Sum Problem of Maximal Monotone Operators

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"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is"

—John von Neumann

Abstract

Maximal monotone operators play an important role in nonlinear modern analysis and partial differential equations. Here, we focus on the most famous and significant open problem that is "Sum problem" in monotone operator theory in the nonreflexive Banach spaces.

Key words: Maximal monotone operator; Sum Problems, (FPV) operator, Normal cone, Rockafellar's constraints,
AMS class: 90C29; 90C46; 49J40; 49J52; 49J53

1 Introduction

Monotone operators are an important class of operators used in the study of modern non-linear analysis and various classes of optimization problems. Originally, the definition of the monotone operator was found in Kachurovskii [27]. Then the theories of monotone operators (multifunctions) were introduced by George Minty [31] to aid the abstract study of electrical networks. Later it was used substantially in proving the existence results of partial differential equations by Felix Browder and his school [1, 2, 10, 11, 12, 20, 25, 26, 48].

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1.1 Monotone operator

The concept of monotonicity for set-valued operators defined on Banach spaces to its dual was introduced by Minty [34, 36] and Browder [15, 16, 17] around fifty years ago. Some important contributions and surveys of monotone operators may be found in Browder [14, 15, 17], Minty [32, 34, 35], Kachurovskii [28] and Rockafellar [38, 39]. Monotone operators have numerous applications to functional analysis, engineering problems and mathematical physics (see Brézis [13], Pascali and Sburlan [13]). Further, monotone operators (multifunctions) are found in Deimling [24], Cesari [21], Zeidler [47] and Vainberg [43]. Kenderov [30] proved that monotone operators are almost everywhere single-valued. The application of monotone mappings in the setting of the infinite-dimensional spaces to integral equations and differential equations would be found in Kachurovskii [28], in the book of Brézis [13] and Browder [18]. The more recent work on monotone operators are found in Zeidler [48]. In the setting of infinite-dimensional spaces, mappings are usually called operators. But there are many more applications of monotone mappings in the finite dimensional spaces for the numerical optimization (see Chen and Rockafellar [22]).

1.2 Maximal monotone operator

Minty [33] introduced the significance of the maximality of monotone operators and established the maximality of continuous monotone mappings. It has been observed that the notion of monotonicity has huge application in nonlinear analysis, especially in convex analysis. Because the convexity of a proper, lower semi-continuous function can be characterized by monotonicity of its subdifferentials (see [23, 40]). The monotonicity of the subdifferential of a proper closed convex function is studied by Minty in [35]. Further, while studying proximal operator on Hilbert spaces, Rockafellar [4] and Moreau independently developed a tool subgradient which serves as derivative for a convex function. In this sequel, Moreau [37] studied the maximal monotonicity of subdifferentials of any proper lower semi-continuous convex functions. The proof of maximality of subdifferentials of any proper closed convex functions was extended to any general Banach spaces by Rockafellar [41]. Also, he introduced the notion of cyclic monotonicity [40] and proved that maximal cyclically monotone operators are maximal monotone operators and have a subdifferential operator. The proof of maximality of subgradient was a little bit cumbersome. Recently, an easy proof using the basic tools of convex analysis is presented by Alves and Svaiter [7, Chapter 9].

1.3 Application

During the first decades, the concept of monotone operators was applied to many branches of mathematics such as differential equations, economics, engineering, management science, probability theory and other applied sciences. Monotone operators which have no proper monotone extensions are called maximal monotone operators. Maximal monotone operators appear in several branches of applied mathematics, such as optimization, partial differential equations and variational analysis. In particular, maximal monotone operators are applied to study the existence of Eigen vectors of the second-order nonlinear elliptic equation in Sobolev spaces [29]. After the introduction of the monotone operator by Minty to aid the abstract study of electrical networks [31], the foundation of this modern operator theory was established by Minty, Browder and his school (Brézis, Hess [19], Asplund [3], Rockafellar, Zarantonello [45, 46]).

