MATSCIENCE REPORT 28

LECTURES ON

MAGNETIC PROPERTIES OF A SUPERCONDUCTOR

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Visiting Professor, Matscience.

Notes by
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LECTURES ON MAGNETIC PROPERTIES OF A SUPERCONDUCTOR

by

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Chapter I

Introduction

Any microscopic theory of superconductivity should lead to the derivation of London's equation which connects the applied magnetic field and the current in the bulk superconductor. According to London[1] the current in a superconductor consists of two parts, J_n and J_s, the normal and superconducting respectively, i.e.,

$$\vec{J} = \vec{J}_n + \vec{J}_s \tag{1.1}$$

where J_n obeys the usual Ohm law and is given by

$$J_n = \sigma E \tag{1.2}$$

σ is the normal conductivity. The supercurrent J_s obeys the London equation:

$$J_s = -\frac{1}{c\Lambda} A^{tr} \tag{1.3}$$

where A^{tr} is the vector potential such that

$$\operatorname{curl} \vec{A} = \vec{H} \tag{1.4a}$$

and

$$\operatorname{Div} A^{tr} = 0 \tag{1.4b}$$

Since (1.4b) determines the gauge, we can wrige London's equation in a gauge invariant form:

$$\mathrm{Curl}\ c\Lambda J_S = -H \tag{1.5}$$

Λ in the above equation is a constant characteristic of the material which can be calculated from microscopic theory. According to Maxwell's theory

$$\mathrm{Curl}\ H = \frac{4\pi}{c} J_S \tag{1.6}$$

J_S only need be considered since the normal current will die off after a short duration. Using $\mathrm{div}\, H = 0$, we see that H in the material obeys a differential equation of the type

$$\nabla^2 H = \frac{4\pi}{c^2\Lambda} H \tag{1.7}$$

This equation implies the Meissner effect since only the decaying solution of the above equation is physically meaningful. To explain this statement, consider a field H_0 applied parallel to the surface of the superconductor and let x denote the depth or distance perpendicular to the surface. The field inside the specimen at any depth is given by (Fig.1)

$$H = H_0\, e^{-x/\delta} \tag{1.8}$$

where $\delta = c^2\Lambda/4\pi$

This type of solution is independent of the geometry of the specimen

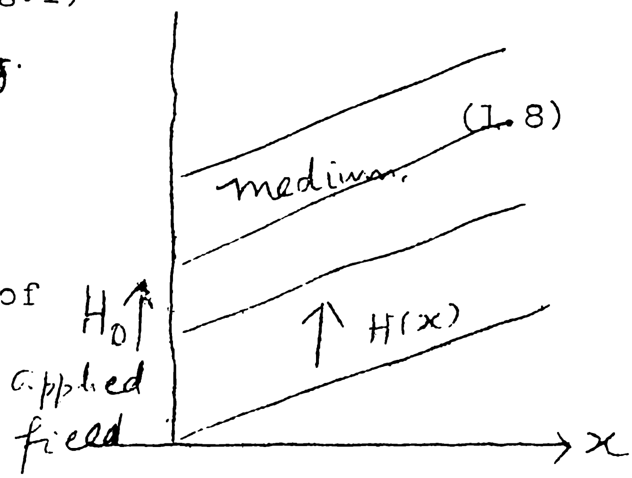


Fig. 1.

But even in a normal metal there is a small current after the establishment of the field. This is a quantum mechanical effect which is called the Landau diamagnetic current. But this is essentially a current of the ampere type as opposed to the super-current or the London current in a superconductor. Therefore our approach to the problem of superconductors should be rather

X different and follow the initiative approach of London, who thought that the wave function of the super electrons should have a certain rigidity and not change with the application of an external magnetic field of vector potential A , to first order. The current $J(R)$ at a point R in the material in an external field is given by the usual quantum mechanical expression

$$J(R) = \sum_{i=1}^{N} \int \frac{e\hbar}{2mi} \left[\psi^*(x_1, \cdots x_N) \nabla_i \psi - \psi \nabla_i \psi^* \right] \delta(R - x_i)\, dx_i - \frac{e^2}{mc} \psi^* \psi A \qquad (1.9)$$

In the above expression $\psi(x_1, \cdots x_N)$ is the wave function of the N super electrons in the specimen. The first term in (1.9) represents the paramagnetic current while the term proportional to $-\vec{A}$ is the diamagnetic current. London assumed that ψ is not altered by the external field and hence there is no paramagnetic current even after the application of the field. On the contrary the diamagnetic term J_D vanishes only for A = 0 , i.e., $J_D \propto A$ and if

$$-\frac{1}{c\Lambda} = -\frac{e^2}{mc} |\psi|^2 \qquad (1.10)$$

where $|\Psi|^2$ represents the number density of super electrons the London equation (1.3) is obtained. Hence

$$\Lambda = \frac{m}{e^2 n_s} = \frac{4\pi}{c^2}\delta^2 \tag{1.11}$$

As opposed to this in the case of a normal metal when the external field A is applied. Ψ gets suitably changed such that the paramagnetic term and the diamagnetic term get almost cancelled. Even then a small term corresponding to diamagnetic part survives, which is attributed to the so called Landau diamagnetism. But this small term is not truly a current with a net outflow. Its nature is that of the ampere type and that is the reason why the Landau diamagnetism is measured in terms of the magnetic moment per unit volume.

If we want to calculate the response in a metal corresponding to an external field A , we can write for the induced current $J_r(x)$ in the linear approximation as

$$J_r(x) = \int K_{rs}(x-x')A_s(x')\,dx' \tag{1.12}$$

where the kernel K cannot be any arbitrary function, but should reflect the physical process in the material. To see this more clearly let us take the fourier transform of both sides of (1.12) and get

$$J_r(q) = K_{rs}(q)\; A(q) \tag{1.13}$$

In an arbitrary gauge for A , i.e. if we allow A to be $A + \nabla\chi$

to preserve the continuity equation $\operatorname{div} J_s = 0$, we have

$$q_s J_s(q) = 0 \tag{1.14}$$

To satisfy the above relation, K_{rs} should satisfy the divergence condition for both r and s and hence it can be written as

$$K_{rs}(q) = (q_r q_s - q^2 \delta_{rs})\, F(|q|^2) \tag{1.15}$$

In the limit when $q \to 0$ we should be able to obtain the London equation $J_s = -\frac{1}{C\Lambda} A^{tr}$ where

$$A^{tr}_r(q) = A_r(q) - \frac{q_r q_s A_s(q)}{q^2} = \left(\delta_{rs} - \frac{q_r q_s}{q^2}\right) A_s(q) \tag{1.16}$$

the second term being gauge dependent. It is easy to see that (1.16) represents the transverse part of $\vec{A}$ since $q_r A^{tr}_r(q) = 0$. Therefore the expression for $J_r(q)$ valid for normal metal as well as a superconductor is

$$J_r(q) = -q^2 F(|q|^2)\, A^{tr}_r(q) \tag{1.17}$$

In a normal metal $-q^2 F(|q|^2)$ goes to zero as q goes to zero (i.e.) $F(|q|^2)$ is regular at $q = 0$, while in a superconductor $q^2 F(|q|^2)$ approaches the limit $\frac{1}{C\Lambda}$ as $q \to 0$, i.e. $F(|q|^2)$ becomes singular as q goes to zero and is responsible for the non-zero London current.

Chapter II.

A variation principle for Z_G.

Before we study the electrodynamics of superconductors we shall review briefly statistical mechanics. of a system of free fermions.

As is well known a general system of free fermions corresponding to a Hamiltonian H can be described by the density matrix ρ defined by

$$\rho = e^{-\beta H} \tag{2.1}$$

where $\beta = \frac{1}{kT}$ and $Tr\,\rho = 1$. Hence

$$\rho = e^{-\beta H} / Tr\, e^{-\beta H} \tag{2.2}$$

The expectation value of any operator O is given by

$$\langle O \rangle = Tr\,(\rho O) \tag{2.3}$$

The free energy of the Fermi system, F

$$e^{-\beta F} = Tr\, e^{-\beta H} \tag{2.4}$$

Instead of using H , we can use $\tilde{H}$ given by

$$\tilde{H} = H - \mu N \tag{2.5}$$

where N is the number operator and μ is the chemical potential which enters as an unknown in the problem, and get the thermodynamic potential Ω using the grand ensemble:

$$e^{-\beta \Omega} = Tr\, e^{-\beta \tilde{H}} \tag{2.6}$$

For free fermions we introduce the creation and annihilation operators a_f, a_f^* where f denotes both momentum and spin i.e., $f = (\vec{p}, s)$. These obey the commutation relations

$$[a_f, a_{f'}^*] = \delta_{ff'} \tag{2.7a}$$

$$[a_f, a_{f'}] = [a_f^*, a_{f'}^*] = 0 \tag{2.7b}$$

The Hamiltonian of the system H equals

$$H = \sum_f \omega_f a_f^* a_f \tag{2.8}$$

where

$$\omega_f = \frac{p^2}{2m} \; ; \; \tilde{H} = \sum_f (\omega_f - \mu) a_f^* a_f \tag{2.9}$$

Let us now calculate the thermodynamic potential , Now

$$\begin{aligned} Tr\, e^{-\beta \tilde{H}} &= Tr\, e^{-\beta \sum_f (\omega_f - \mu) a_f^* a_f} \\ &= Tr\, e^{-\beta \sum_f (\omega_f - \mu) n_f} \\ &= \prod_f Tr\, e^{-\beta(\omega_f - \mu) n_f} \\ &= \prod_f \left(1 + e^{-\beta(\omega_f - \mu)}\right) \end{aligned} \tag{2.10}$$

where we have used

$$[n_f, n_{f'}] = 0 \tag{2.11}$$

Hence

$$\Omega = -\frac{1}{\beta}\sum_f \log\left(1+e^{-\beta(\omega_f-\mu)}\right) \tag{2.12}$$

We shall calculate

$$e^{-\beta\tilde{H}} = e^{-\beta\sum_f(\omega_f-\mu)n_f} = \prod_f e^{-\beta(\omega_f-\mu)n_f} \tag{2.13}$$

In the case of fermions $n^\alpha = n$ for $\alpha \geq 2$.

Hence (2.13) becomes

$$\prod_f \left[1+\left(e^{-\beta(\omega_f-\mu)}-1\right)n_f\right] \tag{2.14}$$

From (2.11) and (2.14) we find ρ as

$$\rho = \frac{e^{-\beta\tilde{H}}}{Tr\, e^{-\beta\tilde{H}}} = \prod_f\left[\Gamma_f^0 n_f + (1-\Gamma_f^0)(1-n_f)\right] \tag{2.15}$$

where Γ_f^0 is the Fermi distribution

$$\Gamma_f^0 = 1/1+e^{\beta(\omega_f-\mu)} \tag{2.16}$$

(2.16) gives the density matrix for non-interacting system of fermions. For calculating the ground state of the interacting fermions we shall use a variational principle to determine the ground state wave function and ground state energies(2). To study the system at finite temperature we shall introduce a trial density matrix ρ_t which after minimisation will become the correct ρ . Now the free energy at non-zero temperature is given by

$$\langle\tilde{H}\rangle - TS = Tr\,\tilde{H}\rho_t + \frac{1}{\beta}Tr\,\rho_t\log\rho_t \tag{2.17}$$

with $Tr\, \rho_t = 1$

$$-TS = \frac{1}{\beta} \langle \log \rho_t \rangle \tag{2.18}$$

The exact density matrix is given by that $\rho_t = \rho$ which satisfies the minimum principle given below.

$$\begin{aligned} \Omega &= Tr\, \tilde{H}\rho + \frac{1}{\beta}\, \rho \log \rho \\ &\leq Tr\, \tilde{H}\rho_t + \frac{1}{\beta}\, Tr\, \rho_t \log \rho_t \end{aligned} \tag{2.19}$$

