On the Difference of Progression Series in Ancient Indian Mathematics

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Abstract

The present article is based on some remarkable progression series found in Ancient Indian mathematics. A new sequence of numbers is proposed, and their properties are studied. Taking the difference of above series, new series are established. Suitable examples are given to support the present investigation.

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1 Introduction and preliminaries

During the early Vedic age, Indian began their interest in the series(Śreḍhi). A large number of Vedic series are found in the Taittirīya Saṃhitā, Vājasaneya Saṃhitā and Pañcaviṃsa Brāhmaṇa ([1]). Some of the series are mentioned in the Bṛahaddebatā, Śatapatha Brāhmaṇa, Baudhāyana Śulba, Buddhist work Digha Nikāya and a Jaina work Antagoḍa Dasāo ([1]).The general formula for the summation of an A.P. found in the Bakhshālī Manuscript(200-400 A.D.) ([1, 6, 11]) is:

$$S = a + (a + b) + ... + [a + (n - 1)b] = n[a + (n - 1)\frac{b}{2}],$$

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where S denotes the sum, a = first term, b = common difference, and n = number of terms.

Āryabhaṭa I (476 A.D.) gave the above form as well as the form: $S = \frac{n}{2}(a+l)$. The same rule appears in the works of Bramhagupta (628 A.D.), Mahāvīra (850 A.D.),Śrīdhara (991 A.D.) and Bhāskara II (1150 A.D.) ([1, 2, 3, 4, 8, 9, 10, 11, 12]). Āryabhaṭa I and all the above writers also gave the sum of the following particular case:

$$1 + 2 + 3 + ... + n = \sum_{k=1}^{n} k = \frac{n}{2}(n+1).$$

Presently a number of the form $\frac{n}{2}(n+1)$, where n is a natural number, is known as a triangular number and is generally denoted by $S(n)$.

Āryabhaṭa I calculated the value of the unknown number of terms n for an A.P. series when its sum is S, first term a, common difference b as follows:

$$n = \frac{1}{2}[\frac{\sqrt{8bS + (2a-b)^2} - 2a}{b} + 1].$$

The same was also given by Bramhagupta, Mahāvīra and Bhāskara II. Āryabhaṭa I, Bramhagupta, Mahāvīra and Bhāskara II ([1, 2, 3, 4, 8, 9, 10, 11, 12, 13]) gave the summation of the series of squares and cubes of n natural numbers as follows:

$$\begin{aligned} 1^2 + 2^2 + 3^2 + ... + n^2 &= \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \\ 1^3 + 2^3 + 3^3 + ... + n^3 &= \sum_{k=1}^{n} k^3 = [\frac{n(n+1)}{2}]^2. \end{aligned}$$

Āryabhaṭa I gave the summation of the sums of n times the 1st term of n naural numbers, $(n-1)$ times of the 2nd term of n naural numbers, $(n-2)$ times of the 3rd term of n naural numbers, ...etc. Then

$$\begin{aligned} S_1 + S_2 + S_3 + ... + S_n &= n.1 + (n-1).2 + (n-2).3 + ... + [n-(n-1)].n \\ = 1 + (1+2) + (1+2+3) + ... + (1+2+3+...+n) &= \frac{n(n+1)(n+2)}{6}. \end{aligned}$$

Nārāyana(1356 A.D.) calls this as repeated sum or Vārasaṃalita. According to him, Vārasaṃalita of 1st order of n natural numbers

$$= 1 + 2 + 3 + ... + n = \frac{n}{2}(n+1).$$

Vārasaṃalita of 2nd order of n natural numbers

$$= 1 + (1+2) + (1+2+3) + ... + (1+2+3+...+n) \ = \frac{n(n+1)(n+2)}{6},$$

which is presently known as a tetrahedral number. Āryabhaṭa I gave the result only up to Vārasaṃalita of 2nd order, but Nārāyana proceed further. His result for Vārasaṃalita of the rth order of n natural numbers is

$$\frac{n(n+1)(n+2)...(n+r)}{1.2.3...(r+1)} \ = \frac{n(n+1)(n+2)...(n+r)}{(r+1)!}.$$

The earliest expression of an idea representing G.P. occurs in the Chandaḥ-sūtra (rule of metres) of Piṅgala (c.200 B.C.), where the series: $1, 2^1, 2^2, 2^3, \ldots$ is found. Mahāvīra gives the generalised result of the sum of the n terms of the G.P. as follows:

$$a + ar + ar^2 + ... + ar^{(n-1)} \ = \frac{a(r^n - 1)}{(r-1)}.$$

This formula also reappears in the works of Pṛuthudakasvāmī (864 A.D.) and Nemicandra (978 A.D.). Mahāvīra also gave the form

$$S = \frac{rar^{(n-1)} - a}{(r-1)}.$$

In this paper, some important progression series of Ancient Indian mathematics are discussed. A new sequence of numbers is introduced and some properties of these numbers are established.