2 Preliminaries and Notion

2.1 Space

we suppose that X is a real Banach space with norm, $\|.\|$ and X^* is the continuous dual of X. X and X^* are paired by $\langle x, x^* \rangle = x^*(x)$, for $x \in X$ and $x^* \in X^*$. If necessary, we identify $X \subset X^{**}$ with its image under the canonical embedding of X into X^{**}. A sequence (x_n) is said to be converge weakly to $x \in X$ if $x^*(x_n) \to x^*(x)$, $\forall x^* \in X^*$. Similarly, a sequence $x_n^* \in X^*$ is said to be weak star convergence to x^*, if $x_n^*(x) \to x^*(x)$, $\forall x \in X$. Weak and weak star convergence are denoted by the notations $\overset{w}{\to}$ and $\overset{w^*}{\to}$, respectively.

For a given subset C of X we denote the interior of C as int$C$, the closure of C as $\overline{C}$ and the boundary of C as bdry C. conv$C$, aff$C$ is the convex and affine hull of C. For $0 \in \text{Core}C$ if and only if $\bigcup_{\lambda>0} \lambda C = X$. For any $C, D \subseteq X$, $C - D = \{x - y |\ x \in C,\ y \in D\}$. For $x, y \in X$, we denote $[x, y] := \{tx + (1-t)y |\quad 0 \leq t \leq 1\}$ and star or center of C as $\text{star}C := \{x \in C |\ \ [x, c] \subseteq C, \quad \forall c \in C\}$.

2.2 Monotone Operators

Let $A : X \rightrightarrows X^*$ be a set-valued operator (also known as multifunction or point-to-set mapping) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$. Domain of A is denoted as dom$A := \{x \in X |\ Ax \neq \phi\}$ and range of A is ran$A = \{x^* \in Ax | \ x \in \text{dom}A\}$. Graph of A is denoted as gra$A = \{(x, x^*) \in X \times X^* | \ x^* \in Ax\}$. A is said to be linear relation if gra$A$ is a linear subspace.

Definition 2.1 (Monotone Operator) *A set-valued mapping $A : X \rightrightarrows X^*$ is said to be monotone if*

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in graA.$$

Example 2.2 *A function $f : \mathbb{R} \to \mathbb{R}$ defines a monotone operator if and only if f is a monotonic increasing function in the usual sense: that is,*

$$[f(x_2) - f(x_1)]\dot{(}x_2 - x_1) \geq 0, \ \forall x_2, x_1 \in \mathbb{R} \ \textit{ iff } \ f(x_2) \geq f(x_1) \ \textit{ whenever } \ x_2 > x_1.$$

The following example provides an example of a set-valued monotone operator.

Example 2.3 *Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$ be a multifunction defined by*

$$f(x) = \begin{cases} 0, & \textit{if } \ x < 0 \\ 1 & \textit{if } \ x > 0 \\ [0, 1] & \textit{if } \ x = 0 \end{cases}.$$

Example 2.4 *Let $f : X \to]-\infty, +\infty]$ be any proper convex function, then the subdifferential operator of f is defined as $\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* | \ \langle y - x, x^* \rangle + f(x) \leq f(y), \forall y \in X\}$ is a monotone operator.*

If $x^* \in \partial f(x), y^* \in \partial f(y)$ then $\langle x^*, y - x \rangle \leq f(y) - f(x)$ and $-\langle y^*, y - x \rangle = \langle y^*, x - y \rangle \leq f(x) - f(y)$. By adding these two inequalities,

$$\langle y^* - x^*, y - x \rangle \geq 0.$$

Definition 2.5 *Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$ we say that (x, x^*) is monotonically related to graA if*

$$\langle x - y, x^* - y^* \rangle \geq 0, \ \ \forall (y, y^*) \in graA.$$

And a set-valued mapping A is said to be maximal monotone if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). In the other words, A is maximal monotone if for any $(x, x^*) \in X \times X^*$ is monotonically related to graA then $(x, x^*) \in graA$.

Example 2.3 gives an example of a set-valued maximal monotone operator. For maximality of the subdifferential operator in Example 2.4, one may refer [41].

Definition 2.6 *We say that A is of type (FPV) if for every open set $U \subseteq X$ such that $U \cap domA \neq \phi$, $x \in U$ and (x, x^*) is monotonically related to $graA \cap (U \times X^*)$, then $(x, x^*) \in\ graA$.*

All monotone operators of type (FPV) are maximal monotone operators [42]. However, converse is a still conjecture one. Subdifferential operators are of type (FPV) [42]. For every $x \in X$, the normal cone operator at x is defined by $N_C(x) = \{x^* \in X^* |\ \sup_{c \in C}\langle c - x, x^*\rangle \leq 0\}$, if $x \in C$; and $N_C(x) = \phi$, if $x \notin C$. Also, it may be verified that the normal cone operator is of type (FPV) [42].