We can check that when $\rho = \dfrac{e^{-\beta\tilde{H}}}{Tr\, e^{-\beta\tilde{H}}}$, the above expression yields Ω:

$$\Omega = -\frac{1}{\beta} \log Tr\, e^{-\beta\tilde{H}} \tag{2.20}$$

We shall mathematically prove the variational principle used in (2.19). If the Hamiltonian of a system is a Hermitian operator, the ground state energy of the system is the smallest possible value of the expectation of the Hamiltonian with respect to a normalized wave function that is arbitrary, except that it satisfies the boundary conditions and the symmetries of the system. This principle can be used to find an upper bound for the ground state energy. A similar principle for Hemholtz's free energy can be proved using a theorem due to Peierls[3]. Peierls theorem can be stated as follows. Let A be an Hermitian operator. The theorem asserts that

$$Tr\, e^{A} \geq \sum_{\alpha} e^{\langle \alpha | A | \alpha \rangle} \tag{2.21}$$

where α 's are complete set of arbitrary, orthonormal functions. The equality holds if α's are the complete set of eigenfunctions of A. The proof of this theorem is based on the property of Convex funct A function $f(x)$ is supposed to be convex downwards if $f''(x) > 0$. Let $\{x_n\}$ be a set of real numbers and $\{C_n\}$ be a set of real numbers such that $C_n \geq 0$ and $\sum_n C_n = 1$. If

$$\bar{f}(x) = \sum_n C_n f(x_n) \tag{2.22}$$

and

$$\bar{x} = \sum_n C_n x_n \tag{2.23}$$

then

$$\bar{f}(x) \geq f(\bar{x}) \tag{2.24}$$

This is true since by mean value theorem we have

$$f(x) = f(\bar{x}) + (x-\bar{x}) f'(\bar{x}) + \frac{1}{2}(x-\bar{x})^2 f''(x_1) \tag{2.25}$$

where x_1, is a fixed real number. Using the fact that $\sum_n C_n = 1$, $C_n \geq 0$ and $f''(x) \geq 0$

$$\bar{f}(x) = f(\bar{x}) + \frac{1}{2}(x-\bar{x})^2 f''(x_1) \tag{2.26}$$

which proves (2.24). If $|\beta\rangle$'s are the eigenstates of A,

$$Tr(e^A) = \sum_\beta e^{A_\beta} \tag{2.27}$$

where A_β are the eigenvalues of A. Let $S_{\alpha\beta}$ be a unitary transformation which connects α to β. It is easy to see that

$$\langle\alpha|A|\alpha\rangle = \sum |S_{\alpha\beta}|^2 A_\beta \tag{2.28}$$

It is well known that

$$\sum_{\alpha} |S_{\alpha\beta}|^2 = 1 \tag{2.29}$$

Hence

$$Tr\, e^{A} - \sum_{\alpha} e^{\langle\alpha|A|\alpha\rangle} = \sum_{\alpha}\left(\sum_{\beta}|S_{\alpha\beta}|^2 e^{A_\beta} - e^{\sum_{\beta}|S_{\alpha\beta}|^2 A_\beta}\right) \tag{2.30}$$

Identifying

$$x = A \quad , \quad f(x) = e^{x}$$

and

$$c_\beta = |S_{\alpha\beta}|^2 \tag{2.31}$$

From this it is clear that L.H.S. of (2.30) is ≥ 0 i.e.,

$$Tr\, e^{A} - \sum_{\alpha} e^{\langle\alpha|A|\alpha\rangle} \geq 0 \tag{2.32}$$

Actually we have used a slight generalization of the above variational principle in writing down (2.10). To prove

$$Tr\, e^{A+B} \geq Tr\, e^{A} \exp\left(\frac{Tr\, e^{A} B}{Tr\, e^{A}}\right) \tag{2.33}$$

Let $|\alpha\rangle$ be the eigenstates of A

$$Tr\, e^{A+B} \geq \sum_{\alpha} e^{A_\alpha + \langle\alpha|B|\alpha\rangle} \tag{2.34}$$

On setting

$$c_\alpha = \frac{e^{A_\alpha}}{Tr\, e^{A}} \tag{2.35}$$

(2.36)

and

$$f(x) = e^{x}$$

(2.37)

it is easy to see

$$\sum_{\alpha} \frac{e^{A_\alpha}}{Tr\, e^{A}}\, e^{\langle \alpha | B | \alpha \rangle} \geq e^{\sum_{\alpha} \frac{e^{A_\alpha}}{Tr\, e^{A}} \langle \alpha | B | \alpha \rangle} \qquad (2.38)$$

Hence

$$\sum_{\alpha} e^{A_\alpha + \langle \alpha | B | \alpha \rangle} \geq Tr\, e^{A}\, e^{\frac{Tr\, e^{A} B}{Tr\, e^{A}}}$$

and so

$$Tr\, e^{A+B} \geq Tr\, e^{A} \cdot e^{\frac{Tr\, e^{A} B}{Tr\, e^{A}}} \qquad (2.39)$$

i.e.,

$$\log Tr\, e^{A+B} \geq \log Tr\, e^{A} + Tr\, e^{A} B / Tr\, e^{A} \qquad (2.40)$$

In order to obtain (2.19), we simply set

$$\rho_t = e^{x} / Tr\, e^{x} \qquad (2.41)$$

and

$$-\beta \tilde{H} = x + B \qquad (2.42)$$

Now

$$\Omega = -\frac{1}{\beta} \log Tr\, e^{x+B} \qquad (2.43)$$

using (2.40), it is easy to see

$$-\frac{1}{\beta} \log Tr\, e^{-\beta \tilde{H}} \leq \frac{Tr\, e^{x} (H + \frac{1}{\beta} x)}{Tr\, e^{x}} - \frac{1}{\beta} \log Tr\, e^{x} \qquad (2.44)$$

Chapter III.

Basic Equations[4].

We shall be dealing with a soft pure superconductor with a hermitian Hamiltonian in the second quantized form given by (in units $\hbar = c = 2m = 1$)

$$\tilde{H} = \sum_{f_1 f_2} K_{f_1 f_2} a^*_{f_1} a_{f_2} + \frac{1}{2} \sum P_{f_1 f_2 f_3 f_4} a^*_{f_1} a^*_{f_2} a_{f_3} a_{f_4} \tag{3.1}$$

Since the super conductor is in an external magnetic field whose vector potential is $\vec{A}$, $K_{f_1 f_2}$ has nondiagonal matrix elements in momentum space of the form

$$K_{f_1 f_2} = \langle f_1 | (p - eA)^2 - \mu | f_2 \rangle \tag{3.2}$$

Following Bogoliubov[5] and Valatin[6] we can write the quasi-particle transformation for a_f, a^*_f as

$$\alpha_f = u_f a_f + v_f a^*_f$$
$$\alpha^*_f = u^*_f a^*_f + v^*_f a_f \tag{3.3}$$

with

$$u_f^2 + v_f^2 = 1. \tag{3.4}$$

α_f and α^*_f are quasi particle operators. We assume the quasiparticles to be statistically independent and we make the ansatz for the density matrix as

$$\rho_t = \prod_f \left[\Gamma^0_f \alpha^*_f \alpha_f + (1 - \Gamma^0_f) \alpha_f \alpha^*_f \right] \tag{3.5}$$

However in the presence of magnetic field, the transformation (3.3) has to be generalized by taking u_f and v_f to be matrices rather than scalars. We demand that the new transformation to be canonical. We also vary $\Gamma^0_f, u_{ff'}, v_{ff'}$ to yield the minimum free energy. These operations determine the unknown parameters entering the transformation as well as in the density matrix. The B C S equations are obtained as special cases of these more general equations[7].

Let us now consider gauge invariance of the above Hamiltonian (3.1). The gauge transformation is

$$A(x) \rightarrow A(x) + \nabla \chi(x)$$
$$\psi(x) \rightarrow e^{ie\chi} \psi \tag{3.6}$$

For an infinitesimal gauge transformation

$$A \rightarrow A + \nabla\chi \quad ; \psi \rightarrow \psi(1 + ie\chi) \tag{3.7}$$

and the annihilation and creation operators become

$$a_{p,\sigma} \rightarrow a_{p,\sigma} + ie\int_0^p \bar{\chi}(p-p')\, a_{p'\sigma}\, dp' \tag{3.8}$$

It is easy to see that $K_{f_1 f_2}$ is unaffected. Correspondingly the interaction term undergoes the transformation

$$\iiiint V(x_1,x_2,x_3,x_4)\, \psi^*(x_1)\, \psi^*(x_2)\, \psi(x_3)\, \psi(x_4)$$
$$\times e^{-ie\chi(x_1)}\, e^{-ie\chi(x_2)}\, e^{ie\chi(x_3)}\, e^{ie\chi(x_4)}$$
$$dx_1\, dx_2\, dx_3\, dx_4 \tag{3.9}$$

To make this expression gauge invariant, it is obvious that the interaction should contain some δ-functions i.e., $\delta(x_1 - x_4)$ and $\delta(x_2 - x_3)$ to cancel the exponential factors arising out of gauge transformation. This condition implies that our potential is local. However the B C S interaction in momentum space being limited to the regions around the Fermi surface, is a non-local potential. Hence the B C S interaction is not gauge invariant. We also note that the external magnetic field also destroys the translational invariance.

The B C S interaction arises as a net effect between the coulomb interaction of the electrons and the phonon interactions. The basic Hamiltonian including phonon coordinates will be gauge invariant. However in the presence of the external magnetic field, the effective potential V may depend on A. In such a case the potential may be supposed to have a factor, $\exp\left\{ ie\int_{x_1}^{x_3} \vec{A}\cdot dl \right\}$. In this fashion we can have a non local potential which is also gauge invariant.

Let us write down the generalized Bugoliubov transformations:

$$\alpha_f = \sum_{f'} \left\{ u_{ff'} a_{f'} + v_{ff'} a^*_{f'} \right\}$$

$$\alpha^*_f = \sum_{f'} \left\{ u^*_{ff'} a^*_{f'} + v^*_{ff'} a_{f'} \right\} \qquad (3.10)$$

or in matrix notation

$$\alpha = u a + v a^*$$

$$\alpha^* = u^* a^* + v^* a \qquad (3.11)$$

We find it convenient to express the conditions for the above transformation to be canonical through super matrices and vectors.

$$\mathcal{a} = \begin{pmatrix} a \\ a^* \end{pmatrix} \tag{3.12}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} \tag{3.13}$$

$$\mathcal{C} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} \tag{3.14}$$

Thus we see

$$\mathcal{a} = \mathcal{C}\,\boldsymbol{\alpha} \tag{3.15}$$

In order that this transformation be canonical, it is necessary that $\mathcal{C}$ be unitary, i.e.,

$$\mathcal{C}^{\dagger} = \mathcal{C}^{-1} \tag{3.16}$$

which can be verified by substitutions. There is another property of $\mathcal{C}$ which can be called the minor symmetry. This is defined as follows. Consider a matrix A

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{3.17}$$

The minor symmetric A^m equals

$$A^m = \begin{pmatrix} a_4^* & a_3^* \\ a_2^* & a_1^* \end{pmatrix} \tag{3.18}$$

The matrix $\mathcal{C}$ satisfies minor symmetry, i.e., $\mathcal{C}^m = \mathcal{C}$

We need the following dyadic forms

$$\mathcal{a}\,\mathcal{a}^* = \begin{pmatrix} aa^* & aa \\ a^*a^* & a^*a \end{pmatrix} \tag{3.19}$$

$$\mathbb{\alpha}\,\mathbb{\alpha}^* = \begin{pmatrix} \alpha\alpha^* & \alpha\alpha \\ \alpha^*\alpha^* & \alpha^*\alpha \end{pmatrix} \tag{3.20}$$

Their thermodynamic expectation values are

$$\langle \mathbb{a}\,\mathbb{a}^* \rangle = \begin{pmatrix} (1-G^{0*}) & F \\ -F^* & G^0 \end{pmatrix} = \mathbb{G}^0 \tag{3.21}$$

$$\mathbb{\Gamma}^0 = \begin{pmatrix} 1-\Gamma^0 & \Phi \\ -\Phi^* & \Gamma^0 \end{pmatrix} = \langle \mathbb{\alpha}\,\mathbb{\alpha}^* \rangle \tag{3.22}$$

It is useful to employ slightly modified expressions:

$$\begin{aligned} \mathbb{G} &= \mathbb{G}^0 - \tfrac{1}{2}\mathbb{1} \\ &= \begin{pmatrix} -G^* & F \\ -F^* & G \end{pmatrix} \end{aligned} \tag{3.23}$$

$$\begin{aligned} \mathbb{\Gamma} &= \mathbb{\Gamma}^0 - \tfrac{1}{2}\mathbb{1} \\ &= \begin{pmatrix} -\Gamma & \Phi \\ -\Phi^* & \Gamma \end{pmatrix} \end{aligned} \tag{3.24}$$

It is verified that $\mathbb{G}$ and $\mathbb{\Gamma}$ are hermitian and mirror antisymmetric, i.e.,

$$\mathbb{G} = \mathbb{G}^{\dagger} = -\mathbb{G}^{m} \tag{3.25}$$

$$G = G^{+} \quad , \quad F = -F^{+} \tag{3.26}$$

and
$$\mathbb{\Gamma} = \mathbb{\Gamma}^{+} = -\mathbb{\Gamma}^{m}$$

$$\Gamma = \Gamma^{+} \tag{3.27}$$

$$\phi = -\phi^{+} \tag{3.28}$$

Since $\mathcal{a}$ and α are connected by $\mathcal{C}$, we can verify the relation:-

$$\mathbb{G} = \mathbb{C}\,\mathbb{\Gamma}\,\mathbb{C}^{+} \tag{3.29}$$

We now recall the ansatz for the density matrix

$$\rho = \prod_{f} \left\{ \Gamma^{0}_{f}\, \alpha^{*}_{f}\, \alpha_{f} + \left(1 - \Gamma^{0}_{f}\right) \alpha_{f}\, \alpha^{+}_{f} \right\} \tag{3.30}$$