2 Progression Series in Patiganita of Śridharcarya

In this section, rules for finding the sum of a series of natural numbers, finding the number of terms and difference of natural series found in the Patiganita ([10])are presented.

2.1 Addition and Subtraction of Logistics

In the Logistics (Parikarma) section of Patiganita, Rules of Verses 14 , 15 of Addition (Saṅkalita) and 15 ,17 of Subtraction (Vyavakalita) are discussed.

Rule 2.1 [Verse14(i)] When the first term (âdi) and the common difference (caya) of a series (in arithmetic progression) are (each) unity, the sum (Saṅkalita) is equal to half the number of terms (Pada) multiplied by the number of terms plus one. That is,

$$S(n) = 1 + 2 + 3 + ... + n = \frac{n}{2}(n+1).$$

Rule 2.2 [Verse14(ii)] The number of terms (gaccha) is equal to the (integral) square root of twice the sum of the series, which must be the same as the residue left (after the extraction of the square root). That is,

$$n = \text{ integral part of } \sqrt{2S(n)}.$$

Rule 2.3 [Verse15] The sum of a series of natural numbers is equal to one-half of the square of the number of terms plus the number of terms. And that (sum) multiplied by 8, (then) increased by 1, (then) reuced to its square root, (then) diminished by 1, and (then) halved, is the number of terms (in the series). That is,

$$S(n) = 1 + 2 + 3 + ... + n = \frac{n^2 + n}{2} \quad \text{and } n = \frac{\sqrt{(8S(n) + 1)} - 1}{2}.$$

Rule 2.4 [Verse16] Having added (nidhâya) the number of terms of the subtrahend series (vyavakalita-pada) plus one to the number of terms of the minuend series (saṅkalita-pada), multiply that (sum) by the difference of the number of terms (of the two series): that (product), when halved, become the residue of subtraction (of the given series). That is,

$$S(n) - S(m) = \frac{(n - m)(n + (m + 1))}{2}, \quad \text{where } n > m.$$

Rule 2.5 [Verse17] Having subtracted the residue of subtraction (i.e., the difference of the minuend and subtrahend series) from the sum of the minuend series, and multiplied the remainder (obtained) by 2, the square root there of, which must be equal to the residue left (after the extraction of the square root), should be delared as the number of terms (of the subtrahend series). That is,

$$m = \text{ integral part of } \sqrt{2(S(n) - D)}, \quad \text{where } D = S(n) - S(m).$$

3 Progression Series in Gaṇitasārakaumudī of Ṭhakkura Pherū

In this section, rules for finding the sum of natural series, two rules to find the number of terms and difference of natural series are discussed.

3.1 Sum and Difference of Natural Series

Out of eight fundamental operation of integers, Rules of Verses 16, 17, 18, 20 of Sum and 21, 22, 24, 25 of Difference found in the Gaṇitasārakaumudī ([7]) are studied.

Rule 3.1 [Verse $GSK1.16a$] (Sum of natural series)

$$S(n) = 1 + 2 + 3 + ... + n = \frac{n}{2}(n+1).$$

Note that the n is here called icchā (the requisite).

Rule 3.2 [Verse $GSK1.17$] (Sum of natural series)

$$S(n) = 1 + 2 + 3 + ... + n = \frac{n}{2}.(n+1),$$

if n is even and

$$S(n) = 1 + 2 + 3 + ... + n = n.(\frac{n+1}{2}),$$

if n is odd.

Note that the n is here called diṇa (or the number of days).

Rule 3.3 [Verse $GSK1.18$] (Sum of natural series)

$$S(n) = \frac{(nx+x).n}{2x}.$$

Note that the superfluous x is here called paṇh-akkhara (praśna-akṣara) or '(the number of) the letters in question'.

Rule 3.4 [Verse $GSK1.20$] (number of terms, two rules)

$$n = \frac{\sqrt{(8S(n)+1)} - 1}{2},$$

and

$$n = \text{ integral part of } \sqrt{2S(n)}.$$

Rule 3.5 [Verse $GSK1.21$] (definition of difference) For two natural numbers, n and m with $n > m$, let $S(n, m)$ is defined as:

$$S(n, m) = S(n) - S(m).$$

Then, either $S(m)$ or $S(n,m)$ is called *vimakaliya* (vyavakalita, that which is subtracted), and in either case, the number of terms left in the summation $S(n)$ decreases one by one from n, which is called the root-quantity (mūla-rāsī), as the *vimakaliya* increases.

Rule 3.6 [Verse GSK1.22] (difference of natural series)

$$S(n,m) = \frac{(n-m)(n+(m+1))}{2}, \quad \text{where } n > m.$$

Here $m = vimakaliya - paya$ and $S(n,m) = vimakaliya - sesa$.