3 The Sum Problem

The most famous and significant problem in monotone operator theory is the "Sum problem", that is: Let A and B be two maximal monotone operators. Is $A+B$ maximal monotone? Many solutions are available by giving various conditions to the operators as well as to the underlying space. Here we have provided an example for better understanding. Let C and D be two closed disks in $\mathbb{R}^2$ such that C and D intersect at a single point p. Then N_C and N_D are maximally monotone operators. But,

$$(N_C + N_D)(p) = N_C(p) + N_D(p)$$

is a proper subset of $\mathbb{R}^2$. Thus, $gra(N_C + N_D)$ is a proper subset of $gra(N_{\{p\}}) = \{p\} \times \mathbb{R}^2$. Therefore, $N_C + N_D$ is not maximal monotone (see [42]).

Hence, Rockafellar proposed a condition as dom $A \cap$ int dom $B \neq \phi$ called as Rockafellar's constraint qualification [41] and proved that the sum $A+B$ is maximal monotone in the setting of reflexive Banach spaces. This theorem is called as Rockafellar's Sum theorem. He posed the question"Is Rockafellar's constraint condition (dom $A \cap$ int dom $B \neq \phi$) sufficient for maximality of the sum $A + B$ in any general Banach spaces?" The problem is posed as follows:

Let X be a non-reflexive Banach space. Suppose that $A, B : X \rightrightarrows X^*$ are maximal monotone operators with $domA \cap \text{int dom } B \neq \phi$. Is $A+B$ necessarily maximal monotone?

In [41], Rockafellar proved that the maximality of the sum of two maximal monotone operators, namely, Rockafellar's Sum theorem in reflexive Banach spaces. After the introduction of Fitzpatrick function, most of the proof of the results on maximal monotone operators are simpler than earlier proof, and the proof of Rockafellar's Sum theorem [41] becomes simple.

The Sum problem was studied by many researchers like Borwein, Bauschke, Wang, Voisei, Liangjin etc. They have tried to prove the sum problem by giving different conditions within the real Banach spaces setting. The main impetus in the Sum theorem remains to prove it in general real Banach spaces, which is the last huddle to overcome. In [6], Borwein provides a partial answer to the Sum problem by assuming that A and B are maximal monotone operators with int dom$A \cap$ int dom$B \neq \phi$.

Theorem 3.1 *[6] Let $A, B : X \rightrightarrows X^*$ be maximal monotone. Suppose that* $\operatorname{int}\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \neq \phi$. *Then $A + B$ is maximal monotone.*

In [5], Bauschke, Wang and Yao prove that the sum of maximal monotone linear relation and the subdifferential operator of a sub-linear function with Rockafellar's constraint qualification is maximal monotone in general Banach spaces. Yao [44] provide a partial answer to the Sum problem by extending the results of [5] to the subdifferential operator of any proper lower semi-continuous convex function. The result is stated as follows:

Theorem 3.2 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation and let $f : X \to]-\infty, +\infty]$ be a proper lower semi-continuous convex function with* $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} \partial f \neq \phi$. *Then $A + \partial f$ is maximally monotone.*

Further, Borwein and Yao [8] generalized the result (Theorem 3.2) to any arbitrary maximal monotone operator, i.e.,

Theorem 3.3 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $B : X \rightrightarrows X^*$ be maximally monotone. Suppose that* $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \neq \phi$. *Then $A + B$ is maximally monotone.*

Finally, Borwein and Yao [9] provide another partial answer to the Sum problem by relaxing the linearity from the result of [8] and prove the maximality of $A + B$ with the conditions that A and B are maximal monotone operators, star(dom$A$) $\cap$ int dom$B \neq \phi$ and A is of type (FPV). Formally, we state the result in the following.