Using (3.30), we evaluate (3.24). In doing this it is to be remembered that trace operation is over the Fock space and not on the super-matrix space. We now find that (3.24)becomes

$$\mathbb{\Gamma} = \begin{pmatrix} -\Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \tag{3.31}$$

Using (3.29) and (3.31), we obtain from (3.23) the expressions

$$G^{0} = u^{*}\Gamma^{0}u + v^{*}(1-\Gamma^{0})v$$

$$F = v\Gamma^{0}u + u(1-\Gamma^{0})v$$

$$F^{+} = u^{*}\Gamma^{0}v^{*} + v^{*}(1-\Gamma^{0})u^{*} \tag{3.32}$$

The two particle correlation function i.e., $\langle a^*_{f_1} a^*_{f_2} a_{f_3} a_{f_4} \rangle$ can be computed as follows:

$$\langle a^*_{f_1} a^*_{f_2} a_{f_3} a_{f_4} \rangle$$

$$= \langle (u^*_{f_1\nu_1} \alpha^*_{\nu_1} + v_{f_1\nu_1} \alpha_{\nu_1})(u^*_{f_2\nu_2} \alpha^*_{\nu_2} + v^*_{f_2\nu_2} \alpha_{\nu_2})$$

$$(u_{f_3\nu_3} \alpha_{\nu_3} + v_{f_3\nu_3} \alpha^*_{\nu_3})(u_{f_4\nu_4} \alpha_{\nu_4} + v_{f_4\nu_4} \alpha^*_{\nu_4}) \rangle \tag{3.33}$$

$$= G^0_{f_1 f_4} G^0_{f_2 f_3} - G^0_{f_1 f_3} G^0_{f_2 f_4} + F^+_{f_1 f_2} F_{f_3 f_4}$$

It is interesting to note that the two particle correlation function appears in a factorized form. The three terms in (3.33) can be thought to correspond to Hartree, Fock and BCS approximations respectively. We compute $\langle \hat{H} \rangle$ as

$$\langle \hat{H} \rangle = \sum K_{f_1 f_2} \langle a^*_{f_2} a_{f_1} \rangle + \frac{1}{2} \sum P_{f_1 f_2 f_3 f_4} \langle a^*_{f_1} a^*_{f_2} a_{f_3} a_{f_4} \rangle$$

$$= \sum K_{f_1 f_2} G^0_{f_2 f_1} + \frac{1}{2} \sum P_{f_1 f_2 f_3 f_4} (G^0_{f_1 f_4} G^0_{f_2 f_3} - G^0_{f_1 f_3} G^0_{f_2 f_4} + F^+_{f_1 f_2} F_{f_3 f_4}) \tag{3.34}$$

$$= \frac{1}{2} Sp[(E+K) G^0 + P^+ D]$$

where we have the following:

$$Q_{f_1 f_2 f_3 f_4} = P'_{f_1 f_2 f_3 f_4} - P''_{f_1 f_2 f_3 f_4}$$

with

$$P'_{f_1 f_2 f_3 f_4} = P_{f_1 f_3 f_4 f_2} \; ; \; P''_{f_1 f_2 f_3 f_4} = P_{f_1 f_3 f_2 f_4} \tag{3.35}$$

We also introduce a dot operation by

$$\sum A_{\xi_1\xi_2} B_{\xi_2\xi_1} = A\cdot B = Sp(AB)$$
$$\sum A_{\xi_1\xi_2\xi_3\xi_4} B_{\xi_4\xi_3} = (A\cdot B)_{\xi_1\xi_2}$$
$$\sum A_{\xi_1\xi_2} B_{\xi_2\xi_1\xi_3\xi_4} = (A\cdot B)_{\xi_3\xi_4} \qquad (3.36)$$

using (3.35) and (3.36) we can verify

$$E = K + Q\cdot G^{0} \qquad (3.37)$$

$$D = P\cdot F \qquad (3.38)$$

To derive (3.37) and (3.38) we have to keep in mind that

$$P_{\xi_1\xi_2\xi_3\xi_4} = P(\xi_1-\xi_4) = P(\xi_2-\xi_3) \qquad (3.39)$$

As before we again find it convenient to introduce the super matrices:

$$\mathbb{E} = \mathbb{E}^{+} = -\mathbb{E}^{m} = \begin{pmatrix} -E^{*} & D \\ -D^{*} & E \end{pmatrix} \qquad (3.40)$$

and

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{+} = -\boldsymbol{\mathcal{E}}^{m} = \begin{pmatrix} -\mathcal{E}^{*} & \Delta \\ -\Delta^{*} & \mathcal{E} \end{pmatrix} \qquad (3.41)$$

such that

$$\mathbb{E} = \mathbb{C}\,\boldsymbol{\mathcal{E}}\,\mathbb{C}^{+} \qquad (3.42)$$

Now we can express (3.37) and (3.38) as

$$\mathbb{E} = \mathbb{E}(\mathbb{G}^0) = \mathbb{K} + \mathbb{P}\circ G^0 \tag{3.43}$$

where

$$\mathbb{K} = \begin{pmatrix} -K^* & 0 \\ 0 & K \end{pmatrix} \tag{3.44}$$

$$\mathbb{P}\cdot\mathbb{G}^0 = \begin{pmatrix} -Q\cdot G^{0*} & P\cdot F \\ -P\cdot F^* & Q\cdot G^0 \end{pmatrix} \tag{3.45}$$

Using these expressions we can write down for the thermodynamic potential Ω

$$\Omega = \langle \hat{H} \rangle - TS$$

$$= \frac{1}{4} Sp\left[(\mathbb{E}+\mathbb{K})\mathbb{G}^0 + 2\beta^{-1}\left(\Pi^0 \ln \Pi^0 + (1-\Pi^0)\ln(1-\Pi^0)\right)\right] \tag{3.46}$$

where Sp means the trace operation in the super matrix space and this for the factor 1/4. It is also seen that Ω is real.

We have to minimize Ω by varying $\mathbb{C}$ and Γ .
However since $\mathbb{C}$ is unitary and mirror symmetric, we use the method of Lagrangian multipliers, to take account of these restrictions. The restrictions are

$$\mathbb{U}^{(1)} = \mathbb{C}\mathbb{C}^\dagger - 1 = 0$$

$$\mathbb{U}^{(2)} = \mathbb{C}^\dagger\mathbb{C} - 1 = 0$$

$$\mathbb{C} - \mathbb{C}^m = 0 \tag{3.49}$$

The Lagrangian multipliers are

$$\Lambda^{(1)} = \Lambda^{(1)\dagger}$$

Minimum property of Ω can be expressed as that the first order changes of

$$\Omega = \Omega - Sp\,(\Lambda^{(1)} U^{(1)} + \Lambda^{(2)} U^{(2)}) \tag{3.51}$$

with respect to variations of $\mathbb{C}$, $\mathbb{C}^{+}$ and Π have to vanish. From (3.46) we obtain

$$\begin{aligned} 0 = \delta\Omega = Sp\Big[\tfrac{1}{4}\big(\delta E\, G^{0} + (E + K)\,\delta G^{0}\big) \\ + \tfrac{1}{2}\beta^{-1} \ln\Big(\frac{\Pi^{0}}{1-\Pi^{0}}\Big)\,\delta\Pi^{0} \\ - \Lambda^{(1)}\delta U^{(1)} - \Lambda^{(2)}\delta U^{(2)}\Big] \end{aligned} \tag{3.52}$$

We also note

$$\delta E = P \cdot \delta G^{0} \tag{3.53}$$

$$\delta G^{0} = \delta\mathbb{C}\,\Pi^{0}\mathbb{C}^{+} + \mathbb{C}\,\Pi^{0}\delta\mathbb{C}^{+} + \mathbb{C}\,\delta\Pi^{0}\,\mathbb{C}^{+} \tag{3.54}$$

$$\delta U^{(1)} = \delta\mathbb{C}\,\mathbb{C}^{+} + \mathbb{C}\,\delta\mathbb{C}^{+} \tag{3.55}$$

$$\delta U^{(2)} = \delta\mathbb{C}^{+}\mathbb{C} + \mathbb{C}^{+}\delta\mathbb{C} \tag{3.56}$$

Using these equations we obtain for

a) $\delta\Pi^{0} \neq 0$ but $\delta\mathbb{C} = 0$ and $\delta\mathbb{C}^{+} = 0$

$$\begin{aligned} 0 = Sp\Big[\tfrac{1}{4}\big(P \cdot (\mathbb{C}\,\delta\Pi^{0}\mathbb{C}^{+})\,G^{0} + (E + K)(\mathbb{C}\,\delta\Pi^{0}\mathbb{C}^{+})\big) \\ + \tfrac{1}{2}\beta^{-1}\ln\frac{\Pi^{0}}{1-\Pi^{0}}\,\delta\Pi^{0}\Big] \end{aligned}$$

or

$$= \frac{1}{2} Sp\Big[\Big(\mathbb{C}^{+}E\,\mathbb{C} + \beta^{-1}\ln\frac{\Pi^{0}}{1-\Pi^{0}}\Big)\delta\Pi^{0}\Big]$$

$$\Big(\mathbb{C}^{+}E\,\mathbb{C} + \beta^{-1}\ln\frac{\Pi^{0}}{1-\Pi^{0}}\Big)_{\text{diagonal}} = 0 \tag{3.57}$$

This leads to

$$\Gamma^0 = \left[1 + \exp\left(\beta\, \mathcal{E}_{diagonal}\right)\right]^{-1} \tag{3.58a}$$

and
$$\Gamma = -\frac{1}{2}\tanh\left(\frac{\beta}{2}\, \mathcal{E}_d\right) \tag{3.58b}$$

b) $\delta\Gamma^0 = 0$ but $\delta\mathbb{C} \neq 0$ and $\delta\mathbb{C}^{+} = 0$.

Now

$$0 = Sp\left[\tfrac{1}{2}\mathbb{E}\,\delta\mathbb{C}\,\Gamma^0\,\mathbb{C}^{+} - \Lambda^{(1)}\delta\mathbb{C}\,\mathbb{C}^{+} - \Lambda^{(2)}\mathbb{C}^{+}\delta\mathbb{C} + \tfrac{1}{2}\mathbb{E}\,\mathbb{C}\,\delta\Gamma^0\,\delta\mathbb{C}^{+} - \Lambda^{(1)}\mathbb{C}\,\delta\mathbb{C}^{+} - \Lambda^{(2)}\delta\mathbb{C}^{+}\mathbb{C}\right]$$

using
$$\delta\mathbb{C}^{+} = -\mathbb{C}^{+}\,\delta\mathbb{C}\,\mathbb{C}^{+}$$

we find

$$0 = Sp\left[\tfrac{1}{2}\Gamma^0\mathbb{C}^{+}\mathbb{E}\,\delta\mathbb{C} - \tfrac{1}{2}\mathbb{C}\,\mathbb{E}\,\mathbb{C}\,\Gamma^0\mathbb{C}^{+}\delta\mathbb{C}\right]$$
$$= \frac{1}{2}Sp\left[\left(\mathbb{G}^0\mathbb{E} - \mathbb{E}\mathbb{G}^0\right)\delta\mathbb{C}\,\mathbb{C}\right] \tag{3.59}$$

Hence it follows that

$$\left[\mathbb{G}^0, \mathbb{E}\right] = 0\ ; \left[\mathbb{G}, \mathbb{E}\right] = 0 \tag{3.60}$$

because $\left[\mathbb{G}^0, \mathbb{E}\right]$ has the same symmetrics as $\delta\mathbb{C}^{+}\mathbb{C}$ (3.58) and (3.60) are the basic equations. Since $\left[\mathbb{G}, \mathbb{E}\right] = 0$ it should be possible to diagonalise them simultaneously. Now from (3.58b), we have

$$\mathbb{G} = -\frac{1}{2}\tanh\left(\frac{\beta}{2}\,\mathbb{E}_{diagonal}\right) \tag{3.61}$$

Hence we can consider (3.60) and (3.61) as a set of equations to be solved for G and E. Alternatively we can also consider the above set of equations as equations for C and Γ . This is really connected with the choice of $\mathcal{C}$ which will diagonalize G and E , when Γ is diagonal. To the end consider a different set of quasiparticles α' such that

$$
\begin{aligned}
\alpha &= \mathbb{D}\,\alpha' \\
\text{and}\quad \mathbb{D} &= (\mathbb{D}^{+})^{-1} = \mathbb{D}^{m} \\
[\mathbb{D}, \Gamma^{0}] &= 0
\end{aligned}
\qquad (3.62)
$$

It is easy to see that because of (3.62), the physical properties of the system are not affected by this transformation. Hence it is always possible to find a mirror symmetric unitary $\mathbb{C}$ matrix which diagonalises $\mathbb{G}$ and $\mathbb{E}$ simultaneously into Γ and $\mathcal{E}$. For this $\mathbb{C}$, we have automatically

$$
0 = \mathbb{C}\,[\Gamma, \mathcal{E}]\,\mathbb{C}^{+} = [\mathbb{C}\Gamma\mathbb{C}^{+}, \mathbb{C}\mathcal{E}\mathbb{C}^{+}] = [\mathbb{G}, \mathbb{E}] \qquad (3.63)
$$

We can now rephrase the entire problem as determination of $\mathcal{C}$ and Γ .

$$
\mathbb{C}\mathbb{C}^{+} = \mathbb{C}^{+}\mathbb{C} = 1 \qquad (3.64a)
$$

$$
\mathbb{G} = \mathbb{G}^{+} = -\mathbb{G}^{m} = \mathbb{C}\Gamma\mathbb{C}^{+} \qquad (3.64b)
$$

$$
\mathbb{E} = \mathbb{K} + \mathbb{F}\cdot\mathbb{G}^{0} = \mathbb{E}^{+} = -\mathbb{E}^{m} = \mathbb{C}\mathcal{E}\mathbb{C}^{+} \qquad (3.64c)
$$

$$
\Gamma = -\frac{1}{2}\tanh\left(\frac{\beta}{2}\mathcal{E}\right) \qquad (3.64d)
$$

We will refer to the equations (3.64) as $\mathcal{C}$ problem',

Chapter IV.

