, even if all point-eigenvalues of the operator L_0 connect analytically with eigenvalues of the operator $L_0 + \varepsilon L_1$ these eigenvalues need not form the complete spectrum of this operator.

The verification of our statement concerning the two examples is immediate if we realize first that the operators L_0 are the inverses of the differential operator $H_0 = -d^2/dx^2$ acting on continuously differentiable functions $\phi(x)$ possessing a square integrable second derivative and satisfying the boundary condition

$$\phi(0) = 0, \qquad \frac{d}{dx}\phi(1) = 0 \quad \text{in Case 1,}$$

$$\phi(0) = 0, \qquad \phi(1) = 0 \qquad \text{in Case 2,}$$

and secondly that the operators $L_\varepsilon = L_0 + \varepsilon L_1$ are the inverses of again the differential operator $H_\varepsilon = -d^2/dx^2$ acting on functions $\phi(x)$ as before, but this time satisfying the boundary conditions

$$\phi(0) = 0, \qquad \left(\frac{d}{dx} - \frac{\varepsilon}{1+\varepsilon}\right)\phi(1) = 0 \quad \text{in Case 1,}$$

$$\phi(0) = 0, \qquad \left(\frac{d}{dx} + \frac{1-\varepsilon}{\varepsilon}\right)\phi(1) = 0 \quad \text{in Case 2.}$$

Note that these boundary conditions depend on ε.

Now, as is well known, the operator H_ε in both cases possesses the (not normed) eigenfunctions

$$u_n(x) = \sin \mu_n x,$$

with $\mu = \mu_n$ being the nth root of the equation

$$\mu = \sigma \tan \mu,$$

where

$$\sigma = \varepsilon/(1+\varepsilon) \quad \text{in Case 1,}$$

$$\sigma = (\varepsilon - 1)/\varepsilon \quad \text{in Case 2.}$$

The corresponding eigenvalues are $\omega_n = \mu_n^2$ in both cases. The system of these eigenfunctions is complete for $\varepsilon \geqq -1$ in Case 1 and $\varepsilon > 0$ in Case 2; these two conditions correspond to $\sigma \geqq -1$.

If this condition is violated, i.e., if $\sigma < -1$, there is an additional point-eigenvalue $\omega^* = -\lambda^2$, where λ is the root of the equation

$$\lambda = -\sigma \tanh \lambda,$$

with the eigenfunction $u^*(x) = \sinh \lambda x$.

From the fact that $\tanh \lambda/\lambda$ behaves like $-1/\lambda$ at $\lambda \sim -\infty$ we conclude that λ behaves like $\sigma - 1$, hence like $-\varepsilon$ in Case 2, near $\varepsilon = 0$ for $\varepsilon < 0$. Thus the eigenvalue ω^* is seen to behave as $\omega^* \sim -\varepsilon^2$ near $\varepsilon = 0$ for $\varepsilon < 0$ in Case 2.

Our two examples show a peculiar phenomenon, namely, that the disturbed operator H_ε cannot be written in the form $H_\varepsilon = H_0 + \varepsilon V$ although it is possible to write the inverse L_ε in the form $L_0 + \varepsilon L_1$; this is connected with the fact that the two operators H_ε and H_0 have different domains of definition. This phenomenon, met somewhat incidentally, plays an important role in the theory of perturbation of unbounded operators. We have met this phenomenon in §8 and shall meet it again in Chapter III.

A5. The projectors associated with an isolated part of the spectrum. In §4 we derived an equation for the projectors P of an isolated segment of the spectrum of the operator $H = H_0 + V$. In this section we shall first show that the equation, $Q = f(Q)$, has a solution. Here Q is the difference

$$Q = P - P_0$$

of disturbed and undisturbed projector and

$$f(Q) = Z_0 Q P_0 H_0 - Z_0(1 - P_0 - Q)V(P_0 + Q).$$

We apply iterations beginning with $Q_0 = 0$ and defining Q_{n+1} by

$$Q_{n+1} = f(Q_n).$$

Without restriction we assume that the origin $\omega = 0$ lies in the closed interval $\mathscr{I}_0$ which contains the isolated segment of the spectrum of H_0. Then we let α and β be the maxima of $|\omega|$ in $\mathscr{I}_0$ and of $|\omega|^{-1}$ outside of $\mathscr{J}$, where $\mathscr{J}$ is the open interval containing $\mathscr{I}_0$ that was so chosen that the spectrum of H_0 is empty in $\mathscr{J} - \mathscr{I}_0$. Clearly, $\alpha\beta < 1$; in addition,

$$|\omega \eta_{\mathscr{I}_0}(\omega)| \leqq \alpha \quad \text{and} \quad |\zeta(\omega)| = |\omega^{-1}[1 - \eta_{\mathscr{J}}(\omega)]| \leqq \beta$$

so that $\|P_0 H_0\| \leqq \alpha$, $\|Z_0\| \leqq \beta$.

Let $q > 0$ be a number to be chosen later on. Then, whenever $\|Q\| \leqq q$, we obviously have

$$\|f(Q)\| \leqq \alpha\beta q + \beta(1 + q)^2 \|V\|.$$

Moreover, assuming $\|Q\| \leqq q$ and $\|Q'\| \leqq q$ we have

$$\|f(Q) - f(Q')\| \leqq \theta \|Q - Q'\|$$

with $\theta = \alpha\beta + 2\beta(1 + q)\|V\|$. We now want to choose q conveniently and then $\|V\|$ so small that $\theta < 1$ and that, moreover, $\|Q\| \leqq q$ implies $\|f(Q)\| \leqq q$. To this end we simply choose any $q < 1$ and restrict V by

$$\|V\| < \frac{1 - \alpha\beta}{\beta} \frac{q}{(1 + q)^2}.$$

Then we find indeed $\theta < \alpha\beta + (1 - \alpha\beta)(2q/(1 + q)) < 1$ and

$$\|f(Q)\| \leqq \alpha\beta q + (1 - \alpha\beta)q = q,$$

as desired. Evidently, iteration can be carried out and the sequence $\{Q_n\}$ converges to a limit operator Q in the sense of $\|Q_n - Q\| \to 0$ as $n \to \infty$.

By standard arguments one proves that this limit operator satisfies the conditions

$$QP_0 = Q \quad \text{and} \quad P_0Q = 0.$$

These conditions or, equivalently, the conditions

$$PP_0 = P \quad \text{and} \quad P_0P = P_0$$

were imposed on the operator $P = P_0 + Q$ for convenience, since they are linear. To complete the theory it is necessary to construct, in place of the nonorthogonal projectors P, orthoprojectors which project orthogonally into the range $\mathfrak{S}$ of P.

First we note that $\|Q\| \leqq q < 1$ implies

$$\|P_0\Phi\| \leqq \|P\Phi\| + q\|\Phi\|$$

or, with $P_0\Phi$ in place of Φ,

$$\|P_0\Phi\| \leqq \|P\Phi\| + q\|P_0\Phi\| \quad \text{whence} \quad \|P\Phi\| \geqq (1 - q)\|P_0\Phi\|.$$

Introducing the adjoint P^* of the operator P, which exists since P is bounded, we have

$$(\Phi, P^*P\Phi) \geqq (1 - q)^2(\Phi, \Phi) \quad \text{for } \Phi = P_0\Phi.$$

Consequently, the bounded positive-definite operator P^*P has a unique inverse in the range $\mathfrak{S}_0$ of P_0, which we denote by $[P^*P]^{-1}$.

One readily verifies that the operator

$$\hat{P} = P[P^*P]^{-1}P^*$$

is defined in all of $\mathfrak{H}$ and, also, that $\hat{P}$ *is the orthoprojector with the range* $\mathfrak{S}$.

Moreover, we find

$$H\hat{P} = HP\hat{P} = PHP\hat{P} = \hat{P}PHP\hat{P} = \hat{P}HP\hat{P} = \hat{P}H\hat{P}.$$

Since the latter operator is Hermitean, for real ε the same is true of $H\hat{P}$. Consequently,

$$\hat{P}H = H\hat{P} \quad \text{for real } \varepsilon,$$

i.e., for real ε the projector $\hat{P}$ commutes with H as it should.

Instead of the operator $H\hat{P}$ when restricted to act in the space $\mathfrak{S}$ we may introduce an equivalent operator H^0 acting in the space $\mathfrak{S}_0$. We need only introduce the operator

$$S = P[P^*P]^{-1/2}P_0,$$

which is defined in all of $\mathfrak{H}$, and set

$$H^0 = S^*HS.$$

Since, evidently, $S^*S = P_0$ and $SS^* = \hat{P}$, the operator H^0, when acting in $\mathfrak{S}_0$, is indeed equivalent with the operator

$$H\hat{P} = SH^0S^*.$$

Clearly, if the operator H_ε depends analytically on ε, the same is true of the operators $\hat{P} = \hat{P}_\varepsilon$, $H_\varepsilon\hat{P}_\varepsilon$, S_ε, H^0_ε, since it is true of P_ε, as shown earlier.

The last results can be used to give a simpler *derivation of Rellich's Theorem* referred to in Appendix A2.

Accordingly, suppose that the operator $H = H_\varepsilon$ acts in a finite-dimensional space $\mathfrak{H}$, of dimension n say, depends analytically on ε, and is Hermitean for real values of ε. The *statement* then is that for sufficiently small $|\varepsilon|$ there are n eigenvalues $\lambda_\varepsilon^{(1)}, \cdots, \lambda_\varepsilon^{(n)}$ of H_ε which depend analytically on ε, and corresponding n eigenvectors $\Phi_\varepsilon^{(1)} \neq 0$, $\cdots$, $\Phi_\varepsilon^{(n)} \neq 0$, which also depend analytically on ε and span the whole space $\mathfrak{H}$.

Following Sz.-Nagy [3], we prove this statement by induction with respect to the dimension n. Clearly, the statement holds for $n = 1$. We consider the expansion

$$H_\varepsilon = H_0 + \varepsilon H^{(1)} + \varepsilon^2 H^{(2)} + \cdots,$$

in which the operators $H_0, H^{(1)}, \cdots$ are evidently Hermitean.

In case all operators $H_0, H^{(1)}, \cdots$ are multiples of the identity

$H_0 = \lambda_0$, $H^{(1)} = \lambda_1, \cdots$, clearly also H_ε is a multiple of the identity[1] $H_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \cdots$.

In case this is not so, there is a first one, $H^{(r)}$ say, among the operators $H^{(1)}, \cdots$ which is not a multiple of the identity. We then may write

$$H_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \cdots + \varepsilon^r H_\varepsilon^{(r)},$$

where $H_\varepsilon^{(r)}$ depends analytically on ε and is Hermitean for real ε. Clearly, the operator $H_0^{(r)} = H^{(r)}$ has at least two distinct eigenvalues.

Therefore we may select two closed intervals each containing at least one eigenvalue of $H^{(r)}$ whose union contains the whole spectrum of $H^{(r)}$. We take one of these intervals, $\mathscr{I}^{(r)}$, as the interval $\mathscr{I}$ which figures in the theory of §4 and in the beginning of the present section. Accordingly, we introduce for this interval the projectors $P_0^{(r)}$, $P_\varepsilon^{(r)}$, and $\hat{P}_\varepsilon^{(r)}$, associated with the operator $H_\varepsilon^{(r)}$ (in place of H_ε). From the statements made just before we then may conclude that the operator $(H_\varepsilon^{(r)})^0$, which acts in the space $\mathfrak{S}_0^{(r)}$, is analytic in ε and Hermitean for real ε. Moreover, it acts in a space of a dimension $n^{(r)} \leqq n - 1$ since the interval $\mathscr{I}^{(r)}$ does not contain the whole spectrum of $(H_\varepsilon^{(r)})^0$. Therefore, the induction assumption may be applied to H_ε^0. This operator therefore possesses $n^{(r)}$ analytic eigenvalues $\lambda_\varepsilon^{(r)}$ with eigenvectors $\Phi_\varepsilon^{(r)}$ which span the space $\mathfrak{S}_0^{(r)}$. These eigenvalues are then also eigenvalues of the operator

$$H_\varepsilon^{(r)}\hat{P}_\varepsilon^{(r)} = S_\varepsilon^{(r)}(H_\varepsilon^{(r)})^0 S_\varepsilon^{(r)*},$$

with the eigenvectors $\Phi_\varepsilon^{(r)} = S_\varepsilon^{(r)}(\Phi_\varepsilon^{(r)})^0$. These vectors are at the same time eigenvectors of H_ε with the eigenvalues

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \cdots + \varepsilon^r\lambda_\varepsilon^{(r)}.$$

The same argument can be applied to the other one of the two closed intervals selected. As a result we conclude that the statement of Rellich's Theorem indeed holds for the operator H_ε acting in a space of dimension n.

[1] It is at this place that the analytic dependence of H_ε on ε is used. If the expansion of H_ε is only asymptotic, the conclusion need not hold as shown by the first example of Appendix A3.

Appendix to Chapter II

A6. An explicit example. In this first section of Appendix II we shall discuss a specific example of an operator $H = H_0 + V$ for which the intertwining transformations $U^{\pm} = 1 \pm \Gamma R^{\pm}$ can be given explicitly; we take the disturbing operator V such that its representing kernel is the product

$$v(\omega; \omega') = p(\omega)q(\omega')$$

of two functions p and q so chosen that v belongs to one of our admissible classes, which implies (among other properties) that $p(\omega)$ and $q(\omega)$ are bounded. Moreover, p should be square integrable. If both $p(\omega)$ and $\overline{q(\omega)}$ are square integrable, representing vectors Φ_p and Φ_q, we may describe the operator V also by

$$V\Phi = \Phi_p(\Phi_{\bar{q}}, \Phi).$$

One readily verifies that the kernels $r^{\pm}(\omega; \omega')$ of the operators $R^{\pm}$ can be given explicitly by the expressions

$$r^{+}(\omega; \omega') = p(\omega)[g^{+}(\omega)]^{-1}q(\omega'),$$

$$r^{-}(\omega; \omega') = p(\omega)[g^{-}(\omega')]^{-1}q(\omega'),$$

in which

$$g^{\pm}(\omega) = 1 \mp \int [\mp\tilde{\omega} \pm \omega]^{-1}q(\tilde{\omega})p(\tilde{\omega})\, d\tilde{\omega}.$$

Here we have used the abbreviation $[\lambda]^{-1} = \lambda^{-1} + i\pi\delta(\lambda)$ employed in the definition of the operation Γ in §6. Setting

$$l(\omega) = 1 + \int (\tilde{\omega} - \omega)^{-1}q(\tilde{\omega})p(\tilde{\omega})\, d\tilde{\omega},$$

we can also write

$$g^{\pm}(\omega) = l(\omega) \mp i\pi q(\omega)p(\omega).$$

We assume that the functions $p(\omega)$ and $q(\omega)$ are such that $g^{\pm}(\omega) \neq 0$ on their supports, i.e., on the closures of the sets where they do not vanish. Then, clearly, the functions $r^{\pm}(\omega; \omega')$ are defined for all ω, ω',

and in fact belong to the class from which $v(\omega; \omega')$ was taken, as could be verified. Also the operators $U^{\pm} = 1 \pm \Gamma R^{\pm}$ are defined and satisfy the relation $U^{+}U^{-} = 1$. The relation $U^{-}U^{+} = 1$, however, is not always satisfied, as will become clear from the following considerations.

The possibility that $U^{-}U^{+} \neq 1$ is connected with the possibility that the operator $H = H_0 + V$ may have point-eigenvalues.

In case the operator H_0 is simple, so that the "values" of the representers ψ are complex numbers, one easily verifies: If a complex value $\hat{\omega}$ is a point-eigenvalue of H, the function $l(\omega)$ vanishes for $\omega = \hat{\omega}$, i.e.,

$$l(\hat{\omega}) = 1 + \int (\tilde{\omega} - \hat{\omega})^{-1} q(\tilde{\omega}) p(\tilde{\omega})\, d\tilde{\omega} = 0;$$

moreover, $p(\hat{\omega}) = 0$ if ω is real. The latter condition is necessary so that the eigenfunction $(\omega - \hat{\omega})^{-1} p(\omega)$ can be quadratically integrable. Actually, this condition, together with $l(\hat{\omega}) = 0$, is sufficient, since $l(\hat{\omega}) = 0$ and $p(\hat{\omega}) = 0$ for real $\hat{\omega}$ imply that $\hat{\omega}$ is outside of the supports of p and q by virtue of the assumption made above about $p(\omega)$ and $q(\omega)$. If the root $\hat{\omega}$ of $l(\hat{\omega}) = 0$ is nonreal the function $(\omega - \hat{\omega})^{-1} p(\omega)$ is always square integrable since $p(\omega)$ is bounded.

In case the "values" of ψ are vectors in an accessory space $\mathfrak{A}$ of any dimension, the eigenfunctions are of the form $(\omega - \hat{\omega})^{-1} p(\omega)\xi$, where ξ is a vector in $\mathfrak{A}$ and the equation for $\hat{\omega}$ becomes $l(\hat{\omega})\xi = 0$. Thus we may say: The eigenvalue $\hat{\omega}$ should be a number for which the operator

$$\int (\hat{\omega} - \tilde{\omega})^{-1} q(\tilde{\omega}) p(\tilde{\omega})\, d\tilde{\omega},$$

acting in $\mathfrak{A}$, has the eigenvalue 1 and ξ should be a corresponding eigenvector in $\mathfrak{A}$.

In case an eigenvalue $\hat{\omega}$ of H exists, the inverse relation $U^{-}U^{+} = 1$ is violated [12], and the transformations $U^{\pm}$ do not give the complete spectral transformation. More about this question will be said in Appendix A9.

A7. Classes of gentle operators. In our discussion of the perturbation of continuous spectra we have characterized certain operators R by a number of properties which insured that the equations $R^{+} = (1 + \Gamma R^{+})V$, $R^{-} = V(1 - \Gamma R^{-})$ have such operators $R^{\pm}$ as solutions provided V is one of these operators and, in addition, sufficiently small with respect to an appropriate norm. In the present

section we shall describe two classes of operators having these properties. These operators will be represented by integral operators with kernels $r(\omega; \omega')$. We shall first describe classes of functions $r(\omega; \omega')$ [**10**; **12**].

The *first class*, $\mathfrak{r}^\mu$, depends on the choice of a positive number $\mu < 1$. The functions $r(\omega; \omega')$ in this class satisfy a Hölder condition with the exponent μ with respect to each of the two variables ω, ω'. Also, these functions tend to zero as $|\omega| \to \infty$ and as $|\omega'| \to \infty$ and satisfy a Hölder condition with respect to ω^{-1} and $(\omega')^{-1}$. To describe these requirements in a concise form we introduce the function

$$h^\mu(\omega_1, \omega_2) = [1 + |\omega_1|^\mu][1 + |\omega_2|^\mu]|\omega_1 - \omega_2|^{-\mu}$$

of ω_1, ω_2 for $\omega_1 \neq \omega_2$. Because of

$$h^\mu(\omega_1^{-1}, \omega_2^{-1}) = h^\mu(\omega_1, \omega_2),$$

we may define h^μ even for $\omega_1^{-1} = 0$ and for $\omega_2^{-1} = 0$. To the function $r(\omega; \omega')$ we then assign the preliminary norm

$$\|r\|'_\mu = \underset{\omega_1 \neq \omega_2;\, \omega_1' \neq \omega_2'}{\text{l.u.b.}} h^\mu(\omega_1, \omega_2)h^\mu(\omega_1', \omega_2')|r(\omega_1; \omega_1') - r(\omega_2; \omega_2')|,$$

where in forming this least upper bound one should admit infinite values of the variables ω.

The class $\mathfrak{r}^\mu$ *then consists of all those functions* $r(\omega; \omega')$, *defined as zero for* $\omega = \infty$ *and for* $\omega' = \infty$, *for which this norm* $\|r\|'_\mu$ *is finite.*

In this definition we may just as well allow that the representation of vectors Φ by functions $\psi(\omega)$ is not simple; that is to say we may allow the "values" of ψ to be vectors in an accessory space $\mathfrak{N}$. The "values" of the kernels $r(\omega; \omega')$ are then operators acting in this space and the absolute value $|r|$ of r is to be taken as the operator norm

$$|r| = \underset{\xi \neq 0}{\text{l.u.b.}} |r\xi|/|\xi|,$$

where ξ runs over all vectors $\xi \neq 0$ in the Hilbert space $\mathfrak{N}$ with the norm $|\xi|$. The arguments given in this section automatically cover this situation.

Clearly, the functions r of the class $\mathfrak{r}^\mu$ are continuous and bounded. That is to say, the operators $r(\omega; \omega')$ acting in $\mathfrak{N}$ are bounded, uniformly with respect to $(\omega; \omega')$ and they depend continuously on $(\omega; \omega')$ when the distance $|r_1 - r_2|$ is taken as the operator norm of $r_1 - r_2$. Clearly, there is a constant a' such that the inequality $r_0 = \text{l.u.b.}_{\omega_1, \omega_2}|r(\omega_1, \omega_2)| \leqq a'\|r\|'_\mu$ holds for all r in $\mathfrak{r}^\mu$.

Another, rather obvious, property of the class $\mathfrak{r}^\mu$ is that it is a linear space complete with respect to the norm $\| \ \|'_\mu$.

Next we consider the "compound kernels"

$$r_1\gamma r_2(\omega;\omega') = \int r_1(\omega;\tilde{\omega})[\tilde{\omega} - \omega']^{-1}r_2(\tilde{\omega};\omega')\,d\tilde{\omega}$$

and

$$(\gamma r_1)r_2(\omega;\omega') = \int [\omega - \tilde{\omega}]^{-1}r_1(\omega;\tilde{\omega})r_2(\tilde{\omega};\omega')\,d\tilde{\omega}$$

formed with two functions r_1, r_2 from $\mathfrak{v}^\mu$ and $[\omega - \omega']^{-1} = (\omega - \omega')^{-1} + i\pi\delta(\omega - \omega')$. The principal value involved in the formation of these integrals is evidently defined since r_1 and r_2 are Hölder continuous. We now state that *the kernels* $r_1\gamma r_2$ *and* $(\gamma r_1)r_2$ *again belong to the class* $\mathfrak{v}^\mu$. This fact is a consequence of the theorem of Plemelj and Privalov, see, e.g., [38]. In deriving this fact one will determine a constant c such that

$$\|r_1\gamma r_2\|'_\mu,\ \|(\gamma r_1)r_2\|'_\mu \leqq c\|r_1\|'_\mu\|r_2\|'_\mu$$

for all r_1, r_2. We now introduce our final norm $\|\ \|$ in such a way that the last inequality holds without the factor c:

$$\|r_1\gamma r_2\|_\mu,\ \|(\gamma r_1)r_2\|_\mu \leqq \|r_1\|_\mu\|r_2\|_\mu.$$

To this end we evidently need only set

$$\|r\|_\mu = c\|r\|'_\mu.$$

The properties enumerated are sufficient to insure that the equations

$$r^+ = v + (\gamma r^+)v, \qquad r^- = v - v\gamma r^-$$

have solutions in $\mathfrak{v}^\mu$ provided the kernel v is in the class $\mathfrak{v}^\mu$ and $\|v\| < 1$.

In deriving other properties of the functions r in $\mathfrak{v}^\mu$ it is advantageous at first to work not with square integrable functions but with the space $\mathfrak{h}^\mu$ of functions $\psi(\omega)$, defined also for $\omega = \infty$, for which the norm

$$\|\psi\|_\mu = c \operatorname*{l.u.b.}_{\omega_1 \neq \omega_2} h^\mu(\omega_1,\omega_2)|\psi(\omega_1) - \psi(\omega_2)|$$

is finite. It then follows again by the theorem of Plemelj and Privalov that the function

$$\gamma r\psi(\omega) = \int [\omega - \omega']^{-1}r(\omega;\omega')\psi(\omega')\,d\omega'$$

also belongs to $\mathfrak{h}^\mu$ and that

$$\|\gamma r\psi\|_\mu \leqq \|r\|_\mu\|\psi\|_\mu.$$

It is also convenient to use notations such as "r", "γr", for the integral operators with the kernels r and γr.

We are then able to speak of the product operator "(γr_1)" "(γr_2)" for r_1 and r_2 in $\mathfrak{r}^\mu$ and therefore may formulate the identity

$$\text{"}(\gamma r_1)\text{"}\ \text{"}(\gamma r_2)\text{"} = \text{"}\gamma\{r_1\gamma r_2 + (\gamma r_1)r_2\}\text{"},$$

which can be derived as indicated in §6; for details see [**10**; **12**; **40**].

This identity can be used to prove the boundedness of the integral operators "γr" with respect to the square integral norm. Let ψ be a function in $\mathfrak{h}^\mu$ with bounded support. Then we have

$$\begin{aligned}\int |\gamma r\psi(\omega)|^2\, d\omega &= (\gamma r\psi, \gamma r\psi)\\ &= (\psi, (\gamma r)^*\gamma r\psi) = -(\psi, (\gamma r^*)\gamma r\psi)\\ &= -(\psi, \gamma\{r^*\gamma r + (\gamma r^*)r\}\psi).\end{aligned}$$

Now we set $\gamma = \alpha + i\beta$ corresponding to $[\omega - \omega']^{-1} = (\omega - \omega')^{-1} + i\pi\delta(\omega - \omega')$ and observe that the kernel $\{r^*\alpha r + (\alpha r^*)r\}$ vanishes for $\omega' = \omega$; the kernel $\gamma\{r^*\alpha r + (\alpha r^*)r\}$ is therefore continuous everywhere. From this fact one can derive that an inequality

$$|\psi, \gamma\{r^*\alpha r + (\alpha r^*)r\}\psi| \leqq C^2 r_0^2(\psi, \psi)$$

holds with an appropriate number C; see [3]. On the other hand, we have

$$\gamma\{r^*\beta r + (\beta r^*)r\} = (\gamma r^*)\beta r + (\beta r^*)\gamma r$$

and[1]

$$\begin{aligned}-(\psi, \gamma\{r^*\beta r + (\beta r^*)r\}\psi) &= (\gamma r\psi, \beta r\psi) + (\beta r\psi, \gamma r\psi)\\ &\leqq 2\pi r_0\|\psi\|\ \|\gamma r\psi\|,\end{aligned}$$

where r_0 is the maximum of $|r(\omega; \omega')|$. Consequently, we have

$$\|\gamma r\psi\|^2 \leqq 2\pi r_0\|\psi\|\ \|\gamma r\psi\| + C^2 r_0^2\|\psi\|^2,$$

whence,

$$\|\gamma r\psi\| \leqq \gamma_0\|r\|_\mu\|\psi\| \quad \text{with } \gamma_0 \leqq (C + 2\pi)c^{-1}.$$

By virtue of this inequality we can extend the operator "γr" to all square integrable functions with the same bound $\gamma_0\|r\|_\mu$.

We are now ready to describe a class $\mathfrak{R} = \mathfrak{R}^\mu$ of operators R; we simply take the class of those bounded operators acting in the Hilbert space $\mathfrak{H}$ which are represented by integral operators with kernels in $\mathfrak{r}^\mu$ and set $[]R[] = \|r\|_\mu$. In case $\mu > 1/2$ it turns out that to every kernel in $\mathfrak{r}^\mu$ such a bounded operator is assigned, so that $\mathfrak{R}$ is complete in this case; but this is not so if $\mu \leqq 1/2$ since then not every function in $\mathfrak{H}_\mu$

[1] The present argument should replace the incomplete argument on p. 272 of [**10**].

represents a vector in $\mathfrak{H}$. We may, however, assign to each r in $\mathfrak{r}^{\mu}$ an ideal element in the completion $\overline{\mathfrak{R}}$ of the class $\mathfrak{R}$, since $\mathfrak{r}^{\mu}$ is complete. Since the integral operator "γr" is bounded on the square integrable functions it corresponds to a bounded operator ΓR on $\mathfrak{H}$, even if R is only an ideal element. From this fact and the fact that $r_0\gamma r$ and $(\gamma r)r_0$ belong to $\mathfrak{r}^{\mu}$ if r, r_0 do it follows that the operators $R_0\Gamma R$ and $(\Gamma R)R_0$ belong to $\mathfrak{R}$ if R does and if R_0 belongs to $\overline{\mathfrak{R}}$. Thus property $\bar{4}$ is established. Inequality 3, property 1 and property 5 are immediately verified.

O. A. Ladyženskaya and L. D. Faddeev [21] have extended the gentleness conditions described here allowing for a different Hölder exponent at infinity and applied them to differential operators of the form $-\Delta + q$, where Δ is the Laplacean for $n = 3$ variables and q stands for multiplication by a function of these variables of a suitable class.