Rule 3.7 [Verse GSK1.24] (number of terms of the difference)

$$m = \text{ integral part of } \sqrt{2(S(n) - S(n,m))}.$$

Rule 3.8 [Verse GSK1.25] (difference of natural series)

$$S(m) = \frac{2n+1-k}{2}.k,$$

where $k = n - m = vimakaliya - paya$ and $S(m) = vimakaliya - sesa$. Note that here the *vimakaliya* (that which is subtracted) and the $vimakaliya - sesa$ (the remainder of the subtraction) exchange their positions.

Remark 3.1 The above formulae of the progression series are also found in the *Lilavati* of Bhaskara II ([3]).

4 Proposed Sequence of Numbers

Using (2.4 and 3.6), let us introduce a new sequence of numbers called Co-Gadtia numbers defined as follows:

Definition 4.1 A number of the form $T(n)$ with $T(1) = 0$, and satisfying $T(n) = S(n,2) = S(n) - S(2)$, where $n \geq 2$ is a natural number, is called a Co-Gadtia number.

Note 4.1

$$S(n) = 1 + 2 + 3 + ... + n = \sum_{k=1}^{n} k = \frac{n}{2}.(n+1),$$

where n is a natural number, is called a triangular number, where as

$$G(n) = 2 + 3 + ... + n = \sum_{k=2}^{n} k = \frac{(n-1)(n+2)}{2},$$

where n is a natural number, is called a Gadtia number ([5]). The triangular numbers are

$$S(1) = 1, S(2) = 3, S(3) = 6, S(4) = 10, S(5) = 15, S(6) = 21, S(7) = 28,$$
$$S(8) = 36, S(9) = 45, S(10) = 55, S(11) = 66, S(12) = 78, S(13) = 91, ...etc.$$

and the Gadtia numbers are

$$G(1) = 0, G(2) = 2, G(3) = 5, G(4) = 9, G(5) = 14, G(6) = 20, G(7) = 27,$$
$$G(8) = 35, G(9) = 44, G(10) = 54, G(11) = 65, G(12) = 77, G(13) = 90, ...etc.$$

Definition 4.2 A number of the form $T(n)$ that satisfy $T(n) = \frac{(n-2)(n+3)}{2}$, where n is a natural number, is called a Co-Gadtia number.

Note 4.2 Throughout this paper, we shall denote the nth Co-Gadtia number by $T(n)$.

Example 4.1 Evaluation of First Six Co-Gadtia numbers using Definition (4.1).

$$\begin{aligned}
T(1) &= 0, \\
T(2) = S(2) - S(2) = 3 - 3 &= 0, \\
T(3) = S(3) - S(2) = 6 - 3 &= 3, \\
T(4) = S(4) - S(2) = 10 - 3 &= 7, \\
T(5) = S(5) - S(2) = 15 - 3 &= 12, \\
T(6) = S(6) - S(2) = 21 - 3 &= 18.
\end{aligned}$$

Example 4.2 Evaluation of Co-Gadtia numbers within 100 using Definition(4.2).

$$\begin{aligned}
T(1) &= 0, \\
T(2) = \frac{(2-2)(2+3)}{2} = \frac{(0)(5)}{2} &= 0, \\
T(3) = \frac{(3-2)(3+3)}{2} = \frac{(1)(6)}{2} &= 3, \\
T(4) = \frac{(4-2)(4+3)}{2} = \frac{(2)(7)}{2} &= 7, \\
T(5) = \frac{(5-2)(5+3)}{2} = \frac{(3)(8)}{2} &= 12, \\
T(6) = \frac{(6-2)(6+3)}{2} = \frac{(4)(9)}{2} &= 18.
\end{aligned}$$

In a similar manner, other numbers can be found as $T(7) = 25, T(8) = 33, T(9) = 42, T(10) = 52, T(11) = 63, T(12) = 75, T(13) = 88, ...etc.$

5 Some Fascinating Properties of Co-Gadtia Numbers

In this section, some linear and non-linear recurrence relations among the Co-Gadtia numbers are presented, and other properties are also discussed. Two theorems relating to Co-Gadtia numbers are established.