B C S Equations

We now solve the non linear equations for the case $A = 0$ and obtain the gap equations *à la* B C S. When $A = 0$ the spin dependence of K, P and Q are factorizable and diagonal. P is translationally invariant. To take advantage of this we now introduce the following;

$$S^{(1)}_{\sigma_1,\sigma_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^{\sigma_1}_{\sigma_2}$$

$$S^{(2)}_{\sigma_1,\sigma_2} = \sigma_1 \cdot \delta^{\sigma_1}_{\bar{\sigma}_2} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \tag{4.1}$$

$$I^{(1)}_{p,p'} = \delta_{p,p'} \quad ; \quad I^{(2)} = \delta_{p,\bar{p}'} \tag{4.2}$$

In the above we have introduced the 'barring' operation by which we mean

$$\bar{f} = (-p, -\sigma) \quad \text{if} \quad f = (p, \sigma) \tag{4.3}$$

In this notation

$$u_{p,\sigma,p',\sigma'} = u_p \, I^{(1)}_{p,p'} \, S^{(1)}_{\sigma,\sigma'}$$

$$v_{p,\sigma,p',\sigma'} = v_p \, I^{(2)}_{p,p'} \, S^{(2)}_{\sigma,\sigma'} \tag{4.4}$$

In this case we find that

$$\left.\begin{aligned} u &= u_d \\ G &= G_d \\ \Gamma &= \Gamma_d \\ E &= E_d \\ \mathcal{E} &= \mathcal{E}_d \end{aligned}\right\} I^{(1)} S^{(1)} \qquad \left.\begin{aligned} V &= V_d \\ F &= F_d \\ \Phi &= \Phi_d \\ D &= D_d \\ \Delta &= \Delta_d \end{aligned}\right\} I^{(2)} S^{(2)} \tag{4.5}$$

where the subscript d indicates diagonal matrix element. We that (3.64a) becomes

$$u_d^2 + v_d^2 = 1 \tag{4.6}$$

The interaction potential is assumed to have the properties

$$P'^{(\sigma)}_{\sigma_1\sigma_2\sigma_3\sigma_4} = \delta^{\sigma_1}_{\sigma_4}\delta^{\sigma_2}_{\sigma_3}\ ;\ P''^{(\sigma)}_{\sigma_1\sigma_2\sigma_3\sigma_4} = \delta^{\sigma_1}_{\sigma_2}\delta^{\sigma_3}_{\sigma_4}$$

$$P'^{(q)}_{q_1q_2q_3q_4} = \delta^{q_1-q_3}_{q_2+q_4} P_{q_1-q_4}\ ;\ P''^{(q)}_{q_1q_2q_3q_4} = \delta^{q_1+q_3}_{q_2+q_4} P_{q_1-q_2} \tag{4.7}$$

Using these it is easy to verify

$$\begin{aligned} P'^{(\sigma)} \circ S^{(1)} &= S^{(1)} \\ P''^{(\sigma)} \circ S^{(1)} &= 2S^{(1)} \\ P^{(\sigma)} \circ S^{(2)} &= -S^{(2)} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} P'^{(k)}(I^{(1)}A_d) &= I^{(1)}(PA)_d \\ (PA)_k &= \sum_{k'} P_{k-k'} A_{k'} \\ P''^{(k)}(I^{(1)}A_d) &= I^{(1)}(P''^{(k)}A)_d \\ (P''^{(k)}A) &= P_0 \sum_k A_k = \text{Const} \\ P(I^{(2)}A_d) &= I^{(2)}(P\bar{A})_d \end{aligned} \tag{4.9}$$

After this ansatz (4.5) we have using (4.7), (4.8), (4.9)

$$E_k = k^2 - \mu + \sum_{k'} \left(P_{k-k'} - 2P_0\right) G^0_{k'} \tag{4.10}$$

where
$$G^{0}_{k'} = \frac{\Gamma_{k'}}{\mathcal{E}_{k'}} E_{k'} + \frac{1}{2} \tag{4.11}$$

and
$$D_k = -\sum_{k'} P_{k-k'} F_{k'} \tag{4.12}$$

where
$$F_{k'} = \frac{\Gamma_{k'}}{\mathcal{E}_{k'}} D_{k'} \tag{4.13}$$

$$\Gamma_k = -\frac{1}{2} \tanh \frac{\beta}{2} \mathcal{E}_k \tag{4.14}$$

$$\mathcal{E}_k = \sqrt{E_k^{2} + D_k^{2}} \tag{4.15}$$

It is interesting to recognize that $\mathcal{E}_p$ is simply the eigenvalue of $\mathbb{E}$. E_k given in (4.10) is more general than the corresponding BCS expression in that it includes the Hartree Fock term, $\sum_{k'} (P_{kk'} - 2P_0) G^{0}_{k'}$. We will simply omit the Hartree Fock term as it merely complicates the equations.

Let us look at the equation for the gap given by (4.12). A trivial solution $D_k = 0$ always exists which means that $F = 0$. In this case there is no super conductor as is well known from BCS arguments. Assuming that a non trivial solution D_k always exists, we can compute D_k when the potential is factorizable

$$P_{kk'} \longrightarrow \lambda V_k V_{k'} \tag{4.16}$$

where λ is the strength of the interaction. If we put

$$D_k = V_k c \tag{4.17}$$

we obtain a transcendental equation for C :

$$1 = -\lambda \sum_{k'} V_{k'} \frac{\Gamma_{k'}}{\mathcal{E}_{k'}} = -\lambda \sum_{k'} V_{k'}^{2} \Gamma_{k'}^{*} \Big/ \sqrt{(k^2-\mu)^2 + V_{k'}^2 C^2} \qquad (4.18)$$

Only for the attractive case, we can hope to obtain a non trivial solution, since Γ_k is positive definite. In the BCS case the potential is assumed to be constant over a small shell around the Fermi surface of width given by Debye momentum. At $T=0$, (4.18) becomes

$$1 = -\lambda \sum_{(k')^2} \frac{\Gamma_{k'}}{\sqrt{(k^2-\mu)^2 + C^2}} \qquad (4.19)$$

For small λ ,

$$C \sim \omega e^{-\frac{1}{|\lambda| N(0)}} \qquad (4.20)$$

where $N(0)$ is the density of states. It is clear from (4.20) that there is no expansion around $\lambda = 0$ since every differential coefficient vanishes near $\lambda = 0$. It is to be noted that there is an essential singularity at $\lambda = 0$. D_k at any finite T can also be calculated and the critical temperature T_c is obtained for the vanishing of D. In the weak coupling approximation

$$kT_c = 1.14\, \omega e^{-\frac{1}{|\lambda| N(0)}} \qquad (4.21)$$

We can also check that there is no bulk current, i.e.,

$$\begin{aligned}\langle J(q) \rangle &= \frac{e}{V} \sum_{p,\sigma} (2p+q)\, G^{0}_{p,\sigma,\, p+q,\sigma} \\ &= \frac{e}{V} \sum_{p,\sigma} (2p+q)\, \delta^{\sigma}_{\sigma} \delta^{p}_{p+q} G^{0}_{p} \\ &= \frac{4e}{V} \delta^{0}_{q} \sum_{p} p \cdot G^{0}_{p} = 0 , \end{aligned} \qquad (4.22)$$

Chapter V.

Meissner Effect.

Having obtained BCS equations in the absence of the external magnetic we shall study the properties of a superconductor in a weak external magnetic field i.e. when A is small. From thermodynamic considerations we know that the difference in free energy between the superconducting state and the normal state in the absence of magnetic field is

$$F_{so} - F_{no} = - H_c^2 / 8\pi \tag{5.1}$$

where the H_c is the critical field which destroys superconductivity at $T = 0$. By small A we mean that

$$A/\delta \ll H_c \tag{5.2}$$

where δ is the penetration depth.

From equations (3.46) we have to solve for $\mathbb{C}$ when $\mathbb{K}$ contains two terms $\mathbb{K}^0$ and $\mathbb{K}^{(1)}$, $\mathbb{K}^{(1)}$ being treated as a perturbation due to small A. Consider the perturbation ansatz

$$\mathbb{C} = \mathbb{C}^0 (1 + \mathbb{B}) \tag{5.3}$$

such that

$$\mathbb{C}\,\mathbb{C}^{\dagger} = 1 \tag{5.4}$$

which implies $\mathbb{B} = -\mathbb{B}^{\dagger}$. The change in $\mathbb{\Gamma}$ is related to the change in $\mathbb{G}$, both being diagonal is given by

$$\Gamma' = \frac{d\Gamma}{d\mathcal{E}} \cdot \mathcal{E}' = -\frac{\beta}{4}\left[\cosh\left(\frac{\beta}{2}E\right)\right]^{-2} \mathcal{E}' \qquad (5.5)$$

Similarly changes in $\mathbb{C}$, $\mathbb{G}$, and $\mathbb{E}$ are given by

$$\mathbb{C}' = \mathbb{C} \cdot \mathbb{B}$$

$$\mathbb{G}' = \mathbb{C}\,\hat{\Gamma}\,\mathbb{C}^{+} \qquad (5.6)$$

$$\mathbb{E} = \mathbb{C}\,\hat{\mathcal{E}}\,\mathbb{C}^{+} \qquad (5.7)$$

$$(5.8)$$

with

$$\hat{\Gamma} = [\mathbb{B}, \Gamma] + \Gamma' = \hat{\Gamma}^{+} \qquad (5.9)$$

$$\hat{\mathcal{E}} = [\mathbb{B}, \mathcal{E}] + \mathcal{E}' = \hat{\mathcal{E}}^{+} \qquad (5.10)$$

where from now we will use $\mathbb{C}$ to indicate $\mathbb{C}^{0}$. Now

$$\mathbb{E}' = \mathbb{K}' + \mathbb{P} \cdot \mathbb{G}' \qquad (5.11)$$

where

$$K'_{\gamma_1\gamma_2} = \delta^{\sigma_1}_{\sigma_2}\,(q_1 + q_2) \circ A(q_2 - q_1) \qquad (5.12)$$

Substituting equations (5.6) to (5.8) in (5.11) we obtain the basic equation

$$\hat{\mathcal{E}} - \mathbb{C}^{+}[\mathbb{P} \circ (\mathbb{C}\,\hat{\Gamma}\,\mathbb{C}^{+})]\,\mathbb{C} = \mathbb{C}^{+}\,\mathbb{K}'\,\mathbb{C} \qquad (5.13)$$

where the elements $\hat{\Gamma}$ and $\hat{\mathcal{E}}$ are written as

$$\hat{\Gamma} = \begin{pmatrix} -\hat{\rho}^{*} & \hat{\phi} \\ -\hat{\phi}^{*} & \hat{\rho} \end{pmatrix} = \hat{\Gamma}^{+} \qquad (5.14)$$

and
$$\hat{\mathcal{B}} = \begin{pmatrix} -\hat{\mathcal{E}} & \hat{\Delta} \\ -\hat{\Delta}^{+} & \hat{\mathcal{E}} \end{pmatrix} = \hat{\mathcal{B}}^{+} \tag{5.15}$$

Separating as before the spin dependence, we have
$$\mathbb{C} = \begin{pmatrix} u_d I^{(1)} S^{(1)} & v_d I^{(2)} S^{(2)} \\ v_d I^{(2)} S^{(2)} & u_d I^{(1)} S^{(1)} \end{pmatrix} \tag{5.16}$$

and
$$\mathbb{K}' = \begin{pmatrix} -K'^{*} & 0 \\ 0 & K' \end{pmatrix} \tag{5.17}$$