J. Schwartz [27] has used a similar extension and verified that it can be applied to the operator $-\Delta + q$ for $n > 1$ variables. He also noted that for functions of a complex variable boundedness of the kernel is sufficient for its gentleness.

Our *second class* of kernels [12; 40] consists of the Fourier transforms

$$r(\omega; \omega') = \frac{1}{2\pi} \int\int e^{-i\omega\sigma + i\omega'\sigma'} \rho(\sigma; \sigma')\, d\sigma\, d\sigma'$$

of absolutely integrable functions $\rho(\sigma; \sigma')$, i.e., of functions for which the norm

$$[\![\rho]\!] = \int\int |\rho(\sigma; \sigma')|\, d\sigma\, d\sigma' < \infty.$$

Clearly, the space of such functions ρ is complete with respect to the norm $[\![\rho]\!]$.

To each such function we assign the function

$$\gamma\rho(\sigma; \sigma') = i \int_{-\infty}^{0} \rho(\sigma + \tau; \sigma' + \tau)\, d\tau,$$

whose Fourier transform will be the kernel of the operator ΓR, as we shall show. For these functions $\gamma\rho$ the Holmgren norm

$$\|\gamma\rho\|_{\text{Ho}} = \max \left\{ \text{l.u.b.}_{\sigma} \int |\gamma\rho(\sigma; \sigma')|\, d\sigma', \quad \text{l.u.b.}_{\sigma'} \int |\gamma\rho(\sigma; \sigma')|\, d\sigma \right\}$$

is bounded by $[\![\rho]\!]$, i.e.,

$$\|\gamma\rho\|_{\text{Ho}} \leqq [\![\rho]\!].$$

To show this we first let ρ be of bounded support and estimate

$$\int |\gamma\rho(\sigma; \sigma')|\, d\sigma' = \int\int_{\tau \leqq 0} |\rho(\sigma + \tau; \sigma' + \tau)|\, d\tau\, d\sigma' \leqq [\![\rho]\!]$$

and similarly for $\int |\gamma\rho(\sigma; \sigma')|\, d\sigma$. Thus inequality (γ) holds for such functions ρ.

If ρ is approximated by a sequence of functions ρ_n of bounded support such that $[\![\rho - \rho_n]\!] \to 0$, the functions $\gamma\rho_n$ form a Cauchy sequence with respect to $\| \quad \|_{H_0}$. We then define $\gamma\rho(\sigma; \sigma')$ as the limit function of this Cauchy sequence. Clearly, inequality (γ) is valid for these functions $\gamma\rho$.

Next we state that the compound kernels

$$\rho_{12}(\sigma; \sigma') = \int \rho_1(\sigma; \tilde{\sigma})\gamma\rho_2(\tilde{\sigma}; \sigma')\, d\tilde{\sigma}, \qquad \rho_{12}(\sigma; \sigma') = \int \gamma\rho_1(\sigma; \tilde{\sigma})\rho_2(\tilde{\sigma}; \sigma')\, d\sigma,$$

are absolutely integrable if ρ_1 and ρ_2 are and that the inequalities

$$[\![\rho_{12}]\!] \leqq [\![\rho_1]\!]\, [\![\rho_2]\!]$$

hold; one easily verifies this fact after having verified it for kernels with bounded support.

Furthermore, one may verify the identity

$$\int \gamma\rho_1(\sigma; \tilde{\sigma})\gamma\rho_2(\tilde{\sigma}; \sigma')\, d\tilde{\sigma} = \gamma\left\{\int \rho_1(\sigma; \tilde{\sigma})\gamma\rho_2(\tilde{\sigma}; \sigma')\, d\tilde{\sigma} + \int \gamma\rho_1(\sigma; \tilde{\sigma})\rho_2(\tilde{\sigma}; \sigma')\, d\tilde{\sigma}\right\},$$

or

$$\begin{aligned}\int\int\int_{\tau_1 \leqq 0, \tau_2 \leqq 0} &\rho_1(\sigma + \tau_1, \tilde{\sigma} + \tau_1)\rho_2(\tilde{\sigma} + \tau_2; \sigma' + \tau_2)\, d\tau_1\, d\tau_2\, d\tilde{\sigma} \\ &= \int\int\int_{\tau \leqq 0, \tau_2' \leqq 0} \rho_1(\sigma + \tau; \sigma_1)\rho_2(\sigma_1 + \tau_2'; \sigma' + \tau_2' + \tau)\, d\tau\, d\tau_2'\, d\sigma_1 \\ &\quad + \int\int\int_{\tau \leqq 0, \tau_1' \leqq 0} \rho_1(\sigma + \tau + \tau_1'; \sigma_2 + \tau_1')\rho_2(\sigma_2; \sigma' + \tau)\, d\tau\, d\tau_2'\, d\sigma_2\end{aligned}$$

by setting $\tau = \tau_1$, $\sigma_1 = \tilde{\sigma} + \tau_1$, $\tau_2' = \tau_2 - \tau_1$ in the first integral on the right and $\tau = \tau_2$, $\sigma_2 = \tilde{\sigma} + \tau_2$, $\tau_1' = \tau_1 - \tau_2$ in the last integral.

Suppose the kernel $v(\sigma; \sigma')$ is absolutely integrable and has norm $[\![v]\!] < 1$. Then, clearly, the two equations

$$\rho^+(\sigma; \sigma') = v(\sigma; \sigma') + \int \gamma\rho^+(\sigma; \tilde{\sigma})v(\tilde{\sigma}; \sigma')\, d\tilde{\sigma},$$

$$\rho^-(\sigma; \sigma') = v(\sigma; \sigma') - \int v(\sigma; \tilde{\sigma})\gamma\rho^-(\tilde{\sigma}; \sigma')\, d\tilde{\sigma}$$

have absolutely integrable solutions.

The facts stated make it possible to set up an admissible class of operators R. The Fourier transforms

$$\chi(\sigma) = \frac{1}{\sqrt{(2\pi)}} \int e^{i\sigma\omega}\psi(\omega)\, d\omega$$

of the H_0-representers ψ of the vectors Φ in $\mathfrak{H}$ may be regarded as new representers of these vectors,

$$\Phi \underset{0}{\overset{\wedge}{\leftrightarrow}} \chi(\sigma);$$

and accordingly we may regard functions $\rho(\sigma; \sigma')$ as kernels of integral operators which represent operators R acting in $\mathfrak{H}$. The class of bounded such operators R which can be represented in this way by absolutely integrable kernels ρ now forms our second admissible class $\mathfrak{R}$. To those functions ρ with $[]\rho[] < \infty$ to which no such operator corresponds we assign an "ideal element" R. In any case we set $[]R[] = []\rho[]$.

As is well known, cf., e.g., [9], a kernel with a bounded Holmgren norm represents a bounded operator in Hilbert space; hence $\gamma\rho$ represents such an operator ΓR, no matter whether R is a proper or ideal operator. Moreover, $\|\Gamma R\| \leqq \|\gamma\rho\|_{H_0}$ holds, so that $\|\Gamma R\| \leqq []R[]$. Thus property 6 obtains. It is also clear that the kernels

$$\int \rho_0(\sigma; \tilde{\sigma})\gamma\rho(\tilde{\sigma}; \sigma')\, d\tilde{\sigma} \quad \text{and} \quad \int \gamma\rho(\sigma; \tilde{\sigma})\rho_0(\tilde{\sigma}; \sigma')\, d\tilde{\sigma}$$

represent operators $R_0\Gamma R$ and $(\Gamma R)R_0$ when R_0 is in $\mathfrak{R}$ and R in $\bar{\mathfrak{R}}$. Also

$$[]R_0\Gamma R[],\ [](\Gamma R)R_0\| \leqq []R\|[]R_0[].$$

Thus properties $\bar{4}$ and 3 obtain. The same is clearly true of property 5.

All that is left for us to show is the relation $[H_0, \Gamma R] = R$ for R in $\mathfrak{R}$, formulated as part of property 1. To this end we note that when the vector Φ is represented by the function $\chi(\sigma)$, the operator H_0 is applicable exactly when $\chi(\sigma)$ has a square integrable derivative and is represented by $-id\chi(\sigma)/d\sigma = -i\chi'(\sigma)$. The vector $[H_0, \Gamma R]\Phi$ is represented by

$$\int\int_{\tau \leqq 0} r(\sigma + \tau; \sigma' + \tau)\chi'(\sigma')\, d\tau\, d\sigma'$$
$$= \int\int_{\sigma_1 \leqq \sigma} r(\sigma_1; \sigma_2)\chi'(\sigma_2 - \sigma_1 + \sigma)\, d\sigma_1\, d\sigma_2.$$

For such a Φ the vector $\Gamma R\Phi$ is represented by

$$\Gamma R\Phi \underset{0}{\hat{\leftrightarrow}} i\iint_{\tau\leqq 0} r(\sigma+\tau;\sigma'+\tau)\chi(\sigma)\,d\tau\,d\sigma'$$
$$= i\iint_{\sigma_1\leqq\sigma} r(\sigma_1;\sigma_2)\chi(\sigma_2-\sigma_1+\sigma)\,d\sigma_1\,d\sigma_2,$$

and hence admits the operator H_0 with the result

$$H_0\Gamma R\Phi \underset{0}{\hat{\leftrightarrow}} \iint_{\sigma_1\leqq\sigma} r(\sigma_1;\sigma_2)\chi'(\sigma_2-\sigma_1+\sigma)\,d\sigma_1\,d\sigma_2 + \int r(\sigma;\sigma_2)\chi(\sigma_2)\,d\sigma_2$$
$$= \iint_{\tau\leqq 0} r(\sigma+\tau;\sigma'+\tau)\chi'(\sigma')\,d\tau\,d\sigma' + \int r(\sigma;\sigma')\chi(\sigma')\,d\sigma'$$

so that $H_0\Gamma R\Phi = (\Gamma R)H_0\Phi + R\Phi$ holds.

Thus we have shown that the class $\mathfrak{R}$ here defined is admissible.

A8. Existence of wave operators for gentle disturbances. In this section we first want to show that the operators $\Gamma R(t) = e^{itH_0}\Gamma R e^{-itH_0}$ strongly tend to limits $\Gamma_\infty R$ and 0 as $t \to +\infty$ and $-\infty$ provided R belongs to one of the two admissible classes of gentle operators. This statement was the main ingredient of our treatment of scattering in §7.

Suppose R belongs to the first class $\mathfrak{R}_\mu$. Since the operators ΓR are bounded, it is sufficient to establish the relations

$$\Gamma R\Phi \underset{t\to-\infty}{\to} 0, \qquad \Gamma R\Phi \underset{t\to\infty}{\to} \Gamma_\infty R\Phi$$

for a dense class of vectors Φ. We choose the vectors which are represented by Hölder continuous functions $\psi(\omega)$ with bounded support. The expression

$$\psi(\omega;t) = \int e^{it(\omega-\omega')}[\omega-\omega']^{-1}r(\omega;\omega')\psi(\omega')\,d\omega'$$

for the representer of $\Gamma R(t)\Phi$ may then be written as

$$\psi(\omega;t) = \int e^{it(\omega-\omega')}(\omega-\omega')^{-1}r(\omega;\omega')\psi(\omega')\,d\omega'$$

if ω lies outside of the support ψ. Let this support be contained in the interval $|\omega| < a$; then there is a number c such that $|\psi(\omega;t)| \leqq c\omega^{-1}$ for all $|\omega| \geqq a$ and all t. Consequently, to a given $\varepsilon > 0$ we can find a $b \geqq a$ such that

$$\int_{|\omega|\geqq b} |\psi(\omega;t)|^2\,d\omega \leqq \varepsilon/2.$$

For $|\omega| \leqq b$ we write $\psi(\omega; t)$ in the form

$$\psi(\omega; t) = \int_{|\omega'| \leqq 2b} e^{it(\omega - \omega')}[\omega - \omega']^{-1}\{r(\omega; \omega')\psi(\omega') - r(\omega; \omega)\psi(\omega)\}\, d\omega' + r(\omega; \omega)\psi(\omega) \int_{|\omega'| \leqq 2b} e^{it(\omega - \omega')}[\omega - \omega']^{-1}\, d\omega'.$$

After extending the path of integration into the upper half plane we find for the last integral the expression

$$I_b(\omega; t) = \int_{-2b}^{2b} e^{it(\omega - \omega')}(\omega - \omega')^{-1}\, d\omega'.$$

For $t < 0$ the real part of the exponent is negative; hence, we may replace the path by the segment $|\omega| \geqq b$, run through in the opposite direction. The integral can therefore be estimated by

$$|I_b(\omega; t)| \leqq 2/b|t| \quad \text{for } |\omega| \leqq b.$$

The first term in the expression for $\psi(\omega; t)$ tends to zero as $t \to -\infty$ by virtue of the Riemann-Lebesgue Lemma, in fact uniformly so for $|\omega| \leqq b$. Therefore we can find a $t_\varepsilon > 0$ such that

$$\int_{|\omega| \leqq b} |\psi(\omega; t)|^2\, d\omega \leqq \varepsilon/2 \quad \text{for } t \leqq -t_\varepsilon.$$

In other words, $\int |\psi(\omega; t)|^2\, d\omega \to 0$ as $t \to -\infty$.

For $t > 0$ we must push the path of integration of $I_b(\omega; t)$ into the negative half plane. Taking account of the residue $2\pi i$ we find

$$\int |\psi(\omega; t) - 2\pi i r(\omega; \omega)\psi(\omega)|^2\, d\omega \to \infty \quad \text{as } t \to \infty.$$

Thus we have proved our contention about the limits of ΓR, as $t \to \pm\infty$.

We proceed to establish the existence of strong limits, as $t \to \pm\infty$, of the operator $e^{itH_0}e^{-itH}$ in case the disturbance $V = H - H_0$ belongs to the *second admissible class* of gentle operators.

To do this we must determine the Fourier transformed kernel representing the operator $\Gamma R^+(t)$. If an operator R with the direct representer $r(\omega; \omega')$ has the transformed representer

$$\rho(\sigma; \sigma') = \frac{1}{2\pi} \iint e^{i\sigma\omega - i\sigma'\omega'} r(\omega; \omega')\, d\omega\, d\omega',$$

the operator $R(t) = e^{itH_0}Re^{-itH_0}$ has the direct representer $e^{it(\omega - \omega')}r(\omega; \omega')$ and the transformed representer

$$\frac{1}{2\pi} \iint e^{i(\sigma + t)\omega - i(\sigma' + t)\omega'} r(\omega; \omega')\, d\omega\, d\omega' = \rho(\sigma + t; \sigma' + t).$$

Applying this result to the operator ΓR with the transformed representer $\gamma\rho(\sigma;\sigma') = i\int_{-\infty}^{0}\rho(\sigma+\tau;\sigma'+\tau)\,dt$ we find that the transformed representer of $\Gamma R(t)$ is

$$\gamma_t\rho(\sigma;\sigma') = i\int_{-\infty}^{0}\rho(\sigma+t+\tau;\sigma'+t+\tau)\,dt = i\int_{-\infty}^{t}\rho(\sigma+\tau;\sigma'+\tau)\,dt.$$

The Holmgren norm of this kernel can be estimated by

$$\|\gamma_t\rho\|_{H_0} \leqq \int\int|\rho(\sigma;\sigma')|\,d\sigma\,d\sigma' = \|\rho\|.$$

Our first aim is to prove

$$\int\left|\int\gamma_t\rho(\sigma;\sigma')\chi(\sigma')\,d\sigma'\right|^2 d\sigma \to 0 \quad \text{as } t\to-\infty$$

for every square integrable function $\chi(\sigma)$.

We maintain it is sufficient to do this for functions $\rho(\sigma;\sigma')$ and $\chi(\sigma)$ of bounded support. For, every kernel ρ can be approximated by a kernel $\dot\rho$ of bounded support such that $[]\dot\rho - \rho[]$ is arbitrarily small and the same is then true of

$$\|\Gamma\dot R(t) - \Gamma R(t)\| \leqq \|\gamma_t\dot\rho - \gamma_t\rho\|_{H_0} \leqq []\dot\rho - \rho[].$$

Also χ can be approximated by a function $\dot\chi$ of bounded support such that $\|\dot\chi - \chi\|$ is small.

Now let the support of $\dot\rho$ be contained in $|\sigma| \leqq \sigma_0$, $|\sigma'| \leqq \sigma_0$; then the support of $\gamma_t\rho$ is contained in the quarter plane $\sigma \geqq -\sigma_0 - t$, $\sigma' \geqq -\sigma_0 - t$ and therefore moves away from the support of $\dot\chi$ as $t\to-\infty$. Hence $\int\gamma_t\dot\rho(\sigma;\sigma')\dot\chi(\sigma')\,d\sigma' = 0$ for large $-t$.

If we let t tend to $+\infty$ we may apply the same argument to the kernel $\gamma_\infty\rho(\sigma;\sigma') - \gamma_t\rho(\sigma;\sigma')$, where

$$\gamma_\infty\rho(\sigma;\sigma') = i\int_{-\infty}^{\infty}\rho(\sigma+\tau;\sigma'+\tau)\,d\tau;$$

in doing this we must use the obvious relation

$$\|\gamma_\infty\rho\|_{H_0} \leqq \|\rho\|.$$

The result is the relation

$$\int\left|\int[\gamma_t\rho(\sigma;\sigma') - \gamma_\infty\rho(\sigma;\sigma')]\chi(\sigma')\,d\sigma'\right|^2 d\sigma \to 0 \quad \text{as } t\to\infty.$$

We may rewrite this and the previously proved relations

$$\Gamma R(t) \to \begin{cases} \Gamma_\infty R & \text{as } t \to \infty, \\ 0 & \text{as } t \to -\infty, \end{cases}$$

whereby convergence is understood in the strong sense.

Suppose the operator $U^- = 1 - \Gamma R^-$ satisfies the relation $HU^- = U^- H_0$. Then we have

$$e^{itH_0} e^{-itH} U^- = e^{itH_0} U^- e^{itH_0} = 1 - \Gamma R^-(t)$$

and our result establishes the existence of the wave operators $W_\pm = \lim_{t \to \pm\infty} e^{itH_0} e^{-itH}$ when acting on the range of the operator U^-; namely, $W_- U^- \Psi = \Psi$, $W_+ U^- \Psi = [1 - \Gamma_\infty R^-]\Psi$. If the operator U^- possesses a right inverse U^+, the limits $W_\pm$ exist for all Φ in $\mathfrak{H}$.

A different approach to the existence of the wave operators will be discussed in Appendix A15.

A9. Perturbation of operators with restricted spectra. We shall call the spectrum of the operator H_0 "restricted" if it does not just consist of the whole ω-axis from $-\infty$ to $+\infty$. We want to show that the theory developed in §6 for operators H_0 with an unrestricted spectrum automatically includes the case that H_0 has a restricted spectrum under very wide circumstances. In any case, though, we assume the spectrum of H_0 to be continuous, more precisely, to be absolutely continuous.

At first we shall assume that H_0 has an unrestricted spectrum, but that the disturbance V has the property that it affects only a subset of the spectrum of H_0. Later on we shall use the results in this case for a treatment of the case that H_0 has a restricted spectrum to begin with.

Accordingly, let the operator H_0 admit a representation through $\Phi \leftrightarrow \psi(\omega)$ and $H_0\Phi \leftrightarrow \omega\psi(\omega)$ with ψ standing for all functions with $(\Phi, \Phi) = \int |\psi(\omega)|^2 \, d\omega < \infty$.

Let $\mathfrak{J}$ be a closed set on the ω-axis and let $\mathfrak{H}_\mathfrak{J}$ be the subspace of all those vectors Φ in $\mathfrak{H}$ whose H_0-representers $\psi(\omega)$ vanish outside of $\mathfrak{J}$. Then we introduce the subclass $\mathfrak{R}_\mathfrak{J}$ of all those gentle operators R of an admissible class $\mathfrak{R}$ whose kernels $r(\omega; \omega')$ vanish unless ω and ω' are in $\mathfrak{J}$. Clearly, the operator R then transforms $\mathfrak{H}_\mathfrak{J}$ into $\mathfrak{H}_\mathfrak{J}$. Such a subclass $\mathfrak{R}_\mathfrak{J}$, as is rather obvious, has properties 1 to 6. All conclusions drawn in §6 from properties 1 to 6 hold, as verified easily. Hence we may say: If the disturbance V belongs to a subclass $\mathfrak{R}_\mathfrak{J}$ as described, the same is true of the operators $R^\pm$ as constructed in §6.

For later purposes it is desirable to generalize these considerations.

To each value of ω we may assign a closed subspace $\mathfrak{A}_\omega$ of the accessory space $\mathfrak{A}$ and require that the "values" of the representing function $\psi(\omega)$ lie in $\mathfrak{A}_\omega$. We then consider the class $\mathfrak{R}'$ of all those gentle operators of an admissible class $\mathfrak{R}$ whose kernels $r(\omega; \omega')$, as operators acting in the space $\mathfrak{A}$, have their ranges in the intersection $\mathfrak{A}_\omega \cap \mathfrak{A}_{\omega'}$ of the spaces $\mathfrak{A}_\omega$ and $\mathfrak{A}_{\omega'}$. Such a class $\mathfrak{R}'$ is again an admissible subclass of gentle operators; consequently, whenever V belongs to $\mathfrak{R}'$ the operators $R^\pm$ of §6 also do. The case considered in the previous paragraph simply results if $\mathfrak{A}_\omega = \mathfrak{A}$ for ω in $\mathscr{S}$ while $\mathfrak{A}_\omega = 0$ for ω outside $\mathscr{S}$.

We now use these observations to reduce the case in which the spectrum of the operator H_0 is not the whole ω-axis to the case where it is.

Suppose then that the spectrum of H_0 is a closed set of $\mathscr{S}$ and that the H_0-representation of vectors Φ by functions $\psi(\omega)$ is such that

$$(\Phi, \Phi) = \int_{\mathscr{S}} |\psi(\omega)|^2 \, d\omega.$$

Suppose, further, that the disturbing operator V is represented by an integral operator with a kernel $v(\omega; \omega')$ defined in the product $\mathscr{S} \times \mathscr{S}$. Then we can formulate a (sufficient) condition for the existence of operators $U^\pm$ transforming H_0 into $H = H_0 + V$ and vice versa. The condition simply is that the function $v(\omega; \omega')$ can be extended to a function defined for all ω and ω' which belongs to one of our admissible classes of kernels, but vanishes if ω is outside of the set $\mathscr{S}$. For then we need only take the operators $U^\pm$ resulting from our theory and restrict them to the space $\mathfrak{H}_\mathscr{S}$.

This remark can be extended in an obvious way to the case that the disturbance V belongs to one of the admissible subclasses described above, generated by a set of subspaces $\mathfrak{A}_\varepsilon$ of the accessory space $\mathfrak{A}$.

Now, we recall that one of the properties of our kernels $r(\omega; \omega')$ was that they were continuous functions of $(\omega; \omega')$. If such a kernel vanishes for ω outside of $\mathscr{S}$ it vanishes for ω on the boundary of $\mathscr{S}$. Thus we see: *A necessary condition* that the perturbation problem of the operator $H_0 + V$ with a closed set $\mathscr{S}$ as spectrum of H_0 can be handled by our method *is that the kernel* $v(\omega; \omega')$ *of* V *vanishes when* ω *is on the boundary of* $\mathscr{S}$.

The statement just made may serve as a *criterion* to indicate whether or not one should expect *that the operators* H_0 *and* $H_0 + V$ *have the same spectrum.*

To illustrate the significance of this observation we consider the operator $H = H_0 + V$ given by

$$\Phi \underset{0}{\Leftrightarrow} \psi(\omega), \qquad 0 \leqq \omega < \infty, \qquad (\Phi, \Phi) = \int_0^\infty |\psi(\omega)|^2 \, d\omega,$$

$$H_0\Phi \underset{0}{\Leftrightarrow} \omega\psi(\omega),$$

$$V\Phi \underset{0}{\Leftrightarrow} \varepsilon p(\omega) \int_0^\infty q(\omega')\psi(\omega') \, d\omega',$$

where p and q are square integrable and Hölder continuous [3].

This operator differs from that treated in Appendix A6 essentially only inasmuch as the domain of ω is now restricted to $0 \leqq \omega < \infty$. Also we confine ourselves to the case that the values of the representers are complex numbers, although our considerations could easily be carried over to the case that these values are vectors of an accessory space $\mathfrak{A}$.

Suppose $\hat{\omega}$ is an eigenvalue of $H_0 + V$ with the eigenfunction $\hat{\psi}(\omega)$; then, clearly

$$\hat{\psi}(\omega) = cp(\omega)/(\omega - \hat{\omega})$$

and $\hat{\omega}$ satisfies the equation

$$\int_0^\infty \frac{q(\omega')p(\omega')}{\omega' - \hat{\omega}} \, d\omega' = -\frac{1}{\varepsilon}.$$

Thus every value $\hat{\omega}$ outside of the spectrum $0 \leqq \omega \leqq \infty$ of H_0 can be made an eigenvalue of H for appropriate (possibly complex) values of ε.

Suppose the function $q(\omega)p(\omega)$ vanishes at $\omega = 0$. Then the integral on the left side of the last equation is bounded as a function of $\hat{\omega}$. Consequently, the operator $H_0 + V$ has no eigenvalue for sufficiently small ε. This is consistent with the theory of §6, according to which the operators H_0 and $H_0 + V$ are equivalent and hence have the same spectrum $0 \leqq \omega < \infty$ provided the operator V is gentle. Now, if $p(0) = q(0) = 0$, the functions $p(\omega)$, $q(\omega)$ belong to the class $\mathfrak{H}_\mu$ if they are taken as zero for $\omega < 0$. Hence, the operator V with the kernel $p(\omega)q(\omega')$ belongs to the first of our classes of gentle operators.

If, on the other hand, $q(0)p(0) \neq 0$, the function of $\hat{\omega}$ on the left tends to ∞ as $\hat{\omega}$ tends to zero through negative values. In this case then, the operator $H_0 + V$ does have a negative point-eigenvalue for arbitrarily small negative values of ε.

These observations indicate that gentleness requires at least that the function $r(\omega, \omega')$ should vanish if ω or ω' is at an end-point of the spectrum of H_0.

For various singular perturbation phenomena occurring if the perturbation is not gentle, see [12; 13; 15; 28].

A10. Differential operators. To illustrate the perturbation procedure that we have described, we consider the differential operator

$$H = -\frac{d^2}{dx^2} + q(x)$$

acting on functions $\psi(x)$ defined for $0 \leqq x < \infty$ and possessing a finite square integral

$$(\psi, \psi) = \int_0^\infty |\psi(x)|^2 \, dx.$$

The operator $H_0 = -d^2/dx^2$ is known to be selfadjoint (see, e.g., [1; 9]), when it acts on functions ψ with a continuous derivative $d\psi/dx$ and a square integrable second derivative d^2/dx^2 which satisfy a boundary condition such as (I) $\psi(0) = 0$, or (II) $d\psi/dx = 0$ at $x = 0$.

Of the function $q(x)$ we assume boundedness and absolute integrability of $(1 + x^2)q(x)$. The operator H is then selfadjoint in the same space of functions as H_0.

To apply the perturbation procedure described we must set up a spectral representation of the undisturbed operator $H_0 = -d^2/dx^2$. To do this we first carry out a Fourier transformation.

We first consider case (I). The proper Fourier transformation to be taken in this case is the one given by the formulas

$$\hat{\psi}(\mu) = \sqrt{(2/\pi)} \int_0^\infty \sin\mu x \, \psi(x) \, dx,$$

$$\psi(x) = \sqrt{(2/\pi)} \int_0^\infty \sin x\mu \, \hat{\psi}(\mu) \, d\mu,$$

which imply the identity

$$\int_0^\infty |\psi(x)|^2 \, dx = \int_0^\infty |\hat{\psi}(\mu)|^2 \, d\mu.$$

The reason for choosing $\sin \mu x$ as kernel of the transformation is that the function $\psi(x)$ then vanishes for $x = 0$ whenever $(1 + \mu^2)\hat{\psi}(\mu)$ is square integrable. It follows that the transform of the operator $-d^2/dx^2$ with boundary condition $\psi(0) = 0$ consists in multiplication

by μ^2. We write this operator symbolically as integral operator with the kernel $\mu^2\delta(\mu - \mu')$. The Fourier transform of multiplication by $q(x)$ is the integral operator with the kernel

$$\hat{v}(\mu; \mu') = \frac{2}{\pi}\int_0^\infty q(x) \sin \mu x \sin x\mu' \, dx.$$

With reference to the transformed representers $\hat{\psi}$, the operator H is therefore represented as

$$H \;\hat{\leftrightarrow}\; \mu^2\delta(\mu - \mu') + \hat{v}(\mu; \mu'),$$

i.e., as the integral operator with the right member here as kernel.