Theorem 3.4 *[9] Let $A, B : X \rightrightarrows X^*$ be maximally monotone. Assume that A is of type (FPV) Suppose that* $\operatorname{star}(\operatorname{dom} A) \cap \operatorname{int}\operatorname{dom} B \neq \phi$. *Then $A + B$ is maximally monotone.*

Also, in [9], raises a question for further research on relaxing 'star shaped' hypothesis on dom A.

4 Steps needed to solve Sum problem

One may take the following two steps to solve the stated problem.

- To relax the 'star shaped' condition in [9].
- It is known that the closure of the domain of type (FPV) is convex. However, if one proves that the domain of type (FPV) is convex, then 'star shaped' is not required to be relaxed.
- To prove that all the maximal monotone operators are of type (FPV).

References

[1] A. R. Aftabizadeh and N. H. Pavel. Nonlinear boundary value problems for some ordinary and partial differential equations associated with monotone operators. *J. Math. Anal.*, 156:535 – 557, 1991.

[2] N. C. Apreutesei. Second order differential equations on half-line associated with monotone opertors. *J. Math. Anal.*, 223:472 – 493, 1998.

[3] E. Asplund. A monotone convergence theorem for sequences of nonlinear mappings. *Proc. Symp. Pure Math.*, 18:1 – 9, 1970.

[4] E. Asplund and R. T. Rockafellar. Gradients of convex functions. *Trans. Am. Math. Soc.*, 139:443 – 467, 1969.

[5] H. H. Bauschke, X. Wang, and L. Yao. On the maximal monotonicity of the sum of a maximal monotone linear relation and the subdifferential operator of a sublinear function. *Proceedings of the Haifa Workshop on Optimization Theory and Related Topics. Contemp. Math., Amer. Math. Soc.*, 568:19 – 26, 2012.

[6] J. M. Borwein. Maximality of sums of two maximal monotone operators in general banach space. *P. Am. Math. Soc.*, 135:3917 – 3924, 2007.

[7] J. M. Borwein and J. D. Vanderwerff. *Convex functions, Constructions, Characterizations and Counter examples, Encyclopedia of Mathematics and Applications.* Cambridge University Press, UK, 2010.

[8] J. M. Borwein and L. Yao. Maximality of the sum of a maximally monotone linear relation and a maximally monotone operator. *Set-Valued Var Anal.*, 21:603 – 616, 2013.

[9] J.M. Borwein and L. Yao. Sum theorems for maximally monotone operators of type (fpv). *J. Aust. Math. Soc.*, 97:1 – 26, 2014.

[10] H. Brézis. Monotone operators, nonlinear semigroups, and applications. *Proceedings of the International Congress Mathematicians. Vancouver*, 1974.

[11] H. Brézis, M. G. Crandall, and A. Pazy. Perturbations of nonlinear maximal monotone sets in banach spaces. *Communications on pure and applied mathematics*, XXIII:123 – 144, 1970.

[12] H. Brézis and A. Haraux. Image d'une somme d'opérateurs monotones et applications. *Israel J. Math.*, 23:165 – 186, 1976.

[13] H. Brezis. *Operateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert.* North Holland, Amsterdam, 1973.

[14] F. E. Browder. Continuity properties of monotone nonlinear operators in banach spaces. *Bull. of Am. Math. Soc.*, 70:551 – 553, 1964.

[15] F. E. Browder. Multivalued monotone nonlinear mappings. *Trans. AMS*, 118:338 – 551, 1965.

[16] F. E. Browder. The fixed point theory of multi-valued mappings in topological vector spaces. *Math. Ann.*, 177:283 – 301, 1968.

[17] F. E. Browder. Nonlinear maximal monotone mappings in banach spaces. *Math. Ann.*, 175:89 – 113, 1968.

[18] F. E. Browder. Continuity properties of monotone nonlinear operators in banach spaces. *Proceeding of Symposia in Pure Mathematics, American Mathematical Society, Providence, Rhode Island,*, 18, 1976.

[19] F. E. Browder and P. Hess. Nonlinear mappings of monotone type in banach space. *J. Funct. Anal.*, 11:251 – 294, 1972.

[20] B. D. Calvert and C. P. Gupta. Nonlinear elliptic boundary value problems in l^p spaces and sums of ranges of accretive operators. *Nonlinear Analysis: Theory Methods and Applications.*, 2:1 – 26, 1978.