Property 5.1 $T(1) = 0, T(2) = 0$ and $T(n) = n + T(n-1)$,
or $T(n) - T(n-1) = n, \quad \text{for all } n \geq 3.$

Property 5.2 $T(n-1) + T(n+1) = 2T(n) + 1, \quad \text{for all } n \geq 3.$

Property 5.3 $T(n-1) + T(n) + 6 = n^2, \quad \text{for all } n \geq 3.$

Property 5.4 $T(n-1) + T(n) + 2 = (T(n) - T(n-1))^2, \quad \text{for all } n \geq 3.$

Property 5.5

$$n = \frac{\sqrt{(8T(n) + 25)} - 1}{2}.$$

Property 5.6

$$n = \text{ integral part of } \sqrt{2T(n) + 6}.$$

Property 5.7

$$T(n, m) = T(n) - T(m) = \frac{(n-m)(n + (m+1))}{2}, \quad \text{where } n > m, \quad \text{and } n \geq 3.$$

Property 5.8

$$\begin{aligned}
1 + 2 + 3 + ... + n &= \sum_{k=1}^{n} k \\
&= \frac{n}{2}.(n+1) \\
&= T(n) + 3 \\
\implies T(n) &= S(n) - 3. \\
\implies T(n) &= 3 + 4 + ... + n.
\end{aligned}$$

Property 5.9

$$\begin{aligned}
1^2 + 2^2 + 3^2 + ... + n^2 &= \sum_{k=1}^{n} k^2 \\
&= \frac{n(n+1)(2n+1)}{6} \\
&= \frac{(2n+1)(T(n)+3)}{3} \\
\Longrightarrow 3\sum_{k=1}^{n} k^2 &= (2n+1)(T(n)+3).
\end{aligned}$$

Property 5.10

$$1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{k=1}^{n} k^3 = [\frac{n(n+1)}{2}]^2 = [T(n)+3]^2.$$

Presentation 5.1 The Co-Gadtia numbers can be presented in triangular form in the whole number table as follows:

$$\begin{array}{rrrrrrr}
 & & & & & & 0 \\
 & & & & & 1 & 2 \\
 & & & & 3 & 4 & 5 \\
 & & & 6 & 7 & 8 & 9 \\
 & & 10 & 11 & 12 & 13 & 14 \\
 & 15 & 16 & 17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 \\
\multicolumn{7}{r}{\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

Remark 5.1 In the above table of representation, the numbers in the third column from the last represent the Co-Gadtia numbers, where as the last column represent the Gadtia numbers and the uppermost diagonal numbers are triangular numbers.

Theorem 5.1 The generating function relating to the Co-Gadtia numbers is given by

$$S = \frac{2x-1}{(1-x)^2} + \frac{1}{(1-x)^3} = \frac{x(3-2x)}{(1-x)^3}.$$

Proof: Let

$$\begin{aligned} S &= 0 + 3x + 7x^2 + 12x^3 + 18x^4 + \dots \\ xS &= 3x^2 + 7x^3 + 12x^4 + \dots \end{aligned}$$

On subtraction, we get

$$S - xS = 3x + 4x^2 + 5x^3 + 6x^4 + \dots \tag{5.1}$$

$$\implies S(1-x) = 3x + 4x^2 + 5x^3 + 6x^4 + \dots \tag{5.2}$$

Let

$$\begin{aligned} S' &= 3x + 4x^2 + 5x^3 + 6x^4 + \dots \\ xS' &= 3x^2 + 4x^3 + 5x^4 + \dots \end{aligned}$$

On subtraction, we get

$$\begin{aligned} S' - xS' &= 3x + x^2 + x^3 + x^4 + \dots \\ \implies S'(1-x) &= 3x + (1 + x + x^2 + x^3 + x^4 + \dots) - (1+x) \\ \implies S'(1-x) &= 2x - 1 + (1 + x + x^2 + x^3 + x^4 + \dots) \\ \implies S'(1-x) &= 2x - 1 + \frac{1}{1-x} \\ \implies S' &= \frac{2x-1}{1-x} + \frac{1}{(1-x)^2}. \end{aligned}$$

Putting this value of S' in equation (5.2), we get

$$\begin{aligned} S(1-x) &= \frac{2x-1}{1-x} + \frac{1}{(1-x)^2} \\ \implies S &= \frac{2x-1}{(1-x)^2} + \frac{1}{(1-x)^3} \\ \implies S &= \frac{x(3-2x)}{(1-x)^3}. \end{aligned}$$

Theorem 5.2 x is a Co-Gadtia number if and only if $8x + 25$ is a perfect square.

Proof: Let

$$\begin{aligned} x = T(n) \Longleftrightarrow 8x+25 &= 8T(n)+25 \\ \Longleftrightarrow 8x+25 &= \frac{8(n-2)(n+3)}{2}+25 \\ \Longleftrightarrow 8x+25 &= 4(n^2+n-6)+25 \\ \Longleftrightarrow 8x+25 &= 4n^2+4n+1 \\ \Longleftrightarrow 8x+25 &= (2n+1)^2. \end{aligned}$$

6 Conclusion

Some important progression series found in Ancient Indian mathematics are discussed. A new sequence of numbers called Co-Gadtia numbers is introduced, and some properties of these numbers are established. Suitable examples are given in support of the present work.

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