Before we can apply our perturbation theory we must introduce $\omega = \mu^2$ as independent variable. Accordingly, we introduce the representers

$$\tilde{\psi}(\omega) = (4\omega)^{-1/4}\hat{\psi}(\omega^{1/2})$$

and set

$$\tilde{v}(\omega; \omega') = (16\omega\omega')^{-1/4}\hat{v}(\omega^{1/2}; (\omega')^{1/2}).$$

It is an interesting fact, observed by P. Rejto, that this kernel belongs to class (I), i.e., it satisfies certain Hölder conditions and in particular vanishes when $\omega = 0$ and when $\omega' = 0$. For a detailed proof of this fact see [36]. Here we shall only indicate the underlying reasons. First of all we note that $\tilde{v}(\omega; \omega')$ is of the order $O((\omega\omega')^{-1/4})$ at $\omega \sim \infty$ and $\omega' \sim \infty$. To determine the behavior of $\tilde{v}$ at $\omega = 0$ and at $\omega' = 0$ we note that $\hat{v}(\mu; \mu')$ vanishes for $\mu = 0$ and for $\mu' = 0$. A more detailed analysis will show that $\hat{v}(\mu; \mu') = O(\mu\mu')$ at $\mu = 0$ and at $\mu' = 0$. Consequently,

$$\tilde{v}(\omega; \omega') = O((\omega\omega')^{1/4}) \quad \text{for } \omega \sim 0 \text{ and for } \omega' \sim 0.$$

The Hölder conditions characterizing kernels of class (I) can then be seen to be fulfilled for $\theta = \frac{1}{4}$.

The integral equation for the kernels $\tilde{r}^{\pm}(\omega; \omega')$ are

$$\tilde{r}^{+}(\omega; \omega') = \tilde{v}(\omega; \omega') + \int_0^\infty [\omega - \tilde{\omega}]^{-1}\tilde{r}^{+}(\omega; \tilde{\omega})\tilde{v}(\tilde{\omega}; \omega') \, d\tilde{\omega},$$

$$\tilde{r}^{-}(\omega; \omega') = \tilde{v}(\omega; \omega') - \int_0^\infty \tilde{v}(\omega; \tilde{\omega})[\tilde{\omega} - \omega']^{-1}r^{-}(\tilde{\omega}; \omega') \, d\tilde{\omega}.$$

Clearly, $r^{\pm} = v$ is the first approximation and higher approximations can be obtained by expansion.

For purposes of actual evaluation it is more convenient to use

functions of μ rather than ω. The integral equations for the kernels

$$\hat{r}^{\pm}(\mu; \mu') = (4\mu\mu')^{1/2}\tilde{r}^{\pm}(\mu^2; (\mu')^2)$$

are found to be

$$\hat{r}^{+}(\mu; \mu') = \hat{v}(\mu; \mu') + \int_0^{\infty} [\mu^2 - \tilde{\mu}^2]^{-1}\hat{r}^{+}(\mu; \tilde{\mu})\hat{v}(\tilde{\mu}; \mu')\, d\tilde{\mu},$$

$$\hat{r}^{-}(\mu; \mu') = \hat{v}(\mu; \mu') + \int_0^{\infty} \hat{v}(\mu; \tilde{\mu})[\tilde{\mu}^2 - \mu'^1]^{-1}r^{-}(\tilde{\mu}; \mu')\, d\tilde{\mu}.$$

The first approximation is $\hat{r}^{\pm}(\mu; \mu') = \hat{v}(\mu; \mu')$ while the second approximation is obtained by inserting this first approximation in the integrals. These are the formulas used in quantum theory.

The scattering kernels

$$\tilde{s}^{\pm}(\omega; \omega') = [1 + 2\pi i\tilde{r}^{\pm}(\omega; \omega)]\, \delta(\omega - \omega'),$$

correspond to the kernels

$$\hat{s}^{\pm} = (\mu; \mu') = (4\mu\mu')^{1/2}[1 + 2\pi i(2\mu)^{-1}\hat{r}(\mu; \mu')]\delta(\mu^2 - (\mu')^2).$$

The delta function here is of course to be interpreted through the relation

$$\int \delta(\mu^2 - (\mu')^2)f(\mu)\, d\mu^2 = f(\mu') = \int \delta(\mu - \mu')f(\mu)\, d\mu;$$

in other words, we should set

$$\delta(\mu^2 - (\mu')^2) = \frac{1}{2\mu}\, \delta(\mu - \mu').$$

The scattering kernels therefore are

$$\hat{s}^{\pm}(\mu; \mu') = [1 + \pi i\mu^{-}\hat{r}(\mu; \mu')]\delta(\mu - \mu').$$

In other words, the scattering operators $S^{\pm}$ consist simply in multiplying the function $\hat{\psi}(\mu)$ by the factor

$$1 + i\pi\mu^{-1}\hat{r}^{\pm}(\mu; \mu).$$

In case (II) the boundary condition imposed on $\psi(x)$ to make the operator $H_0 = -d^2/dx^2$ applicable is $d\psi/dx = 0$ at $x = 0$. The appropriate Fourier transform in this case is given by

$$\hat{\psi}(\mu) = \sqrt{(2/\pi)} \int_0^{\infty} \cos \mu x\psi(x)\, dx$$

and its inverse. The kernel $\hat{v}(\mu; \mu)$, given by

$$\hat{v}(\mu; \mu') = \frac{2}{\pi} \int_0^\infty q(x) \cos \mu x \cos x\mu' \, dx,$$

does not vanish for $\mu = 0$ or $\mu' = 0$. Consequently, the corresponding kernel $\tilde{v}(\omega; \omega')$ does not belong to one of our classes. In fact, the operator $H_0 + \varepsilon V$ has a negative point-eigenvalue for every value of ε, provided $\int_0^\infty q(x)\, dx \neq 0$ and $\int_0^\infty |q(x)|\, dx < \infty$. To verify this statement one need only show that $(\psi, (H_0 + \varepsilon V)\psi)$ can be made negative for arbitrary $\varepsilon \neq 0$ by choice of ψ. One may simply take

$$\psi(x) = \begin{cases} 1, & 0 \leqq x \leqq x_1, \\ (x_2 - x)/(x_2 - x_1), & x_1 \leqq x \leqq x_2, \\ 0, & x_2 \leqq x < \infty, \end{cases}$$

and then choose x_1 and x_2 large enough.

A treatment of the perturbation theory of some ordinary differential operators using the method of resolvents was given by Moser [14]. For differential operators of functions of n variables it was shown by Jack Schwartz [27] that a variant of the theory presented here becomes applicable for $n \geqq 2$.

An extension of the results presented to cases where the kernel $v(\omega; \omega')$ does not vanish at the boundary of the domain, will be given by P. Rejto in a forthcoming paper.

A11. Enlargement of the spectrum. In §6 we showed that for gentle disturbances V the operator $H_0 + V$ can be transformed into the operator H_0 by operators $U^\pm = 1 \pm \Gamma R^\pm$ with gentle $R^\pm$ provided the operator V is small enough, namely, if $[\![V]\!] < 1$, where $[\![V]\!]$ is the norm of V associated with an admissible class of gentle operators. Specifically, the operators $R^\pm$ were shown to satisfy the relations $R^+ = U^+V$, $R^- = VU^-$ and $U^+U^- = 1$. Two operators $U^\pm$ with this property will be referred to as a "pair". The condition $[\![V]\!] < 1$, which was imposed for convenience, is not at all necessary; a pair $U^\pm$ may exist under less restrictive circumstances.

This existence question was treated by Ladyženskaya and Faddeev [21] and P. Rejto [36]. The results of Rejto imply the following

EXISTENCE THEOREM. *Suppose the operator H_0 possesses an absolutely continuous spectrum of uniform multiplicity (as we have always assumed here) and let V be a gentle Hermitean operator of the class $\mathfrak{R}_\mu$ with $\mu > \frac{1}{2}$. Suppose further that the operator $H_0 + V$*

possesses no point-eigenvalue in the (closed) spectrum of H_0. *Then a pair of gentle operators* $R^{\pm}$ *associated with the disturbance* V *exist, provided the accessory space is finite-dimensional. If the accessory space is infinite-dimensional the same statement holds if the operator* V *is "completely gentle", in the sense of Appendix* A12.

In the discussions of the present section we shall assume that such a gentle pair $R^{\pm}$ exists without requiring that V be Hermitean and without imposing any condition on the eigenvalues of $H_0 + V_0$.

The first question we shall deal with refers to the inverse relation $U^- U^+ = 1$. In §6 we had shown that this relation is implied by $U^+ U^- = 1$ provided $[\![V]\!] < (1 + \gamma_0)^{-1}$. Certainly this condition is too strong; there are cases where $[\![V]\!] \geqq (1 + \gamma_0)^{-1}$ and still $U^- U^+ = 1$ holds. On the other hand there are cases in which $U^- U^+ \neq 1$. In such a case the pair $U^{\pm}$ will be called incomplete since then only a part of the spectrum of the operator $H = H_0 + V$ (the "main part") is equivalent to that of H_0 and the complete spectrum of H contains additional parts. We therefore say that the disturbance V effects an "enlargement" of the spectrum of H_0.

In the present and the next section we shall study this enlargement phenomenon.

Let us then assume that the operator V belongs to one of our admissible classes $\mathfrak{N}$ of gentle operators and that there is a pair of operators $R^{\pm}$ belonging to $\mathfrak{N}$ which satisfy the relation $R^+ = U^+ V$, $R^- = VU^-$ and hence $U^+ U^- = 1$ with $U^{\pm} = 1 \pm \Gamma R^{\pm}$.

We first observe that the transformation U^+ supplies the spectral representation of the operator H in the range of U^- since $HU^- \Psi = U^- H_0 \Psi$; but it fails to do so in the complementary space of all vectors of the form $\Phi = (1 - U^- U^+)X$. Also, the process of scattering can be described in the range $U^- \mathfrak{H}$ since

$$e^{itH_0} e^{-itH} U^- \Psi \to \Phi_{\pm} \quad \text{as } t \to \pm \infty$$

with $\Phi_- = \Psi$, $\Phi_+ = S_0^- \Psi$, where $S_0^- = 1 - \Gamma_\infty R^-$ and $\Gamma_\alpha R$ is the operator with the kernel $2\pi i \delta(\omega - \omega') r(\omega; \omega')$. Clearly, then, S_0^- can be regarded as a "partial" scattering operator. The operator $S_0^+ = 1 + \Gamma_\infty R^+$ enters the limit relation

$$U^+ e^{itH} e^{-itH_0} \to \begin{cases} S_0^+ & \text{as } t \to \infty, \\ 1 & \text{as } t \to -\infty. \end{cases}$$

The relation

$$S_0^+ S_0^- = 1$$

is thus implied by $U^+U^- = 1$. Our first major aim is to show that the inverse relation $S_0^-S_0^+ = 1$ always holds under the present circumstances, never mind whether or not the inverse relation $U^-U^+ = 1$ holds.

Before doing this it is necessary to derive the relation $S_0^+S_0^- = 1$ directly from $U^+U^- = 1$, without making use of the scattering process. We recall that in §6 the equations $R^+U^- = V = U^+R^-$ led to the relation

$$(*) \qquad R^+ - R^- - R^+\Gamma R^- - (\Gamma R^+)R^- = 0.$$

We shall apply the operation Γ_∞ to it. Writing $\Gamma = \tilde{\Gamma} + \frac{1}{2}\Gamma_\infty$, where $\tilde{\Gamma}R \underset{0}{:::} (\omega - \omega')^{-1}r(\omega; \omega')$, we first observe the relation

$$\Gamma_\infty\{R^+\tilde{\Gamma}R^- + (\tilde{\Gamma}R^+)R^-\} = 0,$$

which we had already derived in Appendix A7. Because of

$$\Gamma_\infty(R^+\Gamma_\infty R^-) = (\Gamma_\infty R^+)\Gamma_\infty R^- \quad \text{and} \quad \Gamma_\infty(\Gamma_\infty R^+)R^- = (\Gamma_0 R^+)\Gamma_\infty R^-$$

it can be written as

$$(\#) \qquad \Gamma_\infty\{R^+\Gamma R^- + (\Gamma R^+)R^-\} = \Gamma_\infty R^+\Gamma_\infty R^-.$$

From relation $(*)$ we then conclude that

$$(\triangle) \qquad \Gamma_\infty(R^+ - R^-) = \Gamma_\infty R^+\Gamma_\infty R^-;$$

and this is exactly the desired relation $S_0^+S_0^- = 1$.

The remarkable fact that also the inverse relation

$$S_0^-S_0^+ = 1$$

holds for the partial scattering operators $S_0^\pm$, even if the inverse relation does not hold for the spectral transformations $U^\pm$, was first observed by Møller [**11**] in connection with Schrödinger operators of the form $-\nabla^2 + q$; therefore we shall refer to the relation $S_0^-S_0^+ = 1$ as the "Møller relation".

Under the circumstances here considered this relation is readily proved. We note that the kernel of the operator $P_\infty = 1 - S_0^-S_0^+$ is given by $\delta(\omega - \omega')p_\infty(\omega)$ with

$$\begin{aligned} p_\infty(\omega) &= 2\pi i r^-(\omega; \omega) - 2\pi i r^+(\omega; \omega) + (2\pi i)^2 r^-(\omega; \omega)r^+(\omega; \omega) \\ &= (2\pi i)^2[r^-(\omega; \omega), r^+(\omega; \omega)]. \end{aligned}$$

Suppose first that the spectrum of H_0 is simple, so that the values of the representers ψ of the states Φ are complex numbers. Then the

same is true of the values of r^+ and r^-. Therefore r^- and r^+ commute and the kernel of $1 - S_0^- S_0^+$ is the same as that of $1 - S_0^+ S_0^- = 0$. Thus $1 - S_0^- S_0^+ = 0$ follows.

If, however, the "values" of ψ are vectors in an accessory space, the "values" of $r^\pm$ are operators in this accessory space and do not commute in general. In this case we note that the operator P_∞ is a projector by virtue of $S_0^+ S_0^- = 1$. From this fact one may deduce that for each value of ω the value $p_\infty(\omega)$, considered an operator acting in the accessory space, is also a projector. For, the operators $p_\infty(\omega)$ are continuous functions of ω with respect to the accessory operator norm, since this was required of the operators $r^\pm(\omega; \omega')$. The range of a projector which depends continuously on a variable ω evidently has the same dimension for all values of ω. Now, as $|\omega| \to \infty$ the operators $r^\pm(\omega; \omega)$ tend to zero by virtue of the definition of the class $\mathfrak{R}$. Hence $p_\infty(\omega) \to 0$ as $|\omega| \to 0$. Since the dimension of $p_\infty(\omega)$ is thus zero in the limit, it is zero for all values of ω. That is to say, $p_\infty(\omega) = 0$ for all ω. Thus, Møller's relation $S_0^- S_0^+ = 1$ is proved.

The Møller relation has various consequences. First of all it enables us to interchange the roles of $+\infty$ and $-\infty$ in the time limit process and of πi and $-\pi i$ in the definition of the operation Γ.

Suppose we define the operation Γ_* with the aid of the factor $(\omega - \omega')^{-1} - \pi i \delta(\omega - \omega')$ and set $R_*^- = R^- S_0^+$ and $R_*^+ = S_0^- R^+$. Then, by virtue of the Møller relation, the identities

$$U_*^+ = 1 + \Gamma^* R_*^+ = S_0^- U^+, \qquad U_*^- = 1 - \Gamma^* R_*^- = U^- S_0^+$$

hold, as easily verified. Since $S_0^\pm$ commutes with H_0, the operators $U_*^\pm$ may serve as a pair of spectral transformations in place of $U^\pm$. Relation $U_*^+ U_*^- = 1$, which thus is implied, also follows directly from $U^+ U^- = 1$ by virtue of $S_0^- S_0^+ = 1$.

We should observe that it is the operator U_*^+ which appeared as "wave operator" for $t \to \infty$ in the description of scattering given in §7, while U^+ was the wave operator for $t \to -\infty$; this indicates that the transition from U^+ to U_*^+ corresponds to the interchange of $+\infty$ and $-\infty$ as limits of the time variable t.

Another important consequence of the Møller relation is that it will enable us to make definite statements about the "new" part of the spectrum of H produced by the perturbation V. To this end we shall study the projector $P = 1 - U^- U^+$, which projects into the eigenspace of this new spectrum. This projector may be written in the form

$$P = \Gamma Q$$

with

$$Q = R^- - R^+ + R^-\Gamma R^+ + (\Gamma R^-)R^+,$$

as follows from our admissibility property 5, described in §6 (p. 23). Now we maintain that

$$\Gamma_\infty Q = 0$$

for this operator Q. To show this we make use of relation (#) with R^+, R^- interchanged and relation ($\triangle$); the result is

$$\Gamma_\infty Q = [\Gamma_\infty R^-, \Gamma_\infty R^+].$$

Now, the Møller relation just states that the right-hand side here vanishes; thus formula $\Gamma_\infty Q = 0$ is proved.

This formula shows that the kernel $q(\omega; \omega')$ of the operator Q vanishes on the diagonal $\omega = \omega'$ of the $(\omega; \omega')$-plane. The singularity of the kernel of the projector $P = \Gamma Q$ is thus less severe than that of operators ΓR in general. As a matter of fact, under circumstances to be specified, the operator ΓQ is completely continuous.

Let us assume that the operators $R^\pm$ belong to the class $\mathfrak{R}^\mu$ with Hölder continuous kernels $r(\omega; \omega')$, as described in Appendix A7. It then follows that also the operator Q belongs to this class $\mathfrak{R}^\mu$. Consequently, the kernel $q(\omega; \omega')$ of Q vanishes at least like $|\omega - \omega'|^\mu$ as $\omega' \to \omega$ and, therefore, the kernel of $P = \Gamma Q = \tilde{\Gamma} Q$ becomes infinite at most like $|\omega - \omega'|^{\mu-1}$ there. For large ω and ω' we may similarly say that the kernel of P becomes infinite at most like $|\omega^{-1} - (\omega')^{-1}|^{\mu-1}$ as $(\omega')^{-1} \to \omega^{-1}$.

Suppose now the accessory space $\mathfrak{A}$ is finite-dimensional. Then, standard arguments are sufficient to show that the operator P is completely continuous. To reach the same conclusion in case $\mathfrak{A}$ is infinite-dimensional, the class of gentle operators admitted must be restricted. We shall discuss such a restriction in the next section. At present we shall draw a conclusion from the complete continuity of the operator P.

As is well known, the complete continuity of an operator implies that the eigenspace associated with one of its eigenvalues is finite-dimensional provided this eigenvalue is not zero. A projector, such as $P = \Gamma Q$, has at most one eigenvalue $\neq 0$, viz., the value 1, and the associated eigenspace is its range. Thus it follows that *the range $P\mathfrak{H}$ of the projector P is finite-dimensional.*

Clearly, the operator H commutes with the projector $P = 1 - U^- U^+$ since $HU^- U^+ = U^- H_0 U^+ = U^- U^+ H$; hence H transforms the

range $P\mathfrak{H}$ of P into itself. Assuming that the operator H is selfadjoint, or at least diagonalizable, we may conclude that *H possesses at most a finite number of eigenvalues $\omega^1, \cdots, \omega^n$ whose eigenvectors $X^1, \cdots, X^n$* span the space $P\mathfrak{H}$. The projector P can therefore be described by

$$P\Phi = (1 - U^- U^+)\Phi = \sum_{\nu=1}^{n} X^\nu (X_*^\nu, \Phi)$$

with the aid of appropriate eigenvectors X_*^ν of the operator $H^* = H_0 + V^*$ satisfying the relations

$$(X_*^\nu, X^\mu) = \delta_{\nu\mu}.$$

Consequently, we may describe the complete spectral representation of the operator H in the following manner: *Let $\mathfrak{H}$ be the n-dimensional space of vectors Ψ_1 with components $c_1, \cdots, c_n$. Then any vector Φ in $\mathfrak{H}$ admits the representation*

$$\Phi = U^- \Psi + \sum_{\nu=1}^{n} X^\nu c_\nu,$$

where the vectors Ψ in $\mathfrak{H}$ and Ψ_1 in $\mathfrak{H}_1$ are given by

$$\Psi = U^+ \Phi, \qquad c_\nu = (X_*^\nu, \Phi).$$

Since

$$H\Phi = U^- H_0 \Psi + \sum_{\nu=1}^{n} X^\nu \omega^\nu c_\nu,$$

it is clear that the *H_0-representer $\phi(\omega)$ of Ψ and the numbers $c_1, \cdots, c_n$ form the spectral representers of the vector Φ for the operator H.*

A few remarks should be added concerning the process of *scattering* under the present circumstances. From the representation

$$H\Phi = U^- H_0 U^+ \Phi + \sum_\nu X^\nu \omega^\nu (X_*^\nu, \Phi),$$

we deduce

$$e^{itH}\Phi = U^- e^{itH_0} U^+ \Phi + \sum_\nu X^\nu e^{it\omega^\nu} (X_*^\nu, \Phi),$$

whence

$$e^{itH} e^{-itH_0} \Phi = U^- U^+(t)\Phi + \sum_\nu X^\nu e^{it\omega^\nu} (X_*^\nu, e^{-itH_0}\Phi)$$

and

$$e^{itH_0} e^{-itH} \Phi = U^-(t) U^+ \Phi + \sum_\nu e^{itH_0} X^\nu e^{-it\omega^\nu} (X_*^\nu, \Phi).$$

Now, according to the statements made in §6 and Appendix A8 (and already used at the beginning of this section) the operators

$U^{\pm}(t) = e^{itH_0}U^{\pm}e^{-itH_0}$ tend to $S_0^{\pm}$ as $t \to \infty$ and to 1 as $t \to -\infty$. The terms $(X_*^{\nu}, e^{-itH_0}\Phi)$ tend to zero by the Riemann-Lebesgue Lemma. Hence we have

$$e^{itH}e^{-itH_0} \to \begin{cases} U^- S_0^+ & \text{as } t \to \infty, \\ U^- & \text{as } t \to -\infty. \end{cases}$$

On the other hand, the contributions from the point-eigenvalues in $e^{itH_0}e^{-itH}\Phi$ do not converge unless all (X_*^{ν}, Φ) vanish. Thus, the operator $e^{itH_0}e^{-itH}$ does not converge unless Φ lies in the space $U^-\mathfrak{H}$ of all Φ of the form $\Phi = U^-\Psi$. Nevertheless, the operator $e^{itH_0}e^{-itH}$, being the adjoint of $e^{itH}e^{-itH_0}$, converges weakly to $S_0^- U^+$ and U^+, respectively. This is an important fact which we shall frequently meet in analogous situations later on.

In any case, the operator S_0^- may be used to describe the process of scattering for states $\Phi \neq 0$ in $U^-\mathfrak{H}$, i.e., in states Φ with initial states $\Phi_- = U^+\Phi \neq 0$ and $P\Phi = 0$. But, the description of scattering with the aid of the operator S_0^- does not take account of the effect of the point-eigenvalues. If one wants to do this one must split Φ into $\Phi = U^-\Phi_- + \sum_{\nu} X^{\nu}c_{\nu}$ and just state that every contribution $e^{-itH}X^{\nu}c_{\nu}$ to $e^{-itH}\Phi$ is given by $e^{-it\omega_{\nu}}X^{\nu}c_{\nu}$.

A12. Completely gentle operators. In deriving the complete continuity of the projector $P = 1 - U^-U^+$ in the last section we observed that an additional restriction must be imposed on the gentle operators of our admissible class $\mathfrak{R}$ in case the accessory space $\mathfrak{A}$ is infinite-dimensional. This is the restriction that for each pair of values ω; ω' the operators $r(\omega; \omega')$, acting in $\mathfrak{A}$, should be completely continuous. Gentle operators R whose kernels have this property will be called "completely gentle". This property was first employed by Ladyženskaya and Faddeev in connection with the question of the existence of operators $R^{\pm}$. In the present short section we *first* want to show that the completely gentle operators of a class $\mathfrak{R}$ form an admissible subclass in the sense described in Appendix A9. *Secondly*, we want to show that the operator ΓQ is completely continuous in $\mathfrak{H}$ whenever the kernel $q(\omega; \omega')$ of a completely gentle operator Q vanishes for $\omega' = \omega$.

To prove the first statement we must show that the operators $R_1\Gamma R_2$ and $(\Gamma R_1)R_2$ are completely gentle if R_1 and R_2 are. To this end we write $\tilde{\Gamma} = \Gamma + \frac{1}{2}\Gamma_{\infty}$, as we have done in Appendix A11, and observe that obviously $R_1\Gamma_{\infty}R_2$ is completely gentle since for each $(\omega; \omega')$ the

kernel $(2\pi i)r_1(\omega;\omega')r_2(\omega';\omega')$ of this operator is the product of two completely continuous operators. The contribution

$$r_0(\omega;\omega') = \int r_1(\omega;\bar{\omega})(\bar{\omega}-\omega')^{-1}r_1(\bar{\omega}-\omega')\,d\bar{\omega},$$

resulting from $R_1\bar{\Gamma}R_2$, is the limit (in the $\mathfrak{A}$-operator norm) of the integral

$$r_\delta(\omega;\omega') = \int_{|\omega-\omega'|\geqq\delta} r_1(\omega;\bar{\omega})(\bar{\omega}-\omega')^{-1}r_1(\bar{\omega}-\omega')\,d\bar{\omega}$$

as $\delta\to 0$. It is thus sufficient to show that the operator $r_\delta(\omega;\omega')$ in $\mathfrak{A}$ is completely continuous. To this end one first approximates in the standard manner the integral by a sum

$$\sum_\nu r_1(\omega;\omega_\nu)(\omega_\nu-\omega')^{-1}r_2(\omega_\nu;\omega')\,\Delta\omega_\nu$$

(with respect to the $\mathfrak{A}$-operator norm) and then uses that each of the operators $r_1(\omega;\omega_\nu)$ and $r_2(\omega_\nu;\omega')$ is completely continuous in it.

To prove the second statement we introduce the function

$$\eta_\delta(\omega'-\omega) = \begin{cases} 1 & \text{for } |\omega'-\omega|\leqq\delta,\\ 0 & \text{for } |\omega'-\omega|>\delta \end{cases}$$

and set $\Gamma Q = P_\delta + P'_\delta$, where the kernels of P_δ and P'_δ are

$$(\omega-\omega')^{-1}\{q(\omega;\omega')-q(\omega;\omega)\}\eta_\delta(\omega'-\omega)$$

and

$$(\omega-\omega')^{-1}q(\omega;\omega')\{1-\eta_\delta(\omega'-\omega)\}.$$

Since q satisfies a Hölder condition, the Holmgren norm [A7, p. **107**] of P_δ can be made arbitrarily small by making δ sufficiently small. The kernel of P'_δ can then be approximated with respect to the Holmgren norm by a piecewise constant kernel in the standard manner. Since each term

$$(\omega_\mu-\omega'_\nu)^{-1}q(\omega_\mu;\omega'_\nu)\{1-\eta_\delta(\omega_\mu-\omega'_\nu)\}$$

is completely continuous with respect to $\mathfrak{A}$, the sum is completely continuous in $\mathfrak{H}$. Thus the validity of the second statement follows.

This result can now be applied to the operator $P=\Gamma Q$ with

$$Q = [R^-,\Gamma R^+] + [\Gamma R^-, R^+],$$

where $R^\pm$, the R-operators associated with the completely gentle disturbance V, are assumed completely gentle themselves. For, we

have $q(\omega; \omega) = 0$ for this operator as proved in Appendix A11. Thus we have established the statement made in Appendix A11 that the new spectrum produced by a completely gentle disturbance V with completely gentle $R^{\pm}$ consists of at most a finite number of point-eigenvalues with finite multiplicity.

In the next section we shall show that the new spectrum may be continuous if the disturbance is not completely gentle even if it is gentle.

A13. Enlargement by continuous spectra. Under the circumstances described in §11, enlargement of the spectrum of the operator H_0 resulting from the disturbance V consists in the appearance of a finite number of point-eigenvalues. There are other circumstances, however, in which the enlargement involves the addition of infinitely many point-eigenvalues or the addition of a continuous spectrum. Such cases will arise if the disturbing operator is not completely gentle. The type of enlargement will then depend intimately on the nature of the operator V. We shall select three specific, though typical, circumstances in which the spectral enlargement can be described completely.