Writing
$$\hat{\Gamma} = \begin{pmatrix} -S^{(1)}\hat{\Gamma}^{*} & S^{(2)}\hat{\Phi} \\ -S^{(2)}\hat{\Phi}^{*} & S^{(1)}\hat{\Gamma} \end{pmatrix} \tag{5.18}$$

and
$$\hat{\mathcal{B}} = \begin{pmatrix} -S^{(1)}\hat{\mathcal{E}}^{*} & S^{(2)}\hat{\Delta} \\ -S^{(2)}\hat{\Delta}^{*} & S^{(1)}\hat{\mathcal{E}} \end{pmatrix}, \tag{5.19}$$

we have
$$G' = -v\hat{\Gamma}^{*}v - u\hat{\Phi}^{*}v - v\hat{\Phi}u + u\hat{\Gamma}u \tag{5.20}$$
$$F' = u\hat{\Gamma}v - v\hat{\Phi}^{*}v + u\hat{\Phi}u + u\hat{\Gamma}u \tag{5.21}$$

Using the above equations, we obtain from (5.13)
$$\hat{\mathcal{E}} + vQ_1G'^{*}v + vP_1F'u + uP_1F'^{*}v - uQ_1G'u = uK'u - vK'v \tag{5.22}$$
$$\hat{\Delta} + uQ_1G'^{*}v + uP_1F'u - vP_1F'^{*}v + vQ_1G'u = -uK'^{*}v - vK'u \tag{5.23}$$

Sincerity of these equations permits an easy separation of the real and imaginary parts. To this end, it is convenient to use

the definitions

$$A_\sigma = \frac{1}{2}\left(A + \sigma A^*\right); \quad A = \sum_\sigma A_\sigma \tag{5.24}$$

where σ = + real

= - imaginary. Also let us define

$$A_{\tau\, KK'} = \frac{1}{2}\left(A_{KK'} + \tau A_{-K,-K'}\right)$$

$$A_{KK'} = \sum_\tau A_{\tau\, KK'}$$

with τ = +, even

= -, odd (5.25)

Writing the separated equations in terms of matrix elements we observe that unknowns occur in pairs, i.e.,

$$\hat{\Gamma}_{K,K'} \longleftrightarrow \hat{\Gamma}_{-K,-K'}$$

$$\hat{\phi}_{K,-K'} \longleftrightarrow \hat{\phi}_{-K,K'} \tag{5.26}$$

Introducing the abbreviations

$$g_{KK'} = u_K u_{K'} - \sigma\tau V_K V_{K'}; \quad f_{KK'} = \sigma u_K u_{K'} + \tau V_K u_{K'}$$

$$Q_{KK'mm'} = P_{Kmm'K'} - 2P_{Km\,K'm'}$$

we have

$$\hat{\varepsilon}_{KK'} + \left[-g_{KK'}Q_{KK'm'm}\,g_{mm'} + f_{KK'}P_{K,-K'm,-m'}\,f_{mm'}\right]\hat{\Gamma}_{mm'}$$

$$+\left[g_{KK'}Q_{KK'm'm}\,f_{mm'} + f_{KK'}P_{K,-K',m,-m'}\,g_{mm'}\right]\hat{\phi}_{m,-m'}$$

$$= g_{KK'}\,K'_{KK'} \tag{5.27}$$

$$\hat{\Delta}_{k,-k'} + \left[f_{kk'} Q_{kk'm'm} g_{mm'} + g_{kk'} P_{k,-k',m,-m'} f_{mm'} \right] \hat{\Gamma}_{m,m'}$$

$$+ \left[-f_{kk'} Q_{kk'm'm} f_{mm'} + g_{kk'} P_{k,-k',m,-m'} g_{mm'} \right] \hat{\Phi}_{m,}$$

$$= - f_{kk'} K'_{kk'} \tag{5.28}$$

We see from the properties of P and Q, that the quantity $q = k' - k$ enters parametrically into (5.27) and (5.28) and thus the problem reduces to solving equations for fixed q.

Introducing the vector and matrix notation:

Vectors

$$\check{\mathcal{E}} = \left(\hat{\mathcal{E}}^{q}_{\sigma\tau\, k} \right)$$

$$\check{\Delta} = \left(\hat{\Delta}^{q}_{\sigma\tau\, k} \right)$$

$$\check{\Gamma} = \left(\hat{\Gamma}^{q}_{\sigma\tau\, k} \right)$$

$$\check{\Phi} = \left(\hat{\Phi}^{q}_{\sigma\tau\, k} \right)$$

$$\check{K} = \left(K'^{q}_{\sigma\tau\, k} \right) \tag{5.29}$$

diagonal matrices

$$g = \left(g^{q}_{k} \right) \; ; \quad f = \left(f^{q}_{k} \right) \tag{5.30}$$

and matrices

$$P = \left(P^{q}_{km} \right) \; ; \; Q = \left(Q^{q}_{km} \right) \tag{5.31}$$

where

$$P_{k,-k',m,-m'} = \delta^{k'-k}_{m'-m} P^{q}_{km} = \delta^{k'-k}_{m'-m} P_{k-m}$$

$$Q_{kk'm'm} = \delta^{k'-k}_{m'-m} Q^{q}_{km} = \delta^{k'-k}_{m'-m} \left(P_{k-m} - 2P_0 \right) \tag{5.32}$$

We can write (5.27) and (5.28) as

$$\check{\mathcal{E}} + [fPf - gQg]\check{\Gamma} + [fPg + gQf]\check{\Phi} = g\check{K}'$$

$$\check{\Delta} + [gPf + fQg]\check{\Gamma} + [gPg - fQf]\check{\Phi} = -f\check{K}' \tag{5.33}$$

By examining the equations (5.33), we find that

$$[1 + \rho^{-}(-gQg + fPf)]\check{\Gamma} + [\rho^{-}(gQf + fPg)]\check{\Phi} = \rho^{-} g \check{K}'$$

$$[1 + \rho^{+}(-fQf + gPg)]\check{\Phi} + [\rho^{+}(fQg + gPf)]\check{\Gamma} = -\rho^{+} f \check{K}' \tag{5.34}$$

where

$$\mathcal{E}^{\pm} = \mathcal{E}_{k} \pm \mathcal{E}_{k+q}$$

$$\Gamma^{\pm} = \Gamma_{k} \pm \Gamma_{k+q}$$

$$\rho^{\pm} = (\mathcal{E}^{\pm})^{-1} \Gamma^{\pm} = \Gamma^{\pm} (\mathcal{E}^{\pm})^{-1} \tag{5.35}$$

and
$$\check{\Gamma} = \rho^{-} \check{\mathcal{E}}$$
$$\check{\phi} = \rho^{+} \check{\Delta}$$

(5.36)

To establish Meissner effect we have to study the induced current $\langle J(q) \rangle$:

$$\langle J(q) \rangle' = \langle J^{p}(q) \rangle' + \langle J^{d}(q) \rangle' \tag{5.37}$$

where

$$\langle J^{p}(q) \rangle' = \frac{e}{V} \sum_{p,\sigma} (2p+q) G'_{p,\sigma\,,\,p+q,\sigma}$$
$$\langle J^{d}(q) \rangle' = \frac{2eN}{\hat{V}} A_q \tag{5.38}$$

Since G' depends on $\hat{\Gamma}$, we see from above

$$\langle J^{p}(q) \rangle' = \frac{2e}{V} \sum_{p,\sigma,T} (2p+q) \left(g_p \check{\Gamma}_p - f_p \check{\phi}_p \right) \tag{5.39}$$

Since gauge invariance and Meissner effect are linked, we shall discuss gauge invariance of the above equations. The gauge transformation is:

$$\vec{A}(x) \to \vec{A}(x) + \nabla \chi(x)$$

or in k space

$$A(q) = A(q) + q \cdot \eta(q) \tag{5.30}$$

Accordingly the change in perturbation $K^{(1)}$ is

$$K^{(1)}_{ff'} = K^{(1)}_{ff'} + \delta^{\sigma}_{\sigma'} (k'^2 - k^2) \eta_{\vec{k}' - \vec{k}} \tag{5.31}$$

The associated transformation in ψ is

$$\psi(x,\sigma) = e^{i\chi(x)}\,\psi(x,\sigma)$$

This implies for the annihilation and creation operators

$$a_{q\sigma} = \sum_{q'} (e^{\chi})_{q-q'}\; a_{q'\sigma}$$

For infinitesimal gauge transformation

$$a = (1+\eta)\,a$$
$$a^{+} = (1+\eta^{*})\,a^{+}$$

Corresponding to (5.34), the change in G and F denoted by G' and F' are:

$$G' = [G, \eta^{*}]$$
$$F' = F\eta^{*} - \eta F$$

or

$$\mathbb{G}' = [\mathbb{C}\,\mathbb{B}\mathbb{C}^{+}, \mathbb{G}]$$
$$= \begin{bmatrix} [\eta, G^{*}] & , F\eta^{*} - \eta F \\ -F^{*}\eta + \eta^{*}F^{*} & , [G, \eta^{*}] \end{bmatrix} \qquad (5.41)$$

which yields

$$\mathbb{C}\,\mathbb{B}^{+}\mathbb{C} = \begin{pmatrix} -\eta & 0 \\ 0 & -\eta^{*} \end{pmatrix}$$

The gauge term being a pure longitudinal potential, we shall study the effects of a pure longitudinal potential. It is seen that the solution of the $\mathcal{B}$ problem leads to a current which vanishes. In this case equation (5.34) can be written as

$$(1+\mathbb{Z})\,\check{\mathbb{\Gamma}}^{l} = \check{\mathbb{K}}^{l} \tag{5.42}$$

where

$$\mathbb{Z} = \begin{pmatrix} \rho^{-}[-g\,\mathcal{Q}\,g + f\,P\,f] & \rho^{-}[g\,\mathcal{Q}\,\tilde{f} + f\,P\,g] \\ \rho^{+}[f\,\mathcal{Q}\,g + g\,P\,f] & \rho^{+}[-\tilde{f}\,\mathcal{Q}\,\tilde{f} + g\,P\,g] \end{pmatrix};$$

$$\check{\mathbb{\Gamma}}^{l} = \begin{pmatrix} \check{\Gamma}^{l} \\ \check{\phi}^{l} \end{pmatrix}$$

$$\check{\mathbb{K}}^{l} = \begin{pmatrix} \rho^{-}g\,(k'^{2}-k^{2})\,\eta_{q} \\ -\rho^{+}f\,(k'^{2}-k^{2})\,\eta_{q} \end{pmatrix}$$

where we have introduced the abbreviations

$$g = uu' + vv'; \quad \tilde{g} = uu' - vv'$$

$$f = \sigma(uv' - vu'); \quad \tilde{f} = \sigma(uv' + vu')$$

In the above the unprimed quantities are evaluated at the argument k and the primed ones at $k' = k + q$. Since we are studying the effect of a longitudinal potential, Γ' is zero and hence

$$\hat{\mathbb{\Gamma}} = [\mathbb{B}, \mathbb{\Pi}]$$

$$= \begin{pmatrix} -\hat{\Gamma}^{*} & \hat{\phi} \\ -\hat{\phi}^{*} & \hat{\Gamma} \end{pmatrix} = \begin{pmatrix} -[B^{*},\Gamma] & \{C,\Gamma\} \\ -\{C^{*},\Gamma\} & [B,\Gamma] \end{pmatrix} \tag{5.43}$$

i.e., $$\hat{\phi} = \{C, \Gamma\} = \{\Gamma, -u\eta v + v\eta^* u\}$$

$$\hat{B}\hat{\Gamma} = [B, \Gamma] = [\Gamma, -V\eta V + u\eta^* u] \tag{5.44}$$

As before we can factor off the spin dependence and further introduce the σ and Υ operation. After some manipulation we have

$$\check{\Pi}^{e} = \begin{pmatrix} \Gamma^{+} & \tilde{f} & \eta \\ -\Gamma^{+} & \tilde{g} & \eta \end{pmatrix} \tag{5.45}$$

Now writing out in detail the equation (5.__) reduces to

$$\mathcal{E}^{-}\tilde{g} - g(E-E') + \sigma f\,(D+D') = 0$$

$$\mathcal{E}^{+}\,\tilde{f} - \sigma g(D+D') - f(E-E') = 0 \tag{5.46}$$

where we have made use of the solutions when $A = 0$'.