[21] L. Cesari. *Optimization-Theory and Applications: Problems with Ordinary Differential Equations.* Springer-Verlag, New-York, 1983.

[22] G. H.-G Chen and R. T. Rockafellar. The set-valued monotone mappings are almost every where single-valued. *SIAM Journal on Optimization*, 7, 1997.

[23] Jofre-A. Correa, R. and L. Thibault. Characterization of lower semi continuous convex functions. *Proc. AMS.*, 116:67 – 72, 1992.

[24] K. Demiling. *Nonlinear Functional Analysis.* Springer-Verlag, Berlin, 1973.

[25] C. P. Gupta. *Sum of ranges of operators and applications.* Academic Press, New York, 1997.

[26] C. P. Gupta and P. Hess. Existence theorems for nonlinear noncoercieve operator equations and nonlinear elliptic boundary value problems. *J. Differential Equations*, 22:305 – 313, 1976.

[27] R. I. Kachurovskii. On monotone operators and convex functionals. *Uspekhi Mathemacheskays Nauk*, 15(4), 1960.

[28] R. I. Kachurovskii. Nonlinear monotone operators in banach space. *Russian Mathematical Surveys*, 23(2), 1968.

[29] A. G. Kartsatos and I. V. Skrypnik. On the eigen value problem for perturbed nonlinear maximal monotone operators in reflexive banach spaces. *Trans. Am. Math. Soc.*, 358:3851 – 3881, 2006.

[30] P. S. Kenderov. The set-valued monotone mappings are almost every where single-valued. *Bulgarskaia Akademiia na Naukite Sofia, Doklady*, 27(9), 1974.

[31] G. J. Minty. Monotone networks. *Proc. Roy Soc. London.*, 257:194 – 212, 1960.

[32] G. J. Minty. On the maximal domain of a monotone function. *Mich. Math. J.*, 14:135 – 137, 1961.

[33] G. J. Minty. Monotone (nonlinear) operators in hilbert space. *Duke Math. J.*, 29:341 – 346, 1962.

[34] G. J. Minty. Nonlinear mappings of monotone type in banach space. *Duke Math. J.*, 29:341 – 346, 1962.

[35] G. J. Minty. On the monotonicity of the gradient of a convex function. *Pacific J. Math.*, 14:243 – 247, 1964.

[36] G. J. Minty. On some aspects of theory of monotone operator in theory and application of monotone operators. *Odersi Gubbio*, 1969.

[37] J.-J. Moreau. Proximite et dualite dans un espace hilbertien. *Bulletin de la Societe Mathematique de France*, 93:273 – 299, 1965.

[38] R. T. Rockafellar. Characterization of the subdifferentials of convex functionals. *Pacific J. Math.*, 17(3):497 – 510, 1966.

[39] R. T. Rockafellar. Local boundedness of nonlinear, monotone operators. *Mich. Math. J.*, 16:397 – 407, 1969.

[40] R. T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pacific J. Math.*, 33:209 – 216, 1970.

[41] R. T. Rockafellar. On the maximality of sums of nonlinear monotone operators. *T. Am. Math. Soc.*, 149:75 – 88, 1970.

[42] S. Simons. *From Hahn-Banach to Monotonicity*. Springer-Verlag, 2008.

[43] M. M. Vainberg. *Variational Method and Method of Monotone Operators*. Halsted Press/John Wiley, Jerusalem, 1973.

[44] L. Yao. The sum of a maximally monotone linear relation and the subdifferential of a proper lower semicontinuous convex function is maximally monotone. *Set-Valued Var. Anal.*, 20:155 – 167, 2012.

[45] E. H. Zarantonello. *Projections on Convex Sets in Hilbert Spaces and Spectral Theory I Projections on Convex Sets In: Contributions to Nonlinear Functional Analysis*. Academic Press, Dublin, 1971.

[46] E. H. Zarantonello. Dense single-valuedness of monotone operators. *Israel J. Math.*, 15:158 – 166, 1973.

[47] E. Zeidler. *Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization*. Springer-Verlag, New-York, 1985.

[48] E. Zeidler. *Nonlinear Functional Analysis and Its Applications: Part 2 A: Linear Monotone Operators and Part 2 B: Nonlinear Monotone Operators*. Springer-Verlag, 1990.