We assume that the representers of the vectors Φ in $\mathfrak{H}$ are functions $\psi(\alpha, \beta)$ of two variables such that

$$(\Phi, \Phi) = \int\int |\psi(\alpha, \beta)|^2 \, d\alpha \, d\beta,$$

and that the spectral variable ω is a function $\omega(\alpha, \beta)$ of α and β. For simplicity we assume

$$\omega = \alpha + \beta.$$

We restrict the variables α and β to have lower bounds, $\alpha \geqq \alpha_*$, $\beta \geqq \beta_*$, so that the spectrum of H_0 is given by $\omega \geqq \alpha_* + \beta_*$.

The values of the representers ψ may be vectors of an accessory space $\mathfrak{A}'$; but we shall have no reason to refer to this space.

As disturbing operator we shall consider the sum

$$V = V_a + V_b + V_c$$

of three contributions which can be represented by integral operators acting on functions $\psi(\alpha, \beta)$. The operators V_a and V_b will have kernels of the form

$$v'_a(\alpha, \beta; \alpha', \beta') = \delta(\beta - \beta') v'_a(\beta; \alpha; \alpha'),$$
$$v'_b(\alpha, \beta; \alpha', \beta') = \delta(\alpha - \alpha') v'_b(\alpha; \beta; \beta'),$$

while the kernel $v_c(\alpha, \beta; \alpha', \beta')$ will have smoothness properties which exclude the presence of a delta function. In fact, the smoothness

properties of v'_a, v'_b, v_c will be such that V_c is completely gentle, while V_a and V_b will be gentle but not completely so.

We should like to describe a typical physical situation which involves interaction operators of the form V_a and V_b just described. This is the situation in which two "particles" are in interaction with an immovable source. The associated state vectors Φ may then be represented as functions $\psi(\omega_a, \omega_b)$ of the two energies ω_a, ω_b of the two particles and of accessory variables; application of the kinetic energy operator H_0 is represented through multiplication by the total energy $\omega = \omega_a + \omega_b$. The potential energy of the particle (a) corresponds to our operator V_a, represented by a kernel which affects only the variable ω_a and which thus is of the form $\delta(\omega_b - \omega'_b)v'_a(\omega_a; \omega'_a)$. Similarly, the kernel of V_b is of the form $\delta(\omega_a - \omega'_a)v'_b(\omega_b; \omega'_b)$. In more involved circumstances the functions v'_a and v'_b may depend on $\omega_b = \omega'_b$ and $\omega_a = \omega'_a$, respectively. The operator V_c may stand for the potential energy operator associated with external forces.

In the present section we shall first be concerned only with the operator $H_a = H_0 + V_a$ and later on deal with the operator $H_a + V_c$.

For each value of β, the function $v'_a(\beta; \alpha; \alpha')$ may be regarded as the kernel of an integral operator $V_a(\beta)$ acting on vectors Φ_β of a Hilbert space $\mathfrak{H}_\beta$ represented by square integrable functions $\psi_\beta(\alpha)$. We write

$$\Phi_\beta \underset{0}{\leftrightarrows} \psi_\beta(\alpha)$$

and

$$V_a(\beta)\Phi_\beta \underset{0}{\leftrightarrows} \int v'_a(\beta; \alpha; \alpha')\psi_\beta(\alpha')\, d\alpha'.$$

Also, the operator H_0 induces an operator acting in $\mathfrak{H}_\beta$; this operator, denoted by $H_0(\beta)$, is represented by

$$H_0(\beta)\Phi_\beta \underset{0}{\leftrightarrows} (\alpha + \beta)\psi_\beta(\alpha).$$

We want to apply the theory of §12 to the operator $H_0(\beta) + V_a(\beta)$. This we can do after having introduced $\omega = \alpha + \beta$ as independent variable instead of α. The kernel v'_a of $V_a(\beta)$ is then given by

$$v'_a(\beta; \omega - \beta; \omega' - \beta)$$

as a function of ω, ω' and the accessory variable β.

We now may require of the operator $V_a(\beta)$ that it is completely gentle for each value of β and depends continuously and boundedly on β. Moreover, we require that there is a pair of completely gentle

operators $R_a^{\pm}(\beta)$ associated with transformations $U_a^{\pm}(\beta) = 1 \pm \Gamma R_a^{\pm}(\beta)$ of the operator H_0 into the operator

$$H_a(\beta) = H_0(\beta) + V_a(\beta).$$

We then know from the general theory that the operator $H_a(\beta)$ has a continuous spectrum, given by $\omega \geqq \alpha_* + \beta$ and that, in addition, $H_a(\beta)$ may have a finite number of point-eigenvalues. We now assume, for simplicity, that for every $\beta \geqq \beta_*$ the operator $H_a(\beta)$ has exactly one simple point-eigenvalue

$$\omega_a(\beta) = \alpha_a(\beta) + \beta,$$

which depends continuously on β and stays away a finite distance from the spectrum of H_0, i.e., $\beta_* - \beta(\alpha) \geqq \beta_0 > 0$.

The associated eigenvectors $X_a(\beta)$, represented by the eigenfunctions $\chi_a(\alpha, \beta)$, may be chosen to depend continuously and boundedly on β.

We maintain that the set $\omega = \omega_a(\beta)$ forms a continuous "new" part of the spectrum of the operator H_a.

To see this we first make a general observation. Suppose a vector Φ in $\mathfrak{H}$ is represented by a function $\psi(\alpha, \beta)$ which for a value of β is square integrable with respect to α and hence may be regarded as a vector Φ_β of the space $\mathfrak{H}_\beta$. Then, clearly, the function $\psi^{(1)}(\alpha, \beta)$ which represents the vector $H_a\Phi = (H_o + V_a)\Phi$ may also, for the selected value of β, be regarded as a vector $\Phi_\beta^{(1)}$ in the space $\mathfrak{H}_\beta$, namely, as the vector obtained from Φ_β by applying the operator $H_a(\beta)$. In other words, $\Phi^{(1)} = H_a\Phi$ implies

$$\Phi_\beta^{(1)} = H_a(\beta)\Phi_\beta.$$

We apply this remark to a vector Φ given through the representation

$$\Phi \underset{0}{\Leftrightarrow} \chi_a(\alpha, \beta)\phi_1(\beta),$$

where $\phi_1(\beta)$ is any square integrable function of β and $\chi_a(\alpha, \beta)$ is the eigenfunction of the operator $H_a(\beta)$ associated with the eigenvalue $\omega_a(\beta)$. Since for each β application of $H_a(\alpha)$ on Φ_β means multiplication of $\chi_a(\alpha, \beta)\phi_1(\beta)$ by $\omega_a(\beta)$ it is clear that application of H_a on Φ has the same effect, i.e.,

$$H_a\Phi \underset{0}{\Leftrightarrow} \chi_a(\alpha, \beta)\omega_a(\beta)\phi_1(\beta).$$

In other words, representation of a vector Φ by the function $\chi_a(\alpha, \beta)\phi_1(\beta)$ is a spectral representation with the spectral function $\omega_a(\beta)$.

We proceed to describe the complete spectral representation of the operator H_a. The spectral representer of a vector Φ in $\mathfrak{H}$ will consist

of a pair of square integrable functions $\phi_0(\alpha, \beta)$ and $\phi_1(\beta)$, which may be regarded as elements of two Hilbert spaces $\mathfrak{R}_0$ and $\mathfrak{R}_1$. The representation is then of the form

$$\Phi \Leftrightarrow \{\phi_0(\alpha, \beta), \phi_1(\beta)\},$$
$$H_a\Phi \Leftrightarrow \{\omega(\alpha, \beta)\phi_0(\alpha, \beta), \omega_a(\beta)\phi_1(\beta)\}$$

with $\omega(\alpha, \beta) = \alpha + \beta$, $\omega_a(\beta) = \alpha_a(\beta) + \beta$. This representation will be effected by two transformations U_{a0}^- and U_{a1}^- which transform the spaces $\mathfrak{R}_0$ and $\mathfrak{R}_1$ into $\mathfrak{H}$:

$$\Phi = (U_{a0}^-, U_{a1}^-)\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix},$$

and two inverse transformations U_{0a}^+, U_{1a}^+ mapping $\mathfrak{H}$ into $\mathfrak{R}_0$ and $\mathfrak{R}_1$:

$$\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} = \begin{pmatrix}U_{0a}^+\\ U_{1a}^+\end{pmatrix}\Phi.$$

The statement that those operators are inverse to each other may be expressed by the relations

$$\begin{pmatrix}U_{0a}^+\\ U_{1a}^+\end{pmatrix}(U_{a0}^-, U_{a1}^-) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$U_{a0}^-U_{0a}^+ + U_{a1}^-U_{1a}^+ = 1.$$

The intertwining property of these operators may be written as

$$\begin{pmatrix}U_{0a}^+\\ U_{1a}^+\end{pmatrix}H = \begin{pmatrix}\omega & 0\\ 0 & \omega_a\end{pmatrix}\begin{pmatrix}U_{0a}^+\\ U_{1a}^+\end{pmatrix},$$
$$H(U_{a0}^-, U_{a1}^-) = (U_{a0}^-, U_{a1}^-)\begin{pmatrix}\omega & 0\\ 0 & \omega_a\end{pmatrix},$$

where ω and ω_a denote the operators which consist in multiplying $\phi_0(\alpha, \beta)$ by $\omega = \alpha + \beta$ and $\phi_1(\beta)$ by $\omega_a = \alpha_a(\beta) + \beta$.

The operators $U_{0a}^{\pm}$ mapping $\mathfrak{H}$ into $\mathfrak{R}_0$ are generated in an obvious way by the operators $U_a^{\pm}(\beta)$ acting in $\mathfrak{H}_\beta$ and the relations $U_{0a}^+U_{0a}^- = 1$, $U_{0a}^+H = \omega U_{0a}^+$, $HU_{a0}^- = U_{a0}^-\omega$ follow from the corresponding relations for $U_a^{\pm}(\beta)$.

The operator U_{a1}^- will be defined by

$$U_{a1}^-\phi_1 \Leftrightarrow \chi_a(\alpha, \beta)\phi_1(\beta).$$

The right member here is clearly square integrable since $\int |\chi_a(\alpha, \beta)|^2\, d\alpha$ was assumed bounded in β. According to the theory of Appendix A11,

the eigenvector $X_a(\beta)$ represented by the eigenfunction $\chi_a(\alpha, \beta)$ satisfies the relations $U_a^+(\beta)X_a(\beta) = 0$. Consequently, the relation $U_{0a}^+ U_{a1}^- = 0$ holds.

Before we can define the operator U_{1a}^+ we must enter an additional assumption, which is automatically satisfied if the operator H_a is Hermitean. The assumption is that the adjoint $H_a^*(\beta) = H_0(\beta) + V_a^*(\beta)$ of $H_a(\beta)$ should also possess a single eigenvalue, namely, $\overline{\omega_a(\beta)}$, and an eigenvector $X_a^*(\beta)$ which depend continuously on β and for which $(X_a^*(\beta), X_a(\beta)) \neq 0$. Without restriction we may assume $(X_a^*(\beta), X_a(\beta)) = 1$. We then introduce the vector

$$X_a^+(\beta) = [1 - U_a^-(\beta)(U_a^+(\beta))^*]X_a^*.$$

Since $U_a^-(\beta)U_a^+(\beta)$ commutes with $H_a(\beta)$, the operator $U_a^-(\beta)(U_a^+(\beta))^*$ commutes with $H_a^*(\beta)$; hence $X_a^+(\beta)$ is also an eigenvector of $H_a^*(\beta)$ with the eigenvalue $\overline{\omega_a(\beta)}$. Also, $(X_a^+(\beta), X_a(\beta)) = 1$ since $[1 - U_a^-(\beta)U_a^+(\beta)]X_a(\beta) = X_a(\beta)$. The reason for introducing the eigenvector $X_a^+(\beta)$ instead of $X_a^*(\beta)$ is that it satisfies the relation $(U_a^-(\beta))^* X_a^+(\beta) = 0$, which follows from the definition of $X_a^+(\beta)$ and $U_a^+(\beta)U_a^-(\beta) = 1$.

With the aid of the vector $X_a^+(\beta)$ we can now define the operator U_{1a}^+, which is to transform a vector Φ into a function of β, by

$$U_{1a}^+\Phi = \int \chi_a^+(\alpha, \beta)\psi(\alpha, \beta)\, d\alpha,$$

where $\chi_a^+(\alpha, \beta)$ is the representer of the eigenvector $X_a^+(\beta)$ and $\psi(\alpha, \beta)$ is the representer of the vector Φ. Relations $(X_a^+(\beta), X_a(\beta)) = 1$ and $(U_a^-(\beta))^* X_a^+(\beta) = 0$, derived at the end of the preceding paragraph, lead to the two relations $U_{1a}^+ U_{a1}^- = 1$, $U_{1a}^+ U_{a0}^- = 0$. Thus the first inverse relation, which we simply denote by

$$U_a^+ U_a^- = 1,$$

is established. Also, the intertwining relations are immediately verified. To verify the second inverse relation, or "completeness relation"

$$U_a^- U_a^+ = 1$$

we must use the assumption that $\omega_a(\beta)$ is the only point-eigenvalue of $H_a(\beta)$ and is simple. This assumption implies that every vector of the form $(1 - U_a^-(\beta)U_a^+(\beta))\Phi_\beta$ is a multiple of the eigenvector $X(\beta)$; consequently, to every Φ in $\mathfrak{H}$ there is a ϕ_1 in $\mathfrak{H}_1$ such that $(1 - U_{0a}^+ U_{a0}^-)\Phi = U_{a1}^-\psi_1$. Applying U_{1a}^+ on this relation we find $\psi_1 = U_{1a}^+\Phi$. Insertion of this expression then yields the completeness relation.

Thus we have succeeded in establishing an (indirect) spectral representation of the operator H_a. The spectrum of this operator consists of the "main part", the same as that of the operator H_0, and an enlargement, the range of the function $\omega = \omega_a(\beta)$. The fact that the new spectrum is continuous is consistent with the fact that the operator V_a, by virtue of the presence of the factor $\delta(\beta - \beta')$ in its kernel, is not completely gentle.

Having set up the spectral transformation for the operator H_a we can establish the spectral transformation for the operator

$$H = H_a + V_c,$$

modeled after a special operator studied by Kato and Kuroda [23], with V_c assumed gentle and sufficiently small, provided the kernel $v'_a(\beta; \alpha; \alpha')$ of V_a satisfies additional conditions. In case we are working with the first of our two classes of gentle operators the additional condition should require $v'_a(\beta; \alpha; \alpha')$ to satisfy a Hölder condition with respect to β, α, α' (and not just with respect to α and α'); but v'_a need not tend to zero as $|\beta| \to \infty$. As a consequence of such a condition the eigenfunctions $\chi_a^{\pm}(\alpha, \beta)$ satisfy Hölder conditions with respect to α and β.

We now apply the transformations $U_a^{\pm}$ to the operator $H_a + V_c$, obtaining an operator

$$\begin{aligned} U_a^+(H_a + V_c)U_a^- &= \begin{pmatrix} U_{0a}^+ \\ U_{1a}^+ \end{pmatrix}(H_a + V_c)(U_{a0}^-, U_{a1}^-) \\ &= \begin{pmatrix} \omega & 0 \\ 0 & \omega_a \end{pmatrix} + \begin{pmatrix} \tilde{V}_{0c0} & \tilde{V}_{0c1} \\ \tilde{V}_{1c0} & \tilde{V}_{1c1} \end{pmatrix} = \tilde{H}_0 + \tilde{V}_b \end{aligned}$$

in obvious notation. This operator acts on the direct sum of the spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ whose vectors are represented by functions $\psi_0(\alpha, \beta)$ with values in $\mathfrak{N}'$ and complex-valued functions $\psi_1(\alpha, \beta)$. Since the eigenvalue $\omega = \omega_a(\beta)$ was assumed monotone in β, it could be introduced as the independent variable in place of β. Once this is done, the problem becomes just one of those treated in §6, the accessory space consisting of pairs $\{\psi_0, \psi_1\}$, with ψ_1 being a complex number and ψ_0 being a vector in $\mathfrak{N}'$. We maintain that the new disturbing operator $\tilde{V}_c$ is gentle. This is evidently the case for $\tilde{V}_{0c0}$; it is also the case for the other components of $\tilde{V}_c$ by virtue of the requirements imposed in the preceding paragraph. If then the operator V_c is sufficiently small, transformations

$$\tilde{U}_c^{\pm} = \begin{pmatrix} \tilde{U}_{0c0}^{\pm} & \tilde{U}_{0c1}^{\pm} \\ \tilde{U}_{1c0}^{\pm} & \tilde{U}_{1b1}^{\pm} \end{pmatrix}$$

exist transforming $\tilde{H}_0 + \tilde{V}_c$ into $\tilde{H}_0$. The operators

$$\begin{pmatrix} U_0^+ \\ U_1^+ \end{pmatrix} = \begin{pmatrix} \tilde{U}_{0c0}^+ & \tilde{U}_{0c1}^+ \\ \tilde{U}_{1c0}^+ & \tilde{U}_{1c1}^+ \end{pmatrix} \begin{pmatrix} U_{0a}^+ \\ U_{1a}^+ \end{pmatrix}$$

and

$$(U_0^-, U_1^-) = (U_{a0}^-, U_{a1}^-) \begin{pmatrix} \tilde{U}_{0c0}^- & \tilde{U}_{0c1}^- \\ \tilde{U}_{1c0}^- & \tilde{U}_{1c1}^- \end{pmatrix},$$

or simply

$$\tilde{U}^+ = \tilde{U}_c^+ U_a^+ \quad \text{and} \quad U^- = U_a^- \tilde{U}_c^-,$$

then transform the operator $H = H_a + V_c$ into the operator

$$\tilde{H}_0 = \begin{pmatrix} \omega & 0 \\ 0 & \omega_a \end{pmatrix}$$

acting in $(\mathfrak{H}_0, \mathfrak{H}_1)$. Thus the spectral representation of the operator $H = H_0 + V_a + V_c$ is attained.

It is to be noted that the transformations $U^\pm$ are not of the form $1 + \Gamma R^\pm$, not even the components $U_{00}^\pm$ are of this form. The component U_{00}^+, for example is given by

$$U_{00}^+ = \tilde{U}_{0c0}^+ U_{0a}^+ + \tilde{U}_{0c1}^+ U_{1a}^+$$

as the sum of two terms of which the first is of the form $1 + \Gamma R$, while the kernel of the second carries the singular factor $[\omega(\alpha, \beta) - \omega_a(\beta')]^{-1}$ rather than $[\omega(\alpha, \beta) - \omega(\alpha', \beta')]^{-1}$. This is of course due to the presence of the "new" continuous spectrum. As we shall see in the next section, this fact is connected with the phenomenon of multi-channel scattering.

Before discussing these scattering problems we should just make a remark about the spectral problem of the operator $H = H_0 + V_a + V_b$. In case the kernels V_a' and V_b' do not depend on β and α, respectively, the transformations $U_a^\pm$ and $U_b^\pm$ commute so that there is no problem; if this is not the case the perturbation problem for the operator H remains to be investigated. The same applies to the general operator $H_0 + V_a + V_b + V_c$.

A14. Multiple channel scattering. The most important question in connection with the problem considered in the preceding section concerns the description of the process of scattering associated with the operator H. We shall consider the operator $H = H_0 + V_a + V_c$; for, it is the operator V_c which induces multi-channel scattering.

It was shown by Kuroda [25] that the scattering problem of the special operator considered by Kato and himself should be subsumed

under the general theory of multiple channel scattering formulated by Jauch [19]. For the class of operators $H_0 + V_a + V_c$, here considered, we shall be able to establish the complete description of the multiple channel scattering with the aid of unitary operators.

It is necessary to be somewhat more pedantic in the description of scattering than we had been before. The scattering operator should be regarded as a transformation of a spectral representation of the operator H assigned to the time $t = -\infty$ into such a representation assigned to $t = \infty$. A spectral representation of H is given by two functions $\phi_0(\omega, \alpha)$ and $\phi_1(\alpha)$ such that application of H corresponds to multiplication by ω and ω_a.

We shall extract this assignment of representers ϕ_0, ϕ_1 to $t = \pm\infty$ from the asymptotic description of the Schrödinger state $e^{-itH}\Phi$ resulting from a given state Φ. We express Φ in terms of spectral representers $\phi_0^{(\text{in})}$, $\phi_1^{(\text{in})}$ through

$$\Phi = U^- \begin{pmatrix} \phi_0^{(\text{in})} \\ \phi_1^{(\text{in})} \end{pmatrix};$$

we have attached the superscript "(in)" to these representers because they will be assigned to $t = -\infty$ later on. Using the intertwining property of H with U^- we find

$$\begin{aligned} e^{-itH}\Phi &= U^- e^{-it\tilde{H}_0} \begin{pmatrix} \phi_0^{(\text{in})} \\ \phi_1^{(\text{in})} \end{pmatrix} \\ &= U_a^- \tilde{U}_c^- e^{-it\tilde{H}_0} \begin{pmatrix} \phi_0^{(\text{in})} \\ \phi_1^{(\text{in})} \end{pmatrix}, \end{aligned}$$

where

$$\tilde{H}_0 = \begin{pmatrix} \omega & 0 \\ 0 & \omega_a \end{pmatrix}.$$

Since the operators U_c^- are of the form $\tilde{U}_c^- = 1 - \Gamma\tilde{R}_c^-$, the operator

$$e^{it\tilde{H}_0}\tilde{U}_c^- e^{-it\tilde{H}_0}$$

tends to the scattering operator $\tilde{S}_c^- = 1 - \Gamma_\infty \tilde{S}_c^-$ associated with the disturbance $\tilde{V}_c$ of the operator $\tilde{H}_0$. Hence we may set

$$\tilde{U}_c^- e^{-it\tilde{H}_0} \sim e^{-it\tilde{H}_0}\tilde{S}_c^- \quad \text{for } t \sim +\infty,$$

with $\tilde{S}_c^-$ replaced by 1 for $t \sim -\infty$.

Next we must derive the asymptotic description of the operators

$$U_a^- e^{-it\tilde{H}_0} = (U_{0a}^- e^{-it\omega}, U_{1a}^- e^{-it\omega_a}).$$

Since the operator U_{a0}^- was generated from the operators of the form $U_a^-(\beta) = 1 - \Gamma R_a^-(\beta)$ with gentle $R_a^-(\beta)$ acting on functions of α, it is clear that $e^{itw}U_a^-(\beta)e^{itw}$ tends to the operator $S_a^-(\beta) = 1 - \Gamma_\infty R_a^-(\beta)$ This operator acting on $\mathfrak{H}_\beta$ induces an operator S_{a0}^- acting on $\mathfrak{K}_0$. Now, we must recall that the transformation U_{a0}^-, the way we have introduced it, transforms a square integrable function into a vector in $\mathfrak{H}$. We want the scattering operator S_{a0}^- associated with U_{a0}^- to be a transformation of a function into a function. Therefore we introduce the transformation "1" which takes a function $\psi(\alpha, \beta)$ into the vector Φ in $\mathfrak{H}$ of which it is the H_0-representer,

$$\text{“1”}\psi = \Phi.$$

Clearly, then, we have

$$e^{-itH_0}\text{“1”} = \text{“1”}e^{-itw}$$

and the transformation $e^{itH_0}U_a^-e^{-itw}$ tends to "1" S_{a0}^- as $t \to \infty$ and to "1" as $t \to -\infty$. Hence the asymptotic description of $U_{a0}^-e^{-itw}$ is

$$U_{a0}^-e^{-itw} \sim e^{-itH_0}\text{“1”}S_{a0}^- = \text{“1”}e^{-itw}S_{a0}^-$$

as $t \sim \infty$, with S_{a0}^- replaced by 1 for $t \sim -\infty$.

No asymptotic simplification is found by letting $t \to \pm\infty$ in $U_{a1}^-e^{-it\omega_a}$; we simply have

$$U_{a1}^-e^{-it\omega_a} = e^{-itH_a}U_{a1}^-,$$

since $\omega_a = \omega_a(\beta)$ is the eigenvalue of H_a associated with the eigenfunction $\chi_a(\alpha, \beta)$ which enters the definition of U_{a1}^-. Thus we may write

$$\begin{aligned} U_a^-e^{-it\tilde{H}_0} &\sim (\text{“1”}e^{-itw}S_{a0}^-, U_{a0}^-e^{-it\omega_a}) \\ &= (\text{“1”}e^{-itw}, U_{a0}^-e^{-it\omega_a})\begin{pmatrix} S_{a0}^- & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Combining these formulas with that for $\tilde{U}_c^-$ we obtain the asymptotic description of $e^{-itH}\Phi$:

$$e^{-itH}\Phi \sim (\text{“1”}e^{-itw}, U_{a0}^-e^{-it\omega_a})S^-\begin{pmatrix} \phi_0^{(\text{in})} \\ \phi_1^{(\text{in})} \end{pmatrix} \quad \text{as } t \sim \infty,$$

with

$$S^- = \begin{pmatrix} S_{a0}^- & 0 \\ 0 & 1 \end{pmatrix}\tilde{S}_c^- = \begin{pmatrix} S_{a0}^-\tilde{S}_{0c0}^- & S_{a0}^-\tilde{S}_{0c1}^- \\ \tilde{S}_{1c0}^- & \tilde{S}_{1c1}^- \end{pmatrix}.$$

For $t \sim -\infty$ we find

$$e^{-itH}\Phi \sim (\text{“1”}e^{-itw}, U_{a0}^-e^{-it\omega_a})\begin{pmatrix} \phi_0^{(\text{in})} \\ \phi_1^{(\text{in})} \end{pmatrix}.$$

Clearly, then, the spectral representers $\phi_0^{(\mathrm{in})}$ and $\phi_1^{(\mathrm{in})}$ are to be assigned to $t = -\infty$, while the representers

$$\begin{pmatrix} \phi_0^{(\mathrm{out})} \\ \phi_1^{(\mathrm{out})} \end{pmatrix} = S^- \begin{pmatrix} \phi_0^{(\mathrm{in})} \\ \phi_1^{(\mathrm{out})} \end{pmatrix}$$

are to be assigned to $t = +\infty$. One verifies that e^{-itH_0} commutes with S^- so that, indeed, $\phi_0^{(\mathrm{out})}$, $\phi_1^{(\mathrm{out})}$ may serve as spectral representers of the vector Φ for the operator H.

The operator S^+ will naturally be defined by

$$S^+ = \tilde{S}_c^+ \begin{pmatrix} S_{0a}^+ & 0 \\ 0 & 1 \end{pmatrix};$$

the relation

$$S^+ S^- = 1$$

is then an immediate consequence of $S_{0a}^+ S_{a0}^- = 1$ and $\tilde{S}_c^+ \tilde{S}_c^- = 1$. Also, the inverse relation

$$S^- S^+ = 1$$

holds, as follows from the inverse relation $\tilde{S}_c^- \tilde{S}_c^+ = 1$ and from the relation $S_{a0}^- S_{0a}^+ = 1$ which follows from Møller's relation $S_a^-(\beta) S_a^+(\beta) = 1$. The validity of this relation, in spite of the fact that $U_a^-(\beta) U_a^+(\beta) \neq 1$, is essential for the complete description of the multi-channel scattering process.

Vectors Φ with representers $\{\phi_0^{(\mathrm{in})}, 0\}$ and $\{0, \phi_1^{(\mathrm{in})}\}$ are said to come out of the "main" and the "side" initial channels; vectors with representers $\{\phi_0^{(\mathrm{out})}, 0\}$ and $\{0, \phi_1^{(\mathrm{out})}\}$ are said to enter "main" and "side" end channels. These vectors are of the form

$$\Phi = U_0^- U_0^+ \Phi \quad \text{and} \quad \Phi = U_1^- U_1^+ \Phi$$

for the incoming channels; for the outgoing channels they will be described below.