$$f\tilde{f}\Gamma^{+} + g\tilde{g}\,\Gamma^{-} = \Gamma(u^2 - v^2) - \Gamma'(u'^2 - v'^2)$$
$$= G - G' \tag{5.47}$$

$$g\tilde{f}\Gamma^{+} - f\,\tilde{g}\Gamma^{-} = \sigma(\Gamma 2uv + \Gamma' 2u'v')$$
$$= \sigma(F + F') \tag{5.48}$$

and $$k^2 - k'^2 = E - E' - Q(G - G') \tag{5.49}$$

$$0 = D + D' + P(F + F') \tag{5.50}$$

Hence $\check{\Pi}^{e}$ given by equation (5.45) is seen to satisfy (5.33).
From this we have if

$$\tilde{\mathbb{\Gamma}} = \begin{pmatrix} -\tilde{\Gamma} & \tilde{g} \\ \Gamma^+ & \tilde{f} \end{pmatrix} \tag{5.51}$$

$$\tilde{\mathbb{K}} = \begin{pmatrix} \rho^- & g\,(k'^2 - k^2) \\ -\rho^+ & f(k'^2 - k^2) \end{pmatrix} \tag{5.52}$$

we still have

$$(1 + \mathbb{Z})\,\tilde{\mathbb{\Gamma}} = \tilde{\mathbb{K}}. \tag{5.53}$$

Now we can conclude that the operator $1+\mathbb{Z}$ has an eigenvalue zero for $q=0$ provided the gap has a non-trivial solution. To see this

$$(1 + \mathbb{Z})_{q=0}\, \mathbb{\Gamma}_{q=0} = \mathbb{K}_{q=0} \tag{5.54}$$

or

$$\begin{pmatrix} 1 - \frac{d\Gamma}{d\varepsilon} Q & 0 \\ 0 & 1 + \frac{\Gamma}{\varepsilon} P \end{pmatrix} \begin{pmatrix} 0 \\ -2\Gamma \frac{D}{\varepsilon}\sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.55}$$

(5.55) can be solved if $F = \frac{D}{\varepsilon} \neq 0$. Thus in general non singular operator $1+\mathbb{Z}$ tends to a singular one in the limit of $\vec{q} = 0$. Using (5.45), (5.23) and (5.29), we find that

$$\langle J(q) \rangle = 2eNV^{-1}\eta_q\Big(q - N^{-1}\sum_p (2p+q)(f\tilde{f}\Gamma^+ + g\tilde{g}\Gamma^-)\Big)$$

$$= 2eNV^{-1}\eta_q\Big(q - N^{-1}\sum_p (2p+q)\,(G^0 - G^{0\prime})\Big)$$

$$= 2eNV^{-1}\eta_q\left(q - qN^{-1}\sum_p G^o_p - qN^{-1}\sum_{p-q} G^o_{p-q}\right)$$

$$= 2eNV^{-1}\eta_q q\left(1 - \frac{1}{N}\frac{N}{2} - \frac{1}{N}\frac{N}{2}\right)$$

$$= 0 \tag{5.56}$$

Thus we have shown that for a purely longitudinal potential we have vanishing current.

Any vector potential A_q can be split into transverse and longitudinal parts

$$\vec{A}_q = \vec{A}_q^{\,tr} + \vec{A}_q^{\,l} \tag{5.57}$$

where $\quad q \cdot A_q^{tr} = 0;$

$$A_q^{l} = q \cdot \eta_q \tag{5.58}$$

Similarly the perturbation K splits into

$$K_k^{q} = K_k^{tr\,q} + K_k^{l\,q} \tag{5.59}$$

with $\quad K_k^{tr\,q} = 2k \cdot A_q^{tr}$

$$K_k^{l\,q} = (2k+q)\,A_q^{l} = \left[(k+q)^2 - k^2\right]\eta_q \tag{5.60}$$

To demonstrate how the system responds to longitudinal and transverse fields, we find it convenient to introduce parity operator in k space $\Pi_{\vec{A}_q^{tr}}$ which reverses the k vector components in the direction of $\vec{A}_q^{tr}$. Under this operation quantities like k^2, $(k+q)^2$, $P_{k-k'}$ (by assumption) are even while $\vec{k} \cdot A_q^{tr}$ is odd. Since Z is composed of quantities like

depending on $k^2, (k+q)^2, P_{k-k'}$ it follows that

$$\left[\, 1 + \mathbb{Z} \,,\ \Pi_{\vec{A}_q}^{tr} \,\right] = 0 \tag{5.61}$$

However

$$\Pi_{\vec{A}_q}^{tr} \ \overset{\vee}{\mathbb{K}}{}^{\ell} = \overset{\vee}{\mathbb{K}}{}^{\ell} \tag{5.62}$$

and

$$\Pi_{\vec{A}_q}^{tr} \ \overset{\vee}{\mathbb{K}}{}^{tr} = - \overset{\vee}{\mathbb{K}}{}^{tr} \tag{5.63}$$

Similarly

$$\mathcal{G} \qquad \overset{\vee}{\Gamma} = \overset{\vee}{\Gamma}{}^{tr} + \overset{\vee}{\Gamma}{}^{\ell} \tag{5.65}$$

$$\Pi_{\vec{A}_q}^{tr} \ \overset{\vee}{\Gamma}{}^{tr} = - \overset{\vee}{\Gamma}{}^{tr}$$

but

$$\Pi_{\vec{A}_q}^{tr} \ \overset{\vee}{\Gamma}{}^{\ell} = \overset{\vee}{\Gamma}{}^{\ell} \tag{5.66}$$

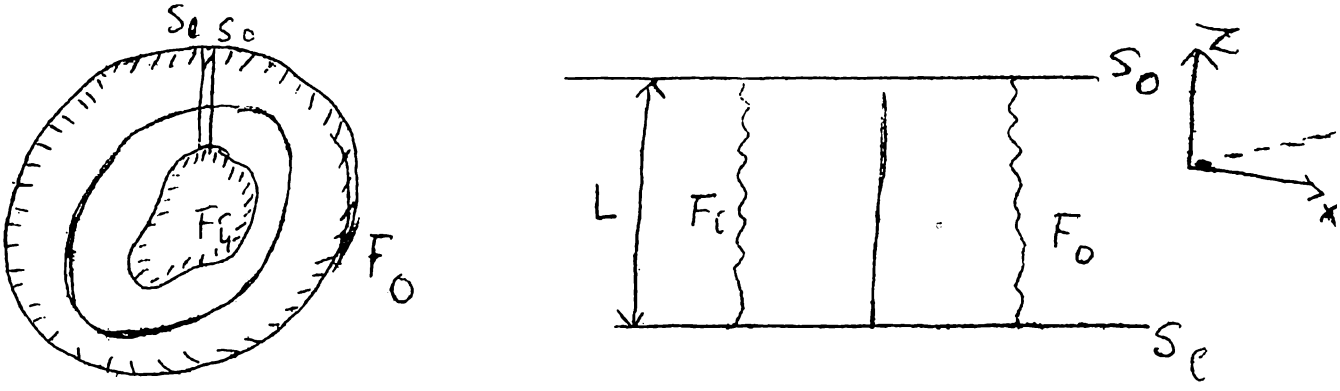
If $1 + \mathbb{Z}$ is non singular for $q \neq 0$, it is expected to remain so in the case of $\overset{\vee}{\Gamma}{}^{tr}$ subspace for even $q = 0$ because all the quantities are assumed to be continuous in q. However we know that the $\overset{\vee}{\Gamma}{}^{\ell}$ satisfies the equation (5.38) with zero eigenvalue for $q = 0$. This is possible only for $\overset{\vee}{\Gamma}{}^{\ell}$ since it is even under $\Pi_{A_q}^{tr}$. This illustrates the completely different response of the system to longitudinal and transverse vector potential perturbations.

Chapter VI.

Flux Quantization(8).

It has been recently found that when a multiply connected superconductor in a magnetic field, is cooled below the transition temperature the expelled flux trapped inside the hole does not have arbitrary values but assumes a value which are multiples of a basic unit which seems to be $\frac{hc}{e}$. This phenomenon throws a lot of light on the microscopic theory of superconductivity. Flux quantization has been anticipated by London and Onsager. However the e occurring in $\frac{hc}{e}$ should be interpreted as $2e$ as is to be the case for paired electrons of the condensed system.

To study this phenomenon, it is convenient to distort the typical doughnut geometry which is used in the experiments on flux quantization, by cutting it open and stretching into a straight shape



It is easy to see that the boundary conditions at the surfaces S_o and S_ℓ are periodic ones. Neglecting surface effects, it is convenient to construct a L - periodic box within which the

usual box quantization description with the periodic boundary condition is adopted. Periodicity in Z direction corresponds to the actual situation while the periodicity in x and y directions is sheer mathematically convenience.

The flux passing through the interior in the doughnutt to the left of F_ℓ is measured by the line integral of vector potential: i.e.,

$$\varphi = -e^{-1} \int_{s_\ell}^{S_0} A \cdot ds \qquad \text{(in doughnut)}$$

$$= -e^{-1} \int_0^L A_z \, dz \qquad \text{(in L Box)} \tag{6.1}$$

Since any change in the gauge does not affect φ, it is convenient to choose a gauge where

$$\vec{A}(x) = A \hat{z}^0 \tag{6.2}$$

with A = Const. Then

$$A = -\frac{e\varphi}{L} \tag{6.3}$$

Now

$$K_{ff} = \delta_{\sigma'}^{\sigma} \, \delta_{k'}^{k} \left[(k+A)^2 - \mu \right] \tag{6.4}$$

Our idea is to show that when A is such that the trapped flux is the correct quantized value, the corresponding bulk current vanishes. However when A is slightly different there exists a bulk current which is proportional to the deviation from the correct A.

Any general $\vec{A}$ can be decomposed according to

$$\vec{A} = \vec{A}' + \vec{A}'' \tag{6.5}$$

such that $2A'$ is always an 'allowed' lattice vector corresponding to L- periodic box. This is true for any A' corresponding to a trapped flux since $A' = n\pi/L$. k being an allowed vector, $-k-2A'$ is also an allowed vector. Using this new vector which means only a shift in the momentum and the only modification in the previous equations for $A \neq 0$ and $A'' = 0$ is in $I^{(2)}$ i.e.,

$$I^{(2)}_{k,k'} = \delta^{k+k'}_{-2A'} \tag{6.6}$$

It is clear from the above that for $A' = 0$, we recover the BCS type equations. When $A' \neq 0$, there is only a shift in momentum lattice vector and the integral equations are as simple as in BCS theory. It is important to note that we can interpret effect of A' as a shift in momentum vector only when $A' = n\pi/L$ in which case the trapped flux assumes only quantized value. In this case it is to see that the bulk current vanishes as in the absence of any external field.

In general we find for the expectation value of the current for $T=0$.

$$\langle J(q) \rangle = 2eNV^{-1}\left\{ N^{-1} \sum_p (2p+q)\, G^0_{p,p+q} + A_q \right\},$$

$$= 2e\,Nv^{-1}\delta^{q}_{0}\left\{N^{-1}\sum_{p}(2p+q)G^{0}_{p}+A\right\}$$

$$= 2e\,Nv^{-1}\delta^{q}_{0}\left\{\sum_{p}(pG^{0}_{p}+\bar{p}\,G^{0}_{p})+\underline{A}\right\}$$

$$= 2e\,Nv^{-1}\delta^{q}_{0}\left\{N^{-1}\sum_{p}(-2A')\,G^{0\prime}_{p}+A'+A''\right\}$$

$$= 2e\,Nv^{-1}\delta^{q}_{0}\,A'' \tag{6.7}$$

where
$$\vec{A}_q = \delta^{q}_{0}\,\vec{A}$$

Thus only when $A''\neq 0$, we obtain a uniform bulk current proportional to A''. Hence only the solutions with i.e. quantized flux values, are also electromagnetically stable.

Finally we will compute the thermodynamic grand potential. For $A''=0$, the only difference with respect to the thermal equilibrium state problem is the shift in the kinetic energy origin. With $A''\neq 0$, We have for

$$\Omega = \sum_{k}(K_k+E_k)G^{0}_{k}+F_k D_k+2\beta^{-1}\Big[(\tfrac{1}{2}+\Gamma_k)\log(\tfrac{1}{2}+\Gamma_k) + (\tfrac{1}{2}-\Gamma_k)\log(\tfrac{1}{2}-\Gamma_k)\Big] \tag{6.8}$$

Let us consider Ω at $T=0$ for simplicity. Then we have

$$K'_{k} = (k+A')^2-(\mu-A''^2)$$
$$K''_{k} = 2(k+A')\cdot A''$$

$$G_k^{0\,\prime\prime} = G_k^{\prime\prime} = \Gamma_k^{\prime\prime} = 0$$

$$\Omega = \sum_k \left[\left(K_k' + E_k' \right) G_k^{0\,\prime} + F_k D_k \right] \tag{6.9}$$

The change in $\Omega(\mu)$ due to A'' can be interpreted as due to a change in μ

We ~~concludes~~ conclude therefore that

$$\Omega(A'') = \Omega(A''=0) + \frac{\partial \Omega}{\partial \mu} \Delta \mu \tag{6.10}$$

Therefore

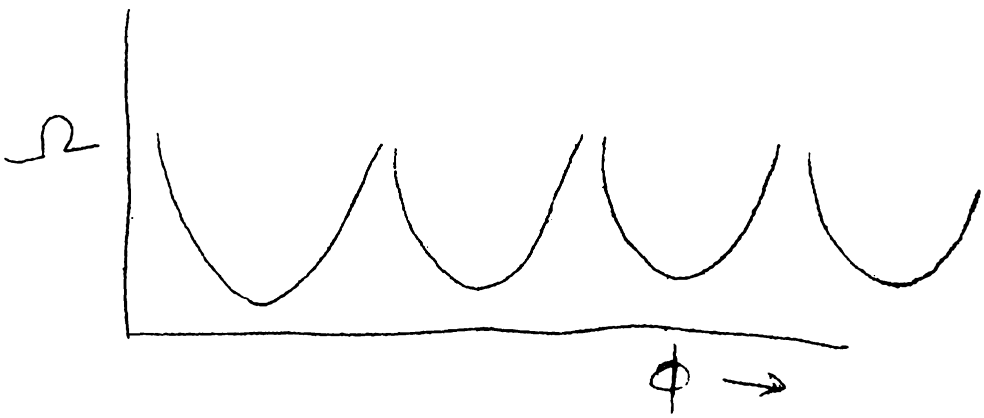
$$\Delta \Omega \cong \frac{\partial \Omega}{\partial \mu} \Delta \mu$$

$$= (-N)\left(-A''^{2}\right) \tag{6.11}$$

since from the definition of $\Omega(\mu)$

$$-\frac{\partial \Omega}{\partial \mu} = N \tag{6.12}$$

Thus the behaviour of Ω in the vicinity of the stationary points corresponding to the flux quantized solutions is parabolic as predicted by Byers and Yang[9] (see Fig. 3).