We should note that the scattering operators here denoted by $S^\pm$ correspond to those that were denoted by $S_0^\pm$ in §7, since they transform H_0-representers into H_0-representers. The operators there denoted by $S^\pm$ will now be denoted by $\hat{S}^\pm$. They are given by

$$\hat{S}^\pm = U^- S^\pm U^+$$

and evidently transform vectors into vectors. In particular, the vector $\Phi' = \hat{S}^- \Phi$ is the one whose incoming representers are the same as the outgoing representers of Φ,

$$\phi_0^{(\mathrm{out})} = \phi_0'^{(\mathrm{in})}, \qquad \phi_1^{(\mathrm{out})} = \phi_1'^{(\mathrm{in})}.$$

It follows that the states associated with the outgoing main and side channels are those that are of the form

$$\Phi = \hat{S}^+ U_0^- U_0^+ \hat{S}^- \Phi \quad \text{and} \quad \Phi = \hat{S}^+ U_0^- U_0^+ \hat{S} \Phi.$$

Finally we discuss the question whether or not the scattering operator S^- can be derived from "wave operators". According to the theory of Jauch [**19; 20**] such wave operators would have to be defined as the limits of the operators $e^{itH}*$, e^{-itH} with the aid of various "channel energy operators" H_*. Clearly, the operator H_0 will correspond to the main channel. In fact, from our asymptotic description of the operator $e^{-itH}U^-$ we find

$$\begin{aligned} e^{itH_0}e^{-itH} &\sim e^{itH_0}(\text{"1"}e^{-it\omega}, U_{1a}^+ e^{-it\omega_a})S^- U^+ \\ &= (\text{"1"}, U_{1a}^+ e^{it(\omega-\omega_a)})S^- U^+ \\ &\to (\text{"1"}, 0)S^- U^+ \quad \text{as } t \to \infty \end{aligned}$$

since $\omega(\alpha, \beta) - \omega_a(\alpha) \geqq \omega_0 > 0$. The convergence here is weak; if one prefers to work with strong convergence one need only take the inverse $e^{itH}e^{-itH_0}$. Clearly then, the wave operator W_{0+} associated with the channel energy H_0 is

$$W_{0+} = (\text{"1"}, 0)S^- U^+.$$

The corresponding operator for $t = -\infty$ is evidently

$$W_{0-} = (\text{"1"}, 0)U^+.$$

The channel energy associated with the side channel is *not* H_a; rather, it is the operator H_1 defined by the H_0-representation

$$H_1\Phi \underset{0}{\Leftrightarrow} \omega_a(\beta)\psi(\alpha, \beta), \quad \text{when } \Phi \underset{0}{\Leftrightarrow} \psi(\alpha, \beta).$$

Of course, when applied on U_{1a}^+ application of H_1 gives the same result as application of H_a, but that is not the case when applied to "1". Our asymptotic formula shows that

$$e^{itH_1}e^{-itH} \sim (\text{"1"}e^{it(\omega_a-\omega)}, U_{1a}^+)S^- U^+$$

so that the wave operators $W_{1\pm}$ defined as weak limits for $t \to \pm\infty$ are

$$W_{1+} = (0, U_{1a}^+)S^- U^+$$

and

$$W_{1-} = (0, U_{1a}^+)U^+.$$

From the combined wave operators

$$W_+ = \begin{pmatrix} W_{0+} \\ W_{1+} \end{pmatrix} = \begin{pmatrix} \text{"1"} & 0 \\ 0 & U_{1a}^+ \end{pmatrix} S^- U^+$$

and

$$W_- = \begin{pmatrix} W_{0-} \\ W_{1-} \end{pmatrix} = \begin{pmatrix} \text{“1”} & 0 \\ 0 & U_{1a}^+ \end{pmatrix} U^+$$

we can indeed construct our scattering operator $\hat{S}^-$ in the form

$$\hat{S}^- = (W_-)^{-1} W_+.$$

A similar formula plays a basic role in the method of the wave operator to be discussed in subsequent sections.

A15. The wave operator method in scattering. Up to now we have been considering the perturbation of an operator with a continuous spectrum by a class of "gentle" disturbances and proved rather complete results. Another approach was initiated by Rosenblum, Kato, Cook, and Kuroda for perturbations which are not restricted by a gentleness requirement. As is to be expected, weaker results are then obtained. Here, we shall only sketch some high points of this approach, referring to the literature [**16**; **17**; **18**; **22**] for proofs and detailed discussion. For simplicity we assume (A) that the spectrum of H_0 has uniform multiplicity.

Let H_0 and $H = H_0 + V$ be selfadjoint operators acting on the Hilbert space $\mathfrak{H}$. In relating the spectrum of H to that of H_0, the main idea in the work of the authors mentioned is to introduce the one parameter family of unitary operators

$$W_t = e^{itH} e^{-itH_0} \tag{1}$$

and investigate their limit as $t \to \pm \infty$. Thus, in the present approach, essential use is made of the fact that the unitary operators e^{itH} and e^{-itH_0} are well defined when H and H_0 are strictly selfadjoint, while in the approach described earlier this fact was not used and it was unnecessary to require H to be selfadjoint.

In the approach to be discussed now it will be shown that under appropriate conditions the operators W_t tend strongly to limits as $t \to \pm \infty$. These limits,

$$W_\pm = \lim_{t \to \pm \infty} W_t,$$

are called the "wave operators". (This definition differs slightly from that given in §7.)

Assuming that these wave operators exist one can show easily that they lead to a partial spectral representation of the operator H.

Clearly, from $(W_t\Phi, W_t\Psi) = (\Phi, \Psi)$ we may conclude that the relation $(W_\pm\Phi, W_\pm\Psi) = (\Phi, \Psi)$ holds for all vectors Φ, Ψ, whence, the relations

$$W_\pm^* W_\pm = 1$$

follow, which show that the operators $W_\pm$ are isometric. From

$$(e^{-i\tau H}\Phi, W_\pm e^{-i\tau H_0}\Psi) = \lim_{t\to\pm\infty} (\Phi, e^{i(\tau+t)H}e^{-(t+\tau)H_0}\Psi) = (\Phi, W_\pm\Psi)$$

we may conclude that the relation

$$W_\pm e^{-i\tau H_0} = e^{-i\tau H}W_\pm$$

holds, which, by virtue of the strict selfadjointness of H and H_0, implies the intertwining relations

$$W_\pm H_0 = HW_\pm, \qquad H_0 W_\pm^* = W_\pm^* H$$

in the domains of H_0 and H, respectively. Thus both, W_+ and W_-, give a partial spectral representation of H. This representation would be complete if these operators were unitary, i.e., if $W_\pm W_\pm^* = 1$; but that need not be the case. We shall come back to this question at the end of this section.

The simplest condition under which the wave operators have been shown to exist was described by Cook [9]. For every vector Φ in an appropriate dense subset of $\mathfrak{H}$ the vector $e^{-itH_0}\Phi$ should admit V and satisfy the inequality

$$\int_{-\infty}^{\infty} \|Ve^{-itH_0}\Phi\|\, dt < \infty.$$

Clearly, this inequality implies the relation

$$\int_s^t \|Ve^{-i\tau H_0}\Phi\|\, dt \to 0 \quad \text{as } s, t \to \infty.$$

Now, from the relation $(d/dt)e^{itH}e^{-itH_0} = ie^{itH}Ve^{-itH_0}$ we have

$$W_t - W_s = \int_s^t (d/d\tau)W_\tau\, d\tau = i\int_s^t e^{i\tau H}Ve^{-i\tau H_0}\, d\tau$$

and hence

$$\|(W_t - W_s)\Phi\| \leqq \int_s^t \|e^{i\tau H}Ve^{-i\tau H_0}\|\, d\tau$$

$$= \int_s^t \|Ve^{-i\tau H_0}\|\, d\tau \to 0 \quad \text{as } s, t \to \infty.$$

Thus Cook's statement follows.

In case H_0 is the Laplacean and V multiplication by a function, simple properties of V can be given [9; 11] so that this sufficient condition is satisfied.

Stronger results are obtained from the approach of Rosenblum, Kato, and Kuroda. The conditions imposed by these authors on the disturbance V will be mentioned below. In general, also these results do not describe the spectral representation of the operator H completely, but they do describe completely the spectral representation of the part of H in the space $W_+\mathfrak{H}$ (or $W_-\mathfrak{H}$), inasmuch as it is shown that it is unitarily equivalent to the "absolutely continuous part" of H_0.

The "part" of an operator, such as H, in a subspace $\mathfrak{H}' = P'\mathfrak{H}$ is the operator $P'H$ provided the projector P' commutes with the operator H in its domain.

To explain the meaning of "absolutely continuous part" let $\mu(\omega)$ be a spectral measure function of H. As is well known such a function is the sum, $\mu = \mu^a + \mu^s$, of a measure function μ^a which is "absolutely continuous" in the sense that it vanishes on every set whose Lebesgue measure vanishes, and a measure function μ^s which is singular inasmuch as it vanishes on every set whose Lebesgue measure does not vanish. Denoting by $\mathfrak{M}$ the subspace of all vectors in $\mathfrak{H}$ which admit an H-representation by functions $\phi(\omega)$ for which $\int |\phi(\omega)|^2 \, d\mu^s(\omega) = 0$, we may describe the absolutely continuous part of H as the part of H in $\mathfrak{M}$, that is to say, as the operator $PH = HP$, where P is the orthoprojector projecting on $\mathfrak{M}$. Similarly we introduce the absolutely continuous part $P_0H_0 = H_0P_0$ of the operator H_0 with the aid of the projector P_0 which projects into the subspace $\mathfrak{M}_0$ of vectors Φ with H_0-representer ψ satisfying $\int |\psi(\omega)|^2 \, d\mu_0^s(\omega) = 0$.

"Absolutely continuous" parts of the wave operators are now introduced as the strong limits

$$W_\pm^a = W_\pm P_0 = \lim_{t \to \pm\infty} W_t P_0.$$

If these limit operators exist they transform $\mathfrak{M}_0$ into $\mathfrak{M}$ and, hence, give the complete spectral representation of H in the subspaces $W_+^a\mathfrak{H} = W_+^a\mathfrak{M}_0$ and $W_-^a\mathfrak{H} = W_-^a\mathfrak{M}_0$ of $\mathfrak{M}$.

With the aid of the operators $W_\pm^a$ one may introduce the operators

$$S_0^{a-} = W_+^{a*}W_-^a, \qquad S_0^{a+} = W_-^{a*}W_+^a.$$

In case $W_\pm^a\mathfrak{M}_0 = W_\mp^a\mathfrak{M}_0$ these operators are unitary when acting in $\mathfrak{M}_0$ and thus may be regarded as scattering operators in $\mathfrak{M}_0$.

The conditions under which Rosenblum and Kato prove the existence

of the operators $W_{\pm}^{a}$ are that the operator V should be selfadjoint and of the trace class. (Rosenblum, in addition, assumed that the spectra of both H_0 and H are absolutely continuous.)

A trace class operator may be defined as the product $V = T_1 T_2$ of two operators of the Schmidt class; and a Schmidt class operator T may be defined as one that has a finite square trace

$$\|T\|_2^2 = \sum_{m=1}^{\infty} \|T\Phi^{(m)}\|^2,$$

where $\{\Phi^{(m)}\}$ is any complete orthonormal system of vectors. For V of trace class one may, in particular, take $T_2 = |V|^{1/2} = [V^*V]^{1/4}$ and $T_1 = Y|V|^{1/2}$, where Y is unitary, so that $\|T_1\|_2 = \| \, |V|^{1/2}\|_2$. The quantity

$$\|V\|_1 = \| \, |V|^{1/2} \, \|_2^2$$

is called the "trace norm" of V.

The Rosenblum-Kato result was extended by Kuroda [**14**] who required of V only that $H_0 + V$ be selfadjoint and that the operator $|V|^{1/2}(H_0 + i)^{-1}$ be in the Schmidt class; note then that V need not be bounded.

A most remarkable feature of Rosenblum's and Kato's result is that the condition imposed on the disturbance V, to be of the trace class, makes no reference to the undisturbed operator H_0. This is in contrast to what has been emphasized in the approach presented in these notes, in which the gentleness property of V referred essentially to the operator H_0. Kuroda's condition does again involve H_0.

It may be mentioned incidentally that operators V could be constructed which satisfy one of our gentleness conditions without being of the trace class.

We do not intend to give a detailed description of the wave operator theory; but we should like to give an outline of Kato's proof of his result concerning perturbation by a trace class operator (under our assumption *A*). The role of the absolute continuity of the component μ^a of the spectral measure of the operator H is embodied in a lemma due to Rosenblum, which we formulate in a slightly modified way.

We note that a spectral representation of the space $\mathfrak{M}$ involving an absolutely continuous spectral measure μ^a can always be replaced by one involving the Lebesgue measure, simply by replacing the original representers, $\psi^{(\mu)}(\omega)$, by the new representers

$$\psi(\omega) = (d\mu^a(\omega)/d\omega)^{1/2} \, \psi^{(\mu)}(\omega).$$

Note that such a representer vanishes outside the support $\mathscr{B}$ of the measure μ^a. The manifold of all vectors Φ in $\mathfrak{M}$ which possess representers with bounded absolute values $|\psi(\omega)|$ is dense in $\mathfrak{M}$ since this manifold comprises the vectors with piecewise constant representers. Denoting by $\mathfrak{M}^m$ the class of those vectors for which $|\psi(\omega)| \leqq m$ we may formulate

ROSENBLUM'S LEMMA. *Let R be an operator of the Schmidt class and let Φ be a vector in $\mathfrak{M}^m$ with reference to selfadjoint operator H. Then the inequality*

$$\int_{-\infty}^{\infty} \|Re^{-itH}\Phi\|^2 \, dt \leqq 2\pi m^2 \|R\|_2^2$$

holds.

To prove it (in somewhat abbreviated form) we first note that the assumption made on R implies that for almost every ω there is an operator $R(\omega)$ such that

$$R\Phi = \int R(\omega)\psi(\omega)\, d\omega$$

when ψ is the representer of Φ in $\mathfrak{M}^m$. We then have

$$Re^{-itH}\Phi = \int R(\omega)\psi(\omega)e^{-it\omega}\, d\omega$$

and, by Parseval's formula,

$$\int \|Re^{-itH}\Phi\|^2 \, dt = 2\pi \int \|R(\omega)\|^2 |\psi(\omega)|^2 \, d\omega \leqq 2\pi m^2 \int_{\mathscr{B}} \|R(\omega)\|^2 \, d\omega$$

$$\leqq 2\pi m^2 \|R\|_2^2,$$

where $\mathscr{B}$ is the support of μ^a. Thus the lemma is proved.

For simplicity we shall restrict ourselves to the case in which the whole spectrum of H_0 is absolutely continuous, so that $\mathfrak{M} = \mathfrak{H}$ and $W_\pm^a = W_\pm$.

The next step then consists in establishing an a priori inequality, i.e., one that holds provided the wave operator W_+ exists. We write $V = T_1 T$ with $T = |V|^{1/2}$ and $T_1 = YT$, where Y is a unitary operator and employ the relation

$$W_t - W_s = i \int_s^t e^{i\tau H} V e^{-i\tau H_0} \, d\tau,$$

used in deriving Cook's statement. Having assumed that W_t has a limit W_+ we may let t tend to infinity. Since $\|W_+\Phi\|^2 = \|W_s\Phi\|^2 = \|\Phi\|^2$ we have

$$\|(W_+ - W_s)\Phi\|^2 = 2Re(W_+\Phi, (W_+ - W_s)\Phi)$$

$$= 2iRe \int_s^\infty (\Phi, W_+^* e^{i\tau H} T_1 T e^{-i\tau H_0}\Phi)\, d\tau$$

$$= 2iRe \int_s^\infty (\Phi, e^{i\tau H_0} W_+^* T_1 T e^{-i\tau H_0}\Phi\, d\tau)$$

$$= 2iRe \int_s^\infty (T_1^* W_+ e^{-i\tau H_0}\Phi, T e^{-i\tau H_0}\Phi\, d\tau)$$

$$\leqq 2\left[\int_s^\infty \|T_1^* W_+ e^{-i\tau H_0}\Phi\|^2\, d\tau \int_s^\infty \|T e^{-i\tau H_0}\Phi\|^2\, d\tau\right]^{1/2}.$$

Applying now Rosenblum's Lemma for H_0 in place of H and $R = T_1^* W_+$ we obtain

$$\|(W_+ - W_s)\Phi\|^2 \leqq 2k\left[\int_s^\infty \|T e^{-i\tau H_0}\Phi\|^2\, d\tau\right]^{1/2}$$

with $k^2 = 2\pi m^2 \|V\|_1$.

Using the triangle inequality we are then led to the inequality

$$(\#) \qquad \|(W_t - W_s)\Phi\|^2 \leqq 2k\left[\int_t^\infty \|T e^{-i\tau H_0}\Phi\|^2\, d\tau\right]^{1/2} + 2k\left[\int_s^\infty \|T e^{-i\tau H_0}\Phi\|^2\, d\tau\right]^{1/2}.$$

This formula does no longer involve the operator W_+. Although it was derived under the assumption that W_+ exists, it can be shown to hold if this assumption is not made. To this end, Rosenblum and Kato show first that the wave operators exist for disturbances V of finite rank. Formula (#) therefore holds for such operators. Since a trace class operator V can be approximated by operators of finite rank with respect to the trace norm, this formula carries over to any trace class operator V.

Letting s and t tend to infinity one deduces from formula (#) the existence of the wave operator W_+. In the same way, the existence of W_- is established. Similarly, one may show that the operators $W_\pm^a$ in $\mathfrak{M}_0$ exist without making the assumption $\mathfrak{M}_0 = \mathfrak{H}$.

Because of the relation $HW_+ = W_+ H_0$ in $\mathfrak{M}_0$, it is clear that the H_0-representer ϕ of Φ in $\mathfrak{M}_0$ may be regarded as the H-representer of the vector $W_+\Phi$. Hence the vectors $W_+\Phi$ lie in $\mathfrak{M}$ whenever Φ lies in $\mathfrak{M}_0$. That is,

$$W_+ \mathfrak{M}_0 \subset \mathfrak{M},$$

as was announced earlier.

Now Rosenblum and Kato use the fact that the condition imposed on V, viz., the condition that V be of the trace class, does not involve any reference to H_0. By virtue of this fact H may be regarded as the undisturbed operator and $-V$ the disturbance.

The role of the operator W_t is then played by W_t^*. This operator therefore converges strongly to the operator W_+^*, which now plays the role of the wave operator in place of W_+. In place of the preceding relation we obtain relation

$$W_+^* \mathfrak{M} \subset \mathfrak{M}_0.$$

Because of relation $W_+^* W_+ = P_0$ we may conclude

$$W_+ \mathfrak{M}_0 = \mathfrak{M}, \qquad W_+^* \mathfrak{M} = \mathfrak{M}_0.$$

Thus the operators W_+ and W_+^* yield the spectral representation of the absolutely continuous part of the operator H. This is the essential part of the Rosenblum-Kato theorem.

In conclusion let us remark that still another approach yielding transformations W_+, W_+^* was initiated by L. deBranges [29].

Appendix to Chapter III

A16. The nongentle character of the operators K. It was stated in §9 that the operators K described there differ essentially from the gentle operators treated in Chapter II, no matter how smooth the functions $k^{\prime}_{lm}((\omega)_l; (\omega')_m)$ may be chosen, and that this is due to the symmetrization involved in the definition of K. We want to make this fact explicit here.

To show this we introduce spectral and accessory variables as indicated earlier in §9; that is, we take the number n together with $(\omega)_{n-1} = (\omega_1, \cdots, \omega_{n-1})$ as accessory variables, and $\omega = \sum^n \omega_\nu = \omega_1 + \cdots + \omega_n$ as spectral variable. Instead of the function $\psi_n((\omega)_n)$ we introduce the function

$$\chi_n(\omega) = \chi_n((\omega)_{n-1}; \omega) = \psi_n\Big((\omega)_{n-1}, \omega - \sum^{n-1} \omega_\nu\Big)$$

as representer of the state ψ_n. The class of these functions χ_n is of course restricted by conditions such as

$$\chi_n((\omega)_{n-1}; \omega) = \chi_n\Big((\omega)_{n-2}, \omega - \sum^{n-1} \omega_\nu; \omega\Big),$$

which express the symmetry of $\psi_n((\omega)_n)$.

We now consider a component K_{lm} of the operator K with $m > 0$ and express the χ-representer $\chi^1_{l-m+n}(\omega)$ of the state $\Psi^1 = K_{lm}\Psi_n$ in terms of the χ-representer $\chi_n(\omega)$ of the state Ψ_n in the form

$$\chi^1_{l-m+n}(\omega) = \int k_{lmn}(\omega; \omega')\chi_n(\omega')\, d\omega'.$$

(There is no restriction in assuming $m > 0$; for, K_{l0} is no integral operator, hence no candidate for gentleness.) The kernel $k_{lmn}(\omega; \omega')$ of this integral transformation is an operator acting on functions of the accessory variables $\omega_1, \cdots, \omega_{n-1}$ and produces functions of the $l - m + n - 1$ accessory variables of χ^1.

In order to describe this kernel explicitly one should first express the ψ-representer of $K_{ln}\Psi_n$ in terms of that of Ψ_n. Setting $n - m = j$ we have

$$(j + l)!\,\psi^1_{j+l}((\omega)_{j+l}) = \sum \text{Perm} \pm \int^m k^j_{lm}((\omega)^j_{j+l};\,(\tilde{\omega})_m)\,\psi_{j+m}((\omega)_j(\tilde{\omega})_m)\,d(\tilde{\omega})_m,$$

where $(\omega)^b_a$ stands for $\{\omega_{b+1}, \cdots, \omega_a\}$. Note that two types may be distinguished among the terms resulting from the permutations involved in the definition of ψ^1: Those for which the variable ω'_{j+l} remains an argument of k_{lm} and those for which ω_{r+l} becomes an argument of ψ_{j+m}. It is the latter terms that cause the nongentle character of the operator K_{lm}. Note that such terms occur only for $j > 0$, i.e., if $\psi_n((\omega)_n)$ is not fully contracted.

We shall be satisfied with discussing in detail the kernel k_{112} which describes how the operator K_{11} acts on a state Φ_2. Setting $\chi_2(\omega;\,\omega) = \psi_2(\omega_1,\,\omega - \omega_1)$ and taking account of $\chi_2(\omega_1;\,\omega) = \chi_2(\omega - \omega_1;\,\omega)$ we readily find that the function $\chi^{(1)}_2(\omega_1,\,\omega)$ corresponding to $K_{11}\Phi_2$ is given as

$$\chi^{(1)}_2(\omega_1;\,\omega) = \int\int k_{112}(\omega_1;\,\omega;\,\omega'_1,\,\omega')\chi_2(\omega'_1;\,\omega')\,d\omega'_1\,d\omega$$

with

$$2k_{112}(\omega_1;\,\omega;\,\omega'_1) = k_{11}(\omega - \omega_1;\,\omega' - \omega_1)\,\delta(\omega_1 - \omega'_1) + k_{11}(\omega_1;\,\omega' - \omega + \omega_1)\,\delta(\omega - \omega_1 - \omega'_1).$$

It is the second term here that causes the nongentleness of the kernel k_{112} because of the presence of the variable ω in the delta function. Clearly, this kernel does not belong to either of our two classes of gentle kernels.

The nongentleness of the operators K_{lm} can be inferred directly from the fact that the product $P_{jk}\Gamma Q_{lm}$ contains the Wick product $:P_{jk}\Gamma Q_{lm}:$ as contribution. For in this Wick product the singularity of the denominator of the kernel of ΓQ_{lm} is not smeared out by integration.

The attached product $P_{jk} \multimap \Gamma Q_{lm}$, on the other hand, may well belong to the same class of operators to which P_{jk} and Q_{lm} belong. Classes for which this is the case will be described in the next section.

A17. Annihilation-creation operators of class $\mathfrak{Q}$. In §11 we assumed that a class $\mathfrak{Q}$ of annihilation-creation operators $Q = \sum_{lm} Q_{lm}$ could be so defined that the operation Γ is applicable on the operators Q in $\mathfrak{Q}'$, that is on operators Q with $Q_{00} = 0$, and that the attached products

$P \multimap \Gamma Q$ and $\Gamma P \multimap Q$ belong to $\mathfrak{Q}$ if P and Q belong to $\mathfrak{Q}'$. In the present section we want to exhibit a class $\mathfrak{Q}$ having these properties.

Naturally we shall try to describe such a class in essentially the same manner in which classes of gentle operators were described in the Appendix A7 to Chapter II. It is not obvious whether or not this can be done since annihilation-creation operators are not gentle, as was shown in Appendix A16. Nevertheless, we shall be able to construct a class $\mathfrak{Q}$ with the desired properties by employing Fourier transforms of absolutely integrable functions.[1] This can be done in a relatively clear-cut way if no pure creation operators, i.e., terms with $m = 0$, are present in the series $Q = \sum_{lm} Q_{lm}$. In case terms Q_{l0} are present we are forced to employ a rather involved treatment. (This is an instance of the general experience that pure creation operators cause the greatest trouble.) We therefore confine ourselves to the case in which $Q_{l0} = 0$ and also $Q_{0m} = 0$.

Our first task will be to introduce classes $\mathfrak{Q}_{lm}$ of basic annihilation-creation operators Q_{lm} which admit the operation Γ. Moreover, these classes will be so chosen that the attached products with r contractions,

$$Q_{jk} \overset{r}{\multimap} \Gamma Q_{lm} \quad \text{and} \quad \Gamma Q_{jk} \overset{r}{\multimap} Q_{lm},$$

belong to the class $\mathfrak{Q}_{(j+l-r)(k+m-r)}$ if Q_{jk} and Q_{lm} belong to $\mathfrak{Q}_{jk}$ and $\mathfrak{Q}_{lm}$. In doing this we shall pay no attention to the accessory variables, keeping in mind, though, that the values of the "functions" referred to are bounded operators acting on functions of these accessory variables. We shall also introduce a dense subspace $\ddot{\mathfrak{H}}$ of the Hilbert space $\mathfrak{H}$ in which the operators Q_{lm} are applicable.

This space $\ddot{\mathfrak{H}}$ will consist of vectors in $\mathfrak{H}$, i.e., of vectors Φ in $\mathfrak{H}$ having only a finite number of components $\psi_n \not\equiv 0$. For $n > 0$ these components, assumed to vanish unless $\omega_1, \cdots, \omega_n > \omega_*$, are to be Fourier transforms

$$\psi_n((\omega)_n) = \int^n \chi_n((\lambda)_n) e^{-i(\omega_1\lambda_1 + \cdots + \omega_n\lambda_n)}\, d(\lambda)_n$$

of functions $\chi_n((\lambda)_n)$ which are continuous and bounded, and for which the integral

$$\|\chi_n\|^+ = \int^n |\chi_n((\lambda)_n)|\, d(\lambda)_n$$

[1] The results of J. Schwartz [47] indicate that such a class $\mathfrak{Q}$ could also be constructed with the aid of Hölder continuous functions.

is finite. We also set

$$\|\chi_n\|^- = \text{l.u.b.}_\lambda |\chi_n((\lambda)_n)|.$$

Clearly, such a function χ_n is square integrable. For $n = 0$ we set $\chi_0 = \psi_0$ and $\|\chi_0\| = |\psi_0|$.

We note that any finite sequence $\{\chi_0, \chi_1, \cdots\}$ of functions with these properties corresponds to a vector in $\ddot{\mathfrak{H}}$ provided the antecedent transform $\psi_n((\omega)_n)$ (for $n > 0$) vanishes unless $\omega_1, \cdots, \omega_n > \omega_*$. (We assume $\omega_* > 0$.)

The class $\mathfrak{Q}_m$ is now formed by those operators Q_{lm} whose kernels q_{lm} are the Fourier transforms

$$q_{lm}((\omega)_l; (\omega')_m) = \int^l \int^m \zeta_{lm}((\lambda); (\lambda')_m) e^{-i(\omega_1\lambda_1 + \cdots + \omega_l\lambda_l - \omega'_1\lambda'_1 - \cdots - \omega'_m\lambda'_m)}\, d(\lambda)_l\, d(\lambda')_m$$

of functions $\zeta_{lm}((\lambda); (\lambda'))$ for which the integrals

$$\|\zeta_{lm}\|^+ = \text{l.u.b.}_{\lambda'} \int^{l+1} |\zeta_{lm}((\lambda)_l; (\lambda' + \tau)_m)|\, d(\lambda)_l\, d\tau$$

and

$$\|\zeta_m\|^- = \text{l.u.b.}_\lambda \int^{m+1} |\zeta_{lm}((\lambda + \tau); (\lambda')_m)|\, d(\lambda')_m\, d\tau$$

are finite. Here

$$(\lambda + \tau) = (\lambda_1 + \tau, \cdots, \lambda_l + \tau) \quad \text{and} \quad (\lambda' + \tau)_m = (\lambda'_1 + \tau, \cdots, \lambda'_m + \tau).$$

We now maintain that the operation Γ can be performed on an operator Q_{lm} by replacing the transform ζ_{lm} by the function

$$\gamma\zeta_{lm}((\lambda)_l; (\lambda')_m) = i \int_{-\infty}^0 \zeta_{lm}((\lambda + \tau)_l; (\lambda' + \tau)_m)\, d\tau;$$

this is in close analogy with the manner in which the operation Γ was performed on the operators Q_{11} of the second class of gentle operators described in Appendix A7. To justify this statement one will extend the argument given in that appendix and verify that the transformed kernel of the commutator $[H_0, \Gamma Q_{lm}]$ is the function

$$\int_{-\infty}^0 \left(\frac{\partial}{\partial\lambda_1} + \cdots + \frac{\partial}{\partial\lambda_l} + \frac{\partial}{\partial\lambda'_1} + \cdots + \frac{\partial}{\partial\lambda'_m}\right)\zeta_{lm}((\lambda + \tau)_l; (\lambda' + \tau)_m)\, d\tau$$
$$= \int_{-\infty}^0 \frac{\partial}{\partial\tau}\, \zeta_{lm}((\lambda + \tau)_l; (\lambda' + \tau)_m)\, d\tau = \zeta_{lm}((\lambda)_l; (\lambda')_m);$$

that is to say, the transformed kernel of the operator Q_{lm}.

Next we apply the operator ΓQ_{lm} on an m-state Φ_n with the transformed representer $\chi_n((\lambda)_n)$. The transformed representer of the

vector $\Gamma Q_{lm}\Phi_n$ is an N-state, with $N = l + n - m$, consisting of $N!(N-l)!$ times the function

$$\chi^1_N((\lambda)_N) = i\mathrm{Sy}\int^m\int_{-\infty}^0 \zeta_{lm}((\lambda+\tau)_l;(\tilde{\lambda}+\tau)_m)\chi_n((\tilde{\lambda})_m(\lambda)^l_N)\,d\tau d(\tilde{\lambda})_m,$$

where $(\lambda)^b_a$ stands for $(\lambda_{b+1},\cdots,\lambda_a)$. Here $n \geqq m$ is implied without restriction since $\Gamma Q_{lm}\Phi_n = 0$ for $n < m$.

Now we should show that the function χ^1_N is square integrable if the norms $\|\zeta_{lm}\|^{\pm}$ and $\|\chi_n\|^{\pm}$ are finite. To this end we estimate the norms $\|\chi^1_N\|^{\pm}$ in terms of $\|\zeta_{lm}\|^{\pm}$. We clearly have

$$\begin{aligned}\|\chi^1_R\|^+ &\leqq \int^{l+n+1} |\zeta_{lm}((\lambda+\tau)_l;(\tilde{\lambda}+\tau)_m)|\,|\chi_n((\tilde{\lambda})_n)|\,d\tau d(\tilde{\lambda})_m\,d(\lambda)_l\\ &\leqq \text{l.u.b.}_{\lambda'}\int^{l+1}|\zeta_{lm}((\lambda+\tau)_l;(\lambda'+\tau)_m)|\,d\tau d(\lambda)_l\int^n|\chi_n((\lambda)_n)|\,d(\lambda_n)\\ &= \|\zeta_{lm}\|^+\|\chi_n\|^+\end{aligned}$$

and

$$\begin{aligned}\|\chi^1_N\|^- &\leqq \text{l.u.b.}_{\lambda,\lambda'}\int^{m+1}\zeta_m((\lambda+\tau);(\tilde{\lambda}+\tau)_m)|\chi_{ln}((\tilde{\lambda})_m(\lambda'))_{n-m})|\,d\tau d(\tilde{\lambda}')_m\\ &\leqq \text{l.u.b.}_{\lambda}\int^{m+1}|\zeta_{lm}((\lambda+\tau)_l;(\tilde{\lambda}+\tau)_m)|\,d\tau d\tilde{\lambda}\,\text{l.u.b.}_{\lambda'}|\chi_{ln}(\lambda')|\\ &= \|\zeta_{lm}\|^-\|\chi_n\|^-.\end{aligned}$$

It follows that the vector $\Gamma Q_{lm}\Phi_n$ belongs to $\ddot{\mathfrak{H}}$ if Φ_n does.

Our next task is to show that the attached products $Q_{jk}\overset{r}{\circ}\Gamma Q_{lm}$ and $\Gamma Q_{jk}\overset{r}{\circ}Q_{lm}$ belong to the space $\mathfrak{Q}_{LM}$ with $L = j + l - r$, $M = k + m - r$ if Q_{jk} and Q_{lm} belong to $\mathfrak{Q}_{jk}$ and $\mathfrak{Q}_{lm}$. To this end we observe that the Fourier transform of the representer of the second attached product is given by

$$\begin{aligned}Z_{lM}((\lambda)_L;(\lambda')_M) = i\mathrm{Sy}_\lambda\mathrm{Sy}_{\lambda'}\int^r\int_{-\infty}^0 &\zeta_{jk}((\lambda+\tau)_j;(\lambda'+\tau)^m_M(\tilde{\lambda}+\tau)_r)\\ &\cdot\zeta_{lm}((\tilde{\lambda})_r(\lambda)^j_L;(\lambda')_m)\,d\tau d(\tilde{\lambda})_r.\end{aligned}$$

To estimate $[\![Z_{LM}]\!]^-$ we write

$$\begin{aligned}\int^{M+1}|Z_{LM}((\lambda+\sigma)_L;(\lambda')_M)|\,d\sigma d(\lambda')_M &\\ \leqq \mathrm{Sy}_\lambda\int^{r+M+2}&|\zeta_{jk}((\lambda+\tau+\sigma)_L;(\lambda'+\tau+\sigma)^m_M(\tilde{\lambda}+\tau)_r)\\ &\cdot\zeta_{lm}((\tilde{\lambda})_n(\lambda+\tau)^j_L;(\lambda'+\sigma)_m)|\,d\tau d(\tilde{\lambda})_r\,d(\tilde{\lambda})_M\,d\sigma\\ = \mathrm{Sy}\int^{r+M+1}&|\zeta_{jk}(\lambda+\rho)_L;(\lambda'+\rho)^m_M(\tilde{\tilde{\lambda}}+\rho)_r)|\\ &\cdot\zeta_{lm}((\tilde{\tilde{\lambda}}+\sigma)_r(\lambda+\sigma)^j_L;(\lambda'+\sigma)_m)|\,d\tau d\sigma d(\tilde{\tilde{\lambda}})_r\,d(\lambda')_M\\ \leqq [\![\zeta_{jk}]\!]^-&\,[\![\zeta_{lm}]\!]^-,\end{aligned}$$

so that

$$[\![Z_{LM}]\!]^- \leqq [\![\zeta_{jk}]\!]^- [\![\zeta_{lm}]\!]^-.$$

Similarly, one finds

$$[\![Z_{LM}]\!]^+ \leqq [\![\zeta_{jk}]\!]^+ [\![\zeta_{lm}]\!]^+.$$

The same relations hold for $Q_{jk} \overset{r}{\multimap} \Gamma Q_{lm}$ instead of $\Gamma Q_{jk} \overset{r}{\multimap} Q_{lm}$.

The properties of the classes $\mathfrak{Q}_{lm}$, for $l \neq 0$, $m \neq 0$, required in §11 are thus verified for the class described in the present section.

We said in §11 that we should like to introduce a linear class $\mathfrak{Q}$ of operators of the form $Q = \sum_{lm} Q_{lm}$ in $\mathfrak{Q}_{lm}$, which are applicable in $\mathfrak{H}$ and to which the attached products $\Gamma P \multimap Q$ and $P \multimap \Gamma Q$ belong if P and Q belong to $\mathfrak{Q}'$. In addition, we should like to define a norm $[\![Q]\!]$ for the operators in $\mathfrak{Q}$ such that the inequalities $[\![\Gamma P \multimap Q]\!]$, $[\![P \multimap \Gamma Q]\!] \leqq [\![P]\!]\,[\![Q]\!]$ hold. But, as we said in §11, we are not able to do so. We now like to explain the reasons for this difficulty.

We recall that the attached product $P_{jk} \multimap \Gamma Q_{lm}$ is given as the sum

$$P_{jk} \multimap \Gamma Q_{lm} = \sum_{r>0} N_{jk}^r P_{jk} \overset{r}{\multimap} Q_{lm},$$

where $N_{jl}^r = j!\,l!(j-r)!\,r!(l-r)!$, as mentioned in §10. The coefficient N_{jl}^r will be very large if j or l are large. This fact indicates that no estimate of the desired type can be derived solely from the estimate $[\![P_{jk} \overset{r}{\multimap} Q_{lm}]\!]$, $[\![P_{jk} \overset{r}{\multimap} Q_{lm}]\!] \leqq [\![P]\!]\,[\![Q]\!]$ derived above. That this is so was to be expected in view of the remarks made at the end of §9. On the other hand these remarks still allow one to hope that inequalities of the desired type can be derived for pure fermion operators Q (or perhaps also for operators Q which are linear in boson operators $A^{\pm}$); but then, of course, the antisymmetry of the kernels would have to be exploited specifically.

In the absence of the desired norm inequality we may just as well satisfy the requirements imposed on the operators Q in the cheapest possible way. We subject these operators to a drastic restriction: They should be *finite sums* $Q = \sum_{lm} Q_{lm}$ of operators of class $\mathfrak{Q}_{lm}$. It is quite obvious that the class $\mathfrak{Q}$ so defined meets the requirements.

The statements we are going to make in the following will always refer to this class $\mathfrak{Q}$; but in verifying these statements it will be sufficient to assume that Q is a single operator Q_{lm} of a class $\mathfrak{Q}_{lm}$. In doing so we restrict ourselves to $l > 0$ and $m > 0$.

The first of these statements is that the Wick power $:(\Gamma Q)^{\nu}:$ of an operator in $\Gamma\mathfrak{Q}'$ is always applicable in $\ddot{\mathfrak{H}}$. For operators $Q = Q_{lm}$ with $m > 0$ this follows by employing the same argument by which

this was proved for $\nu = 1$, i.e., for ΓQ_{lm} itself; at the same time it follows that $:(\Gamma Q_{lm})^{\nu}:$ is in $\ddot{\mathfrak{H}}$.

As in §17 we find it convenient to introduce operators $Q = Q_{x\cdot}$ without components Q_{l0} and those, Q_{x0}, that have only such components Q_0 and, hence, are pure creation operators.

It is important to note that for each Φ in $\ddot{\mathfrak{H}}$ the number ν can be chosen so large that $:(\Gamma Q_{x\cdot})^{\nu}:\Phi = 0$; for, if n_Φ is so chosen that the components ψ_n of Φ vanish for $n > n_\Phi$, we need only take $\nu > n_\Phi$ since each term in $:(\Gamma Q_{x\cdot})^{\nu}:$ involves at least ν annihilation prongs. It is an immediate consequence of this remark that the power series $1 + \sum_\nu :\nu!(\Gamma Q_{x\cdot})^{\nu}:$ defining the Wick exponential function $:e^{\Gamma Q_{x\cdot}}:$ may be broken off after the term of order n_Φ when applied to a vector Φ in $\ddot{\mathfrak{H}}$. It follows that *the Wick exponential function* $:e^{\Gamma Q_{x\cdot}}:$ *for* $Q_{x\cdot}$ *in* $\mathfrak{Q}'$ *is applicable in* $\ddot{\mathfrak{H}}$ *and produces vectors in* $\ddot{\mathfrak{H}}$.

Finally, we must concern ourselves with connected products such as $:(\Gamma P)^{\nu}: \succ Q$. This product will be different from zero for at most a finite number of exponents ν; for, the operator P, being a finite sum $\sum_{lk} P_{lk}$, has only a finite number of annihilation prongs. It follows that only a finite portion of the exponential series contributes to the connected products $:e^{\Gamma P}: \succ Q$. Similar statements hold for the forward connected products. Consequently, *the connected products* $:e^{\Gamma P} \succ Q$ *and* $P \succ :e^{\Gamma Q}:$ *are operators of the class* $\mathfrak{Q}$ *provided* P *and* Q *belong to* $\mathfrak{Q}'$.

Also the validity of our two basic formulas of §12 is readily established by virtue of the restricted character of our present class $\mathfrak{Q}$.

We mention that all these statements could be established also for $l = 0$ and for $m = 0$; but special considerations would be necessary to do this.

The final conclusion which one may derive from all these statements is that the iteration steps for our basic equations can be carried out provided the disturbing operator V belongs to the class $\mathfrak{Q}$ as defined in this section, and that the solutions of these basic equations, if they exist, lead to operators $T^{\pm}$ which intertwine with H and H_0.

The question of convergence of the iteration series and existence of the solutions still remains open.

A18. Statements about scattering for smooth interaction. In §15 we have made a number of statements concerning the behavior of the operators $U^{\pm}(t)$ as $t \to \pm\infty$. We shall now derive these statements from a few basic assumptions on the operators Q, which can be verified for those of the class $\mathfrak{Q}$ described in Appendix A17.

Whenever in this section reference to a particle representation is made, the B-particles are meant. We shall consider operators, given as functionals of the $B^{\pm}$, defined in the space $\ddot{\mathfrak{H}} = \ddot{\mathfrak{H}}_B$ of all vectors Φ in $\mathfrak{H}$ which have only a finite number of components $\Phi_n \neq 0$ with respect to the $B^{\pm}$-particle representation (see §14) and whose representers $\phi_n(\omega_1, \cdots, \omega_n)$ have bounded support. From each such operator F we form the operator-valued function

$$F(t) = e^{itH} F e^{-itH}.$$

We shall use a somewhat restricted notion of convergence for these operators inasmuch as the relation

$$F(t) \to F_{\pm} \quad \text{as } t \to \pm\infty$$

is to mean that the relation

$$F(t)\Phi \to F_{\pm}\Phi \quad \text{as } t \to \pm\infty$$

should hold for each Φ in $\ddot{\mathfrak{H}}$. This kind of convergence is definitely stronger than weak convergence inasmuch as cloud factors do not converge in this sense although they do converge weakly, as will be shown later.

In the following we shall consider the operators Q_{lm} taken from the class $\mathfrak{Q}$ defined in A17; they are defined in $\ddot{\mathfrak{H}}$, and admit the operator Γ. We recall that the operators ΓQ_{lm} are bounded in each subspace $\ddot{\mathfrak{H}}_n$. The same is true of the operator $\Gamma Q_{lm}(t) = e^{itH}\Gamma Q_{lm} e^{-itH}$.

We also employ the operation Γ_{∞}, which was introduced in §7 for $l = m = 1$, by the formula

$$\Gamma_{\infty} Q_{lm} \quad \Leftrightarrow \quad 2\pi i \delta[\omega - \omega']_{lm} q_{lm}((\omega)_l; (\omega'_m)),$$

where q_{lm} is the kernel of Q_{lm}. If either $m = 0$ or $l = 0$ we set $\Gamma_{\infty} = 0$, i.e.,

$$\Gamma_{\infty} Q_{l0} = 0 \quad \text{and} \quad \Gamma_{\infty} Q_{0m} = 0,$$

which is consistent with $\delta[\omega]_l = 0$ and $\delta[-\omega']_m = 0$. Furthermore, we shall use the symbol $\Gamma_{-\infty}$ to mean $\Gamma_{-\infty} = 0$, so that we can combine limit statements for $t \to \infty$ and $t \to -\infty$ in one formula.

The first of these statements refers to the limit of $\Gamma Q(t)$ with Q in $\mathfrak{Q}$. This operator will have no limit if pure creation terms are involved in it. Therefore we shall employ the decompositions

$$G = G_{x0} + G_{x\cdot} \quad \text{and} \quad G_{x0} = G_{\cdot 0} + G_{00}$$

of any operator of the form $G = \sum_{lm} G_{lm}$, where

$$G_{\cdot 0} = \sum_{l>0} G_{l0}, \qquad G_{x\cdot} = \sum_{m>0,\, l} G_{lm}.$$

Our first limit statement then reads

$$\Gamma Q_{x\bullet}(t) \to \Gamma_{\pm\infty} Q_{x\bullet} \quad \text{as } t \to \pm\infty. \tag{1}$$

It includes the statement $\Gamma Q_{\cdot\cdot}(t) \to \Gamma_{\pm\infty} Q_{\cdot\cdot}$, which can be derived by the same argument by which the formula $\Gamma Q_{11}(t) \to \Gamma_{\pm\infty} Q_{11}$ was derived in Appendix II A8. The remaining part of the statement, $\Gamma Q_{0\cdot}(t) \to 0$, follows from the Riemann-Lebesgue Lemma since $[-\omega']_m \leqq -m\omega_* < 0$. The nonconvergence of the pure creation operators $\Gamma Q_{\cdot 0}(t)$ is due to the fact that application of such an operator on Φ does not involve any integration, so that the exponential factor $e^{it\Sigma'\omega}$ of the kernel of $\Gamma Q_l(t)$ is not smoothed out by integration. Note that, nevertheless, such pure creation operators converge weakly to zero.

The next statement, involving two operators P and Q from $\mathfrak{Q}$ reads

$$(\Gamma P(t) \multimap \Gamma P(t))_{x\bullet} - (\Gamma_{\pm\infty} P \multimap \Gamma_{\pm\infty} Q)_{x\bullet} \quad \text{as } t \to \pm\infty; \tag{2}$$

it follows immediately from (1) since the operators ΓP and ΓQ are bounded in each $\mathfrak{H}_n$.

Next we turn to studying the Wick exponential function $:e^{\Gamma Q(t)}:$ and formulate the statements

$$:e^{\Gamma Q_{x\bullet}(t)}: \to :e^{\Gamma_{\pm\infty} Q_{x\bullet}}: \quad \text{as } t \to \pm\infty, \tag{3}$$

$$(:e^{\Gamma P(t)}: \succ e^{\Gamma Q_{\cdot 0}(t)})_{x\bullet} \to 0 \quad \text{as } t \to \pm\infty. \tag{4}$$

In applying $:e^{\Gamma Q_{x\bullet}(t)}:$ on a vector in $\ddot{\mathfrak{H}}$, only a finite portion of the exponential series will be effective since Q and P are taken from the class $\mathfrak{Q}$ of Appendix A17. Consequently, (3) follows immediately from (1). As to (4) we observe that we may replace P by $P_{x\bullet}$ since P_{x0} does not contract on the right. Hence only a finite portion of the series for $e^{\Gamma Q_{\cdot 0}}$ will contract. The relation then follows from (2) since $\Gamma_{\pm\infty} Q_{\cdot 0} = 0$.

The statements made allow us to investigate the asymptotic behavior of the operator

$$T^{\pm}(t) = :e^{\pm\Gamma Q^{\pm}(t)}: = :e^{\pm\Gamma Q^{\pm}_{\cdot 0}(t)} : e^{\pm\Gamma Q^{\pm}_{x\bullet}(t)}:.$$

Certainly, it does not converge to a limit operator unless the "cloud factor" $e^{\pm\Gamma Q^{\pm}_{\cdot 0}(t)}$ is absent. When this factor is removed, the resulting operator does indeed converge, as follows from (3):

$$:e^{\pm\Gamma Q^{\pm}_{x\bullet}(t)}: \to S^{\pm}_{\pm\infty} \quad \text{as } t \to \pm\infty,$$

where

$$S^{\pm}_{-\infty} = 1 \quad \text{and} \quad S^{\pm}_{+\infty} = S^{\pm},$$

with

$$S^{\pm} = :e^{\pm rQ_{\cdot\cdot}^{\pm}}:.$$

Our next aim is to evaluate the product $S^{+}S^{-}$. In doing this we shall make use of the identity

$$T^{+}T^{-} = (T^{+}T^{-})_{\infty} = \tau^{2}, \tag{5}$$

derived in §13. The validity of this relation implies that the pure creation contributions from the cloud terms of T^{+} and T^{-} cancel. We want to exploit this fact.

Evidently, we may write the last relation in the form $(T^{+}T^{-})_{x0} = \tau^{2}$ or, since $T_{x\cdot}^{-}$ does not contribute to this term, in the form

$$\tau^{2} = (T^{+}e^{-rQ_{\cdot 0}^{-}})_{x0} = (:e^{rQ^{+}}\{:e^{rQ^{+}}:\succ e^{-rQ_{\cdot 0}^{-}}\}:)_{x0},$$

by virtue of our second basic formula of §11. Since of the first exponential function on the right only the factor $e^{rQ_{\cdot 0}^{+}}$ contributes to the $x0$-term, we may write the result as the formula

$$e^{rQ_{\cdot 0}^{+}(t)}(:e^{rQ^{+}(t)}:\succ e^{-rQ_{\cdot 0}(t)})_{x0} = \tau^{2}$$

valid for every value of t. As a consequence of this formula and (4) we shall derive

LEMMA. *For operators $Q^{\pm}$ of our class $\mathfrak{Q}$ the relation*

$$T^{+}(t)e^{-rQ_{\cdot 0}^{-}(t)} \to \tau^{2}S_{\pm\infty}^{+}$$

and

$$T^{+}(t)e^{-rQ_{\cdot 0}^{-}(t)}:e^{-rQ_{x\cdot}^{-}(t)}:\tau^{2}S_{\pm\infty}^{\tau}S_{\pm\infty}^{-}$$

hold as $t \to \pm\infty$.

To prove this statement we use our second basic formula of §11 and write

$$\begin{aligned} T^{+}(t)e^{-rQ_{\cdot 0}^{-}(t)} = {} & :e^{rQ^{+}(t)}\{:e^{rQ^{+}(t)}:\succ e^{-rQ_{\cdot 0}^{-}(t)}\}_{x\cdot}: \\ & + :e^{rQ^{+}(t)}\{:e^{rQ^{+}(t)}:\succ e^{-rQ_{\cdot 0}^{-}(t)}\}_{x0}:, \end{aligned}$$

where we have split the expression in the braces into a $(x\cdot)$- and a $(x0)$-term, we replace the factor $e^{rQ^{+}(t)}$ in the second term by $e^{rQ_{x\cdot}^{+}(t)}e^{rQ_{\cdot 0}^{+}(t)}$ and use the formula derived just before formulating the lemma. The result is the formula

$$\begin{aligned} T^{+}(t)e^{-rQ_{\cdot 0}^{-}(t)} = {} & :e^{rQ^{+}(t)}\{:e^{rQ^{+}(t)}:\succ e^{-rQ_{\cdot 0}^{-}(t)}\}_{x\cdot}: \\ & + :e^{rQ_{x\cdot}^{+}(t)}:\tau^{2}. \end{aligned}$$

In the first term on the right the expression in the braces tends to zero by virtue of (4) while of the first factor in this term only a finite portion is effective in each $\ddot{\mathfrak{H}}_n$, therefore, this whole first term tends to zero in each $\ddot{\mathfrak{H}}_n$ and hence for each Φ in $\ddot{\mathfrak{H}}$.

Combining this fact with the definition of $S^+_{\pm\infty}$ we obtain the first statement of the lemma. The second statement follows at the same time from the definition of $S^-_{\pm\infty}$ since the operator on its left side is a pure $(x\cdot)$-operator and has no cloud terms.

Writing the relation $T^+(t)T^-(t) = \tau^2$ in the form $T^+(t)e^{-\Gamma Q^-_{\cdot 0}(t)} \cdot :e^{-\Gamma Q^-_{x\cdot}(t)}: = \tau^2$ we are thus led to the statement of

THEOREM 1. *For a pair of operators $Q^\pm$ in $\mathfrak{Q}$ for which* (1) *to* (5) *are valid, the relation*

$$S^+S^- = 1$$

holds.

The same argument leads to the

COROLLARY. *The inverse relation*

$$S^-S^+ = 1$$

holds provided the relation $T^-T^+ = \tau^2$ holds.

Next we like to prove the statements made in §15 concerning the limits of the *adjusted operators*

$$\tilde{A}^\pm(t) = \tau^{-2}T^+(t)B^\pm T^-(t).$$

These operators are distributions as functions of ω; limit statements involving operators $\tilde{A}^\pm$ and $B^\pm$ will always be meant as limit statements concerning the proper operators $\tilde{A}^\pm \cdot f = \int \tilde{A}^\pm(\omega)f(\omega)\,d\omega$ and $B^\pm \cdot f = \int B^\pm(\omega)f(\omega)\,d\omega$ for arbitrary square integrable test functions f. The operators $B^\pm \cdot f$ are not bounded. Still the operators $\tilde{A}^\pm \cdot f$ are well defined and applicable in $\ddot{\mathfrak{H}}$; for, this is the case of the contractions of $B^\pm \cdot f$ with $T^\pm$, respectively, as well as of the Wick products $:B^\pm \cdot fT^\pm:$.

Without effort we now shall be able to prove

THEOREM 2. *For a pair of operators $Q^\pm$ in $\mathfrak{Q}$ for which* (1) *to* (5) *are valid the relations*

$$\tilde{A}^\pm(t) \to S^+_{\pm\infty}B^\pm S^-_{\pm\infty} \quad \text{as } t \to \pm\infty$$

hold.

Since the factor $e^{-\Gamma Q^-_{\cdot 0}}$ involves only creation operators, it commutes with B^+; hence, we may write $\tilde{A}^+(t)$ in the form

$$\tilde{A}^+(t) = \tau^{-2}T^+(t)e^{-\Gamma Q^-_{\cdot 0}(t)}B^+ :e^{-\Gamma Q^-_{x\cdot}(t)}:.$$

The lemma, together with the definition of $S^{\pm}$ now, yields immediately the relation

$$\tilde{A}^{+}(t) \to S^{+}_{\pm\infty} B^{+} S^{-}_{\pm\infty} \quad \text{as } t \to \pm\infty.$$

A minor adjustment is needed to handle the operator $\tilde{A}^{-}(t)$. Clearly, we have

$$B^{-} e^{-\Gamma Q^{-}_{\cdot 0}(t)} = e^{-\Gamma Q^{-}_{\cdot 0}(t)} B^{-} - B^{-} \multimap Q^{-}_{\cdot 0}(t),$$

but the term $B^{-} \cdot f \multimap \Gamma Q_{\cdot 0}(t)$ tends to zero by virtue of the Riemann-Lebesgue Lemma because of the presence of exponential factor $e^{it\omega}$ associated with the contracted creation prong of $\Gamma Q^{-}_{\cdot 0}(t)$. Therefore, we may conclude

$$\tilde{A}^{-}(t) \to S^{+}_{\pm\infty} B^{-} S^{-}_{\pm\infty} \quad \text{as } t \to \pm\infty.$$

The statements made in this section, of course, have a very limited scope since they were derived only for operators Q of the very restricted class $\mathfrak{Q}$ introduced in Appendix A17. They may serve as a model for what one may hope could be proved under less restrictive circumstances.

A19. Inverse relations and scattering statements for conservation interaction. In §17 we introduced the transformation operators $T^{\pm} = :e^{\pm\Gamma Q^{\pm}}:$ which intertwine with the operators H_0 and $H = H_0 + V$ and we stated that the product $T^{+}T^{-} = \tau^2$ is an operator of the form

$$\tau^2 = e^{2\Theta_{11}},$$

where the operator Θ_{11} is given as

$$\Theta_{11} = \int A^{+}(s) \cdot \theta(s) A^{-}(s)\, ds,$$

the function $\theta(s)$ being characterized by the condition that the function $\zeta(s) = e^{2\theta(s)} - 1$ is the kernel of the operator

$$\int A^{+}(s)\zeta(s)A^{-}(s)\, ds = -\Gamma Q^{+}_{1\cdot} \times \Gamma Q^{-}_{\cdot 1}.$$

The product $F \times G$ of two operators F and G, referred to as "completely connected", is to consist of all those contributions to the product FG in which all annihilation prongs of F and all creation prongs of G are contracted.

We proceed to give a (purely formal) derivation of this statement.

In evaluating the product $T^{+}T^{-}$ in the case of totally smooth operators in §13 we employed the basic identity $\Gamma\{P\Gamma Q + (\Gamma P)Q\} = \Gamma P \Gamma Q - (\Gamma P \Gamma Q)_{00}$ and its extension to the operators

$$R^{\pm} = :Q^{\pm}e^{\pm\Gamma Q^{\pm}}: \quad \text{and} \quad \Gamma R^{\pm} = :e^{\pm\Gamma Q^{\pm}}: - 1$$

in place of Q, P and ΓQ, ΓP. To evaluate the product T^+T^- in the present case we need the analogue of this extension.

Note that the nth term of the operators $R^\pm$ has a kernel which carries n delta functions and not just one, as does the kernel of the operator Q. This fact must be taken into account in establishing the desired analogue.

We start by determining those contributions to expressions $R^+\Gamma R^- + (\Gamma R^+)R^-$ that do not admit the operation Γ. These are the 11-terms and the Wick products of 11-terms.