Chapter VII.

Deviation of Landau-Ginzburg Equations[10]

In the phenomenological theory due to Landau and Ginzburg[11], the state of the superconductor is described by the wave function $\psi(x,T)$ where T is the temperature. $|\psi(x,T)|^2$ is taken to represent the density of super phase. They also assumed that the current is given

$$J = \frac{-ie\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{e^2}{mc}\psi^*\psi A \tag{7.1}$$

In the presence of the magnetic field, the free energy is assumed to be given by

$$F = F_n - \alpha(T)|\psi|^2 + \frac{\beta}{2}(T)|\psi|^4 + \frac{\hbar^2}{2m}\left|\left(\nabla - \frac{ie}{\hbar c}A\right)\psi\right|^2 + H^2/8\pi \tag{7.2}$$

Varying ψ to obtain a minimum value for F, we obtain an equation for ψ

$$\left\{\frac{-\hbar^2}{2m}\left(\nabla - \frac{ieA}{\hbar c}\right)^2 - \alpha(T) + \beta(T)|\psi|^2\right\}\psi = 0 \tag{7.3}$$

In the absence of field i.e. when $A=0$, we have

$$|\psi|^2 = \alpha(T)/\beta(T) \tag{7.4}$$

which is a constant independent of x. If we assume ψ to be rigid i.e. ψ does not change very much for weak perturbation of A^{tr} to first order, we find that the current is given

by

$$J = -\frac{e^2}{mc}\psi^*\psi A^{tr} = -\frac{n_s e^2}{mc} A^{tr} \tag{7.5}$$

where we have used

$$\psi^*\psi = n_s \tag{7.6}$$

Equations (7.1) and (7.2) are the basic equations of Landau-Ginzburg theory. Gorkov[12] derived these equations from a microscopic theory with the assumption that the magnetic potential and the gap are slowly varying functions of positron. Further more he assumed the temperature of the system to be close to the critical temperature so that the gap is small. More recently Tewordt[13] and Werthamer[14] have derived the generalization of Landau-Ginsburg theory basing themselves on the theory of thermal Green's functions. These generalized equations are not identical although they agree with each other and with Gorkov's equation near the critical temperature.

From Chapter III, we have the equations

$$\mathbb{G} = -\frac{1}{2}\tanh\left(\frac{\beta}{2}\mathbb{E}\right) \tag{7.7}$$

$$\mathbb{E} = \begin{pmatrix} -E^* & D \\ -D^* & E \end{pmatrix} \tag{7.8}$$

$$\mathbb{G} = \begin{pmatrix} -G^* & F \\ -F^* & G \end{pmatrix} \tag{7.9}$$

$$E = K + Q \cdot \left(G + \frac{1}{2} I\right)$$
$$D = P \cdot F \qquad (7.10)$$

(7.11)

We can replace (7.7) by the Cauchy integral.

$$\mathbb{G} = \frac{1}{2\pi i} \oint \frac{f(\lambda)}{\lambda - \mathbb{E}} \, d\lambda \qquad (7.12)$$

where

$$f(\lambda) = -\frac{1}{2} \tanh\left(\frac{\beta}{2} \lambda\right) \qquad (7.13)$$

Let us write F and G as

$$F = \frac{1}{2\pi i} \oint f(\lambda) \, F_\lambda \, d\lambda \qquad (7.14)$$

$$G = \frac{1}{2\pi i} \oint f(\lambda) \, G_\lambda \, d\lambda \qquad (7.15)$$

where

$$F_\lambda = \frac{1}{\lambda + E^*} D \, G_\lambda \qquad (7.16)$$

$$G_\lambda = \frac{1}{\lambda - E - D^* \frac{1}{\lambda + E^*} D} \qquad (7.17)$$

To derive these we need to use

$$\frac{1}{\lambda - \mathbb{E}} = \begin{pmatrix} \frac{1}{\lambda + E^* + D \frac{1}{\lambda - E} D^*} & 0 \\ 0 & \frac{1}{\lambda - E - D^* \frac{1}{\lambda + E} D} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\lambda + E^*} D \frac{1}{\lambda - E + D^* \frac{1}{\lambda + E^*} D} \\ -\frac{1}{\lambda - E} D^* \frac{1}{\lambda + E^* + D \frac{1}{\lambda - E} D^*} & 0 \end{pmatrix} \qquad (7.18)$$

In these equations (7.16) and (7.17) the spin part of the matrices have been separated out as before.

To derive the Landau-Ginzburg equations, let us assume that the magnetic potential A and other quantities entering the problem are slowly varying over distances of the order of the Pippard coherence length. In our derivations we shall assume that A is slowly varying and small or more precisely we can set

$$A(R) = \sigma \tilde{A}(\sigma R) = \sigma \tilde{A}(\rho) \tag{7.19}$$

where $\tilde{A}(\rho)$ is a fixed function and σ is a small parameter. The motivation behind such a choice of parameter is that if we expand any gauge invariant quantity in powers of σ, the individual terms of the expansion will be separately gauge invariant. To see this consider a gauge transformation on A with a slowly varying gauge function:

$$\chi(R) = \tilde{\chi}(\sigma R) = \sigma \tilde{\chi}(R) \tag{7.20}$$

which changes A into

$$A \rightarrow A + \nabla_R \chi = \sigma\left(\tilde{A}(\rho) + \nabla_\rho \tilde{\chi}(\rho)\right) \tag{7.21}$$

It is clear from (7.21) that A continues to be small and slowly varying which was made possible by the explicit factor σ. Such a procedure is obviously dictated by the fact that the Landau-Ginzburg equations are not only gauge invariant but in their derivation they do not mention any gauge at all !

Now under a gauge transformation $A \rightarrow A + \nabla\chi$, $\psi \rightarrow \psi e^{i\chi}$, we have

$$G_\lambda(x,x') \rightarrow G_\lambda(x,x')\, e^{i\chi(x') - i\chi(x)}$$

and $$F_\lambda(x,x') \rightarrow F_\lambda(x,x')\, e^{i\chi(x) + i\chi(x')}$$

(7.22)

However we consider

$$\bar{G}_\lambda(x,x') = G_\lambda(x,x') \exp\left[i\int_{x'}^{x} A\cdot dx\right] \quad (7.23)$$

$$\bar{F}_\lambda(x,x') = F_\lambda(x,x') \exp\left[i\int_{x'}^{R} A\cdot dx + i\int_{x}^{R} A\cdot dx\right] \quad (7.24)$$

where $$R = x + x'/2 \quad (7.25)$$

and the integrals taken along straight lines joining the two end points. It is easy to verify $\bar{G}_\lambda(x,x')$ is gauge invariant while $\bar{F}_\lambda$ transforms according to

$$\bar{F}_\lambda(x,x') \rightarrow \bar{F}_\lambda(x,x') \exp\left[2i\chi(R)\right] \quad (7.26)$$

We will hereinafter refer to $\bar{G}_\lambda(x,x')$ and $\bar{F}_\lambda(x,x')$ as gauge invariant propagators.

To derive the Landau-Ginzburg equations we have to expand $\bar{G}_\lambda$ and $\bar{F}_\lambda$ atleast upto second order in σ as is evident from the presence of the term $(\nabla - ieA)^2$. A systematic expansion of the equations can be achieved by means of a method which was devised by Theis[15]. Baraff and Borowitz[16] have also devised an expansion procedure in varying degrees of derivatives of the potential in an complex atom. This procedure consists in transforming all propagators into the mixed representation in which we use as variables the centre of mass coordinate (7.25) and the

momentum conjugate to the difference coordinate:

$$\xi = (x - x') \tag{7.27}$$

Thus we write e.g.

$$G_\lambda(p, R) = \int d^3\xi \, e^{ip.\xi} \, G_\lambda\left(R + \frac{\xi}{2}, R - \frac{\xi}{2}\right) \tag{7.28}$$

The above procedure can also be motivated as follows. Since $D(x - x')$ will be eventually identified with Landau-Ginzburg Ψ function we note that $D(x, x') \to 0$ as $x - x'$ becomes very large. Also as we are essentially studying the motion of a pair, it is convenient to separate the centre of coordinates. We ~~will~~ shall also discover that the x which occurs in Landau-Ginzburg wave function is a centre of mass coordinate, i.e.,

$$\Psi(R) \sim C \, D(R, p) \tag{7.29}$$

Now the interesting question arises as to how $L(R,p)$, $M(R,p)$ and $N(R,p)$ are related when we have an equation:

$$\int L(x, x') \, M(x', x'') \, dx' = N(x, x'') \tag{7.30}$$

Thds has observed that we have

$$\mathcal{O}\left[L(p,R), M(p',R')\right] = N(p,R) \tag{7.31}$$

where $\mathcal{O}$ means the operator

$$\mathcal{O} = \lim_{\substack{R' \to R \\ p' \to p}} \exp\left[-\frac{i}{2}\left(\nabla_R \cdot \nabla_{p'} - \nabla_p \cdot \nabla_{R'}\right)\right] \tag{7.32}$$

If we introduce $\rho = \sigma R$, we have $\nabla_R = \sigma \nabla_\rho$, it is clear that the expansion of the operator Θ in powers of σ coincides with an expansion in powers of gradient. We now simply expand on both sides of the integral equation in powers of σ and equate on both sides the terms of same powers of σ .

We have previously introduced gauge invariant propagators $\bar{G}$ and $\bar{F}$. Since it is easier to deal with G and F first to a certain order and then use the equations (7.23) and (7.24) to the same order to achieve $\bar{G}$ and $\bar{F}$. Now we have to do the same in mixed representation. To obtain $\bar{G}$ and $\bar{F}$ upto second order in the mixed representation we proceed as follows. The equation (7.24) can be written as

$$\bar{G}(R,\xi) = G(R,\xi)\, e^{+i\int_{-1}^{+1} A(R+\tau\xi/2)\cdot\frac{\xi}{2}\, d\tau} \qquad (7.33)$$

when we have used $x = R + \frac{\xi}{2}$, $x' = R - \frac{\xi}{2}$. The integral

$$I_1 = \int_{-1}^{+1} A\left(R + \tau\frac{\xi}{2}\right)\cdot\frac{\xi}{2}\, d\tau$$

becomes on using (7.19)

$$I_1 = \int_{-\sigma}^{\sigma} \tilde{A}\left(\rho + \tau\xi/2\right)\cdot\frac{\xi}{2}\, d\tau \qquad (7.34)$$

This integral can be easily expanded in powers of σ , by calculating its successive derivatives. Upto second order we have

$$I_1 = \xi\cdot A(R) \qquad (7.35)$$

The corresponding operator in the mixed representation is obtained by means of the substitution

$$\xi \longrightarrow -i\nabla_p \qquad (7.36)$$

This operator $\exp(A\cdot\nabla_p)$ is equivalent to a substitution and has a simple classical meaning, i.e., it can be interpreted as a change of independent variable from canonical momentum p to the kinetic momentum $p-A$. However in higher order this is no longer true.

In a similar way we can perform the expansion of

$$I_2 = \int_{x'}^{R} A\cdot dx + \int_{x}^{R} A\cdot dx$$

$$= -\int_0^{\sigma}\left[\tilde{A}\left(\rho+\frac{\tau\xi}{2}\right)-\tilde{A}\left(\rho-\frac{\tau\xi}{2}\right)\right]\frac{\xi}{2}\,d\tau \qquad (7.37)$$

Expanding in powers of σ, we have upto second order

$$I_2 = -\frac{1}{4}A_{\lambda\sigma}\xi_\lambda\xi_\sigma$$

with

$$A_{\lambda\sigma} \equiv \frac{\partial A_\lambda}{\partial R_\sigma} \qquad (7.38)$$

We thus finally have upto second order

$$\bar{G}(p,R) = \left(1 + A(R)\cdot\nabla_p + \frac{1}{2}\left(A\cdot\nabla(p)\right)^2\right)G(p,R) \qquad (7.39)$$

$$\bar{F}(p,R) = \left(1+\frac{i}{4}\frac{\partial A_\ell}{\partial R_\sigma}\frac{\partial}{\partial p_\ell}\frac{\partial}{\partial p_\sigma}\right)F(p,R) \qquad (7.40)$$

where in the last line, we sum over repeated indices.