First of all, the products $Q^+_{1\cdot}\Gamma Q^-_{\cdot 1}$ and $(\Gamma Q^+_{1\cdot})Q^-_{\cdot 1}$ contain 11-contributions, namely, those contributions in which all annihilation prongs of $Q^+_{1\cdot}$ and all creation prongs of $Q^-_{\cdot 1}$ are contracted. They will be referred to as the "completely connected" products

$$(Q^+_{1\cdot}\Gamma Q^-_{\cdot 1})_{11} = Q^+_{1\cdot} \times \Gamma Q^-_{\cdot 1}, \qquad ((\Gamma Q^+_{1\cdot})Q^-_{\cdot 1})_{11} = \Gamma Q^+_{1\cdot} \times Q^-_{\cdot 1}.$$

Since the kernels of these products carry the factor $\delta(s_1 - s_1')$, the operation Γ is not applicable. In addition, the terms

$$:nQ^+_{1\cdot}(\Gamma Q^+_{1\cdot})^{n-1}: :(\Gamma Q^-_{\cdot 1})^n: \quad \text{and} \quad :(\Gamma Q^+_{1\cdot})^n: :nQ^-_{\cdot 1}(\Gamma Q^-_{\cdot 1})^{n-1}:$$

contain contributions on which Γ is not applicable, namely, those in which each of the n factors of $nQ^+_{1\cdot}(\Gamma Q^+_{1\cdot})^{n-1}$ is completely contracted with one of the n factors of $(\Gamma Q^-_{\cdot 1})^n$ and those in which each of the n factors of $(\Gamma Q^+_{1\cdot})^n$ is completely contracted with one of the n factors of $nQ^-_{\cdot 1}(\Gamma Q^-_{\cdot 1})^{n-1}$. For, the kernels of these contributions carry the factor $\delta(s_1 - s_1') \cdots \delta(s_n - s_n')$. We use the symbol $\times$ to denote also these contributions and observe the identities

$$:nQ^+_{1\cdot}(\Gamma Q^+_{1\cdot})^{n-1}: \times :(\Gamma Q^-_{\cdot 1})^n: = n!:(nQ^+_{1\cdot} \times \Gamma Q^-_{\cdot 1})(\Gamma Q^+_{1\cdot} \times \Gamma Q^-_{\cdot 1})^{n-1}:$$

and

$$:(\Gamma Q^+_{1\cdot})^n: \times :nQ^-_{\cdot 1}(\Gamma Q^-_{\cdot 1})^{n-1}: = n!:(\Gamma Q^+_{1\cdot} \times nQ^-_{\cdot 1})(\Gamma Q^+_{1\cdot} \times Q^-_{\cdot 1})^{n-1}:.$$

Now, in analogy with an earlier identity we have

$$nQ^+_{1\cdot} \times Q^-_{\cdot 1} + \Gamma Q^+_{1\cdot} \times nQ^-_{\cdot 1} = 0; \tag{11}$$

for, the Γ-factors,

$$[\tilde{\omega} - \omega']^{-1}_{\cdot 1} \quad \text{and} \quad [\omega - \tilde{\omega}]^{-1}_{1\cdot},$$

involved in these two products cancel since $\omega_1' = \omega_1$ because of the presence of the factor $\delta(s_1 - s_1')$ and since $\delta[\omega - \omega']_{1\cdot} = 0$ by virtue of the subadditivity of the function $\omega(s)$ stipulated in §17. From

The lemma, together with the definition of $S^{\pm}$ now, yields immediately the relation

$$\tilde{A}^{+}(t) \to S^{+}_{\pm\infty} B^{+} S^{-}_{\pm\infty} \quad \text{as } t \to \pm\infty.$$

A minor adjustment is needed to handle the operator $\tilde{A}^{-}(t)$. Clearly, we have

$$B^{-} e^{-\Gamma Q^{-}_{\cdot 0}(t)} = e^{-\Gamma Q^{-}_{\cdot 0}(t)} B^{-} - B^{-} \multimap Q^{-}_{\cdot 0}(t),$$

but the term $B^{-} \cdot f \multimap \Gamma Q_{\cdot 0}(t)$ tends to zero by virtue of the Riemann-Lebesgue Lemma because of the presence of exponential factor $e^{it\omega}$ associated with the contracted creation prong of $\Gamma Q^{-}_{\cdot 0}(t)$. Therefore, we may conclude

$$\tilde{A}^{-}(t) \to S^{+}_{\pm\infty} B^{-} S^{-}_{\pm\infty} \quad \text{as } t \to \pm\infty.$$

The statements made in this section, of course, have a very limited scope since they were derived only for operators Q of the very restricted class $\mathfrak{Q}$ introduced in Appendix A17. They may serve as a model for what one may hope could be proved under less restrictive circumstances.

A19. Inverse relations and scattering statements for conservation interaction. In §17 we introduced the transformation operators $T^{\pm} = :e^{\pm\Gamma Q^{\pm}}:$ which intertwine with the operators H_0 and $H = H_0 + V$ and we stated that the product $T^{+}T^{-} = \tau^2$ is an operator of the form

$$\tau^2 = e^{2\Theta_{11}},$$

where the operator Θ_{11} is given as

$$\Theta_{11} = \int A^{+}(s) \cdot \theta(s) A^{-}(s)\, ds,$$

the function $\theta(s)$ being characterized by the condition that the function $\zeta(s) = e^{2\theta(s)} - 1$ is the kernel of the operator

$$\int A^{+}(s)\zeta(s)A^{-}(s)\, ds = -\Gamma Q^{+}_{1\cdot} \times \Gamma Q^{-}_{\cdot 1}.$$

The product $F \times G$ of two operators F and G, referred to as "completely connected", is to consist of all those contributions to the product FG in which all annihilation prongs of F and all creation prongs of G are contracted.

We proceed to give a (purely formal) derivation of this statement.

In evaluating the product $T^{+}T^{-}$ in the case of totally smooth operators in §13 we employed the basic identity $\Gamma\{P\Gamma Q + (\Gamma P)Q\} = \Gamma P \Gamma Q - (\Gamma P \Gamma Q)_{00}$ and its extension to the operators

$$R^{\pm} = :Q^{\pm} e^{\pm\Gamma Q^{\pm}}: \quad \text{and} \quad \Gamma R^{\pm} = :e^{\pm\Gamma Q^{\pm}}: - 1$$

in place of Q, P and ΓQ, ΓP. To evaluate the product T^+T^- in the present case we need the analogue of this extension.

Note that the nth term of the operators $R^{\pm}$ has a kernel which carries n delta functions and not just one, as does the kernel of the operator Q. This fact must be taken into account in establishing the desired analogue.

We start by determining those contributions to expressions $R^+\Gamma R^- + (\Gamma R^+)R^-$ that do not admit the operation Γ. These are the 11-terms and the Wick products of 11-terms.

First of all, the products $Q^+_{1\cdot}\Gamma Q^-_{\cdot 1}$ and $(\Gamma Q^+_{1\cdot})Q^-_{\cdot 1}$ contain 11-contributions, namely, those contributions in which all annihilation prongs of $Q^+_{1\cdot}$ and all creation prongs of $Q^-_{\cdot 1}$ are contracted. They will be referred to as the "completely connected" products

$$(Q^+_{1\cdot}\Gamma Q^-_{\cdot 1})_{11} = Q^+_{1\cdot} \times \Gamma Q^-_{\cdot 1}, \qquad ((\Gamma Q^+_{1\cdot})Q^-_{\cdot 1})_{11} = \Gamma Q^+_{1\cdot} \times Q^-_{\cdot 1}.$$

Since the kernels of these products carry the factor $\delta(s_1 - s'_1)$, the operation Γ is not applicable. In addition, the terms

$$:nQ^+_{1\cdot}(\Gamma Q^+_{1\cdot})^{n-1}: :(\Gamma Q^-_{\cdot 1})^n: \quad \text{and} \quad :(\Gamma Q^+_{1\cdot})^n: :nQ^-_{\cdot 1}(\Gamma Q^-_{\cdot 1})^{n-1}:$$

contain contributions on which Γ is not applicable, namely, those in which each of the n factors of $nQ^+_{1\cdot}(\Gamma Q^+_{1\cdot})^{n-1}$ is completely contracted with one of the n factors of $(\Gamma Q^-_{\cdot 1})^n$ and those in which each of the n factors of $(\Gamma Q^+_{1\cdot})^n$ is completely contracted with one of the n factors of $nQ^-_{\cdot 1}(\Gamma Q^-_{\cdot 1})^{n-1}$. For, the kernels of these contributions carry the factor $\delta(s_1 - s'_1) \cdots \delta(s_n - s'_n)$. We use the symbol $\times$ to denote also these contributions and observe the identities

$$:nQ^+_{1\cdot}(\Gamma Q^+_{1\cdot})^{n-1}: \times :(\Gamma Q^-_{\cdot 1})^n: = n!:(nQ^+_{1\cdot} \times \Gamma Q^-_{\cdot 1})(\Gamma Q^+_{1\cdot} \times \Gamma Q^-_{\cdot 1})^{n-1}:$$

and

$$:(\Gamma Q^+_{1\cdot})^n: \times :nQ^-_{\cdot 1}(\Gamma Q^-_{\cdot 1})^{n-1}: = n!:(\Gamma Q^+_{1\cdot} \times nQ^-_{\cdot 1})(\Gamma Q^+_{1\cdot} \times Q^-_{\cdot 1})^{n-1}:.$$

Now, in analogy with an earlier identity we have

$$nQ^+_{1\cdot} \times Q^-_{\cdot 1} + \Gamma Q^+_{1\cdot} \times nQ^-_{\cdot 1} = 0; \tag{11}$$

for, the Γ-factors,

$$[\tilde{\omega} - \omega']^{-1}_{\cdot 1} \quad \text{and} \quad [\omega - \tilde{\omega}]^{-1}_{1\cdot},$$

involved in these two products cancel since $\omega'_1 = \omega_1$ because of the presence of the factor $\delta(s_1 - s'_1)$ and since $\delta[\omega - \omega']_{1\cdot} = 0$ by virtue of the subadditivity of the function $\omega(s)$ stipulated in §17. From

relation (11), together with the identities preceding it, we conclude that more generally the relation

$$:nQ_{1\cdot}^{+}(\Gamma Q_{1\cdot}^{+})^{n+1}: :(\Gamma Q_{\cdot 1}^{-})^{n}: + :(\Gamma Q_{1\cdot}^{+})^{n}: :nQ_{\cdot 1}^{-}(\Gamma Q_{\cdot 1}^{-})^{n-1}: = 0$$

holds, which clearly implies the relation

$$R_{1\cdot}^{+} \times \Gamma R_{\cdot 1}^{-} + (\Gamma R_{1\cdot}^{+}) \times R_{\cdot 1}^{-} = 0,$$

in which $R_{1\cdot}^{+}$ and $R_{\cdot 1}^{-}$, somewhat improperly, stand for

$$R_{1\cdot}^{+} = :Q_{1\cdot}^{+}e^{\Gamma Q_{1\cdot}^{+}}: \quad \text{and} \quad R_{\cdot 1}^{-} = :Q_{\cdot 1}^{-}e^{-\Gamma Q_{\cdot 1}^{-}}:.$$

In other words, those contributions to $R^{+}\Gamma R^{-} + (\Gamma R^{+})R^{-}$ on which the operation Γ is not applicable drop out. Subtracting the contributions to $\Gamma R^{+}\Gamma R^{-}$ which are contracted in the same way we find

$$\Gamma\{R^{+}\Gamma R^{-} + (\Gamma R^{+})R^{-}\} = \Gamma R^{+}\Gamma R^{-} - \Gamma R_{1\cdot}^{+} \times \Gamma R_{\cdot 1}^{-},$$

in analogy with the corresponding formula given for smooth interaction in §13.

The product $T^{+}T^{-}$ can now be written in the form

$$\begin{aligned} T^{+}T^{-} &= (1 + \Gamma R^{+})(1 - \Gamma R^{-}) \\ &= 1 + \Gamma(R^{+} - R^{-}) - \Gamma R^{+}\Gamma R^{-} \\ &= 1 + \Gamma\{R^{+} - R^{-} - R^{+}\Gamma R^{-} - (\Gamma R^{+})R^{-}\} - \Gamma R_{1\cdot}^{+} \times \Gamma R_{\cdot 1}^{-}. \end{aligned}$$

From the relations

$$R^{+} = (1 + \Gamma R^{+})(V'' + V_{11}^{+}) = T^{+}(V'' + V_{11}^{+})$$

and

$$R^{-} = (V'' + V_{11}^{-})(1 - \Gamma R^{-}) = (V'' + V_{11}^{-})T^{-}$$

we infer that the expression in the braces equals

$$T^{+}(V_{11}^{+} - V_{11}^{-})T^{-};$$

since we know that the 11-component of this expression vanishes, we conclude

$$(T^{+}(V_{11}^{+} - V_{11}^{-})T^{-})_{11} = 0.$$

This relation may be regarded as a linear homogeneous equation for the 11-operator $V_{11}^{+} - V_{11}^{-}$ and may be written in the form

$$V_{11}^{+} - V_{11}^{-} = L(V_{11}^{+} - V_{11}^{-}),$$

where the linear operator L is small (in some sense) if V is sufficiently small. We now adopt the assumption that V is so small that the

homogeneous relation above is only valid for $V_{11}^{+} - V_{11}^{-} = 0$. We then may set

$$V_{11}^{+} = V_{11}^{-} = V_{11}.$$

At the same time we may conclude that the expression in the braces vanishes. Thus we find that $T^{+}T^{-}$ is given by

$$T^{+}T^{-} = 1 + \Gamma R_{1\cdot}^{+} \times \Gamma R_{\cdot 1}^{-}.$$

Exploiting the definition of the notion of a completely connected product we find

$$\begin{aligned}\Gamma R_{1\cdot}^{+} \times \Gamma R_{\cdot 1}^{-} &= :e^{\Gamma Q_{1\cdot}^{+}} - 1: \times :e^{\Gamma Q_{\cdot 1}^{-}} - 1: \\ &= \sum_{n=1} (n!)^2 :(\Gamma Q_{1\cdot}^{+})^n: \times :(\Gamma Q_{\cdot 1}^{-})^n: \\ &= \sum_{n=1} n! :(\Gamma Q_{1\cdot}^{+} \times \Gamma Q_{\cdot 1}^{-})^n: \\ &= :e^{\Gamma Q_{1\cdot}^{+} \times \Gamma Q_{\cdot 1}^{-}} - 1:.\end{aligned}$$

Thus we obtain finally

$$T^{+}T^{-} = :e^{Z_{11}}:$$

with

$$Z_{11} = \Gamma Q_{1\cdot}^{+} \times \Gamma Q_{\cdot 1}^{-}.$$

By virtue of the presence of the factor $\delta(s_1 - s_1')$ in its kernel the operator Z_{11} is of the form

$$Z_{11} = \int A^{+}(s)\zeta(s)A^{-}(s)\, ds$$

with an appropriate function $\zeta(s)$ whose value for each s is a matrix acting on the silent variables. Assuming $|\zeta(s)|$ to be sufficiently small we may set

$$1 + \zeta(s) = e^{2\theta(s)}$$

and express the Wick exponential function $:e^{Z_{11}}:$ as a direct exponential function

$$:e^{Z_{11}}: = e^{2\Theta_{11}},$$

where the operator Θ_{11} is given by

$$\Theta_{11} = \int A^{+}(s)\theta(s)A^{-}(s)\, ds;$$

see, e.g., [41]

With

$$\tau = e^{\Theta_{11}}$$

we then have established the evaluation

$$T^+T^- = \tau^2,$$

referred to in §17.

Our next task is to determine, again in a purely formal way, the limit of the operator $T^+(t)T^-(t)$ as $t \to \pm\infty$. To this end we let Φ, $\Phi^{(1)}$ be two state vectors and consider the inner product

$$(\Phi, T^+(t)T^-(t)\Phi^{(1)}) = (\Phi, :e^{\Gamma Q_{\cdot 1}^+(t)}e^{\Gamma Q_{\cdot\cdot}^+(t)}e^{\Gamma Q_{1\cdot}^+(t)}: \\ :e^{-\Gamma Q_{\cdot 1}^-(t)}e^{-\Gamma Q_{\cdot\cdot}^-(t)}e^{-\Gamma Q_{1\cdot}^-(t)}:\Phi^{(1)}).$$

Because of the subadditivity of the function $\omega(s)$ (see §17), the terms $\Gamma Q_{\cdot 1}^+(t)$ and $\Gamma Q_{\cdot 1}^-(t)$ have no singular denominators. Since the dot stands for at least two contractions, their contributions tend to zero as $t \to \pm\infty$, by virtue of the Riemann-Lebesgue Lemma. Also the operators $:e^{\pm\Gamma Q_{\cdot\cdot}^{\pm}(t)}:$ tend to the limits $:e^{\pm\Gamma_{\pm\infty}Q_{\cdot\cdot}^{\pm}}:$. Consequently, the bilinear form $(\Phi, T^+(t)T^-(t)\Phi^{(1)})$ has the same limit as the form $(\Phi, :e^{\Gamma_{\pm\infty}Q_{\cdot\cdot}^+}e^{\Gamma Q_{1\cdot}^+(t)}: :e^{-\Gamma Q_{\cdot 1}^-(t)}e^{-\Gamma_{\pm\infty}Q_{\cdot\cdot}^-}:\Phi^{(1)})$. Evidently, all these contributions to this expression disappear in the limit which contains annihilation prongs from $Q_{1\cdot}^+(t)$ contracted with either $:e^{-\Gamma_{\pm\infty}Q^-}:$ or $\Phi^{(1)}$, or creation prongs from $Q_{\cdot 1}^-(t)$ contracted with either $:e^{\Gamma_{\pm\infty}Q^+}:$ or Φ. In other words, the only contributions that survive are those that form the completely contracted product of $:e^{\Gamma Q_{1\cdot}^+(t)}:$ and $:e^{-\Gamma Q_{\cdot 1}^-(t)}:$. As mentioned above, this product is given by

$$:e^{\Gamma Q_{1\cdot}^+(t)}: \times :e^{-\Gamma Q_{\cdot 1}^-(t)} = \tau^2(t).$$

Now, $\tau(t) = e^{\Theta_{11}(t)}$ and since $\Theta_{11}(t)$, as any 11-operator, commutes with H, it is independent of the time t. The same is therefore true of τ. We are thus led to the relation

$$(\Phi, T^+(t)T^-(t)\Phi^{(1)}) \to (\Phi, :e^{\Gamma_{\pm\infty}Q_{\cdot\cdot}^+}:\tau^2:e^{-\Gamma_{\pm\infty}Q_{\cdot\cdot}^-}:\Phi^{(1)}),$$

as $t \to \pm\infty$. Now since $T^+(t)T^-(t) = \tau^2(t) = \tau^2$, the last relation simply implies the relation

$$:e^{\Gamma_{\pm\infty}Q_{\cdot\cdot}^+}:\tau^2:e^{-\Gamma_{\pm\infty}Q_{\cdot\cdot}^-}: = \tau^2.$$

Using the definition of $S^{\pm}$ given in §17, namely,

$$S_{\pm\infty}^+ = \tau^{-1}:e^{\Gamma_{\pm\infty}Q_{\cdot\cdot}^+}:\tau, \qquad S_{\pm\infty}^- = \tau:e^{-\Gamma_{\pm\infty}Q_{\cdot\cdot}^-}:\tau^{-1}$$

we obtain the desired identity

$$S_{\pm\infty}^+S_{\pm\infty}^- = 1.$$

(We recall that $\Gamma_{-\infty} = 0$.)

The limits of the adjusted operators $A^{\pm}(s, t) = e^{\mp it\omega(s)}A^{\pm}(s, t)$, or rather of $(\Phi, \int A^{\pm}(s, t)f(s)\, ds\Phi'')$, are now easily determined. We have

$$\begin{aligned}\tau\tilde{A}^{-}(t)\tau &= T^{+}(t)B^{-}:e^{-\Gamma Q^{-}(t)}:\\ &= T^{+}(t):(B^{-} \prec :e^{-\Gamma Q^{-}(t)}:)e^{-\Gamma Q^{-}(t)}:\\ &= T^{+}(t):\{B^{-} - B^{-} \multimap \Gamma Q^{-}(t)\}e^{-\Gamma Q^{-}(t)}:.\end{aligned}$$

Now, from the Riemann-Lebesgue Lemma and the basic property of $\omega(s)$ we conclude that in effect $B^{-} \multimap \Gamma Q_{\bar{1}}^{-}(t) \to 0$ and $B^{-} \multimap \Gamma Q_{\bar{1}}^{-}(t) \to 0$.

Furthermore, we have $B^{-} \multimap \Gamma Q_{\bar{\infty}}^{-}(t) \to B^{-} \multimap \Gamma_{\pm\infty}Q_{\bar{\infty}}^{-}$. Hence, the weak limit of $T^{+}(t)B^{-}:e^{-\Gamma Q^{-}(t)}:$ is the same as that of $T^{+}(t):e^{-\Gamma Q_{\bar{1}}^{-}}(B^{-}:e^{-\Gamma_{\pm\infty}Q_{\bar{\infty}}^{-}}:):$. The same arguments that led to the evaluation of the limit of $T^{+}(t)T^{-}(t)$ now lead to the result

$$(\Phi, T^{+}(t):e^{-\Gamma Q_{\bar{1}}^{-}}(B^{-}:e^{-\Gamma_{\pm\infty}Q_{\bar{\infty}}^{-}}:\Phi^{(1)}) \to (\Phi, :e^{\Gamma_{\pm\infty}Q_{\bar{\infty}}^{+}}:\tau^{2}B^{-}:e^{\Gamma_{\pm\infty}Q_{\bar{\infty}}^{-}}:\Phi^{(1)}).$$

Using the definition of $S_{\pm\infty}^{\pm}$ we then find, in the weak sense,

$$\tilde{A}^{-}(t) \to S_{\pm\infty}^{+}\tau B^{-}\tau^{-1}S_{\pm\infty}^{-} \quad \text{as } t \to \pm\infty.$$

In order to evaluate the limit of $\tilde{A}^{+}(t) = T^{+}(t)B^{+}T^{-}(t)$ we must carry out the multiplication of B^{+} with the operator $T^{+}(t)$ in front of it. Repeating the preceding argument, with the roles of Φ and $\Phi^{(1)}$ interchanged, we find

$$\tilde{A}^{+}(t) \to S_{\pm\infty}^{+}\tau^{-1}B^{+}\tau S_{\pm\infty}^{-} \quad \text{as } t \to \pm\infty.$$

The operators

$$\tau^{\mp 1}B^{\pm}\tau^{\pm 1} = e^{\mp\Theta_{11}}B^{\pm}e^{\pm\Theta_{11}},$$

which occur in these formulas, can be given a simpler form. Using the expression

$$\Theta_{11} = \int B^{+}(s')\theta(s')B^{-}(s')\, ds'$$

for Θ_{11} together with the commutation laws for $B^{\pm}$ we find

$$\tau B^{-}(s)\tau^{-1} = e^{-\theta(s)}B^{-}(s)$$

and

$$\tau^{-1}B^{+}(s)\tau = B^{+}(s)e^{-\theta(s)}.$$

Note that for each value of s the factor $e^{-\theta(s)}$ is just a matrix acting on the silent accessory variables on which the $B^{\pm}(s)$ depend; therefore, it

commutes with the operators $S^{\pm}$. Consequently, we can write the limit relation for $\tilde{A}^{\pm} = \tilde{A}^{\pm}(s)$ in the form

$$\left.\begin{matrix}\tilde{A}^{+}(t)e^{0}\\ e^{0}\tilde{A}^{-}(t)\end{matrix}\right\} \to S^{+}_{\pm\infty}B^{\pm}S^{-}_{\pm\infty} \quad \text{as } t \to \pm\infty.$$

Thus we are led to the asymptotic formulas given in §17.

A20. The nondegenerate Lee model. To illustrate the treatment of the perturbation problem with conservation smooth interaction, given in §§17 and 18, we shall discuss a special such problem for which the most important ones of the contributions $Q^{\pm}_{lm}$ to the operators $Q^{\pm}$ can be given explicitly.

The state of a single particle will be described by the momentum k and a quantum number ι, subject to a conservation law, which may assume three values: α, β, and γ. These may be any numbers for which $\gamma = \alpha + \beta$. The single particle representer will be denoted by $\psi_{\iota}(k)$ in place of $\psi(k, \iota)$; similarly, the n-particle representer will be written as $\psi_{\iota_1\cdots\iota_n}(k_1, \cdots, k_n)$ in place of $\psi((k_1, \iota_1), \cdots, (k_n, \iota_n))$.

The interaction operator will be of the form $V' = V_{12} + V_{21}$ and the kernels of the operators are of the form

$$v_{\iota_1;\iota_1'\iota_2'}(k_1; k_1', k_2') = v^0_{\iota_1;\iota_1'\iota_2'}(k_1; k_1', k_2')\delta(k_1 - k_1' - k_2')\delta(\iota_1 - \iota_1' - \iota_2'),$$

$$v_{\iota_1\iota_2;\iota_1'}(k_1, k_2; k_1') = v^0_{\iota_1\iota_2;\iota_1'}(k_1, k_2; k_1')\delta(k_1 + k_2 - k_1')\delta(\iota_1 + \iota_2 - \iota_1').$$

Moreover, it will be assumed that the reduced kernels v^0 vanish unless either $(\iota_1', \iota_2') = (\alpha, \beta)$ or $= (\beta, \alpha)$, and $(\iota_1, \iota_2) = (\alpha, \beta)$ or $= (\beta, \alpha)$, respectively. Then the unreduced kernels will vanish unless $\iota_1 = \gamma$ and $\iota_1' = \gamma$, respectively, as follows from the presence of the ι-conservation factor and the assumed relation $\alpha + \beta = \gamma$.

The three values α, β, γ of ι may characterize three kinds of particles. We say a state Φ is one with a definite number of particles of definite kinds if all its components vanish except Φ_n and if the representer $\psi_{\iota_1\cdots\iota_n}$ of this component vanishes unless n_α, n_β, n_γ of the subscripts $\iota_1, \cdots, \iota_n$ have the value α, β, γ, respectively; clearly, $n_\alpha + n_\beta + n_\gamma = n$. The operator V_{12} then transforms such a particular state into one with one less α- and one less β-particle but one more γ-particle; the operator V_{21} has the reverse effect. To stay in agreement with the situation in the Lee model we assume the α-particle to be a boson, the β- and γ-particles to be fermions. This implies that the two functions v^0 should be taken symmetric in the variables (k_1', k_2') and (k_1, k_2).

For simplicity we set

$$v^0_{\gamma;\alpha\beta}(k'_1 + k'_2; k'_1, k'_2) = v_{12}(k'_1, k'_2),$$
$$v^0_{\alpha\beta;\gamma}(k_1, k_2; k_1 + k_2) = v_{21}(k_1, k_2).$$

In terms of the energies $\omega_\iota(k)$ of the three particles the Hamiltonian $H = H_0 + V_{11} + V'$ can be written in the form

$$H = \sum_\iota \int A^+_\iota(k)\{\omega_\iota(k) + v_\iota(k)\}A^-_\iota(k)\,dk$$
$$+ 2\int\int A^+_\gamma(k'_1 + k'_2)v_{12}(k'_1, k'_2)A^-_\alpha(k'_1)A^-_\beta(k'_2)\,dk'_1dk'_2$$
$$+ 2\int\int A^+_\alpha(k_1)A^+_\beta(k_2)v_{21}(k_1, k_2)A^-_\gamma(k_1 + k_2)\,dk_1dk_2.$$

Here, $\omega^0_\iota(s) = \omega_\iota(s) + v_\iota(s)$ is the "unrenormalized" energy of the ι-particle, so that the $v_\iota(k)$ are the three-energy shifts to be determined.

In fact, from the relations $V_{11} = -V' \prec :e^{-\Gamma Q^-}:$ and $V' = V_{12} + V_{21}$ we may conclude that $-V_{11}$ is given by

$$-V_{11} = V_{12} \times \Gamma Q_{21} - V_{12} \prec :\Gamma Q_{10}\Gamma Q_{11}:$$

and, hence, since $Q_{11} = 0$, by

$$-V_{11} = V_{12} \times \Gamma Q_{21}.$$

Here the symbol "$\times$" indicates that both prongs involved should be contracted. Since the front variable of the reduced kernel of V_{12} has $\iota = \gamma$ the same is true of that of V_{11}, so that $v_\alpha(k) = v_\beta(k) = 0$. In other words, only the γ-particle energy must be renormalized.

Because of the special form $V' = V_{12} + V_{21}$ of V' the basic equations for $Q^\pm$ simplify. That for Q^- becomes

$$Q^- = V' - V' \multimap \Gamma Q^- + \tfrac{1}{2}V' \prec :(\Gamma Q^-)^2: - V_{11} \multimap \Gamma Q' - V_{11}$$

and that for Q^+ is similar. Further simplifications will be explained later on.