In our further calculations we will omit the Hartree Fock term which makes the calculations considerably simple, i.e., E now becomes

$$E = (p - A(R))^2 - \mu . \qquad (7.41)$$

Further we will also neglect the p dependence of $D(p,R)$ which implies that the interaction potential is effectively a constant in momentum space. Finally we have for the current in the mixed representation after performing the sum over spin indices.

$$J(R) = \frac{4e}{V}\sum_p \left(p - A(R)\right)\left[G(p,R) + \tfrac{1}{2}\right] = \frac{4e}{V}\sum_p p\,\overline{G}(p,R) \tag{7.42}$$

We also have

$$\overline{D}(p,R) = -\frac{1}{V}\sum_q P_{p-q}\,\overline{F}(q,R) \tag{7.43}$$

We have to carry out an expansion of G in (7.17). To this end put $N = \frac{1}{\lambda + E^*}$ and call $M = \lambda + E^*$. Thus

$$NM = 1 \tag{7.44}$$

where

$$E^* = (p + A(R))^2 - \mu \tag{7.45}$$

According to (7.37) we have in mixed representation

$$(Q_0 + Q_1 + Q_2 + \cdots)(N_0 + N_1 + \cdots)(M_0 + M_1 + M_2 \cdots) = 1 \tag{7.46}$$

Identifying terms of equal order, we obtain

$$Q_0 N_0 M_0 = 1 \tag{7.47}$$

$$Q_0 N_0 M_1 + Q_0 N_1 M_0 + Q_1 N_0 M_0 = 0 \tag{7.48}$$

where $M_0 = \lambda + p^2 - \mu = \lambda + \omega,\ M_1 = 2p \cdot A(R)$

$$M_2 = A^2(R)$$

There are no higher order terms. We immediately have from (7.47)

$$N_0 = \frac{1}{M_0} \tag{7.49}$$

Since

$$\Theta_1 = -\frac{i}{2}\left(\nabla_R \nabla_{p'} - \nabla_p \nabla_{R'}\right) \tag{7.50}$$

We note

$$\Theta_1 \; N_0 M_0 = 0$$

Hence from (7.48)

$$N_1 = - M_1 / M_0{}^2 \tag{7.51}$$

Further using (7.49) and (7.51) we have

$$\Theta_1 \; N_0 M_1 + \Theta_1 N_1 M_0 = 0. \tag{7.52}$$

and also

$$\Theta_2 \; N_0 M_0 = 0, \tag{7.53}$$

as M_0 and N_0 do not depend upon R. Thus

$$N_2 = - \frac{N_0 M_2 + N_1 M_1}{M_0} = - \frac{M_2}{M_0{}^2} + \frac{M_1{}^2}{M_0{}^3} \tag{7.54}$$

To apply this technique to G_λ, we first compute the denominatore in the r.h.s. of (7.17) and to obtain its reciprocal we use the above method. If we express (omitting the λ index from now)

$$G = G^0 + G^1 + - - - - \tag{7.55}$$

using (7.39) we can obtain the corresponding gauge invariant

expressions. This is really a tedious calculation. Hence we give the result:

$$\overline{G_0} = \lambda+\omega / \lambda^2 - \varepsilon^2 \tag{7.56}$$

$$\overline{G_1} = -p \cdot \frac{[4A|D|^2 + iD^*\nabla_R D - iD\nabla_R D^*]}{(\lambda^2-\varepsilon^2)^2} \tag{7.57}$$

$$\begin{aligned}
\overline{G_2} = &-\frac{1}{4}\frac{\nabla_R^2 |D|^2}{(\lambda^2-\varepsilon^2)^2}\left(1+\frac{4\omega(\lambda+\omega)}{\lambda^2-\varepsilon^2}\right) \\
&-\frac{1}{2}\frac{(\nabla_R |D|^2)^2}{(\lambda^2-\varepsilon^2)^3}\left(1+\frac{2\omega(\lambda+\omega)}{\lambda^2-\varepsilon^2}\right) \\
&-\frac{(\nabla_R+2iA)D^* \cdot (\nabla_R-2iA)D}{(\lambda^2-\varepsilon^2)^2} \\
&+\frac{[p\cdot(4A|D|^2+iD^*\nabla_R D - iD\nabla_R D^*)]^2}{(\lambda+\omega)(\lambda^2-\varepsilon^2)^3} \\
&-2(p\cdot\nabla_R)^2|D|^2\frac{\lambda+\omega}{(\lambda^2-\varepsilon^2)^3}\left(1+\frac{2\omega^2}{\lambda^2-\varepsilon^2}\right) \\
&+\frac{(p\cdot\nabla_R|D|^2)^2}{(\lambda+\omega)(\lambda^2-\varepsilon^2)^3}\left(1-\frac{2(\lambda+\omega)^2}{\lambda^2-\varepsilon^2}\right) \\
&+\frac{4p\cdot(\nabla_R+2iA)D^*\, p\cdot(\nabla_R-2iA)D}{(\lambda+\omega)(\lambda^2-\varepsilon^2)^2},
\end{aligned} \tag{7.58}$$

In a similar way using (7.16) and (7.40) we obtain after a detailed calculation

$$\bar{F}_0 = \frac{D}{\lambda^2-\varepsilon^2} \tag{7.59}$$

$$\bar{F}_1 = -\frac{2i\lambda\, p\cdot(\nabla_R - 2iA)D}{(\lambda^2-\varepsilon^2)^2} \tag{7.60}$$

$$\begin{aligned}
\bar{F}_2 = & -\frac{1}{2}\,\frac{\omega(\nabla_R-2iA)^2 D}{(\lambda^2-\varepsilon^2)^2} - \frac{\omega D\,\nabla_R^2|D|^2}{(\lambda^2-\varepsilon^2)^2} - \frac{\omega D\,(\nabla_R|D|^2)^2}{(\lambda^2-\varepsilon^2)^4} \\
& -[p\cdot(\nabla_R-2iA)]^2 D\left[\frac{2}{(\lambda^2-\varepsilon^2)^2} + \frac{1}{2}\frac{\partial^2}{\partial\omega^2}\frac{1}{\lambda^2-\varepsilon^2}\right] \\
& -4\,\frac{D^*[p\cdot(\nabla_R-2iA)D]^2}{(\lambda^2-\varepsilon^2)^3} - \frac{2}{3}D(p\cdot\nabla_R)^2|D|^2 \\
& \times\left[\frac{2}{(\lambda^2-\varepsilon^2)^3} + \frac{\partial}{\partial\omega}\frac{\omega}{(\lambda^2-\varepsilon^2)^3}\right] - \frac{2D(p\cdot\nabla_R|D|^2)^2}{(\lambda^2-\varepsilon^2)^4}
\end{aligned} \tag{7.61}$$

From (7.42) and using the results above, we can immediately derive the generalized Landau-Ginzburg equations. By symmetric integration, only $\bar{G}_1$ contributes as the contribution from G_0 vanishes. We have to also carry out the λ integration. Typically we have

$$\frac{1}{2\pi i}\oint \frac{f(\lambda)}{(\lambda^2-\varepsilon^2)^2}\,d\lambda \tag{7.62}$$

Noting $\dfrac{1}{(\lambda^2-\varepsilon^2)^2} = \dfrac{1}{(\lambda+\varepsilon)^2(\lambda-\varepsilon)^2}$

(7.62) can be written as

$$\frac{1}{2\varepsilon}\left(\frac{f(\varepsilon)}{\varepsilon}\right)' \tag{7.63}$$

The sum over p because of symmetric integration can be expressed as

$$\int d^3p\ \frac{p^2}{3}\ \frac{1}{2\varepsilon}\left(\frac{f(\varepsilon)}{\varepsilon}\right)' \tag{7.64}$$

Making the usual approximation, i.e. the slowly varying functions are replaced by their values on the Fermi surface and changing the integration variable from p to ω, we have finally

$$n\int_0^\infty \frac{d}{d\varepsilon}\left(\frac{f(\varepsilon)}{\varepsilon}\right)\frac{d\omega}{\varepsilon} \tag{7.65}$$

where n the number density $= p_F^3/3\pi^2$

Finally we can write down the expression for the current

$$J(R) = \frac{e}{m}\left[-\frac{i}{2}\left(D^*\nabla_R D - D\nabla_R D^*\right) - 2eA|D|^2\right] \tag{7.66}$$

where
$$h_1(|D|^2) \equiv \int_0^\infty \frac{d}{d\varepsilon}\left(\frac{f(\varepsilon)}{\varepsilon}\right)\frac{d\omega}{\varepsilon} = \frac{1}{2|D|^2}\ \frac{\Lambda}{\Lambda_T} \tag{7.67}$$

where $1/\Lambda_T$ appearing already in B C S theory is

$$1/\Lambda_T = 2|D|^2 \int_0^\infty \frac{d}{d\varepsilon}\left(\frac{f(\varepsilon)}{\varepsilon}\right)\frac{d\omega}{\varepsilon} \tag{7.68}$$

The equation (7.67) is similar to the one that occur in B C S theory except that expression given by (7.67) varies with the position. Equation (7.66) for current density agrees with the result derived by Tewordt and Werthamer. It can be checked that our expression goes over into Gorkov's result when T is close to T_c .

Finally using (7.43), (7.59), (7.60) and (7.61) we see that $\bar{F}_1$ gives no contribution to the gap equation as the interaction potential is assumed to be momentum independent. As before performing the λ integration, the gap equation can be written as

$$\left[-\frac{6}{v_F^2}h_0 + h_1(\nabla_R - 2ieA)^2\right]D + h_2\left[D^*(\nabla_R - 2ieA)D)^2 + \frac{D}{3}\nabla_R^2|D|^2\right] + h_3\frac{D}{6}(\nabla_R|D|^2)^2 = 0 \tag{7.69}$$

where
$$h_n = \frac{\partial}{\partial|D|^2}h_{n-1} \tag{7.70}$$

$$h_0(|D|^2) = 2\int_0^\infty\left[\frac{f(\varepsilon)}{\varepsilon} - \frac{f(\varepsilon_0)}{\varepsilon_0}\right]d\omega \tag{7.71}$$

with $\mathcal{E} = \sqrt{\omega^2 + D^2}$ and $\mathcal{E}_0 = \sqrt{\omega^2 + D_0^2}$

, D_0 denotes the B C S gap at the given temperature. Equation (7.69) agrees with the corresponding one derived by Werthamer, though it differs from that derived by Tewordt. Near the critical temperature all these expressions agree with that of Gorkov.

We can use the equation (7.69) to study the dependence of the gap on the magnetic potential. We note that in equation (7.69) only σ^2 occurs and hence we can use as an expansion parameter $\sigma^2 \equiv \alpha$. However we should not take seriously the terms in the expansion of order higher than σ^2 as we have already neglected terms of higher order in deriving (7.69).

To expand (7.69), it is convenient to express D in terms of its amplitude e.g., and its phase S

$$D = \varphi e^{iS} \tag{7.72}$$

We can rewrite (7.69) as a set of two equations for and S.
We can expand φ and S as

$$\varphi = \varphi_0 + \alpha \varphi_1 + \alpha^2 \varphi_2 + ----$$

$$S = S_0 + \alpha S_1 + \alpha^2 S_2 + ---- \tag{7.73}$$

We work in a gauge such that D_0 is real. (London Gauge). For D_1, we can obtain the equation

$$\frac{3}{v_F^2} h_1 2D_0 \operatorname{Re} D_1 + 2e^2A^2(h_1 + |D_0|^2 h_2) = 0 \tag{7.74}$$

where h_1 and h_2 are calculated at $|D_0|^2$. Since the function $h_1 + |D|^2$ vanishes exponentially as $T \to 0$ equation (7.69) yields no dependence of the gap on A, a result in agreement with that obtained by Nambu and Tnam, and by Tewordt. However if we had kept only the first two terms in (7.69), we shall be neglecting the h_2 term in (7.74). We shall end up with

$$2 D_0 \, \mathrm{Re} \, D_1 = -\frac{2}{3} e^2 v_F^2 A^2 \tag{7.75}$$

which agrees with an older calculation of Gupta and Mathur. (7.75) does not contain any temperature dependent coefficients. This emphasizes the basic limitation due to the assumed expansion in A. It is clear that local superconductivity cannot be valid at very low temperatures where the penetration depth is small and consequently the assumption that the potential is slowly varying will fail. Werthamer has attempted an estimate of the range of validity of local superconductivity theory.

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