The subsequent discussion will depend on the decisive fact that certain subspaces of the space $\mathfrak{H}$ of states, called "sectors" are invariant under the interaction. These sectors will be assigned to a pair of integers, A and B, and denoted by $\mathfrak{H}_{AB}$; they are composed of those vectors whose representers $\psi_{\iota_1\cdots\iota_n}(k_1, \cdots, k_n)$ vanish unless

$$n + n_\gamma = A, \qquad n_\alpha - n_\beta = B.$$

Clearly, application of the operator V_{12} raises the number n_γ of values γ among $\iota_1, \cdots, \iota_n$ by 1 but diminishes the number n by 1; V_{21} has the opposite effect; in any case, the number $n + n_\gamma$ remains invariant. The same is also true of the number $n_\alpha - n_\beta$ since n_α and n_β change by the same number, either 1 or -1. The sector $\mathfrak{H}_{20}$ will be referred to as the "main sector". Its states are linear combinations of those with just one α- and one β-particle and those with just one γ-particle; they are thus represented just by the two functions $\psi_{\alpha\beta}(k_1, k_2)$ and $\psi_\gamma(k_1)$, while all other components vanish.

Since the operator V transforms each sector into itself one could treat the perturbation problem by restricting the states to be in a particular sector. The resulting problems then belong to those treated in Chapter II. When acting in the main sector the disturbing operator turns out to be gentle; but this is not the case in the other sectors. We shall first treat the main sector independently and later on indicate how the other sectors can be handled.

The only contribution to Q^- which transform the main sector into itself are $Q^-_{\gamma;\alpha\beta}$, $Q^-_{\alpha\beta;\gamma}$, and $Q^-_{\alpha\beta;\alpha\beta}$ in obvious notation. For convenience we shall denote these operators by Q^-_{12}, Q^-_{21}, Q^-_{22}. Actually, the first two of these operators are the complete operators Q^-_{12} and Q^-_{21}, as we shall show later on, but the third one is actually only a part of Q_{22}.

From what was said earlier we know that $V_{11} = V_{\gamma\gamma}$. Moreover, we realize that the Wick product disappears from the simplified form of the basic equation if Q^- has only the three contributions just described. The 21-component of this equation is then seen to be

$$Q_{21} = V_{21},$$

so that V_{11} is the operator

$$V_{11} = V_{12} \times \Gamma V_{21}$$

with the reduced kernel

$$v_\gamma(k) = 2 \int v_{12}^{(+)}(\tilde{k}_1, k - \tilde{k}_1)[\omega_\alpha(\tilde{k}_1) + \omega_\beta(k - \tilde{k}_1) - \omega_\gamma(k)]^{-1} \cdot v_{21}(\tilde{k}_1, k - \tilde{k}_1)\, d\tilde{k}_1.$$

This then is the shift of the γ-particle energy, its "self-energy", and $\omega^0_\gamma(s) = \omega_\gamma(s) + v_\gamma(s)$ is its unrenormalized energy; see §18.

For the 12- and 22-components of Q we obtain the equations

$$Q^-_{12} = V_{12} - V_{11} \multimap \Gamma Q^-_{12} - V_{12} \times \Gamma Q^-_{22},$$
$$Q^-_{22} = - V_{21} \multimap \Gamma Q^-_{12} - V_{11} \multimap \Gamma Q^-_{22}.$$

These equations can be solved explicitly. The reduced kernels of Q_{12}^- and Q_{22}^- are found to be

$$q_{22}^-(k_1', k_2') = [1 + y_\gamma(k_1', k_2')]^{-1} v_{12}(k_1', k_2'),$$

$$q_{22;\alpha}^-(k_1, k_2; k_1') = -v_{21}(k_1, k_2)[\omega_\gamma(k_1' + k_2') - \omega_\alpha(k_1') - \omega_\beta(k_2')]^{-1} \cdot [1 + y_\gamma(k_1', k_2')]^{-1} v_{12}(k_1', k_2')$$

with $k_2' = k_1 + k_2 - k_1'$ and

$$y_\gamma(k_1', k_2') = \left\{ v_\gamma(k_1', k_2') - 2 \int v_{12}(\tilde{k}_1, k - \tilde{k}_1)[\omega_\alpha(\tilde{k}_1) + \omega_\beta(k - \tilde{k}_1) - \omega_\alpha(k_1') - \omega_\beta(k_2')]^{-1} \cdot v_{21}(\tilde{k}_1, k - \tilde{k}_1)\, d\tilde{k}_1 \right\} \cdot [\omega_\gamma(k) - \omega_\alpha(k_1') - \omega_\beta(k_2')]^{-1}$$

with $k = k_1' + k_2'$. After expressing $v_\gamma(k)$ as the integral given earlier we may readily combine the two integrals into one:

$$y_\gamma(k_1', k_2') = 2 \int v_{12}(\tilde{k}_1, k - \tilde{k}_1)[\omega_\alpha(\tilde{k}_1) + \omega_\beta(k - \tilde{k}_1) - \omega_\gamma(k)]^{-1} \cdot [\omega_\alpha(\tilde{k}_1) + \omega_\beta(k - \tilde{k}_1) - \omega_\alpha(k_1') - \omega_\beta(k_2')]^{-1} \cdot v_{21}(\tilde{k}_1, k - \tilde{k}_1)\, d\tilde{k}_1.$$

The function $y_\gamma(k_1', k_2')$ is exactly the function denoted by $y(s, \omega^*)$ in §18, for $s = (k, \gamma) = (k_1' + k_2', \alpha + \beta)$ and $\omega^* = \omega_\alpha(k_1') + \omega_\beta(k_2')$. The process of rewriting y_γ as a single integral by combining two integrals agrees exactly with the corresponding process described there. The Γ-factors involved in the two integrals behave like $|2\tilde{k}_1|^{-1}$ as $|\tilde{k}_1|$ tends to infinity; that of the single integral behaves like $|2\tilde{k}_1|^{-2}$. Therefore, the single integral may converge in cases where the two original integrals do not. In such a case, combining the two integrals into one amounts to removing a divergence with the aid of a term supplied by the renormalization of the single particle energy.

We should also mention the function $\zeta_\gamma(k)$, defined by the relation

$$\int A_\gamma^+(k)\zeta_\gamma(k)A_\gamma^-(k)\, dk = -\Gamma Q_{12}^+ \times \Gamma Q_{21}^-.$$

A simple computation shows that it is given by

$$\zeta_\gamma(k) = 2 \int v_{12}(\tilde{k}_1, k - \tilde{k}_1)[\omega_\alpha(\tilde{k}_1) + \omega_\beta(k - \tilde{k}_1) - \omega_\gamma(k)]^{-2} \cdot v_{21}(\tilde{k}_1, k - \tilde{k}_1)\, d\tilde{k}_1.$$

According to what was said in §17 and at the beginning of §A19 this function and the function $\theta_\gamma(k)$ defined by

$$1 + \zeta_\gamma(k) = e^{2\theta_\gamma(k)}$$

enter the definition of the operator τ^2,

$$\tau^2 = :e^{\int A_\gamma^+(k)\zeta_\gamma(k)A_\gamma^-(k)dk}: = e^{2\int A_\gamma^+(k)\theta_\gamma(k)A_\gamma^-(k)dk},$$

which, in turn, enters the definition of the transformations,

$$U^+ = \tau^{-1}T^+, \qquad U^- = T^-\tau^{-1}$$

with $T^\pm = :e^{\pm\Gamma Q^\pm}:$. Note that the Γ-factor of $\zeta_\gamma(k)$ behaves like that of $y_\gamma(k)$ at infinity.

We shall make only a few remarks about the other sectors. We denote by $G_{l_\alpha l_\beta l_\gamma; m_\alpha m_\beta m_\gamma}$ that contribution to our operator G_{lm} whose kernel is restricted to have l_α, l_β, l_γ creation variables and m_α, m_β, m_γ annihilation variables with $\iota = \alpha, = \beta, = \gamma$. Clearly, $l_\alpha + l_\beta + l_\gamma = l$ and $m_\alpha + m_\beta + m_\gamma = m$. We call such contributions simple. To any such simple operator we assign in an obvious manner numbers A, B, A', B' associated with its creation and annihilation variables, just as we have assigned numbers A, B to states.

We use the notation $G_{AB;A'B'}$ for any linear combination of simple operators having the same values of A, B, A', B'. Actually, we shall not show $A'B'$ explicitly.

The basic equation for a contribution $Q_{AB}^- = Q_{AB;A'B'}^-$ to the operator Q^- can now be written in the form

$$\begin{aligned} Q_{AB}^- + V_{21} \overset{\circ}{\frown} \Gamma Q_{AB}^- + V_{12} \overset{\circ}{\frown}_2 \Gamma Q_{AB}^- + V_{11} \overset{\circ}{\frown} \Gamma Q_{AB}^- \\ = - V_{12} \overset{\circ}{\frown}_1 \Gamma Q_{(A-1)(B\pm1)}^- - \tfrac{1}{2} V_{12} \prec : \sum \Gamma Q_{A_1B_1}^- \Gamma Q_{A_2B_2}^- : + V_{AB}. \end{aligned}$$

Here $V_{AB} = V$ for $A = A' = 2$, $B = B' = 0$, and $= 0$ otherwise. Also, $(A - 1)(B \pm 1)$ should be supplemented by $(A' - 1)(B' \pm 1)$ and, finally, the summation in the Wick product refers to $A_1 + A_2 = A$, $B_1 + B_2 = B$. The sign in $B \pm 1$ depends on whether an α- or a β-contraction is involved in $\overset{\circ}{\frown}_1$.

One readily verifies that for $A = 0$ and $A = 1$ the equations reduce to $Q_{0B}^- = 0$ and $Q_{1B}^- = 0$. Therefore, we may assume $A \geqq 2$ and restrict A_1, A_2 in the Wick product to $A_1, A_2 < A$. Consequently, if the operators Q_{AB}^- are known up to a certain value A (but excluding it) the right member of the equation is known.

We now enter the *assumptions* (1) that the left member of the

equation is such that the homogeneous equation has no solution other than zero and (2) that the nonhomogeneous equation has a solution for a proper class of operators on the right, provided the operator V' is properly bounded.

From assumption (1) we shall conclude that the operators Q^-_{AB} vanish unless $A' = A$, $B' = B$ for them. Operators with this property will be called "admitted". We verify that any product of two admitted operators is also admitted. Consequently, in the Wick product on the right member of the equation for a nonadmitted Q^-_{AB} at least one of the two factors is nonadmitted. If it is known that all nonadmitted Q^-_{AB} vanish up to a particular value of A it follows from the first assumption that it is true also for that particular value. Since $Q^-_{AB} = 0$ for $A = 0$ we conclude that all nonadmitted Q^-_{AB} are zero.

For $A = 2$, the lowest possible value of A, the right member is zero unless $B = 0$ since $V_{AB} = 0$ otherwise. Hence, by the first assumption, we have $Q^-_{2B} = 0$ for $B \neq 0$. The operator Q^-_{20} was already determined before. For higher values of A one should invoke the second assumption to establish the existence of Q^-_{AB}. To be sure that these operators exist simultaneously for all values of A and B one should invoke a third *assumption* (3): there is a bound for V such that for all A and B the statement of assumption (2) holds with this bound.

Whether or not the three assumptions can be shown to be valid for a class of operators V is an open question.

We observe that any admitted operator G_{AB} transforms the sector $\mathfrak{H}_{A_0B_0}$ into itself, and annihilates it if $A > A_0$. Consequently, that part, $T^-_{A_0B_0}$, of the Wick power series $T^- = :e^{-TQ^-}:$ which produces operators G_{AB} with $A \leqq A_0$ depends only on the operators Q^-_{AB} with $A \leqq A_0$ thus only on a finite number of them.

The present way of determining the transformation $T^-_{A_0B_0}$ allows one to treat the perturbation in the sector $\mathfrak{H}_{A_0B_0}$ even though (in general) it is not gentle.

Finally, we shall make some remarks about the relationship between the problem considered and that of the proper Lee model; the latter result through two degeneracies.

1. One allows the masses μ_β and μ_γ of the particles (β) and (γ) to tend to infinity in such a way that the difference $\mu_\beta - \mu_\gamma = \omega_0$ remains fixed. The differences of the energies ω_β, ω_γ then tend to this mass difference:

$$\omega_\beta(k) - \omega_\gamma(k') = \sqrt{(\mu_\beta^2 + |k|^2)} - \sqrt{(\mu_\gamma^2 + |k'|^2)}$$
$$\sim \omega_0 + \tfrac{1}{2}\mu_\beta^{-1}|k|^2 - \tfrac{1}{2}\mu_\gamma^{-1}|k'|^2 \to \omega_0.$$

Since only these energy differences occur in the Γ-denominators they may be replaced by ω_0; in fact one may omit the contribution from (β) to H_0 and replace that from γ by

$$\omega_0 \int A_\gamma^+(k) A_\gamma^-(k)\, dk.$$

2. One assumes that the β- and γ-particles are restrained to remain at a fixed point. It can be shown that this requirement is equivalent to replacing $A_{\beta,\gamma}^{\pm}(k)$ by operators $A_{\beta,\gamma}^{\pm}$ which are independent of k and at the same time omitting integration with respect to its variable k. Also, one will let v_{12} and v_{21} depend only on the α-momentum k. Using the notation $\sqrt{2}v_+(k')$ and $\sqrt{2}v_-(k)$ for these reduced kernels, the total energy $H = H_0 + V$ can be written as

$$\begin{aligned} H = {} & \omega_0 A_\gamma^+ A_\gamma^- + \int A_\alpha^+(k)\omega_\alpha(k) A_\alpha^-(k)\, dk \\ & + A_\gamma^+ \int v_+(k') A_\alpha^-(k')\, dk' A_\beta^- \\ & + A_\beta^+ \int A_\alpha^+(k) v_-(k)\, dk A_\gamma^-. \end{aligned}$$

This is the energy operator of the Lee model. It is readily seen that the problem of the Lee model reduced to the main sector is the problem treated in §8. (See [**39**; **44II**].)

Also one observes that, in particular, the expressions for the reduced kernels $\sqrt{2}q_{\alpha\beta}^-$ and $\sqrt{2}q_{\alpha\beta;\alpha'}^-$ of the main sector reduce to

$$\begin{aligned} q^-(k') &= [1 + \zeta_\gamma(k')]^{-1} v_+(k'), \\ q^-(k; k') &= -v_-(k)[\omega_0 - \omega_\alpha(k')]^{-1} [1 + \zeta_\gamma(k')]^{-1} v_+(k') \end{aligned}$$

with

$$v_\gamma(k) = v_{11} - \int v_+(\tilde{k})[\omega_\alpha(\tilde{k}) - \omega_0]^{-1} v_-(\tilde{k})\, d\tilde{k}$$

and

$$\zeta_\gamma(k') = \left\{ v_{11} - \int v_+(\tilde{k})[\omega_\alpha(\tilde{k}) - \omega_\alpha(k')]^{-1} v_-(\tilde{k})\, d\tilde{k} \right\} [\omega_0 - \omega_\alpha(k')]^{-1}$$

in agreement with the corresponding expressions given in §8. The occurrence of the factor 2 here is due to the fact that the unit form of the state with one α- and one β-particle is $\frac{1}{2}\int |\psi_{\alpha\beta}(k)|^2\, dk$ now while it was $\int |\psi(k)|^2\, dk$ in §8.

A21. A non-negative operator $H_0 + V$ with an extra vacuum. It was mentioned at the end of §9 that it could happen that the spectrum of an

operator $H_0 + V$ differs from that of H_0 for arbitrary small disturbances V. This can happen even in cases in which the operator $H_0 + V$ has a lower bound so that the spectrum cannot extend to $-\infty$ as in the cases treated by Galindo [51]. We shall show this by exhibiting an example.

With the aid of a square integrable real function $p(\omega)$ we introduce the operators

$$A^+ \cdot p = \int A^+(\omega)p(\omega)\,d\omega, \qquad p \cdot A^- = \int p(\omega)A^-(\omega)\,d\omega$$

and set

$$V = V_\varepsilon = -\varepsilon(A^+p)(pA^-)^2 - \varepsilon(A^+p)^2(pA^-) + \varepsilon^2(A^+p)^2(pA^-)^2.$$

Clearly, we may write $H_0 + V$ in the form

$$H_0 + V = H_0 - (A^+p)(pA^-) + A^+p\{1 - \varepsilon A^+p\}\{1 - \varepsilon pA^-\}pA^-.$$

The last term here is evidently non-negative. The same is true of $H_0 - (A^+p)(pA^-)$ provided

$$\int p^2(\omega)\omega^{-1}\,d\omega < 1.$$

For, by Schwarz's inequality we then have

$$\left|\int p(\omega)\psi(\omega)\,d\omega\right|^2 \leqq \int \overline{\psi(\omega)}\omega\psi(\omega)\,d\omega,$$

whence, the statement can be read off from the particle representation of H_0 and $(A^+p)(pA^-)$.

As a well-known property of such positive-definite new properties the operator admits a strictly selfadjoint extension and, hence, a spectral resolution.

With the aid of the function $q(\omega) = \omega^{-1}p(\omega)$ we now consider the state

$$\Phi_\varepsilon = e^{-(q\cdot q)/2\varepsilon^2}e^{(A^+\cdot q)/\varepsilon}\Phi_0,$$

which is defined since

$$q\cdot q = \int q^2(\omega)\omega^{-2}\,d\omega < \omega_*^{-1} < \infty.$$

The factor $e^{-(q\cdot q)/2\varepsilon^2}$ is chosen so that $\|\Phi_*\| = 1$.

Using the relations (see e.g. [41])

$$p\cdot A^{-}e^{(A^{+}\cdot q)/\varepsilon} = e^{(A^{+}\cdot q)/\varepsilon}(p\cdot A^{-} - p\cdot q/\varepsilon)$$

with $p\cdot q = 1$ and

$$A^{+}\cdot\omega A^{-}e^{(A^{+}\cdot q)/\varepsilon} = e^{A^{+}\cdot q/\varepsilon}(A^{+}\cdot\omega A^{-} - A^{+}\omega^{-1}q/\varepsilon)$$

we readily find

$$(H_0 + V_\varepsilon)\Phi_\varepsilon = 0.$$

Thus the state Φ_ε is, so to say, a second vacuum for the operator $H_\varepsilon = H_0 + V_\varepsilon$.

We should mention that the unitary transformation

$$U_\varepsilon = e^{-(q\cdot q)/2\varepsilon^2}e^{(A^{+}\cdot q)/\varepsilon}e^{-(q\cdot A^{-})/\varepsilon}$$

with the inverse

$$U_\varepsilon^{-1} = e^{-(q\cdot q)/2\varepsilon^2}e^{-(A^{+}\cdot q)/\varepsilon}e^{(q\cdot A^{-})/\varepsilon}$$

interchanges the operators $H_\varepsilon = H_0 + V_\varepsilon$ and $H_{-\varepsilon} = H_0 + V_{-\varepsilon}$:

$$U_\varepsilon^{-1}H_\varepsilon U_\varepsilon = H_{-\varepsilon}, \qquad U_\varepsilon H_0 U_\varepsilon^{-1} = H_\varepsilon;$$

at the same time, it interchanges the vacuum states Φ_0 and Φ_ε:

$$U_\varepsilon\Phi_0 = \Phi_\varepsilon, \qquad U_\varepsilon^{-1}\Phi_\varepsilon = \Phi_0.$$

This indicates that there is no essential difference between the roles the states Φ_0 and Φ_ε play in the spectral description of the operator H_ε.

The inner product $(\Phi_0, \Phi_\varepsilon) = e^{-(q\cdot q)/2\varepsilon^2}$ of these two vacuum states tends to zero as $\varepsilon \to 0$; and in the limit there is only one vacuum state.

Bibliography to Chapter I

1. M. H. Stone, *Linear transformation in Hilbert space*, Amer. Math. Soc. Colloq. Publ. Vol. XV, Amer. Math. Soc., Providence, R.I., 1932.

2. a. F. Rellich, *Störungstheorie der Spektralzerlegung*, Math. Ann. I, II, **113** (1936); III, **116** (1939); IV, **117** (1940); V, **118** (1942).

b. ———, *Störungstheorie der Spektralzerlegung*, Proc. Internat. Congr. Mathematicians (Cambridge, Mass., 1950), Vol. 1, pp. 606–613, Amer. Math. Soc., Providence, R.I., 1952.

c. ———, *Perturbation theory of eigenvalue problems*, Lecture notes, New York University, New York, 1953.

3. B. de Sz.-Nagy, *Perturbations des transformations auto-adjoints dans l'espace de Hilbert*, Comment. Math. Helv. **19** (1947), 347–366.

4. T. Kato, *On the convergence of the perturbation method*, J. Fac. Sci. Tokyo Univ. **6** (1951), 198–205.

5. E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.

6. F. Riesz and B. de Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Akad. Kiado, Budapest, Hungary, 1952; English transl.: *Functional analysis*, Ungar, New York, 1955.

7. C. Bloch, *Sur la théorie des perturbations des états liés*, Nuclear Phys. **6** (1958), 329–347.

8. N. Dunford and J. T. Schwartz, *Linear operators*, Vol. II, Interscience, New York, 1963. See Vol. I, Chapter VII, §§6 and 11.

9. K. O. Friedrichs, *Spectral theory of operators in Hilbert space*, Lecture notes, New York University, New York, 1960.

Bibliography to Chapter II

10. K. O. Friedrichs, *Ueber die Spektralzerlegung eines Integraloperators*, Math. Ann. **115** (1938), 249-272.

11. C. Møller, *General properties of the characteristic matrix in the theory of elementary particles.* II, Danske Vid. Selsk. Mat.-Fys. Medd. **22** (1946), no. 19.

12. K. O. Friedrichs, *On the perturbation of continuous spectra*, Comm. Pure Appl. Math. **1** (1948), 361-406.

13. T. Kato, *On the convergence of the perturbation method*, J. Fac. Sci. Tokyo Univ. **6** (1951), 198-205.

14. J. Moser, *Störungstheorie des kontinuierlichen Spektrums für gewöhnliche Differentialgleichungen zweiter Ordnung*, Math. Ann. **125** (1953), 366-393.

15. E. C. Titchmarsh, *Some theorems on perturbation theory.* V, J. Analyse Math. **4** (1954/55), 187-208.

16. M. Rosenblum, *Perturbation of the continuous spectrum and unitary equivalence*, Pacific J. Math. **7** (1957), 997-1010.

17. T. Kato, *Perturbation of continuous spectra by trace class operators*, Proc. Japan Acad. **33** (1957), 260-264.

18. J. M. Cook, *Convergence of the Møller wave-matrix*, J. Mathematical Phys. **36** (1957), 82-87.

19. J. M. Jauch, *Theory of the scattering operator*, Helv. Phys. Acta **31** (1958), 127-158.

20. ———, *Theory of the scattering operator.* II. *Multi-channel scattering*, Helv. Phys. Acta **31** (1958), 661-684.

21. O. A. Ladyženskaya and L. D. Faddeev, *On perturbation theory of a continuous spectrum*, Dokl. Akad. Nauk SSSR **120** (1958), 1187-1190. (Russian)

22. S. T. Kuroda, *On the existence and unitary property of the scattering operator*, Nuovo Cimento **12** (1959), 431-454.

23. T. Kato and S. T. Kuroda, *A remark on the unitary property of the scattering operator*, Nuovo Cimento **14** (1959), 1102-1107.

24. S. T. Kuroda, *Perturbation of continuous spectra by unbounded operators.* I, II, J. Math. Soc. Japan **11** (1959), 247-262; **12** (1960), 243-257.

25. ——, *An example of a scattering system in Jauch's sense*, Progr. Theoret. Phys. **24** (1960), 461.

26. T. Ikebe, *Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory*, Arch. Rational Mech. Anal. **5** (1960), 1-34.

27. J. T. Schwartz, *Some non-selfadjoint operators.* I, II, Comm. Pure Appl. Math. **13** (1960), 609-639; ibid. **14** (1961), 619-626.

28. K. O. Friedrichs and P. A. Rejto, *On a perturbation through which the discrete spectrum becomes continuous*, Comm. Pure Appl. Math. **15** (1962), 218-235.

29. L. de Branges, *Perturbations of self-adjoint transformation*, Amer. J. Math. **84** (1962), 543-560.

30. S. T. Kuroda, *On a stationary approach to scattering problem*, Bull. Amer. Math. Soc. **70** (1964), 556–560.

31. M. Sh. Birman and M. G. Krein, *On the theory of wave operators*, Dokl. Akad. Nauk SSSR **144** (1962), 475–478 = Soviet Math. Dokl. **3** (1962), 740–743.

32. M. Sh. Birman, *Perturbations of continuous spectra of singular elliptic operators by varying the boundary and the boundary conditions*, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **17** (1962), 22–55. (Russian);

———, *Conditions for the existence of wave operators*, Dokl. Akad. Nauk SSSR **143** (1962), 506–509 = Soviet Math. Dokl. **3** (1962), 408–411.

33. F. H. Brownell, *Perturbation theory and an atomic transition model*, Arch. Rational Mech. Anal. **10** (1962), 149–170;

———, *Explicit perturbation formulas and convergence theorems*, Math Anal. Appl. **6** (1963), 190–206.

34. T. Kato, *Wave operators and unitary equivalence*, Tech. Rep. No. 11, University of California, Berkeley, Calif., 1963.

35. L. D. Faddeev, *Mathematical questions in the quantum theory of scattering for a system of three particles*, Trudy Mat. Inst. Steklov. **69** (1963). (Russian)

36. P. A. Rejto, *On gentle perturbations.* I, II, Comm. Pure Appl. Math. **16** (1963), 279–303; **17** (1964), 257–292.

37. P. D. Lax and R. S. Phillips, *Scattering theory*, Bull. Amer. Math. Soc. **70** (1964), 130–142.

38. N. J. Muskhelishvili, *Singular integral equations*, Theorem, p. 45, Noordhoff Groningen, 1953.

39. W. Heisenberg, *Lee model and quantization of nonlinear field equations*, Nuclear Phys. (1957), no. 4.

40. N. Dunford and J. T. Schwartz, *Linear operators*, Vol. III, Interscience, New York (to appear).

Added in proof. L. D. Faddeev, *About Friedrichs' model in the theory of perturbations of continuous spectra*, Trudy Mat. Inst. Steklov. **73** (1964). (Russian)

Bibliography to Chapter III

41. K. O. Friedrichs, *Mathematical aspects of the quantum theory of fields*, Comm. Pure Appl. Math. I, II, **4** (1951), nos. 2/3; III, ibid. **5** (1952), no. 1; IV, ibid. **5** (1952), no. 4; V, ibid. **6** (1953), no. 1. (Collectively reissued 1953)

42. L. van Hove, *Les difficultés de divergences pour un modèle particulier de champ quantifié*, Physica **18** (1952), 145–159.

43. J. M. Jauch and F. Röhrlich, *The theory of photons and electrons, the relativistic quantum field theory of charged particles and spin one-half*, Series in Advanced Physics, Addison-Wesley, Cambridge, 1955.

44. L. van Hove, *Energy corrections and persistent perturbation effects in continuous spectra*, Physica, I, **21** (1955), 901–923; II, ibid. **22** (1956), 257–292.

45. N. M. Hugenholtz, *Perturbation approach to the Fermi gas model of heavy nuclei*, Physica **23** (1957), 533–545.

46. C. Bloch, *Sur la détermination de l'état fondamental d'un système de particules*, Nuclear Phys. **7** (1958), 451–458.

47. J. T. Schwartz, *Mass renormalization and spectral shifts*, New York University, Physics Department, New York, 1960.

48. D. Kastler, *Introduction à l'electrodynamique quantique*, Travaux des Recherches Mathématiques, Dunod, Paris, 1961.

49. S. S. Schweber, *Introduction to relativistic quantum field theory*, Row Peterson, Evanston, Ill., 1961.

50. F. A. Berezin. *O modeli Le* (*On the Lee model*), Mat. Sb. (N.S.) **60 (102)** (1963), 425–446.

51. A. Galindo, *On a class of perturbations in quantum field theory*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1128–1134.

52. I. E. Segal, *Mathematical problems of relativistic physics*, Lectures in Applied Math. Vol. II, Amer. Math. Soc., Providence, R.I., 1963.

53. Y. Kato and N. Mugibayashi, *Regular perturbation and asymptotic limits of operators in quantum field theory*, Progr. Theoret. Phys. **30** (1963), 103–133.

54. R. F. Streater and A. S. Wightman, *PCT, spin and statistics*, and all that, W. A. Benjamin, New York, 1964.

Added in proof. For a direct approach to the scattering operator of paragraph 15, see

55. A. L. Chistyakov, *Ob operatore resseyaniya v prostranstve vtorichnovo kvantovaniya*, Dokl. Akad. Nauk SSSR **158** (1964), 66–70. (Russian)

Author Index

Subject Index